EE 231A
Information Theory
Instructor: Rick Wesel

Handout #17, Problem Set 7
Tuesday May 12, 2020
Due: Tuesday,May 19, 2020

90 pts
Reading: Chapter 10

Lecture 13A-B: Computing $R(D)$ for a discrete-alphabet source

1. (10 pts) *4-ary Hamming distortion.*
A random variable X uniformly takes on values $\{0, 1, 2, 3\}$. The distortion function is the usual Hamming distortion.

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x} \\ 1 & \text{if } x \neq \hat{x} \end{cases} \tag{1}$$

Compute the rate distortion function $R(D)$ by finding a lower bound on $I(x; \hat{x})$ and showing this lower bound to be achievable. *Hint:* Fano's inequality.

2. (10 pts) *Rate distortion for uniform source with Hamming distortion.* Consider a source X uniformly distributed on the set $\{1, 2, \ldots, m\}$. Find the rate distortion function for this source with Hamming distortion, i.e.,

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x}, \\ 1 & \text{if } x \neq \hat{x}. \end{cases}$$

Hint This is a generalization of the previous problem.

3. (10 pts) *Scaled Hamming distortion.*
A random variable X uniformly takes on values $\{1, 2, 3\}$. The distortion function is the usual Hamming distortion scaled by 2.

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x} \\ 2 & \text{if } x \neq \hat{x} \end{cases} \tag{2}$$

Compute the rate distortion function $R(D)$.

4. (10 pts) *Rate distortion function with infinite distortion.* Find the rate distortion function $R(D) = \min I(X; \hat{X})$ for $X \sim$ Bernoulli $(\frac{1}{2})$ and distortion

$$d(x, \hat{x}) = \begin{cases} 0, & x = \hat{x}, \\ 1, & x = 1, \hat{x} = 0, \\ \infty, & x = 0, \hat{x} = 1. \end{cases}$$

5. (10 pts) *Erasure distortion.* Consider $X \sim$ Bernoulli(1/2), and let the distortion measure be given by the matrix

$$d(x, \hat{x}) = \begin{bmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{bmatrix}.$$

   (a) (6 pts) Calculate the rate distortion function for this source. *Hint* Recall your solution to the previous the infinite distortion problem.

   (b) (4 pts) Suggest a simple scheme to achieve any value of the rate distortion function for this source.

Lecture 13C: Computing $R(D)$ for a continuous-alphabet source

6. (10 pts) *Bounds on the rate distortion function for squared error distortion.* For the case of a continuous random variable X with mean zero and variance σ^2 and squared error distortion, show that

$$h(X) - \frac{1}{2}\log(2\pi e)D \le R(D) \le \frac{1}{2}\log\frac{\sigma^2}{D}. \tag{3}$$

For the upper bound, consider the joint distribution shown in the figure at the top of page 339 of Cover & Thomas. Are Gaussian random variables harder or easier to describe than other random variables with the same variance?

Synthesis of ideas about Gaussian channel capacity and lossy compression of a Gaussian source

7. (10 pts) *Simplicity is best.* *Fall 99 oral*

This problem considers transmission of a Gaussian source W with mean zero and variance P over a Gaussian channel $Y = X + Z$ with power constraint P where $Z \sim \mathcal{N}(0, N)$.

   (a) Combine the known results of the capacity of the Gaussian channel and $R(D)$ for a Gaussian source and squared error distortion to derive the smallest distortion possible in this scenario.

   (b) Show that directly transmitting the source $X = W$ and employing the unbiased receiver $\hat{W} = Y$ approaches the theoretical performance limit as $P/N \to \infty$.

   (c) Now show that the proper choice of a produces a *biased* receiver $\hat{W} = aY$ that achieves the performance limit of part a for every value of P/N assuming direct transmission $X = W$ as in part b.

Lecture 14D: Proof of Converse for $R(D)$.

8. (10 pts) *Properties of optimal rate distortion code.* A good (R, D) rate distortion code with $R \approx R(D)$ puts severe constraints on the relationship of the source X^n and the representations $\hat{X}^n$. Examine the chain of inequalities (10.58–10.71) considering the conditions for equality and interpret as properties of a good code. For example, equality in (10.59) implies that $f_n(\mathcal{X}^n)$ is a deterministic function of X^n.