Class 15

Simplex Algorithm

Book notes from: LP with matlab by Ferris et al.

Geometric Interpretation

- The simplex algorithm first determines if the feasible region is empty. If empty it declares the problem is not feasible. If not, it finds a vertex of the feasible region

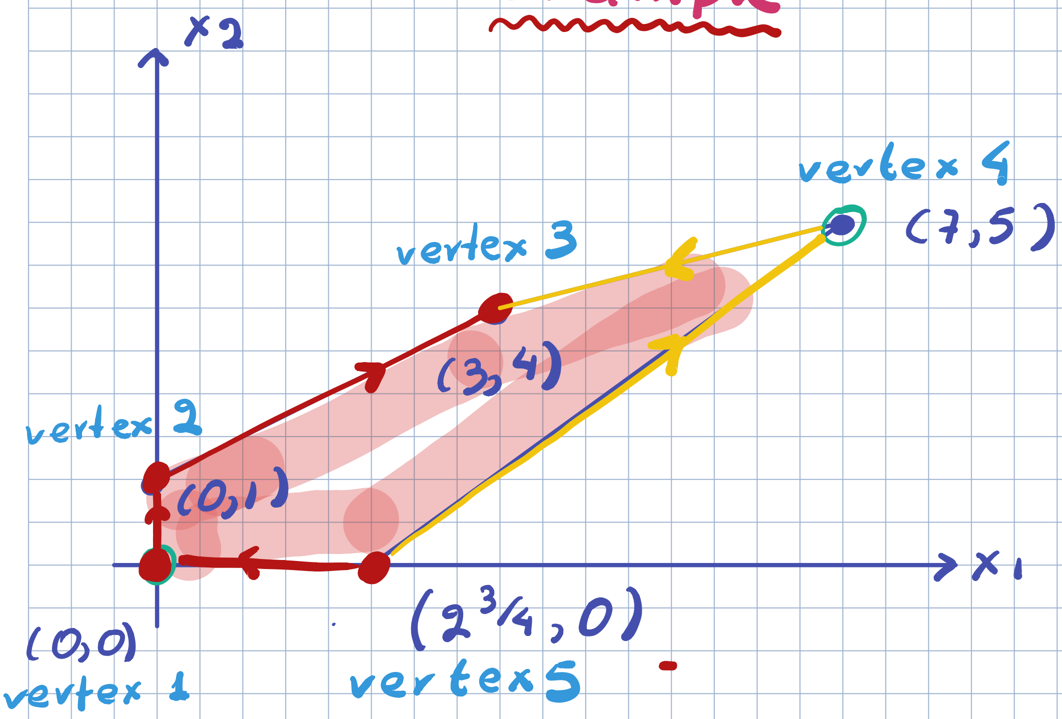
Phase I

checks feasibility
→ outputs vertex if feasible.

- It then starts from a feasible vertex and moves to an adjacent vertex that has a smaller objective value until it can no longer proceed because
(1) it finds a vertex such that all adjacent vertices have a larger objective value, or
(2) detects that the problem is unbounded.

Phase II

Example



Phase II

starts from a vertex, say vertex 5

vertex	cost
5	8.25
1	0
2	-6
3	-15

5	8.25
4	-9
3	-15

$$
\begin{aligned}
\min \quad & z = 3x_1 - 6x_2 \\
\text{st} \quad & x_1 + 2x_2 \geq -1 \\
& 2x_1 + x_2 \geq 0 \\
& x_1 - x_2 \geq -1 \\
& x_1 - 4x_2 \geq -13 \\
& -4x_1 + x_2 \geq -23 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Adjacent vertex → has $n-1$ equalities in common

Note: there may exist multiple paths to follow (hard to say which is the shortest one).

Intuition: because the obj function is linear & feasible region is convex, if we keep following any path with decreasing z value, we will find the opt. value

Digression : Jordan exchange

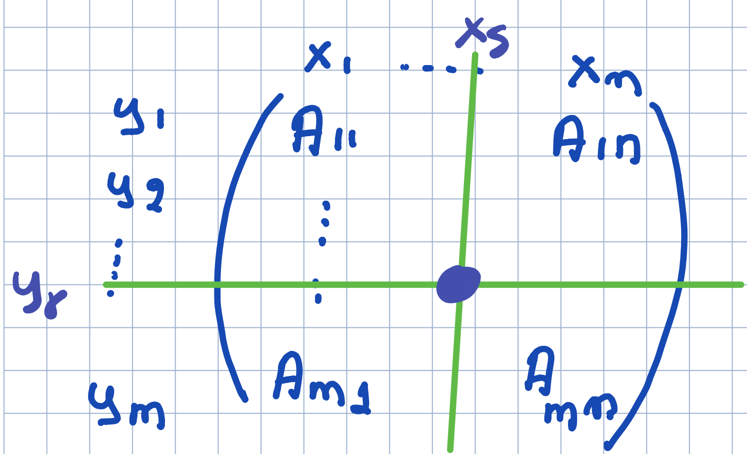
Assume we have a system of equations

$$\underbrace{y}_{\text{dependent variables}} = A \underbrace{x}_{\text{independ variables}}$$

} if we fix x it determines y

Jordan exchange is used to exchange an independent with a dependent variable

eg
$$\begin{aligned} y_1 &= 5 x_1 \\ \underbrace{y_2} &= 2 x_1 + x_2 \end{aligned} \quad \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} x_1 &= \tfrac{1}{5} y_1 \\ \underbrace{y_2} &= \tfrac{2}{5} y_1 + x_2 \end{aligned} \quad \begin{pmatrix} 1/5 & 0 \\ 2/5 & 1 \end{pmatrix}$$



to exchange a variable y_r with a variable x_s we need $A_{rs} \neq 0$

$$y_r = A_{r1} x_1 + \dots + A_{rs} x_s + \dots + A_{rn} x_n$$

$$x_s = \frac{1}{A_{rs}} y_r - \frac{A_{r1}}{A_{rs}} x_1 - \dots - \frac{A_{rn}}{A_{rs}} x_n$$

Substitute x_s in all equations as a function of y_r

Algebraic method for simplex

$$\begin{aligned} \min \quad & p^T x \\ \text{st} \quad & \underset{m\times n}{A}\, x \geq b \\ & x \geq 0 \end{aligned}$$

feasible is a pointed polyhedron $\Rightarrow$ exist vertices

any LP can be expressed in this form $Ax = b$, $x \geq 0$

For now we assume that the LP is feasible and that $x = 0$ is a vertex.

(not a constraining assumption, we will see why when we discuss Phase I)

Introduce slack variables

$z = p^T x$ — objective variable $z \in \mathbb{R}$.

$x_B = Ax - b$ — $x_B \in \mathbb{R}^m$ basic variables

$x_B \geq 0$

$x_N = x$ — $x_N \in \mathbb{R}^n$ non basic variables

So we rewrite the LP as follows

$$\min \quad z = (p^T \;\; 0)\begin{pmatrix} x_N \\ x_B \end{pmatrix}$$

st

$$\rightarrow \; x_B = A x_N - b$$

$$x_B, x_N \geq 0$$

amounts to setting n variables to zero

vertex satisfies n constraints

Create a tableau

x_N

non-basic or independent variables

x_B basic dep. var.

$$\begin{array}{c|cccc|c}
 & x_1 & x_2 & \cdots & x_n & 1 \\
\hline
x_{n+1} & A_{11} & & & A_{1n} & -b_1 \\
x_{n+2} & A_{21} & & & & -b_2 \\
 & & & & & \\
x_{n+m} & A_{m1} & & & A_{mn} & -b_m \\
\hline
z & p_1 & & & p_n & 0
\end{array}$$

→ corresponds to the all zero vertex

We "read a tableau" by setting the non-basic variables to zero, and assigning to the basic and objective variable the values at the last column

Each tableau corresponds to a vertex the vertex that satisfies the constraints corresponding to the non-basic variables

For a tableau to be feasible, the values in the last column have to be positive.

In our example

$$x_1, x_2, \dots, x_7 \geq 0$$

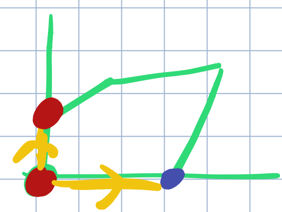
$$z = 3x_1 - 6x_2$$
$$x_3 = x_1 + 2x_2 + 1$$
$$x_4 = 2x_1 + x_2$$
$$x_5 = x_1 - x_2 + 1$$
$$x_6 = x_1 - 4x_2 + 13$$
$$x_7 = -4x_1 + x_2 + 23$$

x_N

	x_1	x_2	
x_3	1	2	1
x_4	2	1	0
x_5	1	-1	1
x_6	1	-4	13
x_7	-4	-1	23
z	3	-6	0

x_B

corresponds to the (0,0) vertex



To go to an adjacent vertex we want to exchange a nonbasic with a basic variable so that the objective value reduces.

Step 1 decide pivot column

Select which variable will become basic (entering variable)

If a variable becomes basic we will allow its value to increase from zero to something positive.

Say we select

$x_1 = \lambda, \quad \lambda \geq 0, \quad x_2 = 0$

$z = 3\lambda > 0$ opposite of what we want

if we select

$x_2 = \lambda, \quad \lambda \geq 0, \quad x_1 = 0$

$z = -6\lambda$ decreases! good choice :)

Generally, all variables that correspond to negative value at last row are good choices for entering variables

"Pricing" : rule we use to select which column with negative P_i to use.

If multiple columns exist select any of them.

Step 2 determine which of the basic variables takes the place of the entering variable (blocking variable)

In our example, $x_2 = \lambda$, $\lambda \geq 0$, $x_1 = 0$

$x_3 = 2\lambda + 1 \geq 0$, holds for all $\lambda \geq 0$

$x_4 = \lambda \geq 0$

$x_5 = -\lambda + 1$, to have $x_5 \geq 0$, we need $\lambda \leq 1$.

$x_6 = -4\lambda + 13$, to have $x_6 \geq 0$, we need $\lambda \leq 13/4$

$x_7 = \lambda + 23 \geq 0$

The first variable that becomes zero as we increase λ is our blocking variable.

The best we can do to increase λ as much as possible and still maintain feasible is to pivot the blocking variable with the entering variable.

Phase II

$$\min \quad z = p^T x_N$$
$$x_B = A x_N - b$$
$$x_B, x_N \geq 0$$

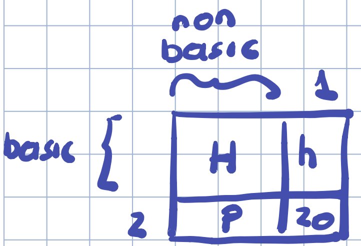


Tableau corresponds to a vertex

"feasible tableau" : all entries in vector in vector h are non-negative

"optimal tableau" : if it is feasible and all entries p in last row are non negative

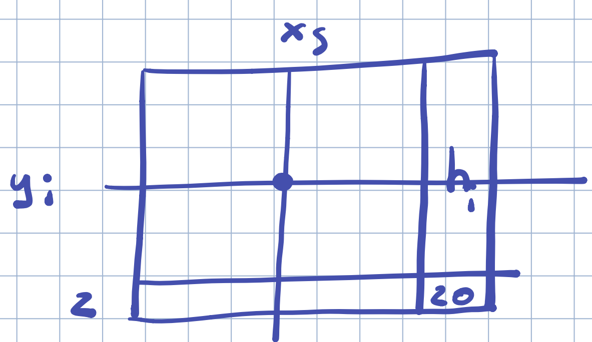
1) select pivot column, entering variable x_s

2) select pivot row, blocking variable x_r

row i: $\quad y_i = H_{is} x_s + h_i \geq 0$

necessary condition for y_i to be blocking variable is that $H_{is} < 0$

if $H_{is} < 0$ then $\quad x_s \leq -\dfrac{h_i}{H_{is}}$



$h_i \geq 0$ for tableau to be feasible

$H_{is} < 0$

ratio test: select pivot row x_r, with

$$-\frac{h_r}{H_{rs}} = \min\left\{-\frac{h_i}{H_{is}},\ H_{is} < 0\right\}$$

Perform pivot between x_s and x_r.
This reduces the obj. value by

$$z = z_0 + \underset{<0}{P_s}\underbrace{\left(-\frac{h_r}{H_{rs}}\right)}_{\geq 0}$$

What happens if all the entries in H_{is} are non-negative?

example

	x_1	x_6	x_3	1
x_4	2	-1	-1	4
x_5	0	-1	-3	5
x_2	1	-1	-2	1
z	-5	3	7	3

$x_6 = x_3 = 0$

$x_1 = \lambda \geq 0$

$z = 3 - 5\lambda$

$x_4 = 2\lambda + 4 \geq 0$

$x_5 = 5 \geq 0$

$x_2 = \lambda + 1 \geq 0$

unbounded problem

no blocking variable!

$z \to -\infty$ as λ increases

direction of
unboundness

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$