EE 231A
Information Theory
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Handout # 21 Solution Set 8
Wednesday,May 27, 2020

100 pts
Reading: Chapter 15 through section 15.6

Lecture 16: Multiple Access Channels

1. (10 pts)

   Unusual multiple access channel.

   (a) It is easy to see how we could send 1 bit/transmission from X_1 to Y—simply set $X_2 = 0$. Then $Y = X_1$, and we can send 1 bit/transmission to from sender 1 to the receiver.

   Alternatively, if we evaluate the achievable region for the degenerate product distribution $p(x_1)p(x_2)$ with $p(x_1) = (\frac{1}{2}, \frac{1}{2})$, $p(x_2) = (1, 0)$, we have $I(X_1; Y|X_2) = 1$, $I(X_2; Y|X_1) = 0$, and $I(X_1, X_2; Y) = 1$. Hence the point $(1, 0)$ lies in the achievable region for the multiple access channel corresponding to this product distribution.

   By symmetry, the point $(0, 1)$ also lies in the achievable region.

   (b) Consider any non-degenerate product distribution, and let $p_1 = p(X_1 = 1)$, and let $p_2 = p(X_2 = 1)$. By non-degenerate we mean that $p_1 \neq 0$ or 1, and $p_2 \neq 0$ or 1. In this case, $Y = 0$ when $(X_1, X_2) = (0, 0)$ and half the time when $(X_1, X_2) = (1, 1)$, i.e., with a probability $(1 - p_1)(1 - p_2) + \frac{1}{2}p_1p_2$. $Y_1 = 1$ for the other input pairs, i.e., with a probability $p_1(1 - p_2) + p_2(1 - p_1) + \frac{1}{2}p_1p_2$. We can evaluate the achievable region of the multiple access channel for this product distribution. In particular,

   $$R_1+R_2 \leq I(X_1, X_2; Y) = H(Y)-H(Y|X_1, X_2) = H((1-p_1)(1-p_2)+\frac{1}{2}p_1p_2)-p_1p_2. \tag{1}$$

   Now $H((1 - p_1)(1 - p_2) + \frac{1}{2}p_1p_2) \leq 1$ (entropy of a binary random variable is at most 1) and $p_1p_2 > 0$ for a non-degenerate distribution. Hence $R_1 + R_2$ is strictly less than 1 for any non-degenerate distribution.

   (c) The degenerate distributions have either R_1 or R_2 equal to 0. Hence all the distributions that achieve rate pairs (R_1, R_2) with both rates positive have $R_1+R_2 < 1$. For example the union of the achievable regions over all product distributions does not include the point $(\frac{1}{2}, \frac{1}{2})$. But this point is clearly achievable by timesharing between the points $(1, 0)$ and $(0, 1)$. Or equivalently, the point $(\frac{1}{2}, \frac{1}{2})$ lies in the convex hull of the union of the achievable regions, but not the union itself. So the operation of taking the convex hull has strictly increased the capacity region for this multiple access channel.

2. (10 pts) *Multiplication Multiple Access.*

Consider the two-user multiplication channel $Y = X_1 \times X_2$ with the following alphabets for X_1 and X_2: $\mathcal{X}_1 = \{0, 1\}$, $\mathcal{X}_2 = \{1, 2, 3\}$. Note that the two alphabets are different. Find and sketch the capacity region for this multiple access channel.

Answer: We need to identify the convex hull of the "pentagons" defined by the following equations:

$$R_1 \leq I(X_1 : Y|X_2) \tag{2}$$
$$R_2 \leq I(X_2 : Y|X_1) \tag{3}$$
$$R_1 + R_2 \leq I(X_1, X_2 : Y)\,. \tag{4}$$

With p as the probability that $X_1 = 1$, these mutual informations are computed as follows:

$$I(X_1 : Y|X_2) = H(p) \tag{5}$$
$$I(X_2 : Y|X_1) = pH(X_2) \leq p \log 3 \tag{6}$$
$$I(X_1, X_2 : Y) = I(X_1; Y) + I(X_2; Y|X_1) = H(p) + pH(X_2) \leq H(p) + p \log 3\,. \tag{7}$$

What is most interesting about this problem is that the "pentagons" are actually rectangles because the condition on $R_1 + R_2$ is always satisfied by rate pairs that meet the other two constraints.

The convex hull can be seen by plotting the (R_1, R_2) curve $(H(p), p \log 3)$. This curve is parametric in p and looks like the familiar plot of $H(p)$ turned on its side. All the rectangles lie inside the top half of this curve, which is the convex hull and therefore the rate region of interest. Figure 1 shows the whole situation.

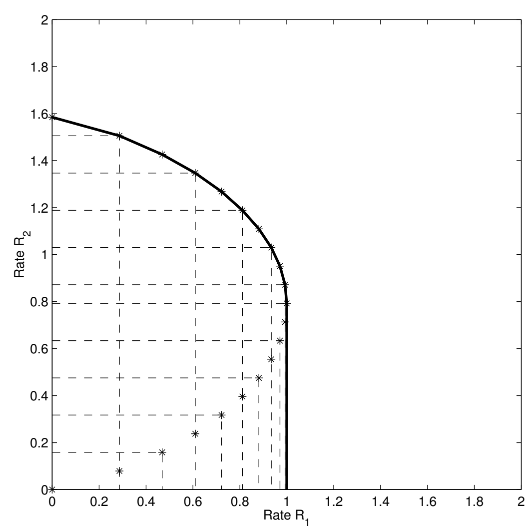


Figure 1: Some of the "pentagons" (really rectangles) and the convex hull that gives the rate region forthis multiple access channel.

3. (10 pts) *Multiple Access for a Modulo Addition Channel.*

Consider the two-user modulo-4 adder channel $Y = X_1 + X_2$ with the following alphabets for X_1 and X_2: $\mathcal{X}_1 = \{0, 1\}$, $\mathcal{X}_2 = \{0, 1, 2, 3\}$. Define $p = P(X_1 = 1)$. Note that addition is modulo-4 so that, for example, $2 + 3 = 1$.

Find AND DRAW the achievable rate region for this multiple access channel.

$$\begin{aligned}
R_1 &\le I(X_1; Y|X_2) = H(p) \le 1 \quad \text{(achieved by } p = 1/2\text{).} \\
R_2 &\le I(X_2; Y|X_1) \le H(X_2) \le 2 \quad \text{(achieved by } X_2 \text{ uniform over } \{0,1,2,3\}\text{.)} \\
R_1 + R_2 &\le I(X_1, X_2; Y) \\
&= I(X_2; Y) + I(X_1; Y|X_2) \\
&\le \big(2 - H(p)\big) + H(p) \quad \text{(achieved by } X_2 \text{ uniform over } \{0,1,2,3\}\text{.)} \\
&= 2
\end{aligned}$$

Where $I(X_2; Y) \le 2 - H(p)$ because the channel between X_2 and Y is simply the 4-key typewriter channel with probability p of choosing the adjacent key. Thus general upper bounds on R_1, R_2 and $R_1 + R_2$ can be simultaneously achieved by choosing $p = 1/2$ and X_2 uniform over $\{0, 1, 2, 3\}$ with the resulting pentagon

$$\begin{aligned}
R_1 &\le 1 \\
R_2 &\le 2 \\
R_1 + R_2 &\le 2\,,
\end{aligned}$$

which is illustrated in Fig. 2.

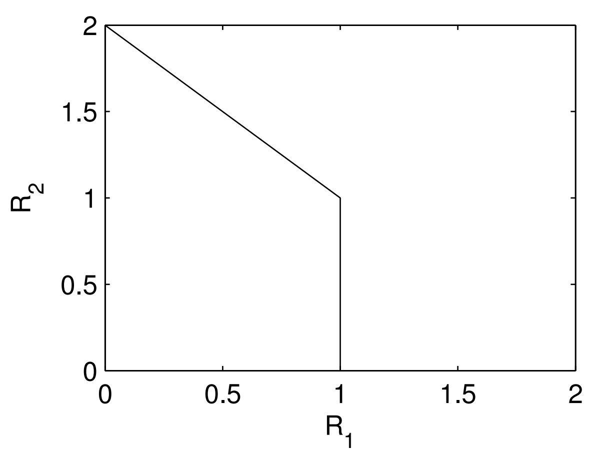


Figure 2: Capacity region for modulo-addition MAC channel.

4. (10 pts) *Multiple Access for Binary Adder Channel.*

Consider the two-user adder channel $Y = X_1 + X_2$ with the following alphabets for X_1 and X_2: $\mathcal{X}_1 = \{0, 1\}$, $\mathcal{X}_2 = \{0, 1\}$. Define $p_1 = P(X_1 = 1)$ and $p_2 = P(X_2 = 1)$.

(a) (4pts) For fixed, specified p_1 and p_2 find the upper bounds on R_1 and R_2 used for constructing a pentagon of the MAC capacity region. The rates R_1 and R_2 are associated with X_1 and X_2 respectively.

Solution:

$$R_1 \leq I(X_1; Y|X_2) = H(X_1) = H(p_1) \tag{8}$$

$$R_2 \leq I(X_2; Y|X_1) = H(X_2) = H(p_2) \tag{9}$$

(b) (3 pts) Use the grouping axiom (perhaps multiple times)to show that

$$H(p_1) + H(p_2) = H\Big(p_1(1-p_2),\ p_1p_2,\ (1-p_1)(1-p_2),\ (1-p_1)p_2\Big). \tag{10}$$

Recall the grouping axiom:

$$H_m(p_1, p_2, \ldots, p_m) = H_{m-1}(p_1+p_2, p_3, \ldots, p_m) + (p_1+p_2)H_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right).$$

Solution:

$$H\Big(p_1(1-p_2),\ p_1p_2,\ (1-p_1)(1-p_2),\ (1-p_1)p_2\Big) \tag{11}$$

$$= H\Big(p_1(1-p_2) + p_1p_2,\ (1-p_1)(1-p_2),\ (1-p_1)p_2\Big) + p_1H(p_2) \tag{12}$$

$$= H\Big(p_1,\ (1-p_1)(1-p_2),\ (1-p_1)p_2\Big) + p_1H(p_2) \tag{13}$$

$$= H\Big(p_1,\ (1-p_1)(1-p_2) + (1-p_1)p_2\Big) + (1-p_1)H(p_2) + p_1H(p_2) \tag{14}$$

$$= H\Big(p_1, (1-p_1)\Big) + (1-p_1)H(p_2) + p_1H(p_2) \tag{15}$$

$$= H(p_1) + H(p_2) \tag{16}$$

(c) (3 pts) For fixed, specified p_1 and p_2 find the upper bound on the sum R_1+R_2 used for constructing a pentagon. To receive full credit for this part (and to make the next part easy) you need to express your answer in the form $H(p_1)+H(p_2)-\alpha H(q)$ for some α and q.

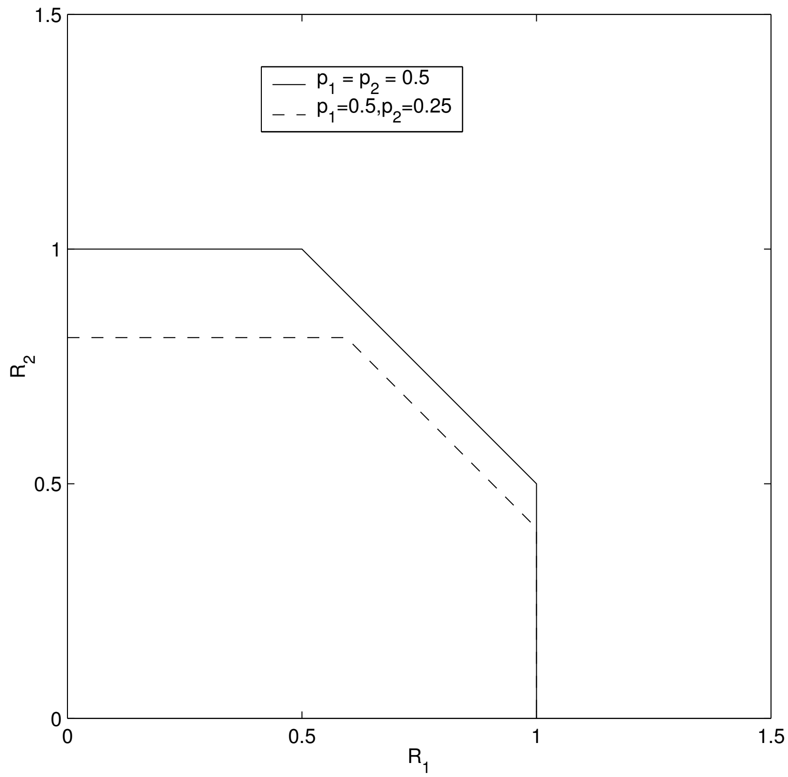
Hint To get the correct form use the grouping axium to relate the appropriate three element entropy expression for the upper bound to the four-element entropy below:

$$H(p_1) + H(p_2) = H\Big(p_1(1-p_2), p_1p_2, (1-p_1)(1-p_2), (1-p_1)p_2\Big). \tag{17}$$

$$\begin{aligned}
R_1 + R_2 &\leq I(X_1, X_2; Y) \\
&= H(Y) - H(Y|X_1, X_2) \\
&= H(Y) \\
&= H\Big(p_1p_2, (1-p_1)p_2 + p_1(1-p_2), (1-p_1)(1-p_2)\Big) \\
&= H\Big(p_1p_2, (1-p_1)p_2, p_1(1-p_2), (1-p_1)(1-p_2)\Big) \\
&\quad - \alpha H\left(\frac{(1-p_1)p_2}{\alpha}, \frac{p_1(1-p_2)}{\alpha}\right) \text{ where } \alpha = (1-p_1)p_2 + p_1(1-p_2) \\
&= H(p_1) + H(p_2) - \alpha H\left(\frac{(1-p_1)p_2}{\alpha}, \frac{p_1(1-p_2)}{\alpha}\right)
\end{aligned}$$

(d) (3 pts) Sketch the pentagon for $p_1 = p_2 = 0.5.$ and for one other choice of (p_1, p_2). For your second pentagon use the binary entropy function provided on the next page. Does your second pentagon lie inside the first pentagon?

Solution: The first pentagon does in fact dominate all the others, but this is rather difficult to show.



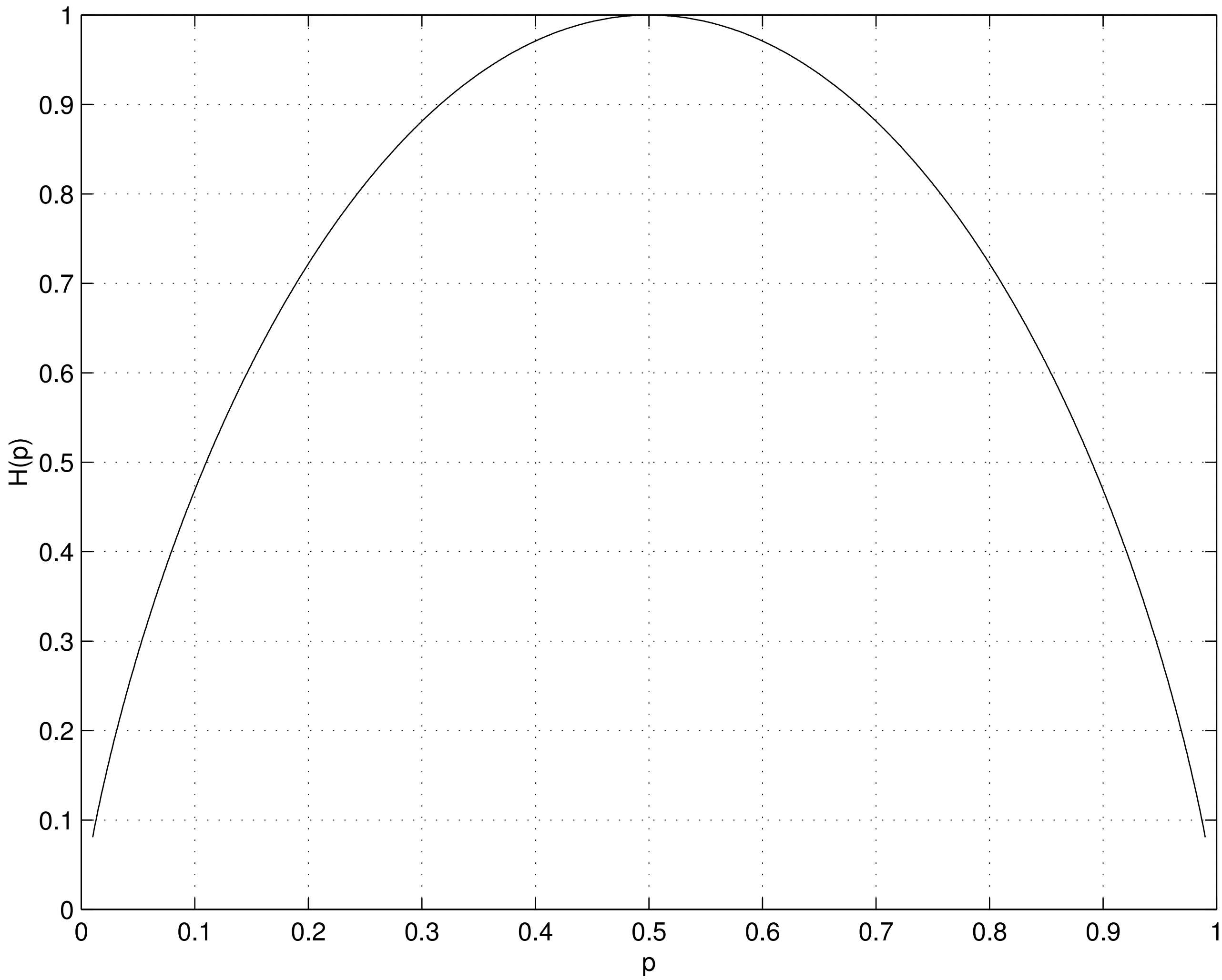
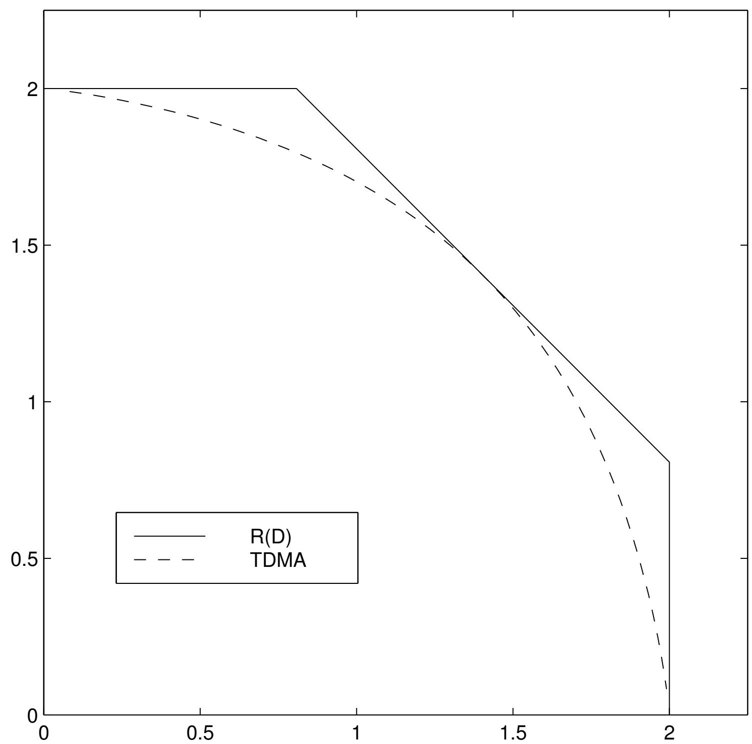


Figure 3: The binary entropy function.

5. (10 pts) *TDMA vs. CDMA*

(a) See Figure below.

(b) See Figure below.



(c) Timesharing achieves optimal rate pairs for $R_1 = 0$, $R_2 = 0$, and $R_1 = R_2 = \log(7)/4$.

(d) Good arguments can be made both ways. With TDMA we can achieve an optimal rate distortion point as long as all users have the same rate and we have a good channel code. However, if user's rates vary (for example, we lower the rate during silent periods), then TDMA won't be optimal. One optimal solution for these situations requires all users to transmit all the time with appropriate codebooks and joint-typical-set decoding. Code division multiple access has all user's transmitting all the time, but the codebooks are not optimal codebooks in the random coding sense. Furthermore, detection schemes that treat all other users as noise are a far cry from the optimal joint decoding required to achieve the optimal rate pairs.

Good orthogonal spreading codes justify treating the other user's as noise since the other users will have small power compared to the desired signal, assuming good power control. Note, however, that if the spreading codes are truly orthogonal then the achievable rate region will be the same as TDMA or FDMA or any other orthogonal decomposition. To reach the optimal non-equal-rate points requires non-orthogonal signaling and joint decoding.

6. (5 pts) *Noiseless Multiple Access Channel.*

Consider the two-user multiple access channel with no noise so that $Y = f(X_1, X_2)$ and f is a deterministic function.

Show that for each pentagon includes the constraint $R_1 + R_2 \leq H(Y)$.

Solution

$$
\begin{aligned}
R_1 + R_2 &\leq I(X_1, X_2; Y) && (18)\\
&= H(Y) - \underbrace{H(Y|X_1, X_2)}_{0} && (19)\\
&= H(Y) && (20)
\end{aligned}
$$

Lecture 17: Slepian-Wolf Encoding

7. (10 pts) *Slepian-Wolf for deterministically related sources.* Find and sketch the Slepian-Wolf rate region for the simultaneous data compression of (X, Y), where $y = f(x)$ is some deterministic function of x. *Slepian Wolf for* $Y = f(X)$.

The quantities defining the Slepian Wolf rate region are $H(X, Y) = H(X)$, $H(Y|X) = 0$ and $H(X|Y) \geq 0$. Hence the rate region is as shown in the Figure 4.

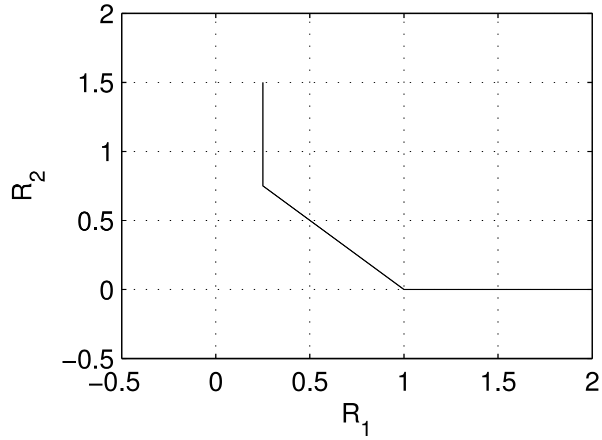


Figure 4: Slepian Wolf rate region for $Y = f(X)$ with $H(x) = 1$ and $H(X|Y) = 0.25$.

8. (10 pts) *Slepian-Wolf.* Let X_i be i.i.d. Bernoulli(p). Let Z_i be i.i.d. $\sim$ Bernoulli(r), and let $\mathbf{Z}$ be independent of $\mathbf{X}$. Finally, let $\mathbf{Y} = \mathbf{X} \oplus \mathbf{Z}$ (mod 2 addition). Let $\mathbf{X}$ be described at rate R_1 and $\mathbf{Y}$ be described at rate R_2. What region of rates allows recovery of $\mathbf{X}, \mathbf{Y}$ with probability of error tending to zero? *Slepian Wolf for binary sources.*

   $X \sim$ Bern(p). $Y = X \oplus Z$, $Z \sim$ Bern(r). Then $Y \sim$ Bern($p*r$), where $p*r = p(1-r)+r(1-p)$. $H(X) = H(p)$. $H(Y) = H(p*r)$, $H(X,Y) = H(X,Z) = H(X) + H(Z) = H(p) + H(r)$. Hence $H(Y|X) = H(r)$ and $H(X|Y) = H(p) + H(r) - H(p*r)$.

   The Slepian Wolf region in this case is shown in Figure 5.

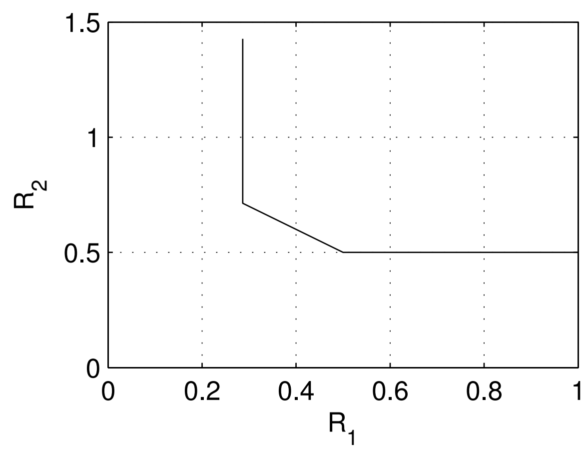


Figure 5: Plot of Slepian-Wolf region boundary for $p = r = 0.89$.

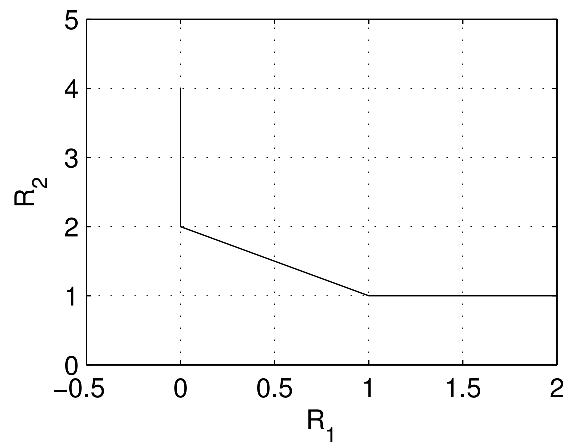
9. (10 pts) *Slepian-Wolf for multiplication.*
   Let X_1 be a binary random variable with $P(X_1 = 1) = p_1$. Let Z be uniformly distributed over the four integers $\{1, 2, 3, 4\}$. Let $X_2 = X_1 \times Z$.

Compute the Slepian-Wolf region of rate pairs that allow the recovery of X_1 and X_2 with probability of error tending to zero as blocklength increases.

Plot your region for $p_1 = 0.5$. Be clear about indicating the region, not only the boundary.

Solution:

$$\begin{aligned} H(X_1|X_2) &= 0 && (21)\\ H(X_2|X_1) &= 2p_1 && (22)\\ H(X_1, X_2) &= H(X_1) + H(X_2|X_1) && (23)\\ &= H(p_1) + 2p_1 && (24) \end{aligned}$$



10. (10 pts) *Slepian-wolf for distributed sensors with erasures.*

Two sensor nodes each observe the same bit $W \in \{0, 1\}$.

W is Bernoulli with $P(W = 1) = p$.

X_i, the obervation of the ith sensor node $i \in \{1, 2\}$ is described by a binary erasure channel with erasure probability α_i. In other words, $X_i = W$ with probability $1 - \alpha_i$ and X_i is an erasure with probability α_i.

The two sensors need to communicate their observations to a central decision-making node. The goal of this problem is to find the Slepian-Wolf region of rates sufficient for the two nodes to communicate their information to the central decision-making node. Please note that the sensors are trying to communicate THEIR OBSERVATIONS rather than simply communicate W. In other words, each sensor must communicate whether it had an erasure as well as any information it might have about W.

Find AND DRAW the Slepian-Wolf region of rate pairs that will accomplish this. Your solution must have a form that is in terms of α_1, α_2 and p. However, your sketch should be for the special case where all these parameters are equal to 1/2.

First we note the entropies of the two sources:

$$H(X_1) = H(\alpha_1) + (1-\alpha_1)H(p) \tag{25}$$
$$H(X_2) = H(\alpha_2) + (1-\alpha_2)H(p) \tag{26}$$

Now, the main computation for this problem is the following:

$$\begin{aligned}
R_1 &\geq H(X_1|X_2) \\
&= \sum_{x_2} H(X_1|X_2 = x_2)P(x_2) \\
&= H(X_1|X_2 = 0)P(X_2 = 0) + H(X_1|X_2 = 1)P(X_2 = 1) + H(X_1|X_2 = e)P(X_2 = e) \\
&= H(\alpha_1)\big((1-\alpha_2)(1-p)\big) + H(\alpha_1)\big((1-\alpha_2)(p)\big) + \big(H(\alpha_1) + (1-\alpha_1)H(p)\big)\alpha_2 \\
&= H(\alpha_1)(1-\alpha_2) + H(\alpha_1)\alpha_2 + (1-\alpha_1)\alpha_2 H(p) \\
&= H(\alpha_1) + (1-\alpha_1)\alpha_2 H(p)
\end{aligned}$$

Similarly, $R_2 \geq H(X_2|X_1) = H(\alpha_2) + (1-\alpha_2)\alpha_1 H(p)$. *Finally,*

$$R_1 + R_2 \geq H(X_1, X_2) \tag{27}$$
$$= H(X_1) + H(X_2|X_1) \tag{28}$$
$$= H(\alpha_1) + (1-\alpha_1)H(p) + H(\alpha_2) + (1-\alpha_2)\alpha_1 H(p) \tag{29}$$
$$= H(\alpha_1) + H(\alpha_2) + (1-\alpha_1)H(p) + (1-\alpha_2)\alpha_1 H(p) \tag{30}$$
$$= H(\alpha_1) + H(\alpha_2) + (1-\alpha_1\alpha_2)H(p)\,. \tag{31}$$

For $\alpha_1 = \alpha_2 = p = 1/2$ *we have*

$$R_1 \geq 1.25 \tag{32}$$
$$R_2 \geq 1.25 \tag{33}$$
$$R_1 + R_2 \geq 2.75\,. \tag{34}$$

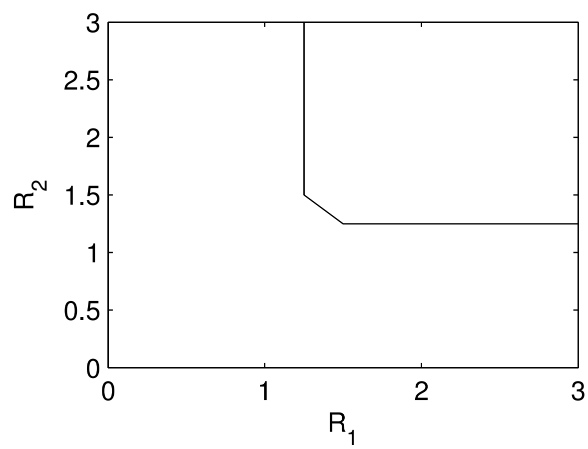


Figure 6: Plot of Slepian-Wolf region boundary for $\alpha_1 = \alpha_2 = p = 1/2$.