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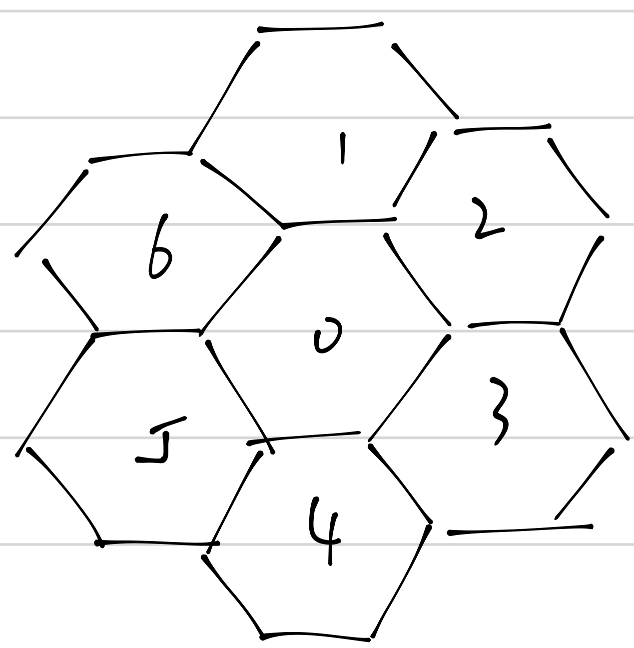
Assessment for part I

1. Harold's Hexagon Hop

what is the entropy rate of Harold's hopping.



There are 7 states, and since Harold always hop to an adjacent hexagon with equal probability, this problem could be solved using weighted graph.

For node "0", it has 6 edges

For node "1", "2", "3", "4", "5", "6", each of them has 3 edges

$\therefore H(X_2 | X_1 = \text{"0"}) = \log_2 6$

$H(X_2 | X_1 \neq \text{"0"}) = \log_2 3$

$2W = \sum_i W_i$ is the sum of all weighted counting each edge twice $\mu_i = W_i / 2W$ is the stationary distribution

$2W = 3\times 6 + 6\times 1 = 24$

For node "0" $\mu = 6/24 = \frac{1}{4}$,

For node $1\sim 6$ $\mu = 3/24 = \frac{1}{8}$

So, the stationary distribution is

$$\mu_0 = \frac{1}{4} \quad \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \frac{1}{8}$$

As we assume that Harold started hopping according to the stationary distribution, and the probability for Harold taking next step is only depended on the current state.

Therefore, this is a stationary Markov process (time_invariant)

The entropy rate:

$$H(\{X_i\}) = \lim_{n\to\infty} H(X_n | X_{n-1}, \dots X_1)$$

$$= \lim_{n\to\infty} H(X_n|X_{n-1}) = \lim_{n\to\infty} H(X_2|X_1)$$

$$= H(X_2|X_1)$$

$$H(X_2|X_1) = \sum_x p(x) H(X_2|X_1 = x)$$

$$= 6\times\frac{1}{8} H(X_2|X_1 \neq \text{"0"}) + \frac{1}{4} H(X_2|X_1 = \text{"0"})$$

$$= 6\times\frac{1}{8}\log_2 3 + \frac{1}{4}\log_2 6 = 1.83496 \text{ bits}$$

Therefore, the entropy rate is 1.183496 bits

2. The Asymptotic composition Property ACP

The sequence x^n is an i.i.d binary sequence with $p(x=1) = 0.75$, we will refer to $p(x=1)$ as the density of x, $d(x)$.

ACP: for any $\varepsilon > 0$ the probability that the fraction of ones in x^n is within ε of $d(x)$ converges to 1 as $n \to \infty$.

Does the sequence x^n satisfy the ACP?

$p(x=1) = 0.75 \quad \therefore \; p(x=0) = 0.25$

Since the random variable x only contains 0, 1

$\therefore$ The number of ones in x^n is:

$$X_1 + X_2 + \cdots + X_n$$

$\therefore$ The fraction of ones in x^n is:

$$\frac{1}{n}(X_1 + X_2 + \cdots + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$ and x^n is i.i.d

According to the weak law of large numbers:

For $n \in N$, $\forall \varepsilon > 0$

$$\lim_{n\to\infty} Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} x_i - E[x]\right| > \varepsilon\right\} = 0$$

$$E[x] = \sum p(x)\cdot x = 1\cdot p(x=1) + 0\cdot p(x=0)$$
$$= p(x=1) = d(x)$$

$$\therefore \lim_{n\to\infty} Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} x_i - d(x)\right| > \varepsilon\right) = 0$$

$$\therefore \lim_{n\to\infty} Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} x_i - d(x)\right| < \varepsilon\right) = 1$$

Therefore, the sequence satisfy the ACP, when $n\to\infty$, $\forall \varepsilon>0$, the probability of the fraction of ones in x^n is within ε of $d(x)$ converges to 1