ECE 231A Discussion 8

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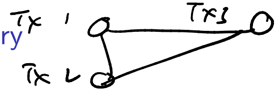
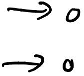
Overview of network information theory

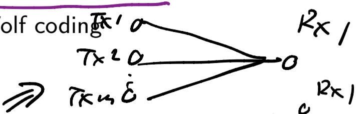


General problem: Given many senders and receivers and a channel transition matrix which describes the effect of the interference and the noise in the network, decide whether or not the sources can be transmitted over the channel.



New ingredients:

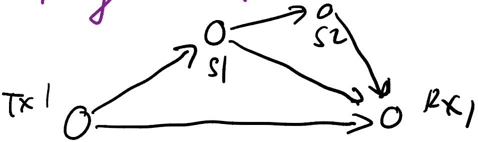
1. distributed communication, e.g., multiple-access channels
2. distributed source coding, e.g., Slepian-Wolf coding



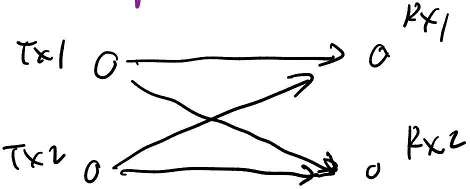
Important special cases:

1. Multiple-access channels: many senders to one receiver
2. Broadcast channels: one sender to many receivers
3. Relay channels: one sender to one receiver, along with one or more intermediate sender-receiver pairs in between to aid the communication
4. Interference channels: two senders and two receivers with crosstalk
5. Two-way channels: two sender-receiver pairs sending info. to each other

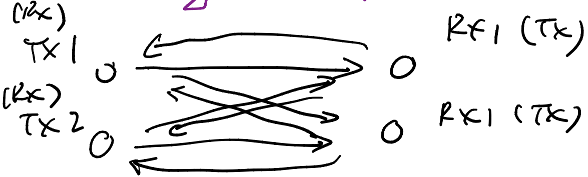
Relay channel



Interference channel



Two-way channel



MAC capacity region

$$\begin{array}{l} \text{Tx 1 } 0 \\ \text{Tx 2 } 0 \end{array} \rightarrow 0 \text{ Rx}$$

Multiple-access channel (MAC): a channel consisted of three alphabets $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{Y}$, and a probability transition matrix $p(y|x_1, x_2)$.

$((2^{nR_1}, 2^{nR_2}), n)$ **code**: two message sets of integers $\mathcal{W}_1 = \{1, 2, \ldots, 2^{nR_1}\}$, $\mathcal{W}_2 = \{1, 2, \ldots, 2^{nR_2}\}$, and two encoding functions, $X_1 : \mathcal{W}_1 \to \mathcal{X}_1^n$, $X_2 : \mathcal{W}_2 \to \mathcal{X}_2^n$ and a decoding function $g : \mathcal{Y}^n \to \mathcal{W}_1 \times \mathcal{W}_2$.

Capacity region: the closure of the set of achievable (R_1, R_2) rate pairs.

MAC capacity: The capacity of a MAC $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$ is the closure of the convex hull of all (R_1, R_2) satisfying

$$\begin{aligned} R_1 &< I(X_1; Y|X_2), \\ R_2 &< I(X_2; Y|X_1), \\ R_1 + R_2 &< I(X_1, X_2; Y) \end{aligned}$$

for some product distribution $p_1(x_1)p_2(x_2)$ on $\mathcal{X}_1 \times \mathcal{X}_2$.

$X_1 \perp\!\!\!\perp X_2$

Remark

1. The convex hull of "pentagons" need not be a pentagon. For example, HW 8, Problem 2.
2. The "pentagon" shape: $I(X_1; Y|X_2) + I(X_2; Y|X_1) \geq I(X_1, X_2; Y)$.

$$\underline{I(X_2;Y|X_1)} = H(X_2|X_1) - H(X_2|Y_1, X_1)$$

$$= H(X_2) - H(X_2|Y_1, X_1)$$

$$= I(X_2; Y, X_1)$$

$$= I(X_2;Y) + I(X_2;X_1|Y)$$

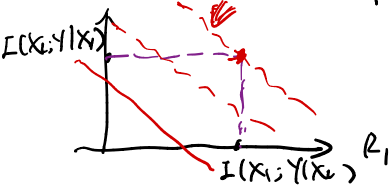
$$\geq \underline{I(X_2;Y)}$$

$$I(X_1;Y|X_2) + I(X_2;Y|X_1)$$

$$\geq I(X_1;Y|X_2) + I(X_2;Y)$$

$$= I(X_1,X_2;Y)$$

$p(x_1)p(x_2)$

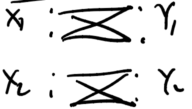


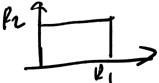
Examples of MACs

$p(y_1, y_2 | x_1, x_2) = p(y_1|x_1)\, p(y_2|x_2)$

$Y = (Y_1, Y_2)$

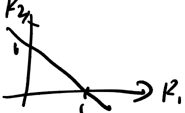
Orthogonal MAC: Consider two independent BSCs $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$, $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1, y_2|x_1, x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed. The capacity region is given by





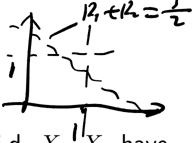
$$R_1 < 1 - H(p_1), \quad R_2 < 1 - H(p_2)$$

Multiplier MAC: Consider a binary multiplier channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$ where $Y = X_1 X_2$. The capacity region is given by



$$R_1 < 1, \quad R_2 < 1, \quad R_1 + R_2 < 1$$

Adder MAC: $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}$, $\mathcal{Y} = \{0, 1, 2\}$ and $Y = X_1 + X_2$ with $Y = 1$ being the erasure. The capacity region is given by



$$R_1 < 1, \quad R_2 < 1, \quad R_1 + R_2 < \frac{3}{2}$$

Gaussian MAC: $Y_i = X_{1,i} + X_{2,i} + Z_i$, $Z_i \sim \mathcal{N}(0, N)$ and i.i.d. X_1, X_2 have power constraint P_1, P_2. The capacity region is

$C(x) = \frac{1}{2} \log(1 + x)$

$$R_1 \le C\left(\frac{P_1}{N}\right), \quad R_2 \le C\left(\frac{P_1}{N}\right), \quad R_1 + R_2 \le C\left(\frac{P_1 + P_2}{N}\right)$$

$C\left(\frac{P_1}{N}\right)$

Orthogonal BSC

$$R_1 < I(X_1; Y | X_2)$$
$$= I(X_1; Y_1, Y_2 | X_2)$$
$$= H(X_1 | X_2) - H(X_1 | X_2, Y) \quad {=}(Y_1, Y_2)$$
$$= H(X_1) - H(X_1 | Y_1)$$
$$= I(X_1; Y_1)$$

$$R_2 < I(X_2; Y_2)$$

FDMA, TDMA, and CDMA

Frequency division multiple access (FDMA): The whole bandwidth is divided into several slots. Each slot is assigned to a particular user. For instance, if $W = W_1 + W_2$, then

$$C_1 = W_1 \log\left(1 + \frac{P_1}{NW_1}\right), \quad C_2 = W_2 \log\left(1 + \frac{P_2}{NW_2}\right).$$

Time division multiple access (TDMA): Time is divided into several slots. In each slot, only one user becomes active and others remain quiet. Since time is reduced, the instantaneous power could be higher than average.

Code division multiple access (CDMA): Spreading codes are used for different senders and receiver decodes them one by one. For example, consider $J = 4$ users accessing the channel simultaneously, the spread code (Walsh code) is

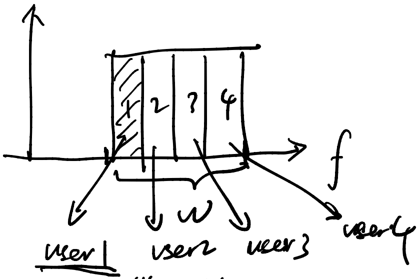
$$\begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

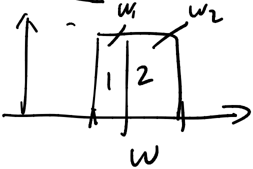
$x = \sum_{i=1}^{4} b_i C_i$

$y = x$ $\quad \frac{1}{J}\langle y, C_i \rangle = b_i$

with property that $\langle C_i, C_j \rangle = J$ if $i = j$ and $\langle C_i, C_j \rangle = 0$ otherwise.

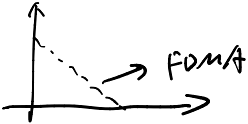
FDMA



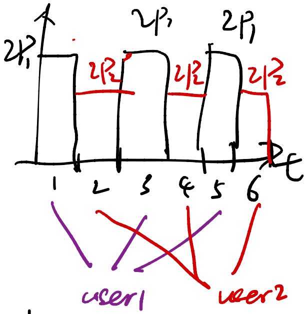


$$w_1 + w_2 = W$$

$$(C_1, C_2) \sim w_1 \in \{1, \cdots W-1\}$$



TDMA

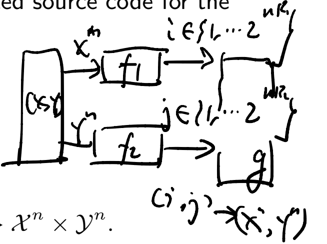


Slepian-Wolf coding

Correlated sources: $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are i.i.d. $\sim p(x, y)$.

Correlated source codes: A $((2^{nR_1}, 2^{nR_2}), n)$ distributed source code for the joint source (X, Y) consists of two encoder maps,

$$f_1 : \mathcal{X}^n \to \{1, 2, \ldots, 2^{nR_1}\}$$
$$f_2 : \mathcal{Y}^n \to \{1, 2, \ldots, 2^{nR_2}\}$$



and a decoder map,

$$g : \{1, 2, \ldots, 2^{nR_1}\} \times \{1, 2, \ldots, 2^{nR_2}\} \to \mathcal{X}^n \times \mathcal{Y}^n.$$

Slepian-Wolf coding: For the distributed source coding problem for the source (X, Y) drawn i.i.d. $\sim p(x, y)$, the achievable rate region is given by

$R_1 + R_2 > H(X_1) + H(X_2)$

$$\begin{aligned} R_1 &\geq H(X|Y), \\ R_2 &\geq H(Y|X), \\ R_1 + R_2 &\geq H(X, Y). \end{aligned}$$

Remark: It is possible to achieve $R_1 < H(X)$ and $R_2 < H(Y)$ simultaneously, which are impossible in single source compression.

Achievability of Slepian-Wolf coding

Achievability: The key idea is the binning procedure.

1. Encoding: Assign every $x^n \in \mathcal{X}^n$ to one of 2^{nR_1} bins independently according to uniform distribution. Similarly, assign every $y^n \in \mathcal{Y}^n$ to one of 2^{nR_2} bins.
2. Decoding: Given the index pair (i_0, j_0), declare $(\hat{x}^n, \hat{y}^n) = (x^n, y^n)$ if there is one and only one (x^n, y^n) s.t. $f_1(x^n) = i_0$, $f_2(y^n) = j_0$ and $(x^n, y^n) \in A_\epsilon^{(n)}$. Otherwise, declare an error.
3. Probability of error:

$$
\begin{aligned}
E_0 =& \{(X^n, Y^n) \notin A_\epsilon^{(n)}\} \\
E_1 =& \{\exists \hat{x}^n \neq X^n : f_1(\hat{x}^n) = f_1(X^n), (\hat{x}^n, Y^n) \in A_\epsilon^{(n)}\} \\
E_2 =& \{\exists \hat{y}^n \neq Y^n : f_2(\hat{y}^n) = f_2(Y^n), (X^n, \hat{y}^n) \in A_\epsilon^{(n)}\} \\
E_{1,2} =& \{\exists (\hat{x}^n, \hat{y}^n) : \hat{x}^n \neq X^n, \hat{y}^n \neq Y^n, f_1(\hat{x}^n) = f_1(X^n), f_2(\hat{y}^n) = f_2(Y^n), \\
& (\hat{x}^n, \hat{y}^n) \in A_\epsilon^{(n)}\}
\end{aligned}
$$

$$
\begin{aligned}
P_e^{(n)} =& P(E_0 \cup E_1 \cup E_2 \cup E_{1,2}) \\
\leq & P(E_0) + P(E_1) + P(E_2) + P(E_{1,2}) \\
\leq & \epsilon + 2^{-n(R_1 - H(X|Y) - \epsilon_1)} + 2^{-n(R_2 - H(Y|X) - \epsilon_2)} + 2^{-n(R_1 + R_2 - H(X,Y) - \epsilon_3)} \\
\leq & 4\epsilon \quad (\text{if } R_1 > H(X|Y), R_2 > H(Y|X), R_1 + R_2 > H(X,Y))
\end{aligned}
$$

Exercise 1: multiplication multiple access

Consider the two-user multiplication channel $Y = X_1 \times X_2$ with the following alphabets for X_1 and X_2: $\mathcal{X}_1 = \{0, 1\}$, $\mathcal{X}_2 = \{1, 2, 3\}$. Note that the two alphabets are different. Find and sketch the capacity region for this MAC.

$$\Pr(X_1 = 1) = p$$

if $X_1 = 0$, $Y = 0$

if $X_1 = 1$, $Y = X_2 > 0$

$\Rightarrow$ knowing Y implies knowing X_1

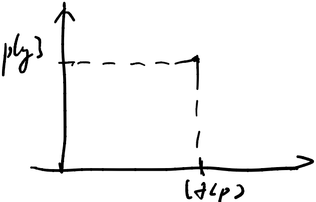
$$\begin{aligned} I(X_1; Y | X_2) &= H(X_1 | X_2) - \underbrace{\cancel{H(X_1 | X_2, Y)}}_{=0} \\ &= H(X_1) \\ &= H(p) \end{aligned}$$

$$\begin{aligned} I(X_2; Y | X_1) &= H(X_2 | X_1) - H(X_2 | X_1, Y) \\ &= H(X_2) - \Pr\{X_1 = 0\} H(X_2 | X_1 = 0, Y) \\ &\quad - \Pr\{X_1 = 1\} H(X_2 | X_1 = 1, Y) \\ &= H(X_2) - (1-p) H(X_2) - p \cdot 0 \\ &= p H(X_2) \\ &\le p \log 3 \end{aligned}$$

$$I(X_1, X_2; Y) = I(X_1; Y) + I(X_2; Y|X_1)$$

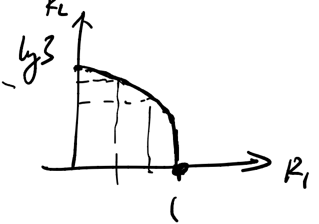
$$= H(p) + pH(X_2)$$

$$\le \underbrace{H(p) + p\log 3}_{p_1(x_1)p_2(x_2)} = I(X_1; Y|X_2) + I(X_2; Y|X_1)$$



$(H(p), p\log 3)$ p $0 \sim 1$

$\Downarrow$



HW 8, Problem 9: Slepian-Wolf for multiplication

Let X_1 be a binary random variable with $P(X_1 = 1) = p_1$. Let Z be uniformly distributed over the four integers $\{1, 2, 3, 4\}$. Let $X_2 = X_1 Z$. Compute the Slepian-Wolf region of rate pairs. Plot the region for $p_1 = 0.5$.

if $X_1 = 0$, $X_2 = 0$

if $X_1 = 1$, $X_2 = Z > 0$

$R_1 > H(X_1|X_2) = 0$

$R_2 > H(X_2|X_1)$

$= p(X_1=0)\, H(X_2|X_1=0) + p(X_1=1)\, H(X_2|X_1=1)$

$= 0 + p_1 \underline{H(Z)}$

$\log 4 = 2$

$X_2 = X_1 Z$, $X_1 = 1$

$\Rightarrow X_2 = Z$

$= 2p_1$

$R_1 + R_2 > H(X_1, X_2) = H(X_1) + H(X_2|X_1)$

$= H(p_1) + 2p_1$

$p_1 = 0.5$

$H(X_1|X_2) = 0$

$H(X_2|X_1) = 1$

$H(X_1, X_2) = 2$