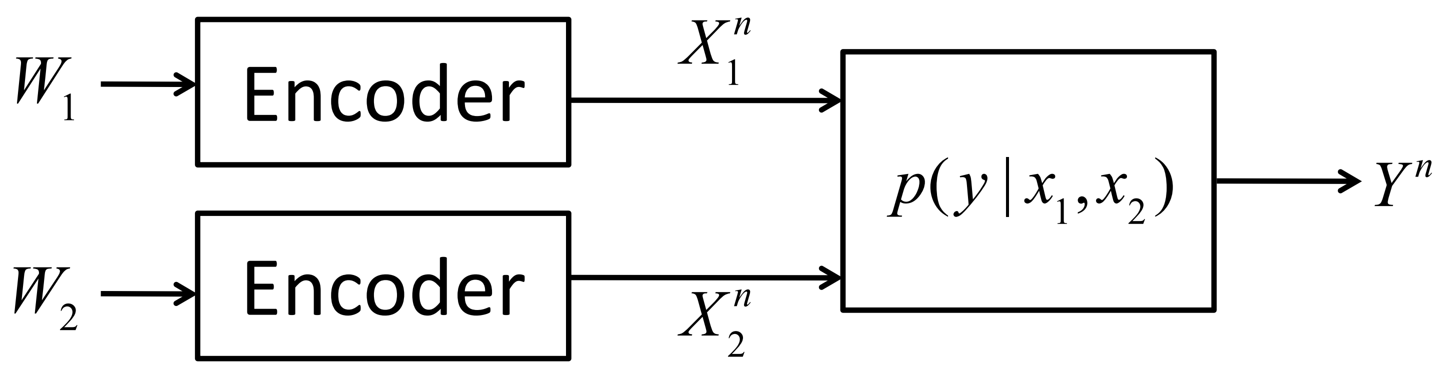
EE 231A Information Theory
Lecture 15
The Multiple Access Channel

A. Capacity Region of Multiple Access Channel and Binary Multiplier MAC example
B. Binary Erasure MAC example
C. Achievability of Multiple Access Channel Capacity Region
D. Gaussian Multiple Access Channel

Part 15A:
Capacity Region
of the
Multiple Access Channel
and
Binary Multiplier MAC

Multiple Access DMC



decode $\quad g(Y^n) = (\hat{W}_1, \hat{W}_2)$

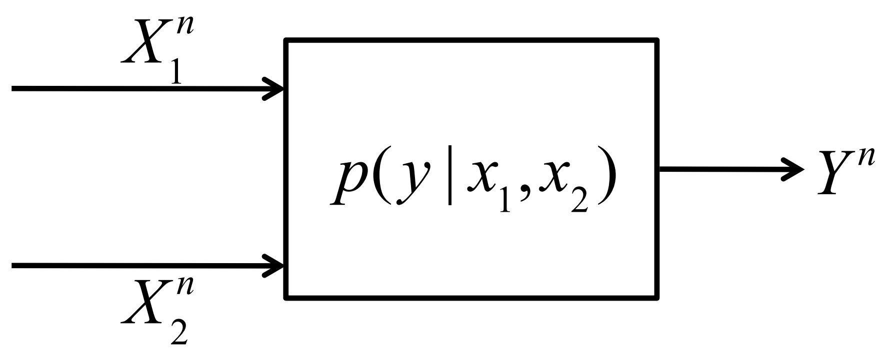
The “pentagon” inequalities

- For a fixed product distribution $p(x_1, x_2) = p(x_1)p(x_2)$, the set of achievable (R_1, R_2) rate pairs (the achievable region) is the set of all (R_1, R_2) pairs satisfying three inequalities (a “pentagon”):

$$R_1 < I(X_1; Y \mid X_2) = I_1$$

$$R_2 < I(X_2; Y \mid X_1) = I_2$$

$$R_1 + R_2 < I(X_1, X_2; Y) = I_3$$

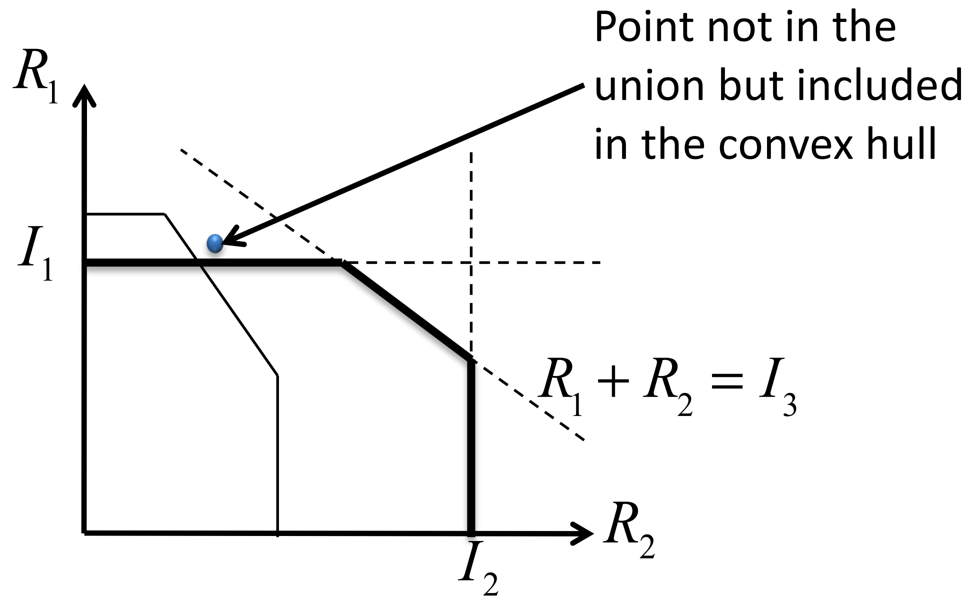


Capacity region (Theorem 15.3.1)

$$R_1 < I(X_1;Y|X_2) = I_1$$

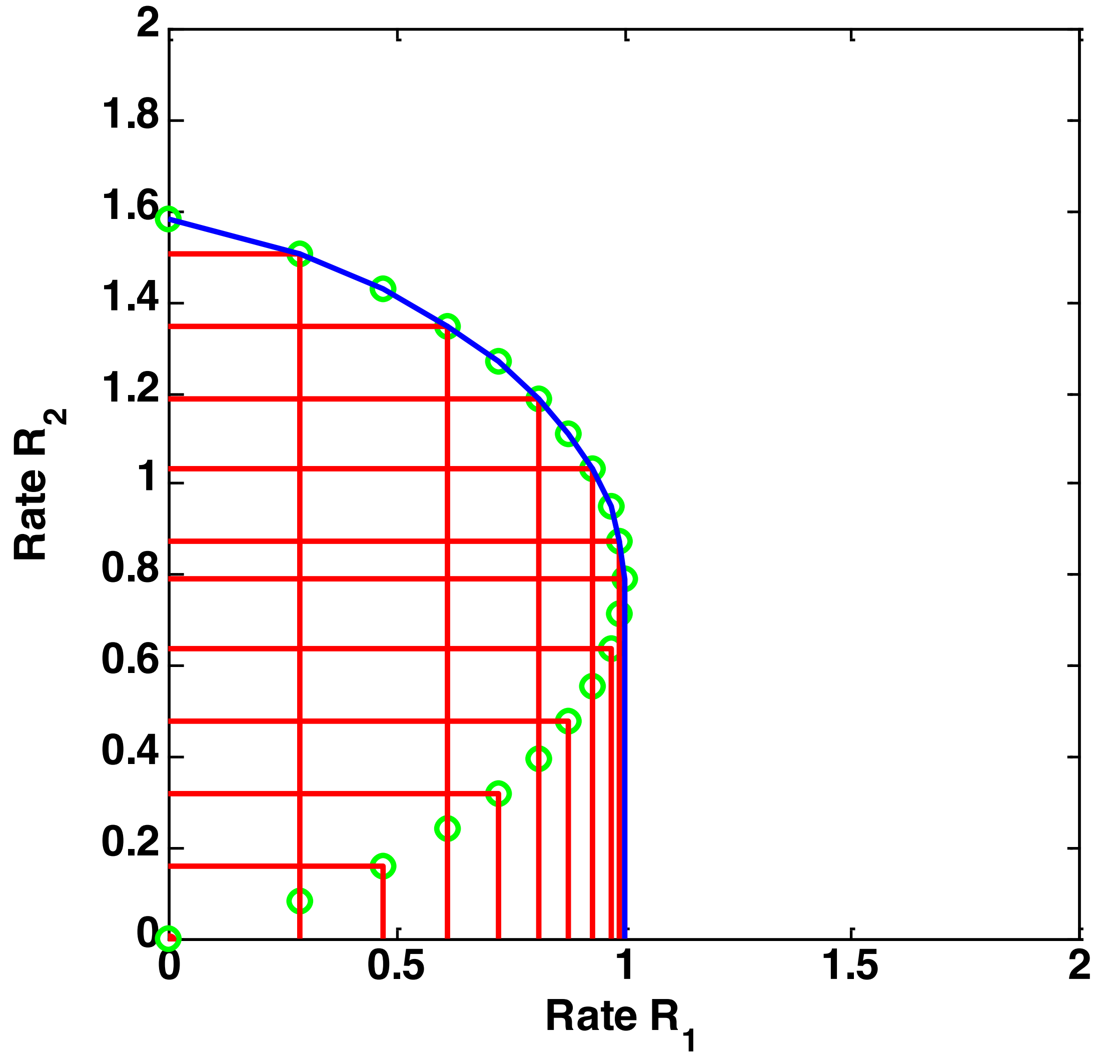
$$R_2 < I(X_2;Y|X_1) = I_2$$

$$R_1 + R_2 < I(X_1,X_2;Y) = I_3$$



- The capacity region is a union of many pentagons, possibly further increased by the closure of a convex hull operation.
  – Points in the convex hull are achieved by time-sharing between two points in pentagons.

Pentagons can be rectangles.



Example 1: Binary Multiplier Channel

- <u>Binary Multiplier Channel</u> $Y = X_1 X_2, \quad X_1, X_2 \in \{0,1\}$

- Describe all pentagons by finding the general pentagon induced by the general product distribution:

$$X_1 \sim \text{Bernoulli}(p_1)$$
$$X_2 \sim \text{Bernoulli}(p_2)$$

Computing I_1, I_2, and I_3

$$R_1 < I(X_1;Y|X_2) = I_1$$
$$R_2 < I(X_2;Y|X_1) = I_2$$
$$R_1 + R_2 < I(X_1,X_2;Y) = I_3$$

$$Y = X_1X_2, \quad X_1, X_2 \in \{0,1\}$$

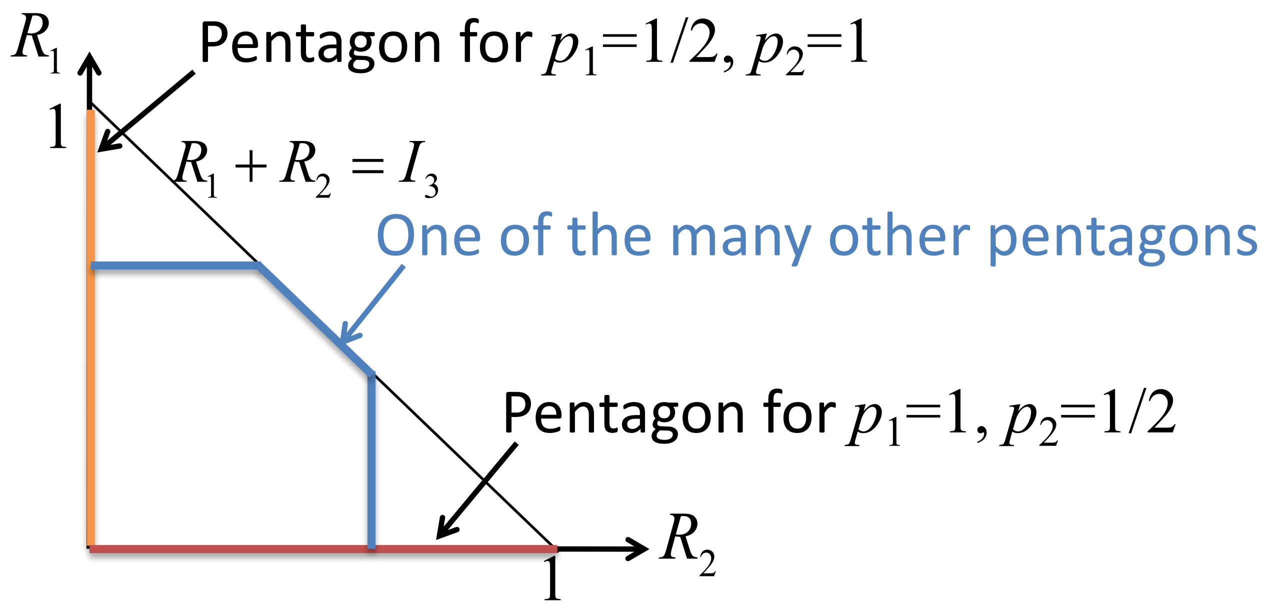
$$\begin{aligned} I_1 &= I(X_1;Y|X_2) \\ &= P(X_2=1)I(X_1;Y|X_2=1) + P(X_2=0)I(X_1;Y|X_2=0) \\ &= p_2H(p_1) + (1-p_2)\cdot 0 \\ &= p_2H(p_1) \end{aligned}$$

$$\begin{aligned} I_2 &= I(X_2;Y|X_1) \\ &= p_1H(p_2) \end{aligned}$$

$$\begin{aligned} I_3 &= I(X_1,X_2;Y) \\ &= H(Y) - H(Y|X_1,X_2) \\ &= H(Y) \quad \text{where } Y = f(X_1,X_2) \\ &= H(p_1p_2) \end{aligned}$$

Binary Multiplier Channel Pentagons, Rate Region

$$R_1 < p_2 H(p_1)$$
$$R_2 < p_1 H(p_2)$$
$$R_3 < H(p_1 p_2)$$



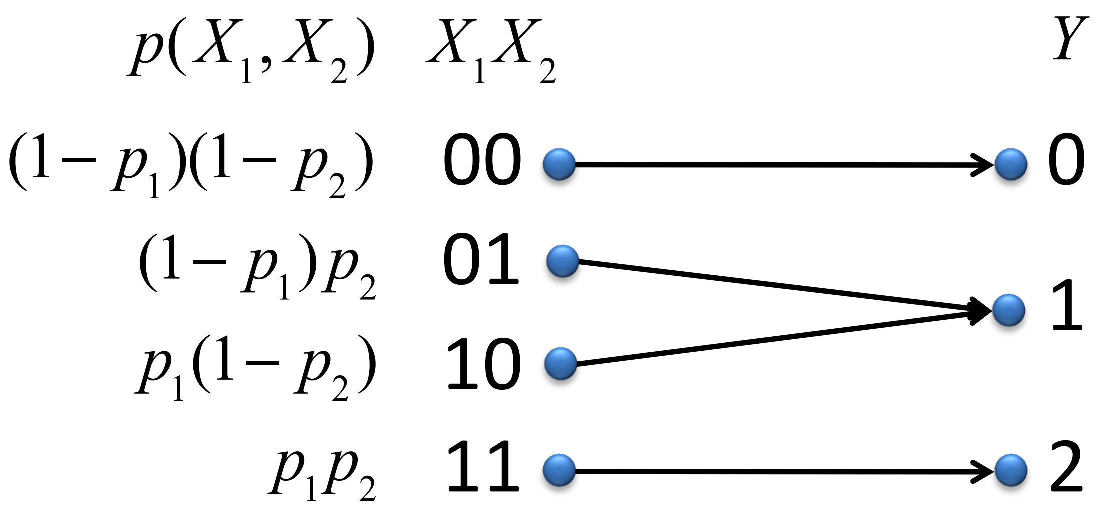
- Convex hull of these two pentagons gives the achievable region since $I_3 = I(X_1, X_2; Y) \le H(Y) \le 1$.

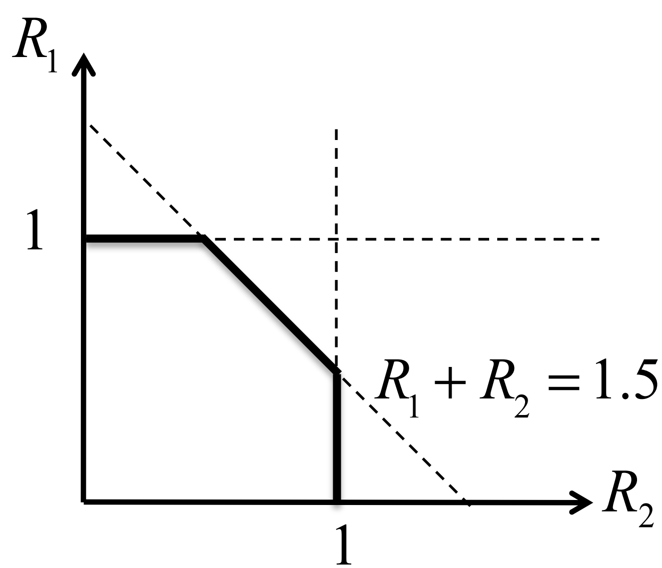
- However this rate region is also completely covered by the union of all pentagons. Convex hull is not necessary.

Part 15B:
Binary Erasure MAC

Example 2: Binary Erasure MAC

$$Y = X_1 + X_2, \qquad X_1, X_2 \in \{0,1\}$$





Pentagon for $p_1 = p_2 = \dfrac{1}{2}$

$$I_1 = I(X_1;Y|X_2) = 1$$

$$I_2 = I(X_2;Y|X_1) = 1$$

$$I_3 = I(X_1,X_2;Y) = H(Y) - H(Y|X_1,X_2) = H(Y) = 1.5$$

One pentagon that contains them all

- Now consider the general pentagon for

$$X_1 \sim \text{Bernoulli}(p_1), X_2 \sim \text{Bernoulli}(p_2)$$

$$I(X_1;Y \mid X_2) = H(p_1) \le 1$$

$$I(X_2;Y \mid X_1) = H(p_2) \le 1$$

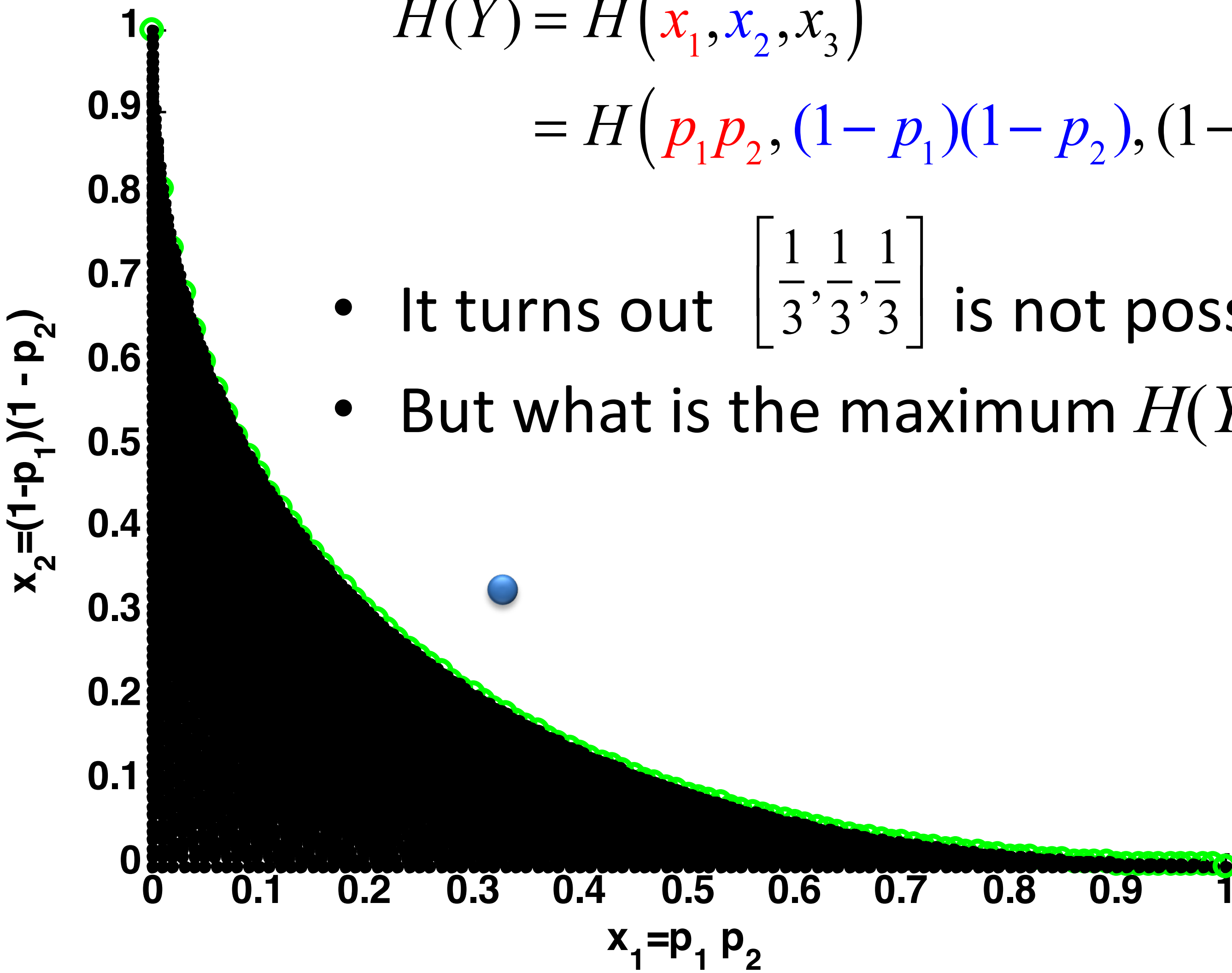
$$I(X_1,X_2;Y) = H\left(Y\right) \le 1.5$$ Need to show this

- So every pentagon is within the pentagon for $p_1 = p_2 = \frac{1}{2}.$

Maximum Possible $H(Y)$

- $H(Y) \le \log 3$, but is this achievable?

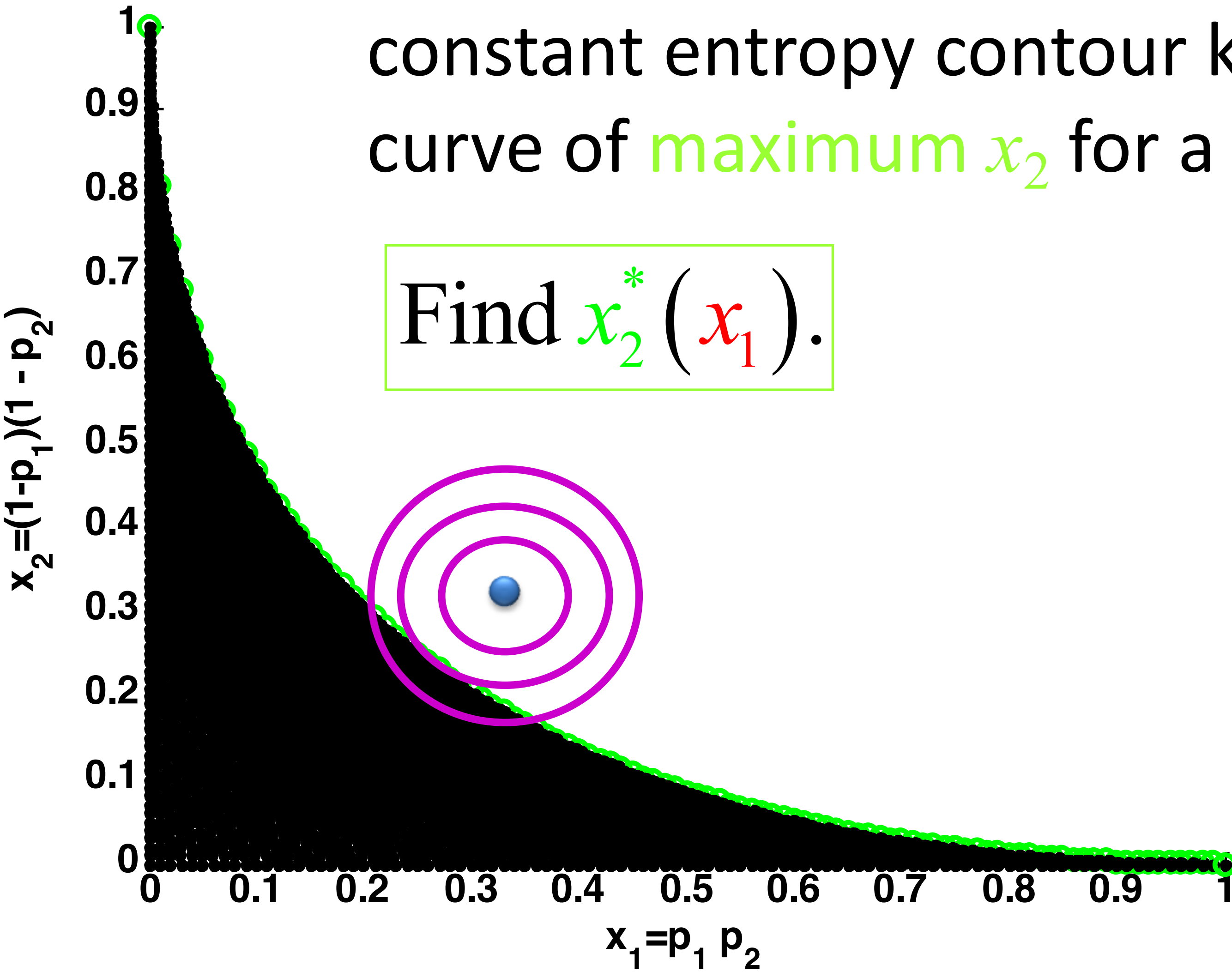
$$H(Y) = H\left(x_1, x_2, x_3\right)$$
$$= H\left(p_1 p_2, (1-p_1)(1-p_2), (1-p_1)p_2 + p_1(1-p_2)\right)$$



- It turns out $\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$ is not possible.
- But what is the maximum $H(Y)$?

Contours of constant $H(Y)$

- Maximum $H(Y)$ will occur when a constant entropy contour kisses the green curve of maximum x_2 for a fixed x_1.



$\boxed{\text{Find } x_2^*(x_1).}$

Finding the border of (x_1, x_2)

$$H(Y) = H\left(x_1, x_2, x_3\right)$$
$$= H\left(p_1 p_2, (1-p_1)(1-p_2), (1-p_1)p_2 + p_1(1-p_2)\right)$$

$$x_2 = \left(1 - p_1\right)\left(1 - \frac{x_1}{p_1}\right)$$

$$= 1 - p_1 - \frac{x_1}{p_1} + x_1$$

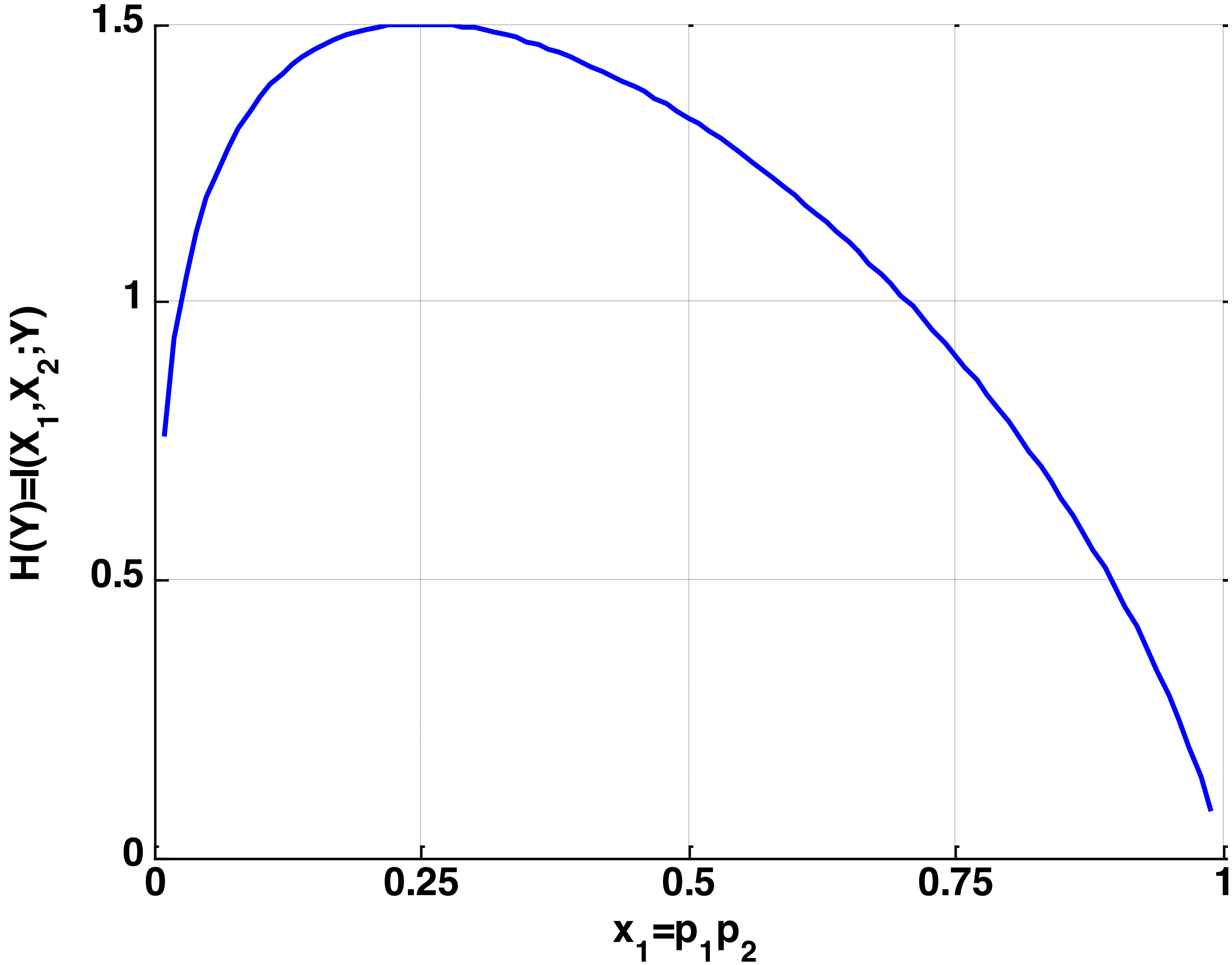
$$\frac{dx_2}{dp_1} = -1 + \frac{x_1}{p_1^2}$$

Setting $\frac{dx_2}{dp_1} = 0 \;\Rightarrow\; p_1 = \sqrt{x_1}$

$$\Rightarrow\; p_2 = \sqrt{x_1}$$

$$x_2^* = \left(1 - \sqrt{x_1}\right)^2$$

Maximum $H(Y)$ is 1.5 at x_1=0.25



Part 15C: Proof of Achievability of Multiple Access Capacity Region

Theorem 15.3.1

- The capacity of the two-user multiple-access channel is the closure of the convex hull of all rate pairs satisfying

$$\boxed{\begin{aligned} R_1 &< I(X_1;Y \mid X_2) = I_1 \\ R_2 &< I(X_2;Y \mid X_1) = I_2 \\ R_1 + R_2 &< I(X_1,X_2;Y) = I_3 \end{aligned}}$$

Random Codebook Generation

- Fix $p(x_1, x_2) = p(x_1)p(x_2)$
- Codebook generation:
  - Generate 2^{nR_1} codewords X_1^n with each element i.i.d. $p(x_1)$.
  - Generate 2^{nR_2} codewords X_2^n with each element i.i.d. $p(x_2)$.

Typical Set Decoding

- Use typical set decoding as follows:
  - Decode Y^n as $W_1=i$, $W_2=j$ if a unique pair (i, j) exists such that

$$\left(X_1^n(i), X_2^n(j), Y^n\right) \in A_\epsilon^{(n)}\left(X_1^n, X_2^n, Y^n\right).$$

  - Otherwise declare an error.

Probability of error

- $$P_e^{(n)} = \frac{1}{2^{n(R_1+R_2)}} \sum_{(W_1,W_2)} \Pr\left\{ g(Y^n) \neq (W_1,W_2) \middle| W_1, W_2 \text{ sent} \right\}$$

- By symmetry, assume $W_1=1$, $W_2=1$

$$E_{ij} = \left\{ \left(X_1^n(i), X_2^n(j), Y^n \right) \in A_\epsilon^{(n)} \left(X_1, X_2, Y \right) \right\}$$

$$P_e = P\left(E_{11}^c \cup \bigcup_{i\neq 1, j=1} P\left(E_{i1}\right) \cup \bigcup_{i=1, j\neq 1} P\left(E_{1j}\right) \cup \bigcup_{i\neq 1, j\neq 1} P\left(E_{ij}\right) \right)$$

$$\leq P\left(E_{11}^c\right) + \sum_{i\neq 1, j=1} P\left(E_{i1}\right) + \sum_{i=1, j\neq 1} P\left(E_{1j}\right) + \sum_{i\neq 1, j\neq 1} P\left(E_{ij}\right)$$

$P\left(E_{i1}\right)$

$$P\left(E_{i1}\right)=P\left\{\left(X_1^n(i),X_2^n(1),Y^n\right)\in A_\epsilon^{(n)}\left(X_1,X_2,Y\right)\right\}$$

$$=\sum_{\left(X_1^n,X_2^n,Y\right)\in A_\epsilon^{(n)}} P\left(X_1^n(i)\right)P\left(X_2^n(1),Y\right)$$

$$\leq \sum_{\left(X_1^n,X_2^n,Y\right)\in A_\epsilon^{(n)}} 2^{-n\left(H(X_1)-\epsilon\right)}2^{-n\left(H(X_2,Y)-\epsilon\right)}$$

$$=\left|A_\epsilon^{(n)}\right|2^{-n\left(H(X_1)-\epsilon\right)}2^{-n\left(H(X_2,Y)-\epsilon\right)}$$

$$=2^{n\left(H(X_1,X_2,Y)+\epsilon\right)}2^{-n\left(H(X_1)-\epsilon\right)}2^{-n\left(H(X_2,Y)-\epsilon\right)}$$

$$P\left(E_{i1}\right), P\left(E_{1j}\right), P\left(E_{ij}\right)$$

$$P\left(E_{i1}\right) \le 2^{n\left(H(X_1,X_2,Y)+\epsilon\right)} 2^{-n\left(H(X_1)-\epsilon\right)} 2^{-n\left(H(X_2,Y)-\epsilon\right)}$$

$$\le 2^{-n\left(H(X_1)+H(X_2,Y)-H(X_1,X_2,Y)-3\epsilon\right)}$$

$$= 2^{-n\left(I(X_1;Y|X_2)-3\epsilon\right)}$$

Similarly $P\left(E_{1j}\right) \le 2^{-n\left(I(X_2;Y|X_1)-3\epsilon\right)}$

and $P\left(E_{ij}\right) \le 2^{-n\left(I(X_1,X_2;Y)-4\epsilon\right)}$

$$\begin{aligned} H(X_1)+H(X_2,Y)-H(X_1,X_2,Y) &= I(X_1;X_2,Y) \\ &= \cancelto{0}{I(X_1;X_2)} + I(X_1;Y \mid X_2) \\ &= I(X_1;Y \mid X_2) \end{aligned}$$

Probability of error conclusion

$$P_e = P\left(E_{11}^c \cup \bigcup_{i\neq 1, j=1} P\left(E_{i1}\right) \cup \bigcup_{i=1, j\neq 1} P\left(E_{1j}\right) \cup \bigcup_{i\neq 1, j\neq 1} P\left(E_{ij}\right) \right)$$

$$\leq P\left(E_{11}^c\right) + \sum_{i\neq 1, j=1} P\left(E_{i1}\right) + \sum_{i=1, j\neq 1} P\left(E_{1j}\right) + \sum_{i\neq 1, j\neq 1} P\left(E_{ij}\right)$$

$$\leq \epsilon + 2^{nR_1} 2^{-n\left(I(X_1;Y|X_2)-3\epsilon\right)}$$
$$+ 2^{nR_2} 2^{-n\left(I(X_2;Y|X_1)-3\epsilon\right)}$$
$$+ 2^{n(R_1+R_2)} 2^{-n\left(I(X_1,X_2;Y)-4\epsilon\right)}$$

$$\boxed{\begin{aligned} R_1 &< I(X_1;Y \mid X_2) = I_1 \\ R_2 &< I(X_2;Y \mid X_1) = I_2 \\ R_1 + R_2 &< I(X_1,X_2;Y) = I_3 \end{aligned}}$$

$$= \epsilon + 2^{n\left(R_1 - I(X_1;Y|X_2)+3\epsilon\right)} + 2^{n\left(R_2 - I(X_2;Y|X_1)+3\epsilon\right)} + 2^{n\left(R_1+R_2-I(X_1,X_2;Y)+4\epsilon\right)}$$

Closure of the convex hull

- What about the closure of the convex hull?

- If $\left(R_1, R_2\right) \in C$ and $\left(R_1^{'}, R_2^{'}\right) \in C$, then

$$\left(\lambda R_1 + (1-\lambda)R_1^{'}, \lambda R_2 + (1-\lambda)R_2^{'}\right) \in C$$

  for $0 \leq \lambda \leq 1$ by time-sharing.

- Thus C is a convex set.

Part 15D:
Gaussian Multiple Access Channel

Gaussian MAC

$$Y = \sum_{i=1}^{m} X_i + Z$$

- Define $C\left(\frac{P}{N}\right) = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$

$$R_i < C\left(\frac{P_i}{N}\right)$$

$$R_i + R_j < C\left(\frac{P_i + P_j}{N}\right) = C\left(\frac{2P}{N}\right) \quad \text{if } P_i = P_j$$

$$\vdots$$

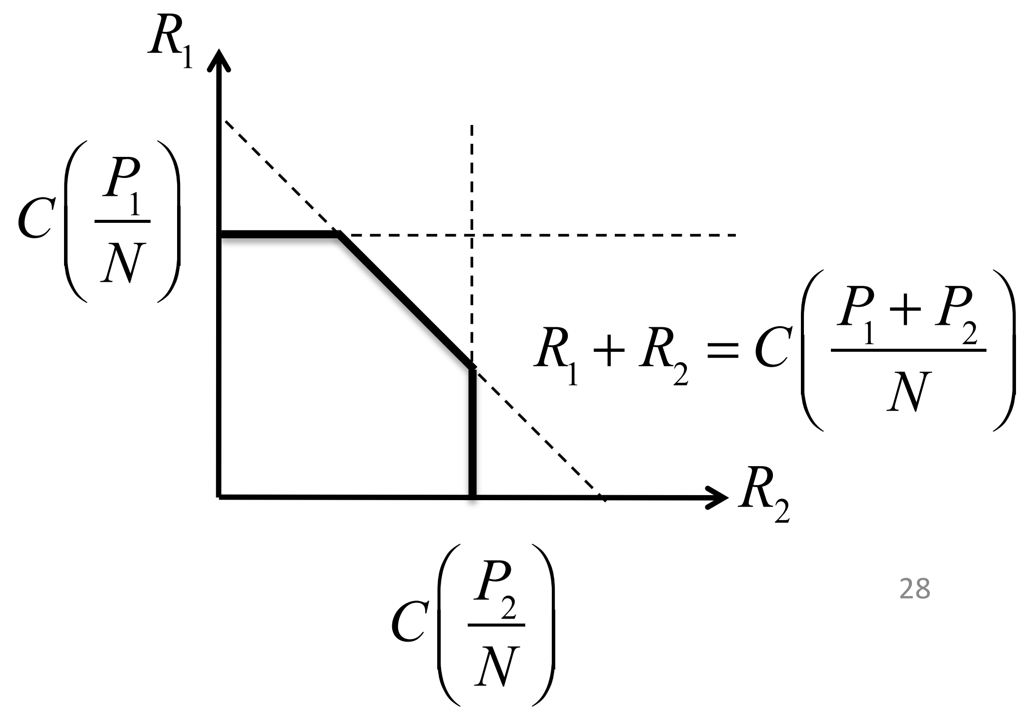
$$\sum_{i=1}^{m} R_i < C\left(\frac{mP}{N}\right) \qquad \text{if } P_i = P_j$$

Two User Gaussian MAC

$$R_1 < C\left(\frac{P_1}{N}\right)$$

$$R_2 < C\left(\frac{P_2}{N}\right)$$

$$R_1 + R_2 < C\left(\frac{P_1 + P_2}{N}\right)$$



Limit of many users on Gaussian MAC

Note: when $R_i = R_j \quad \forall i, j$

$$C\left(\frac{mP}{N}\right) < mC\left(\frac{P}{N}\right)$$

- So the last inequality dominates and

$$C\left(\frac{mP}{N}\right) \to \infty \quad \text{as } m \to \infty$$

$$\frac{1}{m}C\left(\frac{mP}{N}\right) = \frac{1}{m}\frac{1}{2}\log\left(1+\frac{mP}{N}\right) \approx \frac{\log_2\left(\frac{mP}{N}\right)}{2m} \quad \to 0$$