Lecture 9

Duality

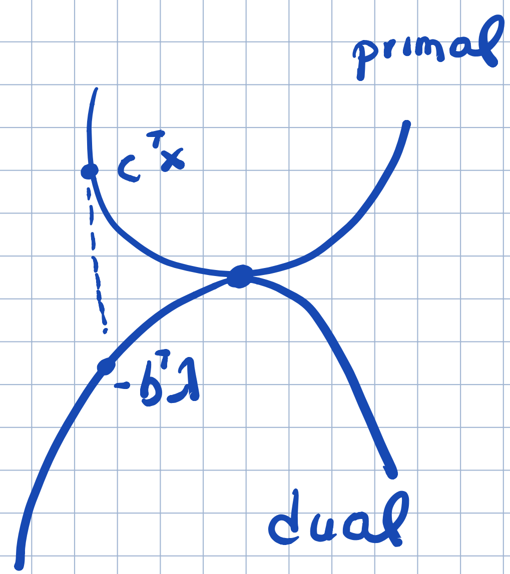
Primal	Dual
$\min c^T x$	$\max -b^T \lambda$
s.t. $\underset{m\times n}{A} x \leq b$	s.t. $A^T \lambda = -c$, $\lambda \geq 0$
$\begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} x \leq b$	$\underset{n\times m}{\begin{bmatrix} \mid & & \mid \\ a_1 & \cdots & a_m \\ \mid & & \mid \end{bmatrix}}$
$x \in \mathbb{R}^n$	$\lambda \in \mathbb{R}^m$
x^*, p^*	λ^*, q^*

($m \geq n$ pointed polyhedra)

1) Weak duality

$c^T x \geq -b^T \lambda$ for any feasible x and λ

$p^* \geq q^*$ because x^*, λ^* are feasible.



2) Strong duality

$p^* = q^*$ provided one of the two LPs is feasible.

3) Complementary slackness (and KKT conditions)

Primal $\longleftrightarrow$ **Dual**

Primal	Dual
$\min\ c^T x$	$\max\ -b^T \lambda$
s.t. $Ax \leq b$	s.t. $A^T \lambda + c = 0$
	$\lambda \geq 0$

$$\left.\begin{array}{l}
\bullet\ x^* \text{ feasible} \quad (Ax^* \leq b) \\
\bullet\ \lambda^* \text{ feasible} \quad (A^T\lambda^* + c = 0,\ \lambda^* \geq 0) \\
\left.\begin{array}{l}
\bullet\ \text{if } \lambda_i^* > 0 \Rightarrow a_i^T x^* - b_i = 0 \\
\ \ \text{if } a_i^T x^* < b_i \Rightarrow \lambda_i^* = 0
\end{array}\right\}
\end{array}\right\} \text{KKT}$$

Primal		dual
$\min\ c^T x$		$\max\ -b^T z - d^T y$
s.t. $Ax \leq b$	z	s.t. $A^T z + G^T y + c = 0$
$Gx = d$	y	$z \geq 0$

- x^* feasible $\Rightarrow$ $Ax^* \leq b$, $Gx^* = d$
- z^*, y^* feasible $\Rightarrow$ $A^T z^* + G^T y^* + c = 0 \Rightarrow -c^T = z^{*T} A + y^{*T} G$, $z^* \geq 0$
- duality gap is zero

$$c^T x^* = -b^T z^* - d^T y^* \Rightarrow$$

$$(-z^{*T} A - \cancel{y^{*T} G})\, x^* + b^T z^* + \cancel{d^T y^*} = 0$$

$$\Rightarrow \quad \underset{\geq 0}{z^{*T}} \underbrace{(Ax^* - b)}_{\leq 0} = 0 \quad \Rightarrow \begin{pmatrix} \text{if } z_i > 0 \Rightarrow a_i^T x - b_i = 0 \\ \text{if } a_i^T x - b_i < 0 \Rightarrow z_i = 0 \end{pmatrix}$$

complementary slackness

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primal
$\min \; c^T x$
st $\quad Ax = b$
$\quad x \geq 0$

dual
$\max \; -b^T \lambda_1$
st $\quad A^T \lambda_1 + c = \lambda_2$
$\quad \lambda_2 \geq 0$

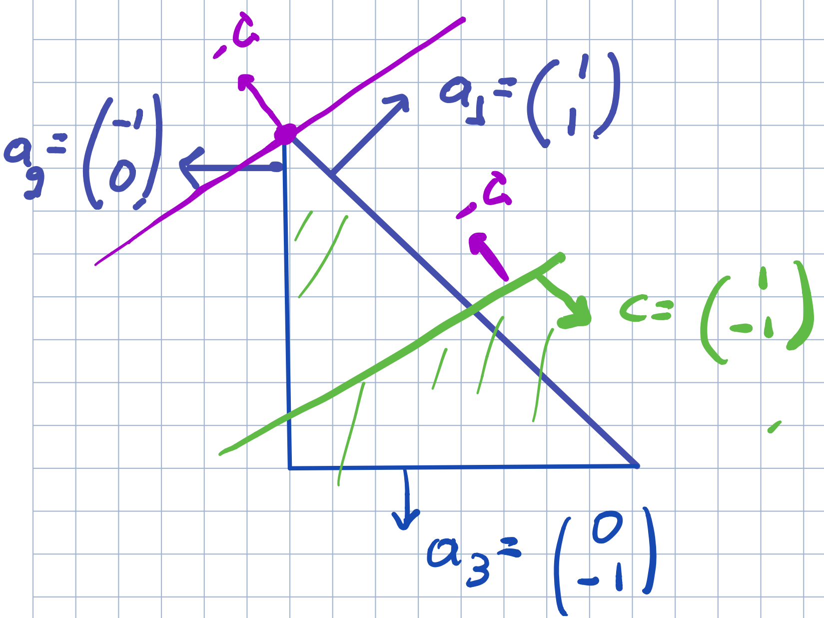
- x^* feasible
- λ_1^*, λ_2^* feasible
- duality gap is zero $\quad c^T x^* = -b^T \lambda_1^*$

⋮

complementary slackness : either $\lambda_{2,i}^* = 0$ or $x_i^* = 0$

Geometric Interpretation.

example we will use.



primal

$$\min \; (1 \;\; -1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t.

$$\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

dual

$$\max \; -\lambda_1$$

s.t. $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \lambda_1 - \lambda_2 \\ \lambda_1 + \lambda_3 \end{pmatrix} = 0$

$\lambda_1, \lambda_2, \lambda_3 \geq 0$

Optimal point vertex

$p(x) = c^T x$

move towards direction of $-c$

We will assume that the feasible set is a pointed polyhedron and the optimal value occurs at a vertex.

- Let x^* and λ^* be optimal (satisfy KKT)
- x^* is a vertex that satisfies with equality the constraints in the set

$$J = \{ i \mid a_i^T x^* = b_i \}$$

2 cases

1) $|J| = n$ nongenerate vertex

2) $|J| > n$ degenerate vertex.

$$\lambda_i^* = \begin{cases} 0 & i \notin J \\ \geq 0 & i \in J \end{cases}$$ from compl. slackness

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<u>Assume</u> primal has a unique solution at a nondegenerate vertex $|J| = n$

λ^*, $\quad A^T \lambda^* = -c \qquad -c = a_1 \lambda_1^* + \dots a_m \lambda_m^*$

$\lambda_i^* \geq 0$

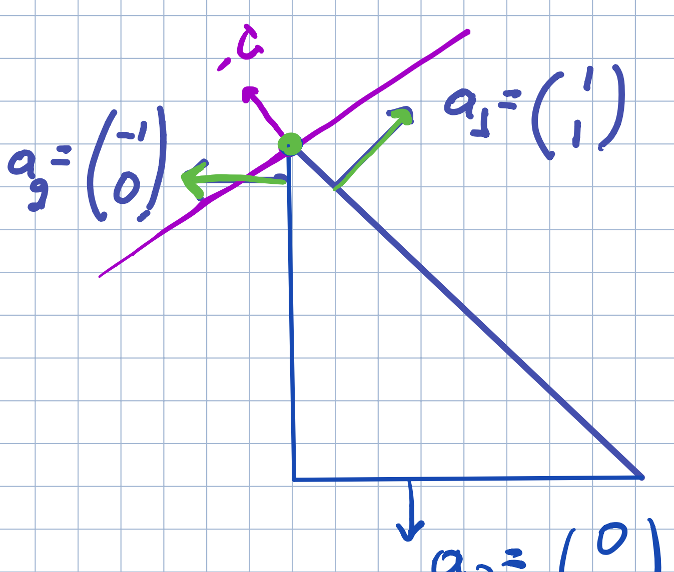
$-c$ belongs in the cone generated by $a_1 \dots a_m$.

$$\left[\underbrace{A_J^T}_{n \times n} \,\middle|\, 0 \right] \begin{bmatrix} \lambda_J^* \\ 0 \end{bmatrix} = -c$$

full rank set of equations
n-variables
n-equations

There is a unique solution λ^* for the dual.

Example



$-c = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\overset{-c}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} = \lambda_1 \overset{a_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} + \lambda_2 \overset{a_2}{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}$$

$\lambda_1, \lambda_2 \geq 0$

unique solution

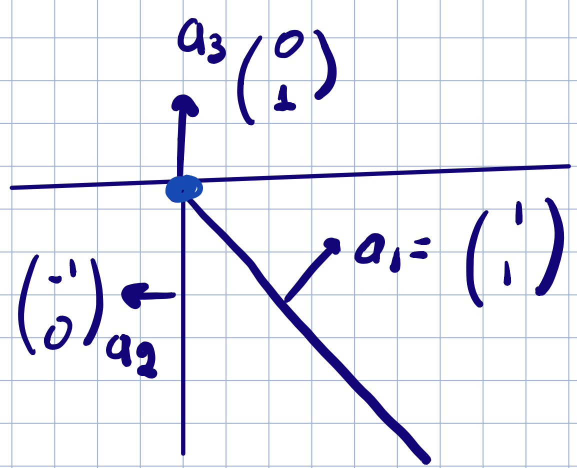
$\lambda_1^* = 1 = \lambda_2^*$

$a_3 \quad (-1)$

Primal		**dual**
Unique solution at a non-degenerate vertex	$\longleftrightarrow$	unique solution at a non-degenerate vertex

Assume a degenerate vertex $|J| > n$

in our example, we have an additional constraint

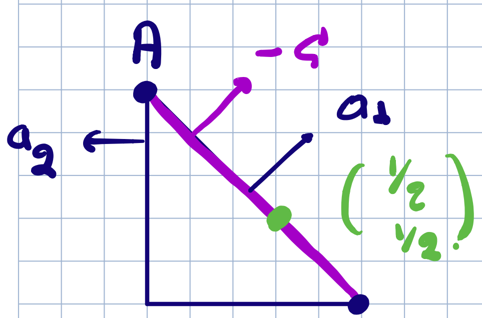


$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda_1^* \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2^* \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_3^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

multiple solutions for the dual

If primal has a unique solution at a degenerate vertex	$\longleftrightarrow$	dual has multiple solutions

Primal has multiple solutions



$-c$ is parallel to $a_1 \Rightarrow$ all points in this face (including the two vertices) will be optimal.

Say $-c = 2\begin{pmatrix}1\\1\end{pmatrix}$

- At vertex A

$$-c = \begin{pmatrix}2\\2\end{pmatrix} = \lambda_1^*\begin{pmatrix}1\\1\end{pmatrix} + \lambda_2^*\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \begin{matrix}\lambda_1 = 2\\ \lambda_2 = 0\end{matrix}$$

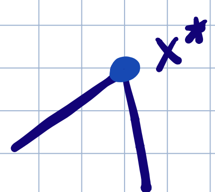
I don't need a_2 to reconstruct $-c$

- At point $\begin{pmatrix}1/2\\1/2\end{pmatrix}$

- $\lambda_1^* \geq 0$, $\lambda_2^* = \lambda_3^* = 0$

$$-c = \begin{pmatrix}2\\2\end{pmatrix} = \lambda_1^*\begin{pmatrix}1\\1\end{pmatrix}, \quad \lambda_1^* = 2$$

If I find that a vertex x^* of the primal is optimal, how can I tell whether this is a unique solution or not?



$$J = \{ i \mid a_i^T x^* = b_i \} \qquad |J| = n$$

$\lambda^* \rightarrow$ has at most n nonzero values

if λ^* has zero at some index $k \in J$ $\lambda_k = 0$

$-c$ can be expressed as a linear comb $\underbrace{\text{of } n-1 \text{ constraints}}_{\text{define a face}}$

for all points in this face we can find dual variables that satisfy $k \notin T$.

Primal will have multiple opt. solutions.

(We will give algebraic criteria for multiple solutions when discussing decoding.)

Primal		dual
• unique solution at a non-deg vertex	$\longleftrightarrow$	unique solution at a non-degenerate vertex
• unique solution at a degenerate vertex	$\longleftrightarrow$	multiple solutions
• multiple solutions	$\longleftrightarrow$	unique solution at a degenerate vertex

What if we start from the dual problem.

$$\max \; -b^T\lambda$$
$$\text{s.t.} \quad c + A^T\lambda = 0 \;\rightarrow\; n \text{ equalities} \qquad \lambda \in \mathbb{R}^m$$
$$\lambda \geq 0 \;\rightarrow\; m \text{ inequalities}$$

Assume λ^* is a unique solution at a <u>non-degenerate</u> vertex of the dual feasible set.

n equalities $c + A^T\lambda = 0$ $\Rightarrow$ exactly n

$m-n$ inequalities become equality $\lambda_i^* = 0$ | λ_i^* are nonzero.

exactly n constraints are active at the primal optimal solution $\Rightarrow$ unique solution in a nondegenerate vertex

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Example 1

- Express as an LP:

$$\min \| A x - b \|_\infty, \qquad A \;\; m \times n, \quad x \in \mathbb{R}^n$$

- Find the dual and show it is equivalent to the problem:

$$\begin{aligned} \max \;& -b^T z \\ \text{st} \;& A^T z = 0 \\ & \|z\|_1 \le 1 \end{aligned}$$

Recall $\|x\|_\infty = \max_i |x_i| = \max_i \{ x_i, -x_i \}$

$$\min \; \overbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}^{c^T} \begin{pmatrix} x \\ t \end{pmatrix} \qquad x \in \mathbb{R}^n, \; t \in \mathbb{R}$$

$$\text{st} \quad \begin{pmatrix} A & -1 \\ -A & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \le \begin{pmatrix} b \\ -b \end{pmatrix} \begin{matrix} \to u \\ \to v \end{matrix} \qquad \lambda = \begin{pmatrix} u \\ v \end{pmatrix}$$

$\mathbb{A}$ $\quad$ $\mathbb{b}$

dual

(1)

$$\begin{aligned} \max \quad & -b^T u + b^T v \\ \text{s.t.} \quad & A^T u - A^T v = 0 \\ & \mathbb{1}^T u + \mathbb{1}^T v = 1 \\ & u, v \geq 0 \end{aligned}$$

(2)

$$\begin{aligned} \max \quad & -b^T z \\ \text{s.t.} \quad & A^T z = 0 \\ & \|z\|_1 \leq 1 \end{aligned}$$

Proof of equivalence

We will show that for each feasible solution of one we find a feasible solution of the other that achieves same obj. va

$(1) \rightarrow (2)$

Let u and v be feasible in (1)

Set $z = u - v$ (this will be our dual variable)

$$A^T(u-v) = 0 \Rightarrow A^T z = 0$$

$$\|z\|_1 = \|u - v\|_1 \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\leq} \|u\|_1 + \|v\|_1 = \underbrace{\sum u_i + \sum v_i}_{=1}$$

z is feasible in (2) and achieves the same obj. value.

$(2) \rightarrow (1)$

$$A^T z = 0$$

Assume z is feasible in (2) $\Rightarrow$ $\|z\|_1 \leq 1$

positive elements of z $\quad u_i = \max\{0, z_i\} + a$ constant

negative elements of z $\quad v_i = \max\{0, -z_i\} + a$

these u and v satisfy $\quad z = u - v$

$$u, v \geq 0$$

We want $\quad 1^T u + 1^T v = 1$

$$1^T u + 1^T v = \sum |z_i| + 2ma \doteq 1$$

select $\quad a = \dfrac{1 - \|z\|_1}{2m} \geq 0$

example

$$z = \begin{pmatrix} 0.1 \\ -0.3 \\ -0.2 \end{pmatrix} \qquad u = \begin{pmatrix} 0.1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ a \\ a \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ +0.3 \\ +0.2 \end{pmatrix} + \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$