# ECE 231A Discussion 6

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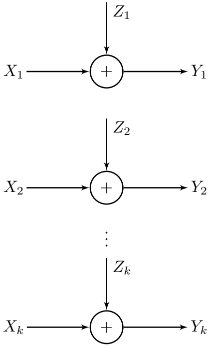
# Parallel Gaussian channels




$\max I(X_1 \cdots X_k ; Y_1 \cdots Y_k)$

s.t. $\sum_{i=1}^{k} P_i \leq P$

**Parallel Gaussian channels**: Assume that there are $k$ parallel Gaussian channels shown above. For channel $j$, $1 \leq j \leq k$,

$$Y_j = X_j + Z_j, \quad Z_j \sim \mathcal{N}(0, N_j)$$

where $Z_j$ is independent from channel to channel.

**Goal**: distribute a total power $P$ among $k$ channels to maximize the capacity.

## Water-filling for parallel Gaussian channels

**Capacity of $k$ parallel Gaussian channels**:

$$C = \max_{f(x_1,\ldots,x_k):\sum_{i=1}^k \mathbb{E}[X_i^2]=P} I(X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_k)$$

$$= \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{(\nu - N_i)^+}{N_i}\right)$$


where $\nu$ is chosen so that $\sum_{i=1}^k (\nu - N_i)^+ = nP$. **(Water-filling process)**

**Proof**: First, we can show that

$I(X_1, \ldots, X_k; Y_1, \ldots, Y_k) \le \sum_{i=1}^k \left(h(Y_i) - h(Z_i)\right) \le \sum_{i=1}^k \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right)$

where $P_i \triangleq \mathbb{E}[X_i^2]$. Next, solving for capacity is equivalent to

$$\max_{P_1,\ldots,P_k} \quad \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right) \qquad J(P_1 \cdots P_k, \lambda)$$

$$= \sum \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right) + \lambda \sum_{i=1}^{k} P_i$$

$$\text{s.t.} \quad \sum_{i=1}^{k} P_i \le P$$

$$P_i \ge 0, \forall i = 1, \ldots, k$$

Differentiating the Lagrangian, we have $\frac{1}{2}\frac{1}{P_i + N_i} + \lambda = 0$ or $P_i = (\nu - N_i)^+$.

# Water-filling algorithm

**Question**: How to efficiently find the "water level" $\nu$?

**Analysis**: Let $N_1 \leq N_2 \leq \cdots \leq N_k$. Define $A \triangleq \{i : P_i \geq 0\}$. If given $A$, $\sum_{i=1}^{k} P_i = P$, then $\sum_{i \in A}(\nu - N_i) = P$. Namely, $\nu = \frac{1}{|A|}(P + \sum_{i \in A} N_i)$.

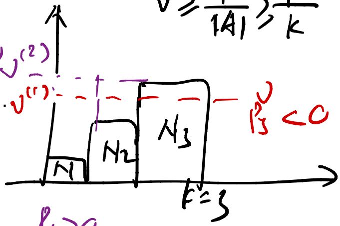
$\nu \geq \frac{P}{|A|} \geq \frac{P}{k}$

Additionally, if $j \in A$, then $\forall i \leq j,\ i \in A$.

$P_j \geq 0$ $\quad P_j = \nu - N_j$

**Water-filling algorithm**: $P_i = \nu - N_i \geq P_j \geq 0$, $\nu^{(2)}$

**Require:** Assume that $N_1 \leq N_2 \leq \cdots \leq N_k$.




Set $A \leftarrow \{1, 2, \ldots, k\}$, $i \leftarrow k$.
$F \leftarrow$ False;
**while** $F =$ False **do**
    $\nu \leftarrow \frac{1}{|A|}\left(P + \sum_{i \in A} N_i\right)$;
    $P_i \leftarrow \nu - N_i$, $i \in A$;
    **if** $P_i \geq 0, \forall i \in A$ **then**
        $F \leftarrow$ True;
    **end if**
    $A \leftarrow A \setminus \{i\}$;
    $i \leftarrow i - 1$;
**end while**

$P_1 > 0$

$P_2 > 0$

# Sufficient statistic

$\theta \to X \to T(X)$

**Sufficient statistic**: $T(X)$ is a sufficient statistic relative to the family $\{f_\theta(x)\}$ if $\theta \to T(X) \to X$ holds.

$I(\theta;X) \geq I(\theta;T(X))$

$I(\theta;X) \leq I(\theta;T(X))$ $\Rightarrow I(\theta;X) = I(\theta;T(X))$

**Corollary**: $I(\theta; X) = I(\theta; T(X))$.

**Gaussian case**: Let $Y_1 = X + Z_1$, $Y_2 = X + Z_2$, where $Z_i \sim \mathcal{N}(0, \sigma^2)$, $Z_1$ and $Z_2$ are independent. Then $T = Y_1 + Y_2$ is a sufficient statistic for $X$.

$Y_1 \sim \mathcal{N}(X, \sigma^2)$ $\quad Y_2 = \mathcal{N}(X, \sigma^2)$

**Proof**: First, $T \sim \mathcal{N}(2X, 2\sigma^2)$. Need to show $f(Y_1, Y_2|T, X) = f(Y_1, Y_2|T)$. Given $X = x$,

$$f(y_1, y_2|x, t) = \frac{f(y_1, y_2, t|x)}{f(t|x)} = \frac{f(y_1, y_2|x) f(t|y_1, y_2, x)}{f(t|x)}$$

$$= \begin{cases} 1 & y_1 + y_2 = t \\ 0 & y_1 + y_2 \neq t \end{cases}$$

$$= \frac{f(y_1|x) f(y_2|x) \delta(y_1 + y_2 - t)}{f(t|x)}$$

$$= \frac{1}{\sqrt{\pi\sigma^2}} \exp\left(-\frac{(y_1 - y_2)^2}{4\sigma^2}\right) \delta(y_1 + y_2 - t).$$

where $f(y_i|x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(y_i - x)^2}{2\sigma^2})$, $i = 1, 2$, $f(t|x) = \frac{1}{\sqrt{4\pi\sigma^2}} \exp(-\frac{(t-2x)^2}{4\sigma^2})$.
Thus, $f(y_1, y_2|x, t)$ does not depend on $x$,

$$f(y_1, y_2|t) = \int_x f(y_1, y_2|x, t) f(x|t)\, \mathrm{d}x = f(y_1, y_2|t, x).$$

$\int f(y_1, y_2, x|t)\, dx$

# HW problem 9: the two-look Gaussian channel

$$X \longrightarrow \boxed{\phantom{xxxxxx}} \longrightarrow (Y_1, Y_2)$$

Consider the ordinary Gaussian channel with two correlated looks at $X$, that is $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1 \qquad Y_1 \sim N(0, P+N)$$
$$Y_2 = X + Z_2 \qquad Y_2 \sim N(0, P+N)$$

with power constraint $P$ on $X$, and $(Z_1, Z_2) \sim \mathcal{N}_2(0, K)$, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix} \qquad (Y_1, Y_2) \sim \begin{bmatrix} P+N & N\rho+P \\ N\rho+P & P+N \end{bmatrix}$$

$$\mathrm{cov}(Y_1, Y_2) = E[Y_1 Y_2] = E[(X+Z_1)(X+Z_2)] = E[X^2 + Z_1 Z_2 + \cancel{XZ_1} + \cancel{XZ_2}] = N\rho + P$$

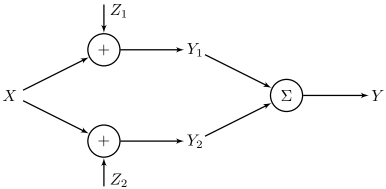
Find the capacity of this channel.

$$I(X; Y_1, Y_2) = h(Y_1, Y_2) - h(Y_1, Y_2 | X)$$
$$= h(Y_1, Y_2) - h(Z_1, Z_2 \cancel{| X})$$

$$C = \max I(X; Y_1, Y_2) = h(Y_1, Y_2) - h(Z_1, Z_2)$$
$$X \sim (0, P) \qquad =$$

# Exercise 1: multipath Gaussian noise channel

Consider a Gaussian noise channel with power constraint $P$, where the signal takes two different paths and the received signals are added together at the antenna.




$Y = Y_1 + Y_2$

$= (X+Z_1) + (X+Z_2)$

$= 2X + (Z_1+Z_2)$

$Z_1 + Z_2$

Find the capacity of this channel if $Z_1$ and $Z_2$ are jointly normal with covariance matrix

$2X \longrightarrow (+) \longrightarrow Y$

$$K_Z = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}.$$

$\mathbb{E}[(2X)^2] = 4\,\mathbb{E}[X^2] = 4P$

$I(X;Y) =$

$\mathrm{Var}(Z_1+Z_2) = \mathbb{E}[(Z_1+Z_2)^2] - \cancel{\mathbb{E}[Z_1]\mathbb{E}[Z_2]}_{=0}$

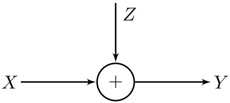
$= \mathbb{E}[Z_1^2 + Z_2^2 + 2Z_1Z_2]$

$= 2\sigma^2 + 2\rho\sigma^2$

$= 2\sigma^2(1+\rho)$

# Exercise 2: a mutual information game

Consider the following channel




In this problem, we shall constrain the signal power and the noise power,

$$\mathbb{E}[X] = 0, \quad \mathbb{E}[X^2] = P,$$
$$\mathbb{E}[Z] = 0, \quad \mathbb{E}[Z^2] = N,$$

and assume that $X$ and $Z$ are independent, the capacity is $I(X; X+Z)$.
Now for the game,

1. the noise player chooses a distribution on $Z$ to minimize $I(X; X+Z)$.
2. the signal player chooses a distribution on $X$ to maximize $I(X; X+Z)$.

Letting $X^* \sim \mathcal{N}(0, P)$, $Z^* \sim \mathcal{N}(0, N)$. Show that Gaussian $X^*$ and $Z^*$ satisfy the saddlepoint conditions,

$$I(X; X+Z^*) \leq I(X^*; X^*+Z^*) \leq I(X^*; X^*+Z).$$

**Hint**: The entropy power inequality: for two $n$-dimensional, independent vectors $X^n, Y^n$,

$$2^{\frac{2}{n}h(X^n+Y^n)} \geq 2^{\frac{2}{n}h(X^n)} + 2^{\frac{2}{n}h(Y^n)}$$



$$2^{2h(X+Z)} \geq 2^{2h(X)} + 2^{2h(Z)}$$

$$I(X; X+Z^*)$$
$$= h(X+Z^*) - h(X+Z^* | X)$$
$$= \underline{h(X+Z^*)} - h(Z^*) \quad E[(X+Z^*)^2] = P+N$$
$$\leq h(X^*+Z^*) - h(Z^*)$$
$$= I(X^*; X^*+Z^*)$$

$$I(X^*; X^*+Z) = h(X^*+Z) - h(X^*+Z | X^*)$$
$$= h(X^*+Z) - h(Z)$$

Recall EPI that $2^{2h(X^*+Z)} \geq 2^{2h(X^*)} + 2^{2h(Z)}$

$$h(X^*+Z) \geq \frac{1}{2}\log\left(2^{2h(X^*)} + 2^{2h(Z)}\right)$$
$$\geq \frac{1}{2}\log\left(2^{2h(X^*)} + 2^{2h(Z)}\right) - h(Z)$$

Write $h(Z) = \frac{1}{2}\log(2\pi e) g(Z)$ (i.e. $g(Z) \triangleq \frac{1}{2\pi e} 2^{2h(Z)}$)

$$h(X^*) = \frac{1}{2}\log(2\pi e) P$$

$$= \frac{1}{2}\log\left((2\pi e)P + (2\pi e)g(Z)\right) - \frac{1}{2}\log(2\pi e)g(Z)$$

$$= \frac{1}{2}\log\left(1+\frac{P}{g(Z)}\right) \geq \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

$$= I(X^*; X^*+Z^*)$$

$1+\frac{P}{g(Z)} \downarrow$ of $g(Z)$

$$\min\left(1+\frac{P}{g(Z)}\right) \Leftrightarrow \max g(Z) = \frac{1}{2\pi e}2^{2h(Z)}$$

$$\Leftrightarrow \max h(Z)$$

$$\Leftrightarrow Z^* \sim N(0, N)$$

$$\Downarrow$$

$$g(Z) = N$$