Lecture 12

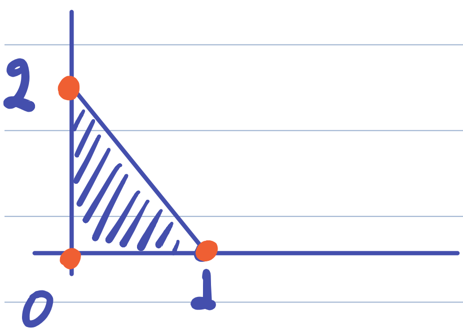
Integer Programming

$$\begin{aligned} &\max\ c^T x \\ &\text{st}\quad Ax \le b \\ &\underbrace{x \in \mathbb{Z}^n}_{\text{integer values}} \end{aligned} \qquad\qquad \begin{aligned} &\max\ c^T x \\ &\text{st}\quad Ax \le b \\ &x \in \mathbb{R}^n \end{aligned}$$

relaxed problem: LP

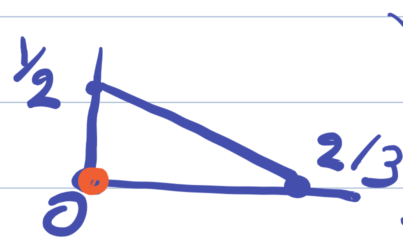
When do these problems have the same solution?

Sufficient condition: all the vertices of the polyhedron $\underbrace{P = \{x \mid Ax \le b\}}_{\text{integral polyhedron}}$ take integer values



convex hull of integer vectors.

(Note that this is not necessary



)

When does this happen?

Theorem: If the matrix A is totally unimodular (TUM) then for all integer vectors b the polyhedron $P = \{x \mid Ax \leq b\}$ is an integral polyhedron.

sufficient

A matrix is TUM if the determinant of every square submatrix in A is in $\{0, 1, -1\}$. In particular, all elements in A are in $\{0, 1, -1\}$

Examples: • identity matrix $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ TUM

• permutation matrix → exactly one 1 per row and per column

TUM

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

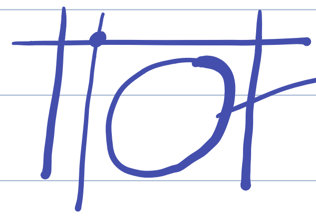
Note: A does not need to be square to be TUM

Proof of theorem.

1) If A is TUM, and U is a nonsingular square submatrix, then U^{-1} is an integral matrix

$$U^{-1} = \frac{1}{\det(U)} \left(\text{adjoint matrix of } U\right) = \frac{1}{\det U} \left(\begin{matrix}\text{cofactor}\\ \text{matrix of } U\end{matrix}\right)^T$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$



all square submatrix have det $\{+1, -1, 0\}$

$\det(U)$ is in $\{+1, -1\}$

U^{-1} has elements in $\{+1, -1, 0\}$ $\rightarrow$ integral matrix.

2) Consider a vertex x of $P = \{x \mid Ax \leq b\}$
it satisfies $J(x)$ inequalities with equality

$$\underbrace{A_{J(x)}}_{n \times n \text{ invertible matrix}} x = b_{J(x)}, \qquad |J(x)| = n$$

$$\Rightarrow \quad x = \underbrace{A^{-1}_{J(x)}}_{\text{integral}} \underbrace{b_{J(x)}}_{\text{integral}} \quad \Rightarrow \underline{x \text{ is integral}}$$

Because A TUM $\Rightarrow$ $A_{J(x)}$ also TUM matrix

$\Rightarrow$ $A^{-1}_{J(x)}$ integral matrix

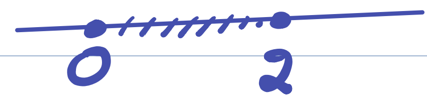
$\square$

Matrix A being TUM is a sufficient condition → is it necessary? <u>NO</u>

eg $\quad 0 \leq x \leq 1$ $\qquad\qquad$ $0 \leq 3x \leq 6$

TUM

integral polyhedron.



There are examples where we have integral polyhedra defined by a system $P = \{x \mid Ax \leq b\}$ where A is <u>not</u> TUM, but P is integral for specific values of the vector b.

$$\underbrace{3}_{A} x \leq \underbrace{6}_{b} \qquad\qquad 3x \leq \boxed{7}$$

$$1 \cdot x \leq 10$$

<u>Proposition</u> If A is TUM then for all integral vector a, b, c, d the polyhedron $P = \{x \mid a \leq x \leq b,\ c \leq Ax \leq d\}$ is integral.

<u>Proof</u>

$$P = \left\{ x \;\middle|\; \underbrace{\begin{pmatrix} A \\ -A \\ I \\ -I \end{pmatrix}}_{\mathbb{A}} x \le \underbrace{\begin{pmatrix} d \\ -c \\ b \\ -a \end{pmatrix}}_{\mathbb{b}} \right\}$$

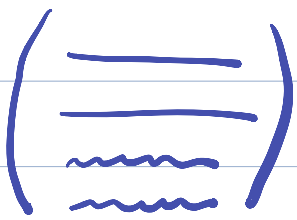
$\mathbb{A}$: sufficient to prove that it is TUM

$\mathbb{b}$: integer values

- square submatrix $\rightsquigarrow$ inside A, or $-A$, or I, or $-I$. then det in $\{0, +1, -1\}$

- some rows from A and some from $-A$

*

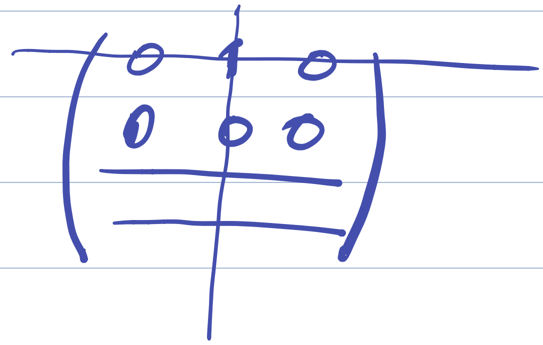


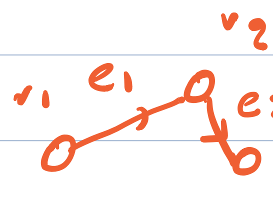
$\to$ if any two rows are "the same" with a different sign $\rightsquigarrow$ det $= 0$

$\to$ if all rows are different $\rightsquigarrow$ same value as det of submatrix of A potentially with diff. sign $\{+1, -1, 0\}$

- Some rows from I and some from A.

*





$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline v_1 & -1 & 0 \\ v_2 & 1 & -1 \\ v_3 & 0 & 1 \\ v_4 & 0 & 0 \end{array}$$

Example Consider the graph adjacency matrix, we will prove it is TUM.

$$M^T = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ \\ v_m \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \\ \begin{pmatrix} +1 & 0 & & \\ 0 & +1 & & \\ \vdots & -1 & & \\ -1 & 0 & & \\ 0 & \vdots & & \end{pmatrix} \end{array} \qquad M_{e,v} = \begin{cases} 1 & \text{if } e \text{ enters } v \\ -1 & \text{if } e \text{ exits } v \\ 0 & \text{otherwise} \end{cases}$$

Note that in each column we have exactly one +1 and exactly one -1.

Consider any square submatrix.

1) if there is any row or column all zero $\Rightarrow \det = 0$
2) if there exists a row or column with a unique +1 or -1 expand along this row or column $\rightsquigarrow$ compute det of a smaller submatrix
3) every column has at least 2 nonzero elements

$$\begin{bmatrix} +1 & 0 & -1 \\ -1 & +1 & 0 \\ 0 & -1 & +1 \end{bmatrix}$$

$\rightsquigarrow$ because every column has exactly one +1 & -1
sum of rows = 0
rows are linearly depend
$\Rightarrow \det = 0$

Min-cut Max-flow problem

Max flow problem

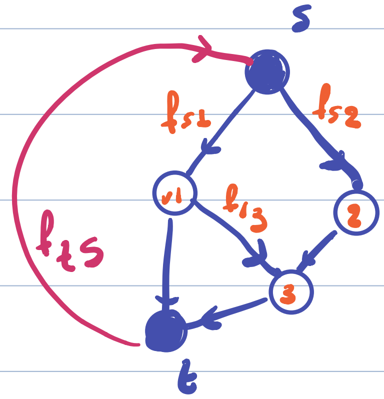
dual variables

$$\min \; -e_m^T f$$

edges d ← s.t. $(I \;\; 0) f \le c$ ←

vertices p ← $M^T f \le 0$ ←

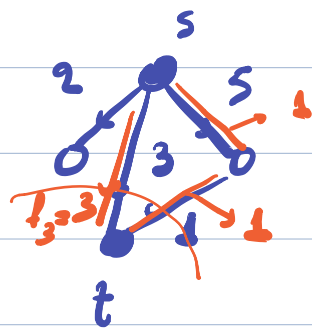
v ← $-I f \le 0$ ←



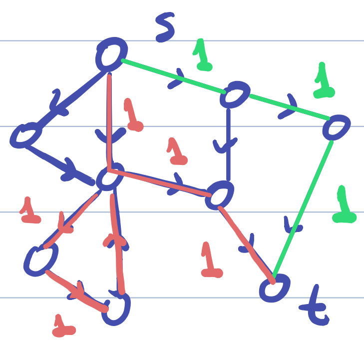
- If the capacities take integer values $\Rightarrow$ feasible set is an integral polyhedron.

$\Downarrow$

max flow will take integer values



- If the capacities are all 1 (unit rate edges)



f_{ij}^* in $\{0, 1\}$

- either an edge is going to be used at capacity or not at all.

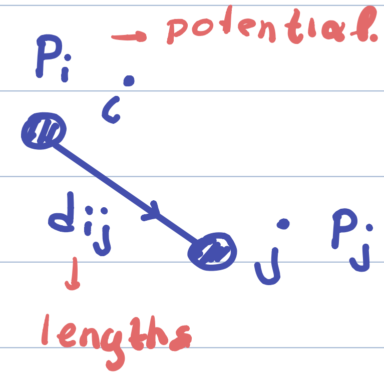
edge disjoint paths → have no common edge.

Claim: in a graph with unit capacity edges we can find h edge-disjoint paths that connect s to t, where h = mincut between s & t.

Dual LP

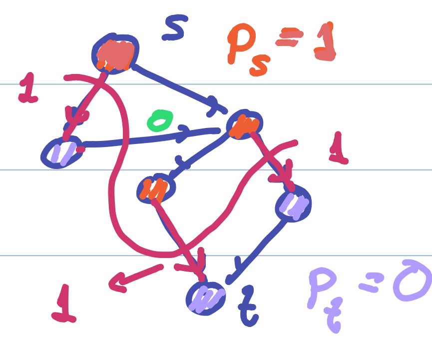
$$\min\ c^T d$$
$$\text{s.t.}\quad p_s - p_t \geq 1$$
$$d_{ij} \geq p_i - p_j$$
$$d_{ij} \geq 0,\ p_i \geq 0$$

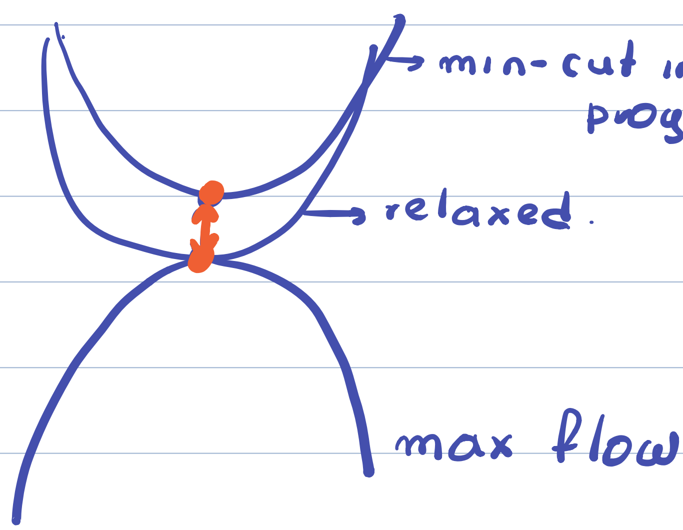
interpretation:



Integer LP

$$\min\ c^T d$$
$$\text{s.t.}\quad p_s - p_t \geq 1 \leftarrow$$
$$d_{ij} \geq p_i - p_j \leftarrow$$
$$d_{ij} \in \{0, 1\}$$
$$p_i \in \{0, 1\}$$

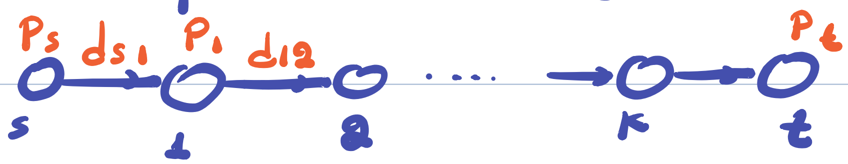




strong duality →
max flow = min cut

max flow LP	dual-relaxed min-cut LP
$\max f_{ts}$ st $\sum f_{ki} - \sum f_{ij} \leq 0$ $f_{ij} \leq c_{ij}$ $f_{ij} \geq 0$	$\min \sum c_{ij} d_{ij}$ st $p_s - p_t \geq 1$ $d_{ij} \geq p_i - p_j$ $d_{ij}, p_i \geq 0$

Consider a path that connects s to t



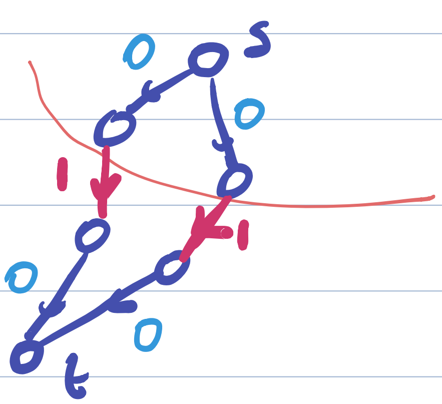
$$\left.\begin{aligned} d_{s1} &\geq p_s - \cancel{p_1} \\ d_{12} &\geq \cancel{p_1} - \cancel{p_2} \\ &\vdots \\ d_{kt} &\geq \cancel{p_k} - p_t \end{aligned}\right\} \quad d_{s1} + d_{s2} + \cdots + d_{kt} \geq p_s - p_t \geq 1$$

"length" of each path in a feasible solution has to be at least 1

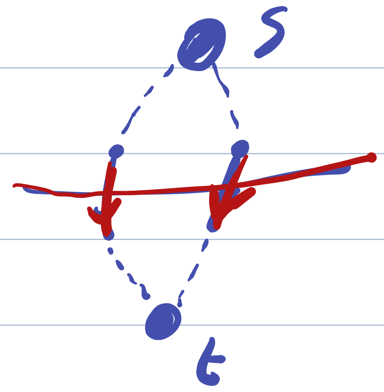
length of path = sum of lengths of edges

"fractional cuts"

integer min-cut



KKT conditions: $d_{ij}(f_{ij} - c_{ij}) = 0$



if $d_{ij} = 1 \Rightarrow f_{ij} = c_{ij}$

$\downarrow$

edge ij is in the mincut

max flow must use this edge at its capacity

Quiz - problem 3

$$P1 \quad \min \; c^T x \quad \text{st} \quad \underset{m\times n}{A}\, x = b, \quad x \geq 0, \quad x \in \mathbb{R}^{n}$$

$$D1 \quad \max \; -b^T y \quad \text{st} \quad A^T y + c \geq 0, \quad y^* \in \mathbb{R}^m$$

$x^* \rightarrow c_I^T x_I^* = -b^T y_I^*$

$n - n_I$

$y^* = y_I^*$

$$P2 \quad \min \; c_I^T x_I \quad \text{st} \quad \underset{m \times n_I}{A_I}\, x_I = b, \quad x_I \geq 0, \quad x_I^* \in \mathbb{R}^{n_I}$$

$$D2 \quad \max \; -b^T y_I \quad \text{st} \quad A_I^T y_I + c_I \geq 0, \quad y_I^* \in \mathbb{R}^m$$

$$c_I^T x_I^* = -b^T y_I^*$$

(*) $c_i + a_i^T y_I^* \geq 0$ then found optimal sol. to P1

Yes, because

- y_I^* is feasible in D1 (optimal in D2) and achieves same value as in D2
- we will prove y_I^* is optimal in D1 by finding an x^* optimal in P1 } certificate of optimality

$$x^* = \begin{cases} x_{iI} & i \in I \\ 0 & \text{otherwise} \end{cases}$$