EE 231A | Handout # 19 Problem Set 8
Information Theory | Tuesday May 19, 2020
Instructor: Rick Wesel | Due: Tuesday, May 26, 2020

100 pts
Reading: Chapter 15 through section 15.6

Lecture 16: Multiple Access Channels

1. (10 pts)

   Unusual multiple access channel. Consider the following multiple access channel: $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0, 1\}$. If $(X_1, X_2) = (0, 0)$, then $Y = 0$. If $(X_1, X_2) = (0, 1)$, then $Y = 1$. If $(X_1, X_2) = (1, 0)$, then $Y = 1$. If $(X_1, X_2) = (1, 1)$, then $Y = 0$ with probability $\frac{1}{2}$ and $Y = 1$ with probability $\frac{1}{2}$.

   (a) Show that the rate pairs (1,0) and (0,1) are achievable.

   (b) Show that for any non-degenerate distribution $p(x_1)p(x_2)$, we have $I(X_1, X_2; Y) < 1$.

   (c) Argue that there are points in the capacity region of this multiple access channel that can only be achieved by timesharing, i.e., there exist achievable rate pairs (R_1, R_2) which lie in the capacity region for the channel but not in the region defined by

   $$\begin{aligned} R_1 &\leq I(X_1; Y|X_2), && (1) \\ R_2 &\leq I(X_2; Y|X_1), && (2) \\ R_1 + R_2 &\leq I(X_1, X_2; Y) && (3) \end{aligned}$$

   for any product distribution $p(x_1)p(x_2)$. Hence the operation of convexification strictly enlarges the capacity region. This channel was introduced independently by Csiszár and Körner[2] and Bierbaum and Wallmeier[1].

2. (10 pts) *Multiple Access on an Adder Channel.*

   Consider the two-user adder channel $Y = X_1 + X_2$ with the following alphabets for X_1 and X_2: $\mathcal{X}_1 = \{0, 1\}$, $\mathcal{X}_2 = \{1, 2, 3\}$. Note that the two alphabets are different. In this problem you will find and sketch the capacity region for this multiple access channel.

   (a) (5 pts) For a fixed value of $p = P(X_1 = 1)$, find the set of equations that decribe the associated "pentagon". *Hint:* This pentagon is actually rectangle. What distribution on X_2 maximizes the area of this pentagon?

   (b) (4 pts) Find the achievable region of this multiple access channel.

   (c) (1 pt) In this case, does the convex hull operation increase the achievable region beyond the union of all pentagons?

3. (10 pts) *Multiple Access for a Modulo Addition Channel.*

   Consider the two-user modulo-4 adder channel $Y = X_1 + X_2$ with the following alphabets for X_1 and X_2: $\mathcal{X}_1 = \{0, 1\}$, $\mathcal{X}_2 = \{0, 1, 2, 3\}$. Define $p = P(X_1 = 1)$. Note that addition is modulo-4 so that, for example, $2 + 3 = 1$.

   Find AND DRAW the achievable rate region for this multiple access channel.

4. (10 pts) *Multiple Access for Binary Adder Channel.*

   Consider the two-user adder channel $Y = X_1 + X_2$ with the following alphabets for X_1 and X_2: $\mathcal{X}_1 = \{0, 1\}$, $\mathcal{X}_2 = \{0, 1\}$. Define $p_1 = P(X_1 = 1)$ and $p_2 = P(X_2 = 1)$.

   (a) (4 pts) For fixed, specified p_1 and p_2 find the upper bounds on R_1 and R_2 used for constructing a pentagon of the MAC capacity region. The rates R_1 and R_2 are associated with X_1 and X_2 respectively.

   (b) (3 pts) Use the grouping axiom (perhaps multiple times)to show that

   $$H(p_1) + H(p_2) = H\Big(p_1(1-p_2),\ p_1p_2,\ (1-p_1)(1-p_2),\ (1-p_1)p_2\Big). \qquad (4)$$

   Recall the grouping axiom:

   $$H_m(p_1, p_2, \ldots, p_m) = H_{m-1}(p_1+p_2, p_3, \ldots, p_m) + (p_1+p_2)H_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right).$$

   (c) (3 pts) For fixed, specified p_1 and p_2 find the upper bound on the sum R_1+R_2 used for constructing a pentagon. To receive full credit for this part (and to make the next part easy) you need to express your answer in the form $H(p_1)+H(p_2)-\alpha H(q)$ for some α and q.

   Hint To get the correct form use the grouping axium (stated in the previous part) to relate the appropriate three element entropy expression for the upper bound to the four-element entropy (below) that you found equal to $H(p_1)+H(p_2)$ in the previous part:

   $$H(p_1) + H(p_2) = H\Big(p_1(1-p_2), p_1p_2, (1-p_1)(1-p_2), (1-p_1)p_2\Big). \qquad (5)$$

   (d) (3 pts) Sketch the pentagon for $p_1 = p_2 = 0.5.$ and for one other choice of (p_1, p_2). For your second pentagon use the binary entropy function provided on the next page. Does your second pentagon lie inside the first pentagon?

5. (10 pts) *TDMA vs. CDMA*

   Consider the two-user ($m = 2$) multiple access Gaussian channel with users both constrained to have $P = 3N$ where the receiver sees $Y = X_1+X_2+Z$ and $Z \sim \mathcal{N}(0, N)$.

   (a) Plot the achievable rate region for this multiple access channel.

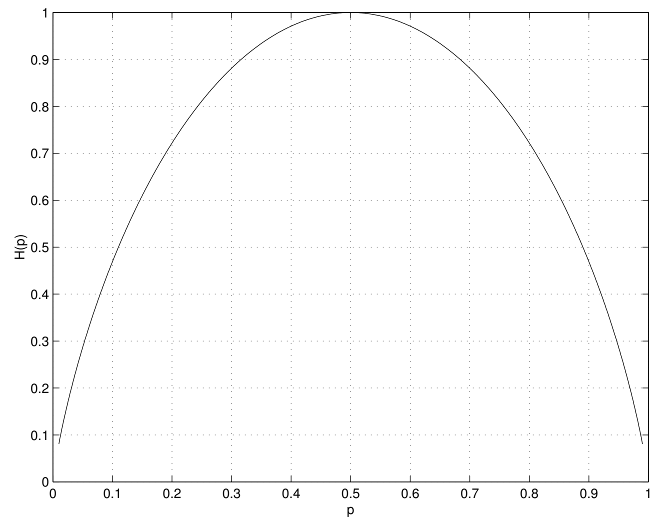

Figure 1: The binary entropy function.

(b) Consider a time-sharing arrangement where λ of the time user one sends on the channel with power $\frac{3N}{\lambda}$ and the other $1-\lambda$ of the time, user two sends on the channel with power $\frac{3N}{1-\lambda}$. Show that both users obey their power constraints and plot the curve of rate pairs achieved by this arrangement on your rate-region from the previous part. You might consider using matlab for this.

(c) When, if ever, does time-sharing achieve optimal pairs (R_1, R_2)?

(d) Suppose you are building a cellular telephone network in which all users will be transmitting the same thing (voice conversations) and you are trying to decide betweem TDMA (essentially the time sharing arrangement we have presented above with $\lambda = 0.5$) and CDMA (a scheme where all users transmit all the time).

An examination of the curves above indicates that either technique should be equally acceptable. However, this particular choice between multiple access schemes has been hotly debated. See if you can make a case for one technique or the other by considering a slightly more complicated model or by considering practical issues. Many answers are acceptable here, do something that interests you.

6. (5 pts) *Noiseless Multiple Access Channel.*

Consider the two-user multiple access channel with no noise so that $Y = f(X_1, X_2)$ and f is a deterministic function.

Show that each pentagon includes the constraint $R_1 + R_2 < H(Y)$.

7. (10 pts) *Slepian-Wolf for deterministically related sources.* Find and sketch the Slepian-Wolf rate region for the simultaneous data compression of (X, Y), where $y = f(x)$ is some deterministic function of x.

8. (10 pts) *Slepian-Wolf.* Let X_i be i.i.d. Bernoulli(p). Let Z_i be i.i.d. $\sim$ Bernoulli(r), and let $\mathbf{Z}$ be independent of $\mathbf{X}$. Finally, let $\mathbf{Y} = \mathbf{X} \oplus \mathbf{Z}$ (mod 2 addition). Let $\mathbf{X}$ be described at rate R_1 and $\mathbf{Y}$ be described at rate R_2. What region of rates allows recovery of $\mathbf{X}, \mathbf{Y}$ with probability of error tending to zero?

9. (10 pts) *Slepian-Wolf for multiplication.*
   Let X_1 be a binary random variable with $P(X_1 = 1) = p_1$. Let Z be uniformly distributed over the four integers $\{1, 2, 3, 4\}$. Let $X_2 = X_1 \times Z$.

   Compute the Slepian-Wolf region of rate pairs that allow the recovery of X_1 and X_2 with probability of error tending to zero as blocklength increases.

   Plot your region for $p_1 = 0.5$. Be clear about indicating the region, not only the boundary.

10. (10 pts) *Slepian-wolf for distributed sensors with erasures.*

    Two sensor nodes each observe the same bit $W \in \{0, 1\}$.

    W is Bernoulli with $P(W = 1) = p$.

    X_i, the obervation of the ith sensor node $i \in \{1, 2\}$ is described by a binary erasure channel with erasure probability α_i. In other words, $X_i = W$ with probability $1 - \alpha_i$ and X_i is an erasure with probability α_i.

    The two sensors need to communicate their observations to a central decision-making node. The goal of this problem is to find the Slepian-Wolf region of rates sufficient for the two nodes to communicate their information to the central decision-making node. Please note that the sensors are trying to communicate THEIR OBSERVATIONS rather than simply communicate W. In other words, each sensor must communicate whether it had an erasure as well as any information it might have about W.

    Find AND DRAW the Slepian-Wolf region of rate pairs that will accomplish this. Your solution must have a form that is in terms of α_1, α_2 and p. However, your sketch should be for the special case where all these parameters are equal to 1/2.

References

[1] M. Bierbaum and H.M. Wallmeier. A note on the capacity region of the multiple access channel. *IEEE Trans. Inform. Theory*, IT-25:484, 1979.

[2] I. Csiszár and J. Körner. *Information Theory: Coding Theorems for Discrete Memoryless Systems.* Academic Press, 1981.