Lecture 2

Today:

1) Equivalence
2) Approximation

Equivalent: by finding the optimal solution of one we can find the solution of the other & vice versa.

Approximate: by solving an LP we get an approximate solution for the original problem

Integer LP (ILP), x_i take integer values

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① General form	② Standard form	③ Inequality form
$\min c^T x$	$\min c^T x$	$\min c^T x$
st $Ax \leq b$	st $Ax = b$	st $Ax \leq b$
$Cx = d$	$x \geq 0$	

Basic equivalence transformations

- transformation of objective function

$$\min c^T x + d \rightsquigarrow \min c^T x$$
$$\max -c^T x \rightsquigarrow \min c^T x$$

- Constraints

1) write $a_i^T x \leq b_i$ as equality:

introduce variable $s_i \geq 0$ slack variable

replace with two constraints:

$$a_i^T x + s_i = b_i$$
$$s_i \geq 0$$

2) Constraints of the form $a_i^T x \geq b$ as equality:

$$a_i^T x - s_i = b_i$$
$$s_i \geq 0$$

3) Constraints of the form $a_i^T x = b_i$ as inequality:

$$a_i^T x \leq b_i$$
$$a_i^T x \geq b_i$$

4) Making all variables non-negative

$$x_i = x_i^+ - x_i^- \,, \quad x_i^+, x_i^- \geq 0$$

Example 1: from ③ inequality form to ② standard form

$$\begin{array}{l} \min\ c^T x \\ \text{st}\ \ \underset{m \times n}{A} x \leq b \end{array} \quad \rightsquigarrow \quad \begin{array}{l} \min\ c^T x \\ \text{st}\ \ Ax = b \\ \quad\ \ x \geq 0 \end{array}$$

using

$$Ax + s = b$$
$$s \geq 0$$

$$\begin{pmatrix} x \\ s \end{pmatrix}$$

$$\min\ [c^T\ \ 0] \begin{pmatrix} x \\ s \end{pmatrix}$$

$$\text{st}\ \ [A\ \ I] \begin{pmatrix} x \\ s \end{pmatrix} = b$$

$$s \geq 0$$

still need non-negativity for all variables

$$x \rightsquigarrow \begin{pmatrix} x^+ \\ x^- \end{pmatrix},\ x^+, x^- \geq 0$$

$$x = x^+ - x^-,\quad c^T x = c^T x^+ - c^T x^-$$

vector of variables becomes

$$\begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix}$$

$$\min\ \tilde{c}^T y$$
$$\text{st}\ \ \tilde{A} y = b$$
$$y \geq 0$$

$$\min\ \overbrace{(c^T\ \ -c^T\ \ \ 0)}^{\tilde{c}^T} \overbrace{\begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix}}^{y}$$

$$\text{st}\ \ \underbrace{[A\ \ -A\ \ \ I]}_{\tilde{A}} \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = b,\ \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} \geq 0$$

Examples

1) $\min\ c^T x$

s.t. $0 \le x \le 1$

$$x_i^* = \begin{cases} 1 & \text{if } c_i < 0 \\ 0 & \text{if } c_i \ge 0 \end{cases}$$

$$p^* = \sum_{i=1}^{n} \min\{c_i, 0\}$$

2) ① general form $\rightsquigarrow$ ② standard form

①	②
$\min\ c^T x$	$\min\ c^T x$
s.t. $Ax \le b$	s.t. $Ax = b$
$Dx = f$	$x \ge 0$

i) make variables nonnegative

$$x_1 = \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \in \mathbb{R}^{2n}$$

$$\min\ \begin{pmatrix} c^T & -c^T \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \end{pmatrix}$$

s.t.

$$\begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \le b$$

$$\begin{pmatrix} D & -D \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \end{pmatrix} = f$$

$$\begin{pmatrix} x^+ \\ x^- \end{pmatrix} \ge 0$$

ii) make inequality $\boxed{x_2 = \begin{pmatrix} x_1 \\ s \end{pmatrix}} \in \mathbb{R}^{2n+m}$

$$c_2 = (c^T \ -c \ \ 0)$$

$$\min \ c_2^T x_2$$

$$\text{st} \quad \underbrace{(A \ -A \ \ I)}_{A_2} \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = b$$

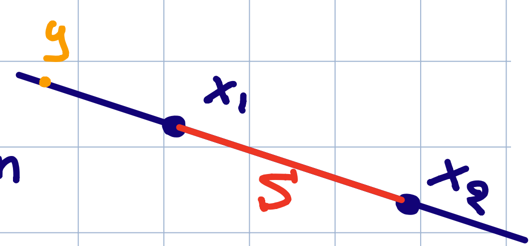
$$\underbrace{(D \ -D \ \ 0)}_{D_2} \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = f$$

$$\begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} \geq 0$$

We will now introduce some classes of functions that lead to equivalent or approximate LPs.

Notation/background

1) Line and line segment



Consider two points, $x_1, x_2 \in \mathbb{R}^n$

Line that goes through x_1 & x_2

$$\mathcal{L} = \{ y \mid y = \theta x_1 + (1-\theta) x_2 \}$$

$\theta = 0, \quad y = x_2$

$\theta = 1 \quad y = x_1$

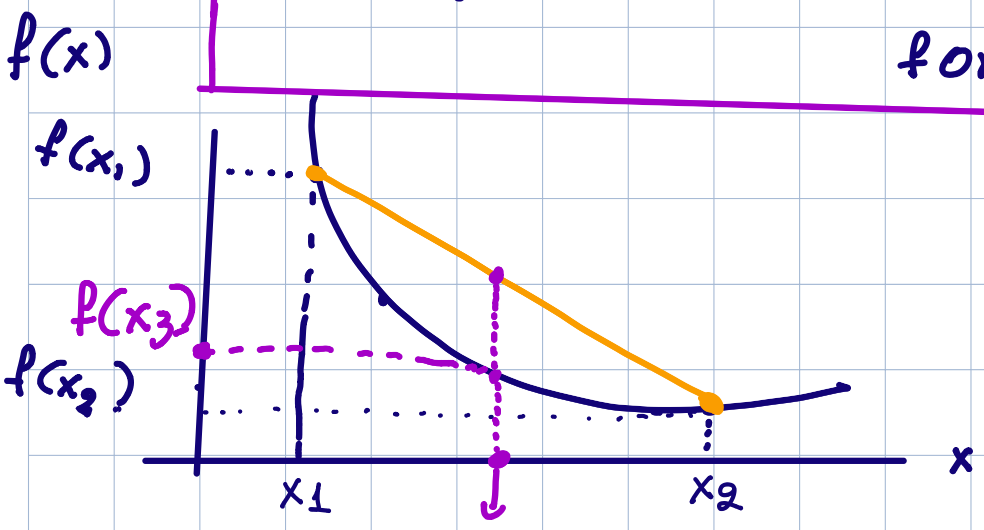
Line segment between x_1 and x_2:

$$S = \{ y \mid y = \theta x_1 + (1-\theta) x_2, \ 0 \leq \theta \leq 1 \}$$

2) Convex function $\quad f: \mathbb{R}^n \to \mathbb{R}$

A function is convex if

$$\boxed{f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2) \quad \text{for } 0 \leq \theta \leq 1 \text{ and all } x_1, x_2}$$



$x_3 = \theta x_1 + (1-\theta)x_2$

3) Linear functions $\quad f(x) = c^T x$

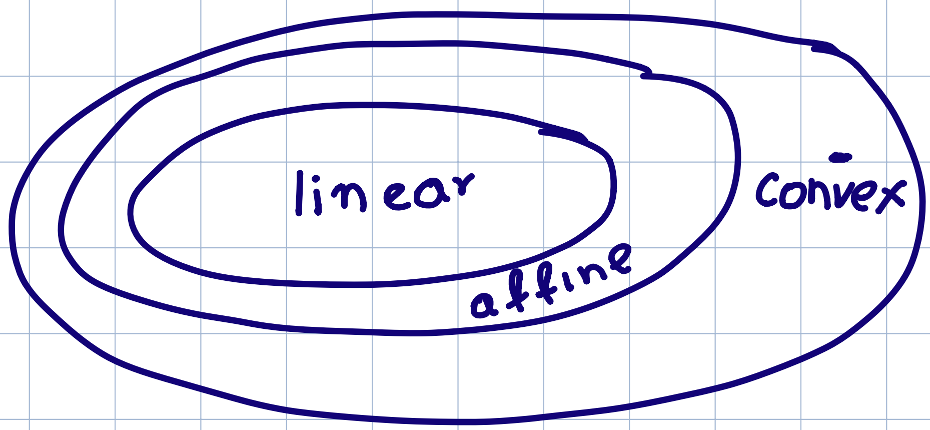
$$f(ax + by) = a f(x) + b f(y)$$ for all $x, y \in \mathbb{R}^n$, $a, b \in \mathbb{R}$

4) Affine functions

$$f(x) = c^T x + d$$

$$f(ax + (1-a)y) = a f(x) + (1-a) f(y)$$

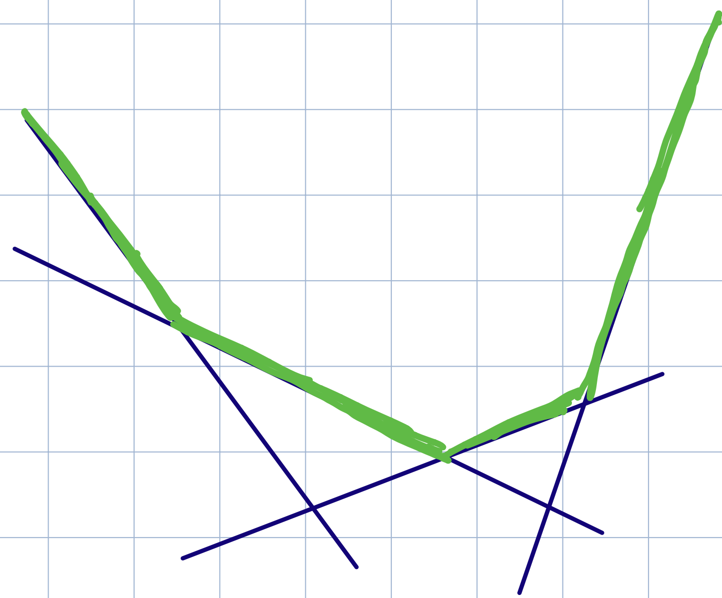
for all $a \in \mathbb{R}$, $x, y \in \mathbb{R}^n$



5) Piecewise linear functions

$f: \mathbb{R}^n \to \mathbb{R}$

$$f(x) = \max_{i=1,\dots,m} (a_i^T x + b_i)$$



Piecewise-linear functions are convex.

Proof

$$0 \leq \theta \leq 1$$

$$f(\theta x_1 + (1-\theta) x_2) \overset{?}{\leq} \theta f(x_1) + (1-\theta) f(x_2)$$

$$f(\theta x_1 + (1-\theta) x_2) = \max_i \{ a_i^T(\theta x_1 + (1-\theta) x_2) + b_i \}$$

$$= \max_i \left\{ \theta (a_i^T x_1 + b_i) + (1-\theta)(a_i^T x_2 + b_i) \right\}$$

$$\leq \theta \max_i (a_i^T x_1 + b_i) + (1-\theta) \max_i (a_i^T x_2 + b_i)$$

$$= \theta f(x_1) + (1-\theta) f(x_2)$$

because

$$\max_i \{ a_i + b_i \} \leq \max_i a_i + \max_i b_i$$

6) Norms

"express distance"

"length"

A function $f: \mathbb{R}^n \to \mathbb{R}$ is called a norm iff

i) $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ nonnegative

ii) $f(x) = 0 \iff x = 0$ definite.

iii) $f(ax) = |a| f(x)$, $a \in \mathbb{R}$, homogeneous

iv) $f(x+y) \leq f(x) + f(y)$ triangle inequality

ℓ_2-norm
Euclidean

$$\|x\| = \|x\|_2 = \sqrt{x^T x} = \sqrt{\sum x_i^2}$$

ℓ_1-norm

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

ℓ_p-norm

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p\right)^{1/p}$$

$$p \geq 1$$

ℓ_∞-norm

$$\|x\|_\infty = \max_i \{|x_1|, |x_2|, \dots, |x_n|\}$$

For finite-dimensional vectors x, all the norms are "equivalent", in the following sense:
There exist constants A and B such that

$$A \|x\|_q \leq \|x\|_p \leq B \|x\|_q$$

break

Prove that $x^T y \le \|x\|_1$ for all y with $\|y\|_\infty \le 1$.

$$x^T y = \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} |x_i||y_i| \le \underbrace{\max_i |y_i|}_{} \underbrace{\sum_{i=1}^{n} |x_i|}_{} =$$

$$= \|y\|_\infty \cdot \|x\|_1$$

$$\le \|x\|_1$$

Piecewise linear minimization can be expressed as (is equivalent to) an LP.

$$(1)\quad \boxed{\min_{x} f(x) = \min_{x}\left\{\max_{i=1..m}(a_i^T x + b_i)\right\}}\quad p_1^*$$

$$(2)\quad \boxed{\begin{aligned} &\min_{x,t}\ t \\ &\text{s.t.}\quad a_i^T x + b_i \le t \qquad i=1,\dots,m \end{aligned}}\quad p_2^*$$

To prove that (1) and (2) are equival.
we will prove that

$$\text{and}\quad \left.\begin{aligned} p_2^* &\le p_1^* \\ p_1^* &\le p_2^* \end{aligned}\right\} \Rightarrow p_1^* = p_2^*$$

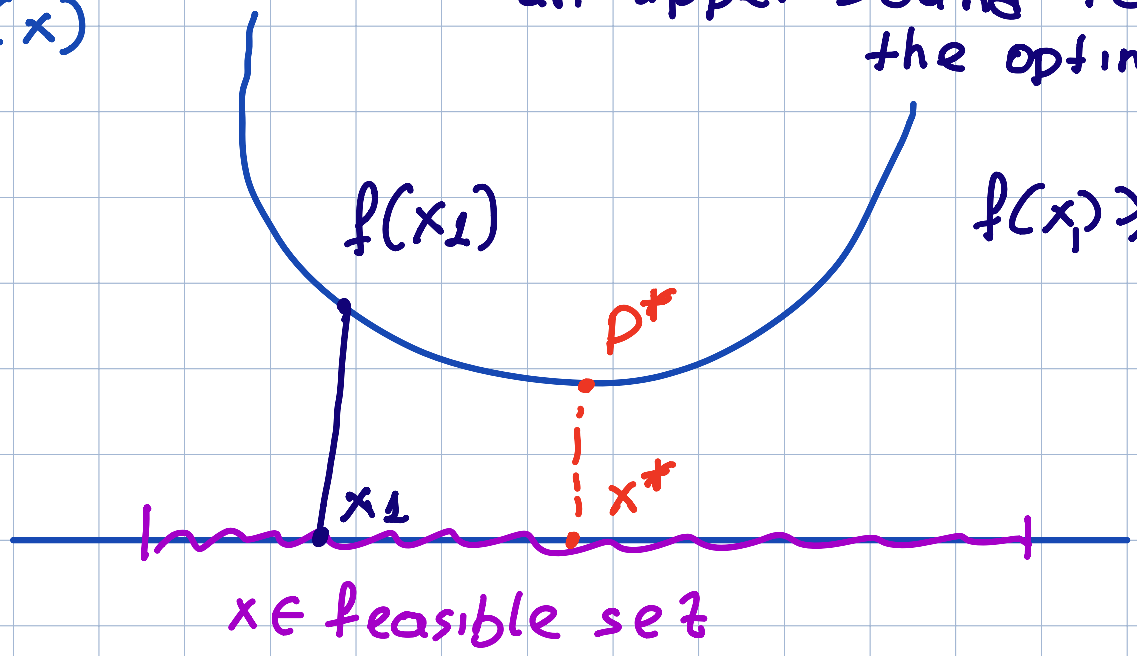
To prove this, we will use the following observation

$\min \ f(x)$

feasible value gives an upper bound to the optimal value

$f(x_1) \geq p^*$

s t



Assume we solve (1) and find some optimal value x^*, $p_1^* = f(x^*)$

Create a feasible solution for the second problem as follows:

let $t = \max_i (a_i^T x^* + b_i) \Rightarrow a_i^T x^* + b_i \leq t$ for all i

$x = x^*$

the above t and x are feasible in (2) and achieve the same obj. value p_1^*.

Thus, when we solve (2) we will find $p_2^* \leq p_1^*$ (because the optimal value for (2) has to be smaller or equal to any feasible value)

Assume we solve (2) and find x^*, t^* and p_2^*

Observe that at least one of the inequalities has to be satisfied with equality (otherwise we could find smaller obj value)

Select for (1)

$$x = x^*$$

this achieves obj value $p_2^* \Rightarrow p_1^* \leq p_2^*$

$$p_1^* = p_2^*$$

We conclude: $\quad P_1^* = P_2^*$

We can write (2) in matrix format as follows:

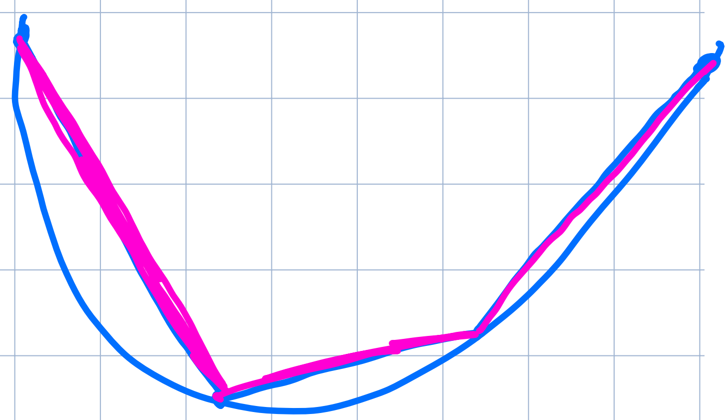
$$\min \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} a_1^T & -1 \\ a_2^T & -1 \\ \vdots & \\ a_m^T & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \leq \begin{pmatrix} -b_1 \\ -b_2 \\ \vdots \\ -b_m \end{pmatrix}$$

Why useful

→ some norms can be expressed as piece-wise linear functions

→ can be used to approximate convex optimization problems



Sum of piecewise functions

$$\min_{x} \Big[\underbrace{\max_{i=1\dots m} (a_i^T x + b_i)}_{t_1} + \underbrace{\max_{i=1\dots K} (c_i^T x + d_i)}_{t_2} \Big]$$

equivalent problem

$$\begin{aligned} \min_{x, t_1, t_2} \quad & t_1 + t_2 \\ \text{s t} \quad & a_i^T x + b_i \leq t_1, \quad i = 1, \dots, m \\ & c_i^T x + d_i \leq t_2, \quad i = 1, \dots, K \end{aligned}$$

Can be expressed in matrix form

$$\min \tilde{c}^T \tilde{x}$$

$$\text{s t} \quad A \tilde{x} \leq \tilde{b}$$

$$\tilde{x} = \begin{pmatrix} x \\ t_1 \\ t_2 \end{pmatrix}, \quad \tilde{c}^T = (0 \quad 1 \quad 1)$$

$$A = \begin{bmatrix} a_1^T & -1 & 0 \\ \vdots & & \\ a_m^T & -1 & 0 \\ c_1^T & 0 & -1 \\ \vdots & & \\ c_K^T & 0 & -1 \end{bmatrix} \qquad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \\ -d_1 \\ \vdots \\ -d_K \end{bmatrix}$$

Norm minimization

Assume $x \in \mathbb{R}$ (scalar). $\|x\|_2 = |x| = \|x\|_\infty = \|x\|_1$

Note that

$|x| = \max\{x, -x\}$

$$\min \; |x|$$
$$\text{st} \quad a_i x \leq b_i, \; i = 1, \dots, m$$

$$\min \; t$$
$$x \leq t$$
$$-x \leq t$$
$$a_i x \leq b_i, \quad i = 1, \dots, m$$

$x \in \mathbb{R}^n$

$$\min_x \|x\|_\infty$$

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$
$$= \max\{x_1, -x_1, x_2, -x_2, \dots, x_n, -x_n\}$$

$$\min \; t$$
$$\text{st} \quad x_i \leq t \qquad i = 1, \dots, n$$
$$-x_i \leq t$$

ℓ_1 norm

$$\min_x \|x\|_1$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} \max\{x_i, -x_i\}$$

$$\begin{aligned} \min_{x, t_1, \dots, t_n} \quad & t_1 + t_2 + \dots + t_n \\ \text{st} \quad & x_i \le t_i \qquad i = 1, \dots n \\ & -x_i \le t_i \end{aligned}$$

$$t = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$$

$$\begin{aligned} \min \quad & 1^T t \\ \text{st} \quad & -t \le x \le t \end{aligned} \qquad \begin{aligned} -t_i \le x_i \le t_i \\ i = 1, \dots, n \end{aligned}$$

ℓ_1, ℓ_∞ minimization equivalent to LP

other norms : use approximation

$$A\|x\|_p \le \|x\|_q \le B\|x\|_p$$

<u>Example</u> : using this relations

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$$

we can appr. solve ℓ_2 minimization problem.