Class 4 (after quiz 1)

Compressed Sensing

$x \in \mathbb{R}^n$ → unknown signal, but we know it is very sparse (most entries are zero)

We make m linear measurements with given matrix A and observe the outcome y, with

$$y = \underset{m \times n}{A}\, x \qquad m << n$$

undetermined set of equations

We want to solve the problem

$$\left.\begin{aligned} &\min_x \ \mathrm{card}(x) \\ &\text{st } Ax = y \end{aligned}\right\} \text{ can we solve it by using} \qquad \begin{aligned} &\min_x \|x\|_1 \\ &\text{st } Ax = y \end{aligned}$$

card(x) = # of nonzero entries of x

Applications : imaging, network tomography, ...

Digression

Given a matrix A of dimension $m \times n$

range of A $\quad R(A) = \{x \in \mathbb{R}^m \,/\, x = Ay, \text{ for some } y \in \mathbb{R}^n\}$

$R(A) \subseteq \mathbb{R}^m$

linear combinations of the colums of A

$$x = a_1 y_1 + a_2 y_2 + \dots + a_n y_n$$

nullspace of A, $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

$N(A) \subseteq \mathbb{R}^n$

all vectors orthogonal to the rows of A

If we have linear equations $Ax = b$

$R(A) \rightarrow$ determines existence of solution.

if $b \in R(A) \Rightarrow$ there exists at least one solution

otherwise not feasible

$N(A) \rightarrow$ determines uniqueness of solution.

$N(A) = 0$ unique solution

otherwise, if x is a solution and $z \in N(A)$, $x + z$ is also a solution

$$A(x+z) = Ax + \underbrace{\cancel{Az}}_{0} = b$$

$$\min_x \ \mathrm{card}(x) \quad \text{st} \quad Ax = b$$

P1

$$\min_x \ \|x\|_1 \quad \text{st} \quad Ax = b$$

P2

Assume we know that the optimal solution of P1 is k-sparse. at most k nonzero components.

Let $z \in \mathcal{N}(A)$, and consider its k largest components

$$|z^{(1)}| \geq |z^{(2)}| \geq \cdots \geq |z^{(k)}|$$

$$\boxed{\begin{array}{l}\underline{\text{Condition}} \quad |z^{(1)}| + |z^{(2)}| + \cdots |z^{(k)}| < \frac{1}{2} \|z\|_1 \\ \quad \text{for all vectors } z \in \mathcal{N}(A) / \{0\}\end{array}} \quad (*)$$

• If I add the magnitude of any k components of z, I always get something $< \frac{1}{2} \|z\|_1$

$$2 \| P_I z \|_1 < \|z\|_1$$

Proof

Sufficiency : if the condition (*) holds, and $\hat{x}$ is the sparsest solution, by solving P_1 we cannot find another solution

$$\hat{x} + z, \quad z \in \mathcal{N}(A) \quad \text{with}$$

$$\|\hat{x} + z\|_1 \leq \|\hat{x}\|_1$$

We want to show that $\|\hat{x} + z\|_1 > \|\hat{x}\|_1$ for all $z \in \mathcal{N}(A)$.

Create a matrix P_I, that only keeps the nonzero coordinates of $\hat{x}$ (projection matrix)

$$(P_I)_{jj} = \begin{cases} 1, & \hat{x}_j \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

I = set of nonzero positions of $\hat{x}$

$$P_I \hat{x} = \hat{x}$$

$$\|\hat{x} + z\|_1 \geq \|\hat{x} + z - P_I z\|_1 - \|P_I z\|_1$$

(follows from triangle inequality
$\|\hat{x} + z - P_I z\| \leq \|\hat{x} + z\|_1 + \|P_I z\|_1$)

Note $z - P_I z$ has zero values at the positions where $\hat{x}$ has non-zero values & v.v.

$$\hat{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \quad P_I = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad z = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \quad z - P_I z = \begin{pmatrix} 0 \\ 6 \\ 7 \end{pmatrix}$$

$$= \sum_i |\hat{x}_i| + \sum_{i \notin I} |z_i| - \|P_I z\|_1$$

$$= \|\hat{x}\|_1 + \underbrace{\sum_i |z_i|} - \underbrace{\sum_{i\in I} |z_i| - \|P_I z\|_1}$$

$$= \|\hat{x}\|_1 + \underbrace{\|z\|_1 - 2\|P_I z\|_1}_{>0} > \|\hat{x}\|_1$$

Necessary if the condition does not hold, we will create a counterexample, a k-sparse solution and a not-k-sparse solution that have the same $\| \ \|_1$.

Assume there exists a matrix A, and vector $z \in N(A)$, such that

$$\| P_I z \|_1 \geq \frac{1}{2} \| z \|_1 \quad \text{for some } P_I \quad |I| = K$$

$$\| z \|_1 \leq 2 \| P_I z \|_1$$

We will compare 2 vectors

$\hat{\omega} = - P_I z$ k-sparse vector.

$\omega = z - P_I z$ not k-sparse.

Select b such that $A \hat{\omega} = b$

Clearly

$$A \omega = A(z + \hat{\omega}) = \underbrace{\cancel{Az}}_{0} + A \hat{\omega} = b$$

$$\|w\|_1 = \|z - P_J z\|_1 = \sum_{i \notin I} |z_i|$$

$$= \sum_i |z_i| - \sum_{i \in J} |z_i| = \|z\|_1 - \|P_J z\|_1$$

$$\leq 2\|P_J z\|_1 - \|P_J z\|_1 = \|P_J z\|_1 = \|\hat{w}\|_1$$