Information Theory
Lecture 16

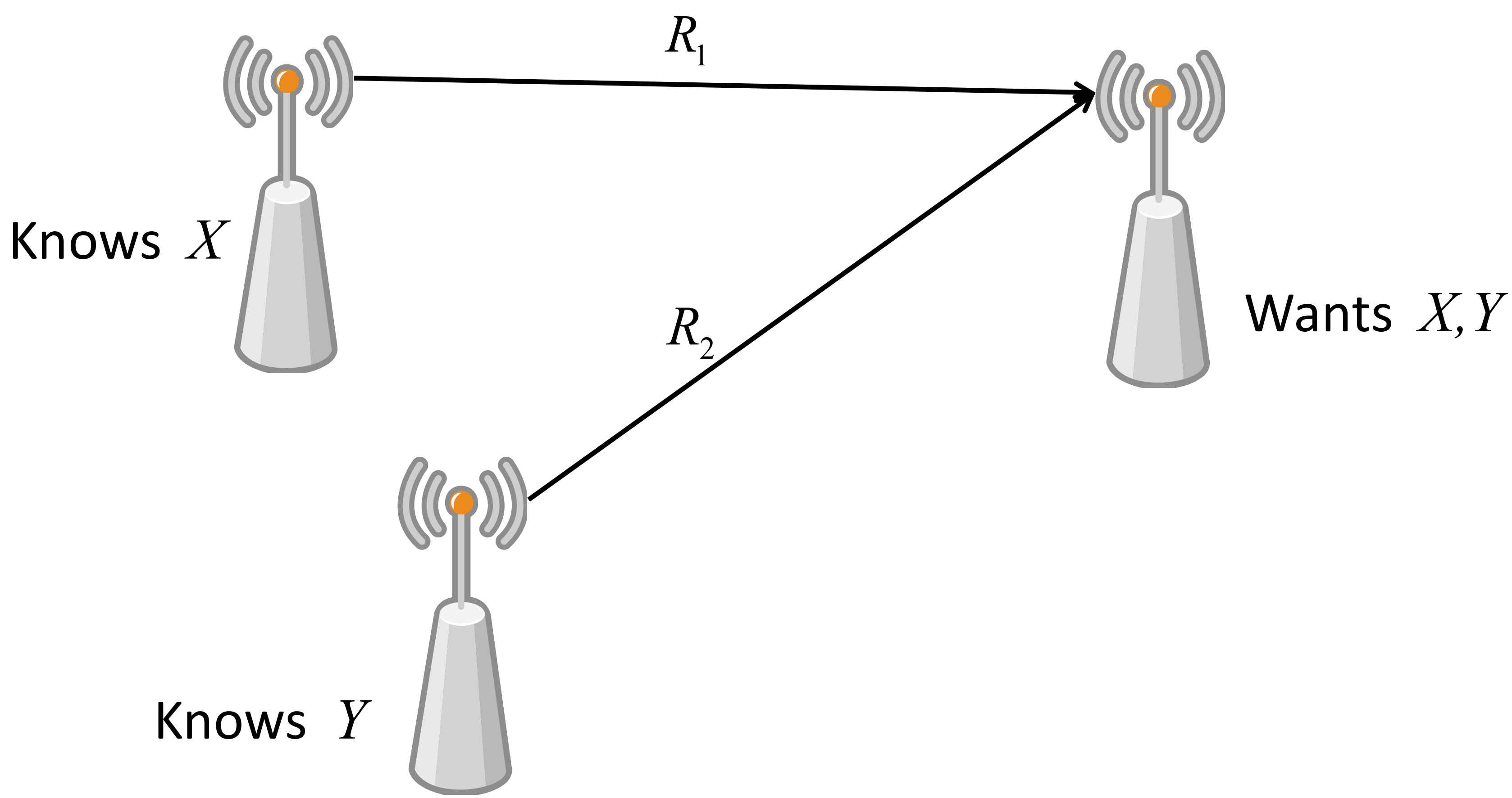
A. Slepian-Wolf Rate Region for Encoding of Correlated Sources and an example

B. Proof of Achievability for Single-Variable Source Coding by Slepian-Wolf

C. Proof of Achievability for Two-Variable Source Coding by Slepian-Wolf

Part 16A:
Slepian-Wolf Rate Region for Encoding of Correlated Sources and an example

Encoding of correlated sources



Encoding of correlated sources (cont.)

- Clearly $R_1 > H(X), R_2 > H(Y)$ will work.

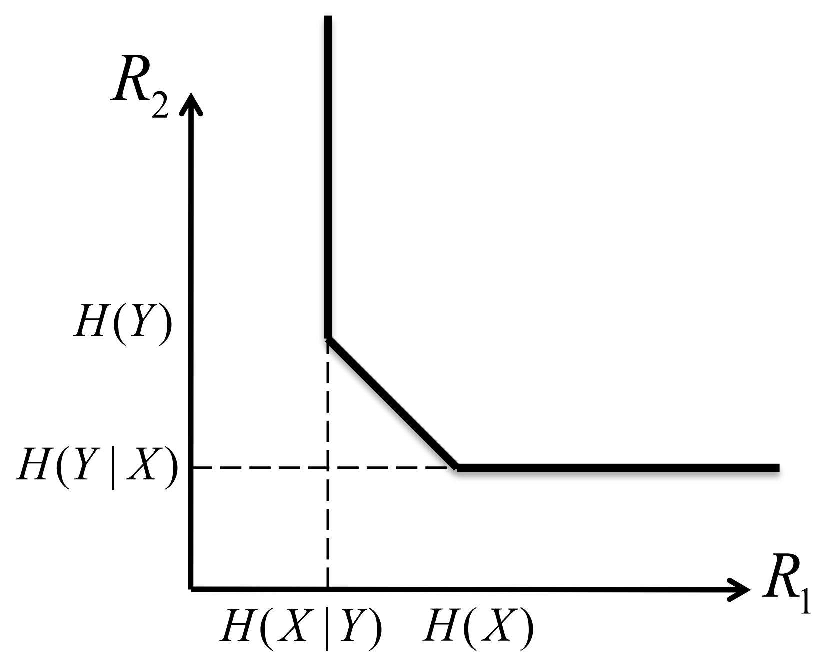
- Can we do better?
  - Yes!

- All we need is

$$R_1 > H(X|Y)$$

$$R_2 > H(Y|X)$$

$$R_1 + R_2 > H(X,Y)$$

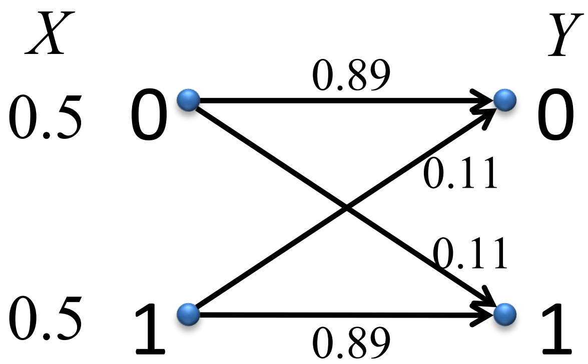


Note

- $$H(X \mid Y) + H(Y \mid X) \le H(X \mid Y) + H(Y)$$
  $$= H(X,Y)$$

- The bound $R_1 + R_2 > H(X,Y)$ is always enough to satisfy the bounds on R_1 and R_2 .

Example: Sending X and Y of a BSC.



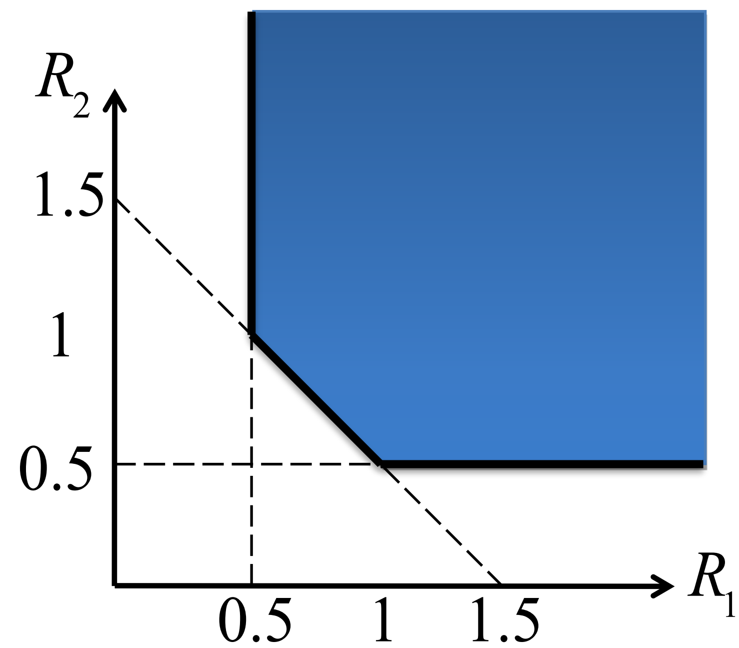
$H(X) = 1$

$H(Y) = 1$

$H(Y \mid X) = H(0.89) = 0.5w$

$H\left(X,Y\right) = H\left(Y \mid X\right) + H\left(X\right) = 1.5$

$H\left(X \mid Y\right) = H\left(X,Y\right) - H\left(Y\right) = 0.5$



- So to encode n (X,Y) pairs we need $2n$ bits by separately encoding X and Y, but we only need $1.5n$ bits if we use Slepian-Wolf.

$R_1 > H(X \mid Y)$

$R_2 > H(Y \mid X)$

$R_1 + R_2 > H(X,Y)$

Part 16B:
Proof of Achievability
for Single-Variable Source Coding
by Slepian-Wolf

Source coding via random index assignment

- A new way to do single-variable source coding

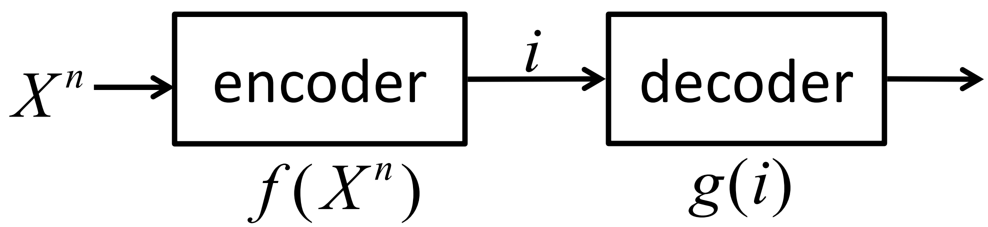
$$X_1, X_2, \ldots, X_n \sim \text{i.i.d. } p(x)$$

- For each possible sequence X^n, randomly select an index $1,\ldots,2^{nR}$.

  – Note: even non-typical X^n's get an index, but indices label multiple X^n's. Hope that each index labels at most one $X^n \in A_\epsilon^{(n)}(x)$.

Typical sequence decoding *with errors*

- Transmitter sends the index. Receiver decodes index to the unique $X^n \in A_\epsilon^{(n)}$ if there is one. Otherwise, an error is declared.



Probability of error

$$E = \left\{ f(\hat{X}^n) = f(X^n) \quad \text{for some } \hat{X}^n \in A_\epsilon^{(n)}(X), \hat{X}^n \neq X^n \right\}$$

$$P\left\{E\right\} = \sum_{\hat{x}^n \in A_\epsilon^{(n)}} P\left(f(\hat{x}^n) = f(x^n)\right)$$

$$= \sum_{\hat{x}^n \in A_\epsilon^{(n)}} 2^{-nR}$$

$$\leq 2^{n\left(H(X)+\epsilon\right)} 2^{-nR}$$

$$= 2^{-n\left(R-H(X)-\epsilon\right)}$$

$$P_e = P\left(\left\{X^n \notin A_\epsilon^{(n)}\right\} \bigcup E\right)$$

$$\leq P\left\{X^n \notin A_\epsilon^{(n)}\right\} + P\left\{E\right\}$$

$$\leq \epsilon + 2^{-n\left(R-H(X)-\epsilon\right)}$$

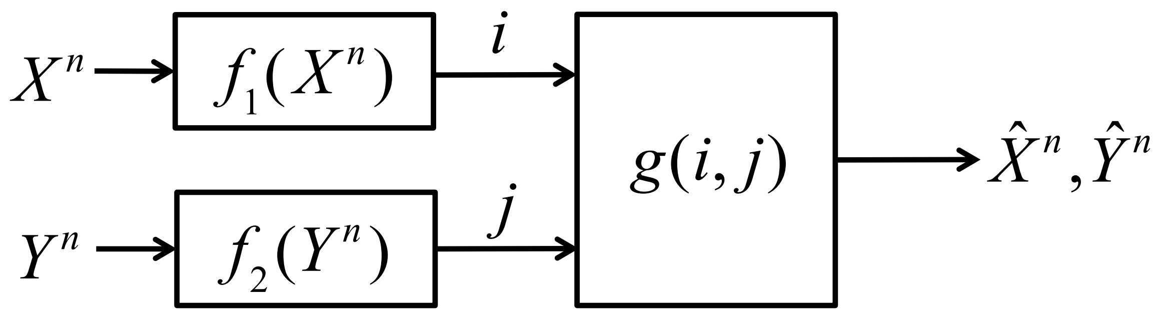
Scheme works asymptotically,
but does not handle non-typical sequences.

Part 16C:
Proof of Achievability for Two-Variable Source Coding by Slepian-Wolf

Random Index Assignment

- Randomly assign X^n's indices i from 2^{nR_1} choices.
- Randomly assign Y^n's indices j from 2^{nR_2} choices.
- X^n is encoded to its index i.
- Y^n is encoded to its index j.

Typical Sequence Decoding



- Decode i,j to the unique X^n, Y^n pair that has index i,j and is jointly typical.

- If two or more pairs are jointly typical, declare an error.

Error Events

$$E_0 = \left\{ X^n, Y^n \notin A_\epsilon^{(n)} \right\}$$

$$E_1 = \left\{ f_1(\hat{X}^n) = f_1(X^n),\ \left(\hat{X}^n, Y^n \right) \in A_\epsilon^{(n)}(X,Y) \right\}$$

$$E_2 = \left\{ f_2(\hat{Y}^n) = f_2(Y^n),\ \left(X^n, \hat{Y}^n \right) \in A_\epsilon^{(n)}(X,Y) \right\}$$

$$E_{12} = \left\{ f_1(\hat{X}^n) = f_1(X^n), f_2(\hat{Y}^n) = f_2(Y^n), \left(\hat{X}^n, \hat{Y}^n \right) \in A_\epsilon^{(n)}(X,Y) \right\}$$

$P(E_1)$

$$E_1 = \left\{ f_1(\hat{X}^n) = f_1(X^n),\ \left(\hat{X}^n, Y^n\right) \in A_\epsilon^{(n)}(X,Y) \right\}$$

$$P\left(E_1\right) = \sum_{y^n \in A_\epsilon^{(n)}(X,Y)} p(y^n) \sum_{\hat{x}^n : (\hat{x}^n, y^n) \in A_\epsilon^{(n)}(X,Y), \hat{x}^n \neq x^n} p\left(f_1(\hat{x}^n) = f_1(x^n)\right)$$

$$\leq \sum_{y^n \in A_\epsilon^{(n)}(X,Y)} 2^{-n\left(H(Y)-\epsilon\right)} \sum_{\hat{x}^n : (\hat{x}^n, y^n) \in A_\epsilon^{(n)}(X,Y), \hat{x}^n \neq x^n} 2^{-nR_1}$$

$$= 2^{-n\left(H(Y)-\epsilon\right)} 2^{-nR_1} \sum_{y^n \in A_\epsilon^{(n)}(X,Y)} \sum_{\hat{x}^n : (\hat{x}^n, y^n) \in A_\epsilon^{(n)}(X,Y), \hat{x}^n \neq x^n} 1$$

$$= 2^{-n\left(H(Y)-\epsilon\right)} 2^{-nR_1} \sum_{(\hat{x}^n, y^n) \in A_\epsilon^{(n)}(X,Y)} 1$$

$$\leq 2^{-n\left(H(Y)-\epsilon\right)} 2^{-nR_1} 2^{n\left(H(X,Y)+\epsilon\right)} = 2^{-n\left(R_1 - H(X|Y) - 2\epsilon\right)}$$

$P(E_{12})$

$$E_{12} = \left\{ f_1(\hat{X}^n) = f_1(X^n), f_2(\hat{Y}^n) = f_2(Y^n), \left(\hat{X}^n, \hat{Y}^n \right) \in A_\epsilon^{(n)}(X,Y) \right\}$$

$$P\left\{ E_{12} \right\} = \sum_{\hat{x}^n \hat{y}^n \in A_\epsilon^{(n)}} P\left(f_1(\hat{x}^n) = f_1(x^n) \right) P\left(f_2(\hat{y}^n) = f_2(y^n) \right)$$

$$= \sum_{\hat{x}^n \hat{y}^n \in A_\epsilon^{(n)}} 2^{-nR_1} 2^{-nR_2}$$

$$\leq 2^{n\left(H(X,Y)+\epsilon\right)} 2^{-nR_1} 2^{-nR_2}$$

$$\leq 2^{-n\left(R_1+R_2-H(X,Y)-\epsilon\right)}$$

Probability Of Error Conclusion

$$P_e = P\left(E_0 \cup E_1 \cup E_2 \cup E_{12}\right)$$
$$\leq P\left(E_0\right) + P\left(E_1\right) + P\left(E_2\right) + P\left(E_{12}\right)$$

$$P\left(E_1\right) \leq 2^{-n\left(R_1 - H(X|Y) - 2\epsilon\right)}$$

$$P\left(E_2\right) \leq 2^{-n\left(R_2 - H(Y|X) - 2\epsilon\right)}$$

$$P\left(E_{12}\right) \leq 2^{-n\left(R_1 + R_2 - H(X,Y) - \epsilon\right)}$$

$$R_1 > H(X \mid Y)$$

$$R_2 > H(Y \mid X)$$

$$R_1 + R_2 > H(X,Y)$$

Slepian-Wolf notation for many sources

$$\left(X_{1,t}, X_{2,t}, \ldots, X_{m,t}\right) \sim \text{i.i.d. } p(x_1, x_2, \ldots, x_m)$$

$$R(S) > H\left(X(S) \mid X(S^c)\right) \quad \text{for all } S \subset \left\{1, 2, \ldots m\right\}$$

$$R(S) = \sum_{i \in S} R_i \qquad X(S) = \left\{X_j : j \in S\right\}$$

$$X(S^c) = \left\{X_j : j \notin S\right\}$$