EE 236 A — Fall 2020

Prof. Christina Fragouli
TAs: Mine Dogan, Kaan Ozkara

Class Outline

1) How can we express problems as LPs.

2) Geometry of LPs

3) Duality

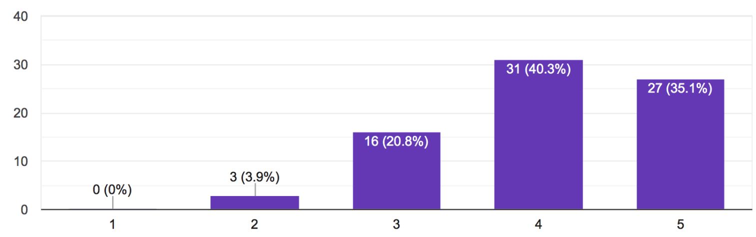
4) Integer Programming and Combinatorial Optimization

5) Algorithms for solving LPs.

Questionnaire

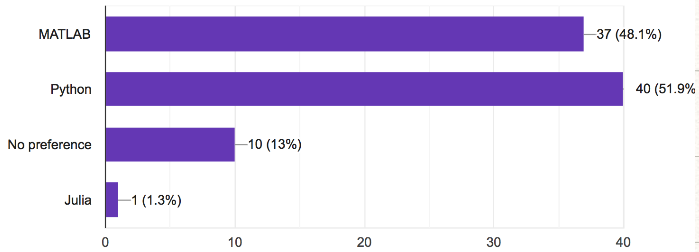
How would you describe your Internet access?

77 responses



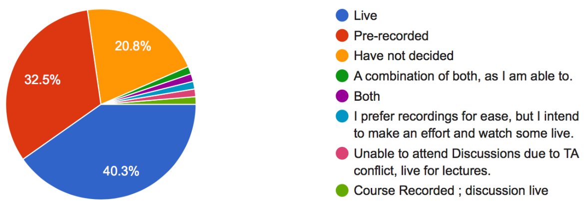
Do you prefer using MATLAB or Python?

77 responses



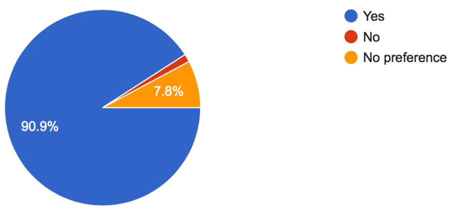
Are you planing to attend the lectures live or see the recordings?

77 responses



Would you prefer us to record the discussion sessions?

77 responses



Platforms: ccle, Gradescope, Campuswire

Logistics

The class evaluation is based on:

10 points — 5 homeworks → individually

30 points — 2 projects → groups 5-6

60 points — 5 quizzes

EE 236A, Fall 2020,
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Schedule

Date	Events	Homeworks	Topics
Th. Oct. 1		Give HW1 (return **Frid. Oct 9** before 9am LA time)	Formulations
Tues. Oct. 6			
Th. Oct. 8			
Tues. Oct. 13	Quiz 1	Give HW2 (return **Frid. Oct 23** before 9am LA time)	
Th. Oct. 15	Give Project 1		Geometry
Tues. Oct. 20			
Th. Oct. 22			
Tues. Oct. 27	Quiz 2	Give HW3 (return **Frid. Nov 6** before 9am LA time)	
Th. Oct. 29			Duality
Tues. Nov. 3	Return Project 1		
Th. Nov. 5			
Tues. Nov. 10	Quiz 3	Give HW4 (return **Frid. Nov 20** before 9am LA time)	
Th. Nov. 12	Give Project 2		ILP
Tues. Nov. 17			
Th. Nov. 19			
Tues. Nov. 24	Quiz 4	Give HW5 (return **Frid. Dec 4** before 9am LA time)	
Th. Nov. 26	Thanksgiving holiday		
Tues. Nov. 30			Decoding
Th. Dec. 3	Return Project 2		
Tues. Dec. 8			
Th. Dec. 10	Quiz 5		

Projects

- Each project will be completed by a group of 5-6 randomly selected students.
- We have two projects, that involve Python programming. A short introduction to Python will be offered on Friday Oct. 2 during the discussion hours.

Quizzes

- We have **5** quizzes as noted in the schedule, one at the end of each module.
- Quizzes cover all material taught, not just the material of the module.
- The quizzes will be offered during the class, at the dates noted in the schedule.
- Only 3 (the best of 5) of the quizzes will count for your grade. Thus you can miss two quizzes and still get a perfect grade.
- Quizzes are open book, open notes, but are completed individually. It is against the code of honor of this class to collaborate on quizzes.
- If you need to miss a quizz because of extenuating circumstances of which you have some evidence, please contact Professor Fragouli to discuss possibilities (this will be a per case discussion).

Homeworks

- We have **5** homeworks in total as noted in the schedule.
- The homework is due by **9:00 AM LA time** on the Fridays noted in the schedule.
- The homeworks are completed individually, however you are welcome to ask public questions or publicly discuss on campuswire. As long as the discussions are available to everyone, and not to small subgroups, it is considered "fair".
- Homeworks are submitted using gradescope.
- Please make sure that your hand-writing is legible if you do not type your solutions!
- Late homeworks do not earn points, but we are happy to correct them.

Grading

- Homeworks: total 10 points (2 points for each homework)
- Quizzes: total 60 points (20 points each but we will keep the 3 highest scores out of the 5 quizzes)
- Projects: total 30 points (15 points each)
- Extra points:
  - The best team wins 5 extra points for each project.
  - You can give (anonymously) 1 extra point to the team member who you think contributed most.
  - The three people most (productively) active on campuswire answering questions get 2 points each (need to not be anonymous).
  - 1 extra point for responding to class evaluation.

What is linear programming

LP: special class of optimization problems.

general form

$$\min f_0(x_1, \dots, x_n)$$
s.t.
$$f_i(x_1, \dots, x_n) \leq b_i \quad i = 1, \dots, m$$

$x_1, x_2, \dots, x_n$: optimization variables $\in \mathbb{R}$

$f_0 : \mathbb{R}^n \to \mathbb{R}$: objective function

$f_i : \mathbb{R}^n \to \mathbb{R}$: constraint functions

Linear Program: both the objective function and the constraint functions are linear.

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \dots, m$$

$$\sum_{j=1} d_{ij} x_j = f_i, \quad i = 1, \dots, p$$

where

$c_j, a_{ij}, b_i, d_{ij}, f_i \rightarrow$ constants problem paramel

$x_1, \dots, x_n \in \mathbb{R}$ optimization variables s

Goal: Given a set of parameters find the optimal values $x_1^*\ x_2^* \dots x_m^*$ that minimize the objective function.

Example

A company produces 2 types of liquid, liquid A and B.
It has to produce at least 100 lt of each everyday. It cannot produce more than 400 lt of liquid A and 300 lt of liquid B.
It has to ship at least 250 lt day.

Liquid A : gains 10 \$/lt
Liquid B : loses 1 \$/lt

} how much should it produce

per day
to maximize gain?

Solution

1) what are the variables?

$x \to$ lt of liquid A

$y \to$ lt of liquid B

2) what is the objective function?

$$\max \ 10x - y$$

3) what are constraints?

$$100 \leq x \leq 400$$

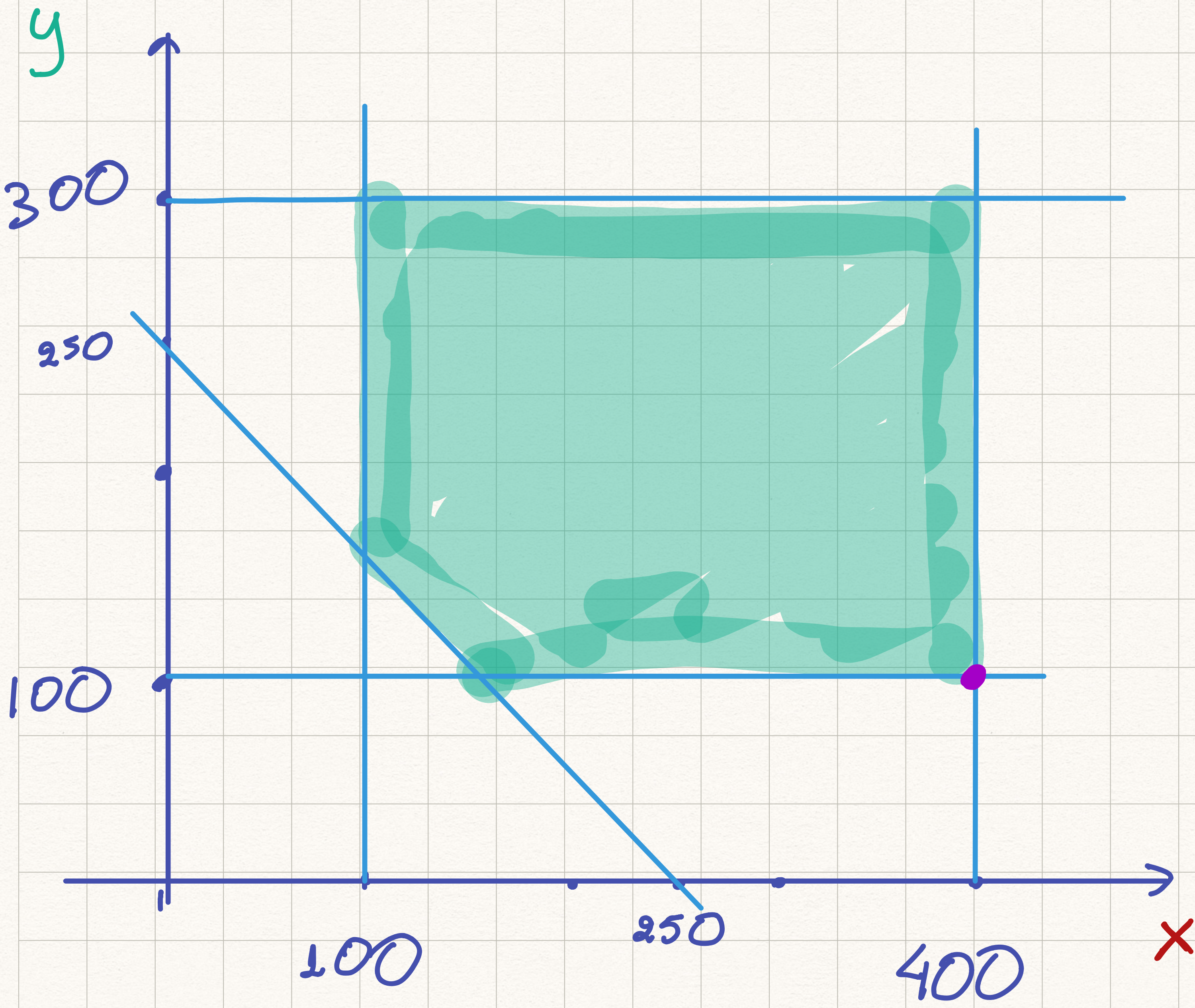
$$100 \leq y \leq 300$$

$$x + y \geq 250$$

$$x, y \geq 0$$

Feasible set: all (x, y) that satisfy the constraints.

$$\max \; 10x - y$$
$$\text{s.t.} \quad 100 \le x \le 400$$
$$100 \le y \le 300$$
$$x + y \ge 250$$



Optimal solution: $(x^*, y^*) = (400, 100)$

Optimal value: $p^* = 3900$

Break

Notation (and review)

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, column vector

$x^T = (x_1 \dots x_n)$ transpose, row vector

special vectors: $\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$x \geq 0$ component wise $\to x_i \geq 0$

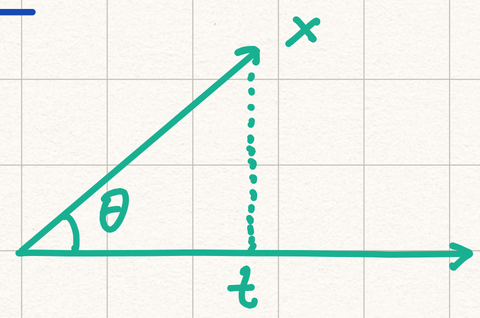
$e_i \to$ orthonormal basis of $\mathbb{R}^n$

$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to$ position i

matrices A, A^T, $A = 0$, $A = I$

Inner product of vectors

$$x^T y = \sum_{i=1}^{n} x_i y_i$$



$$x^T y = \|x\| \cdot \|y\| \cdot \cos\theta$$

magnitude of $\|x\| = \sqrt{x^T x} = \sqrt{\sum x_i^2}$

$x^T y = 0 \Rightarrow \cos\theta = 0$, $\theta = \frac{\pi}{2}$

$x^T y < 0 \Rightarrow \cos\theta < 0$, obtuse angle $\frac{\pi}{2} < \theta < \pi$

$x^T y > 0 \Rightarrow \cos\theta > 0$, $\theta < \pi/2$ acute angle

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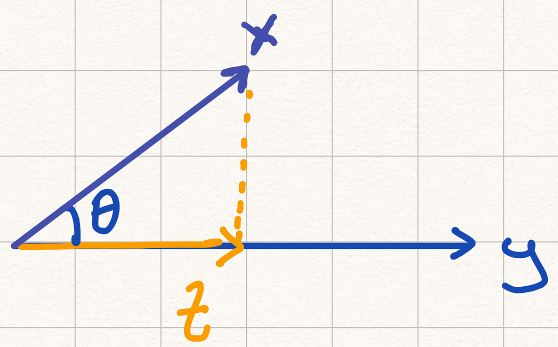


Projection of x on y

Find the expression for vector t

Magnitude of z : $\|z\| = \|x\| \cos\theta$

Direction of z : $\frac{y}{\|y\|}$

Thus

$$z = \frac{\|x\|}{\|y\|} \cos\theta \cdot y = \frac{\cancel{\|x\|}}{\|y\|} \frac{x^T y}{\cancel{\|x\|}\|y\|} \cdot y$$

$$z = \frac{x^T y}{\|y\|^2} y$$

Cauchy-Swartz inequalities

$$-\|x\| \|y\| \leq x^T y \leq \|x\| \|y\|$$

$$\Downarrow \quad y = 1$$

$$-\sqrt{n} \|x\| \leq \sum_{i=1}^{n} x_i \leq \sqrt{n} \|x\|$$

Hyperplane

Set of all points $x \in \mathbb{R}^n$ that satisfy the equation $a^T x = b$

$$H = \{ x \mid a^T x = b \}$$

$a \in \mathbb{R}^n$ normal vector

$b \in \mathbb{R}$

Example

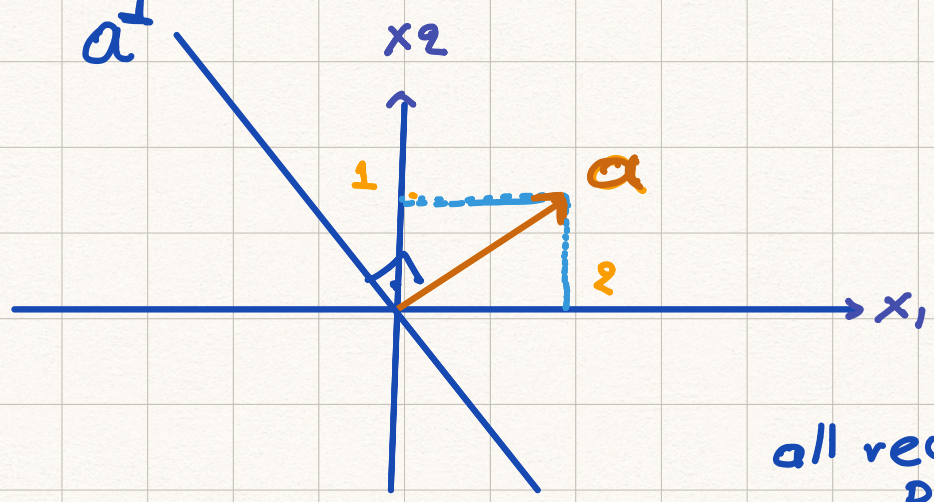
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad b = 11$$

Notation (definition):

orthogonal complement

$$a^{\perp} = \{ \underbrace{x \mid a^T x = 0}_{\text{set of all vectors that are orthogonal}} \}$$

to a

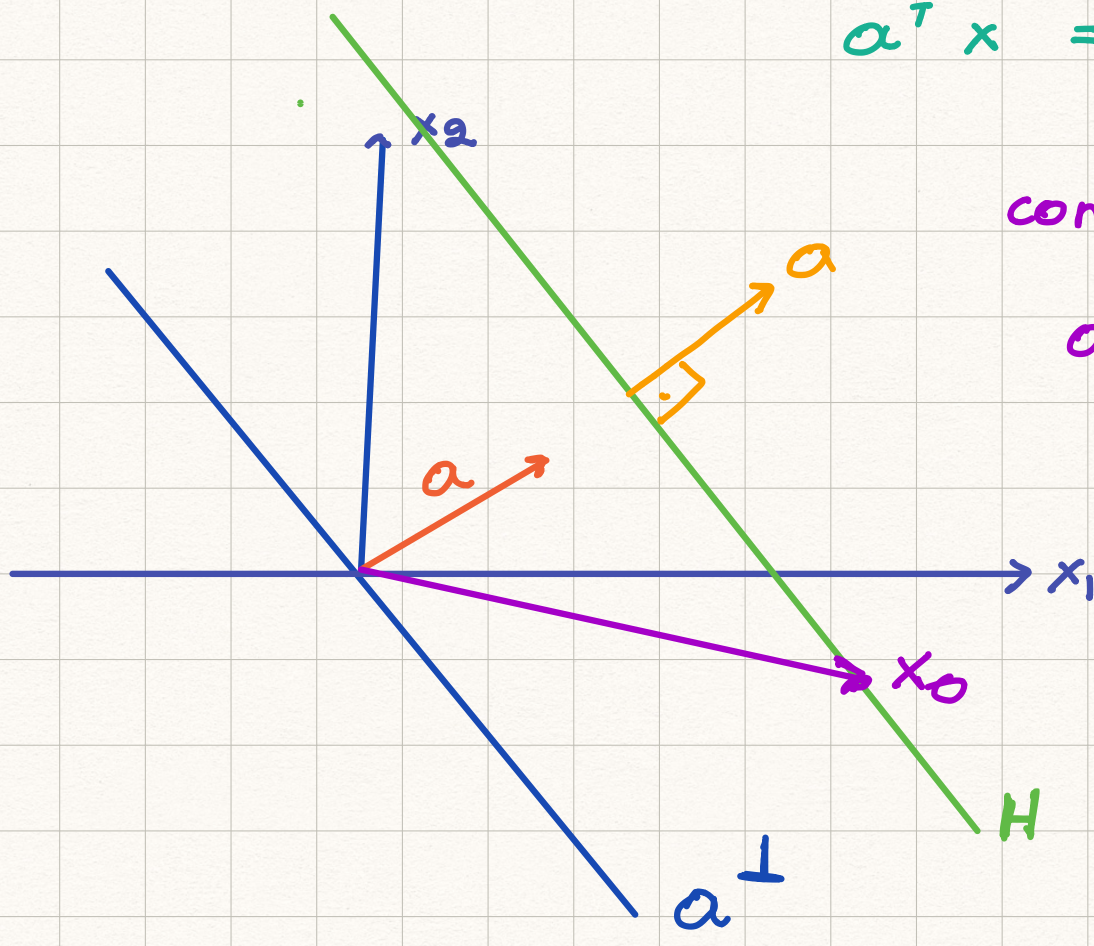


Consider $H = \{ x \mid \overbrace{(2\ \ 1)\begin{pmatrix}x_1\\x_2\end{pmatrix} = 11}^{\text{all vectors whose projection on } a \text{ is } 11.} \}$

$$a^T x = b$$

consider $x_0 \in H$

$$a^T x_0 = b$$



Consider x_0 that belongs in H_2

$$\Rightarrow \quad a^T x_0 = b$$

$$H = \{ x \mid x \in x_0 + a^\perp \}$$

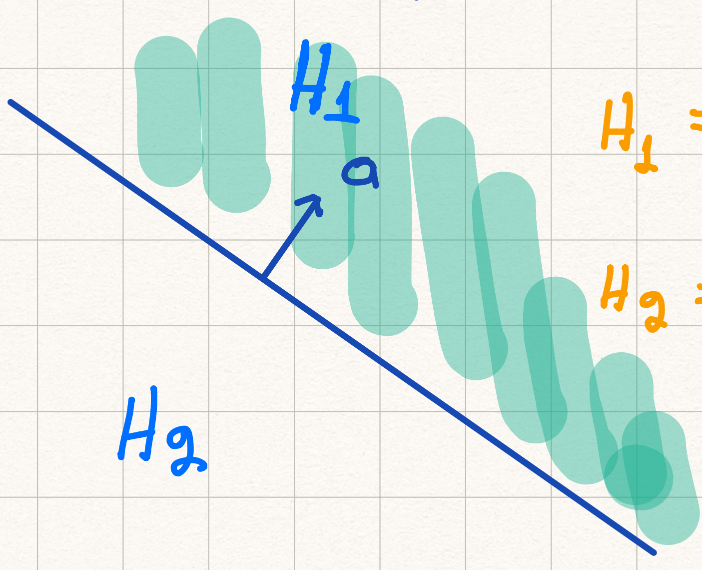
Indeed, $H = \{ x \mid a^T x = b = a^T x_0 \}$

$$= \{ x \mid a^T (x - x_0) = 0 \}$$

$$= \{ x \mid x - x_0 \in a^\perp \}$$

$$= \{ x \mid x \in x_0 + a^\perp \}$$

A hyperplane divides the space $\mathbb{R}^n$ in two <u>halfspaces</u>



$$H_1 = \{ x \mid a^T x \geq b \}$$

$$H_2 = \{ x \mid a^T x \leq b \}$$

Polyhedron: solution of a finite set of linear equalities and inequalities

intersection of halfspaces and hyperplanes

$$P = \{ x \mid a_i^T x \leq b_i, \; c_j^T x = d_j \}$$

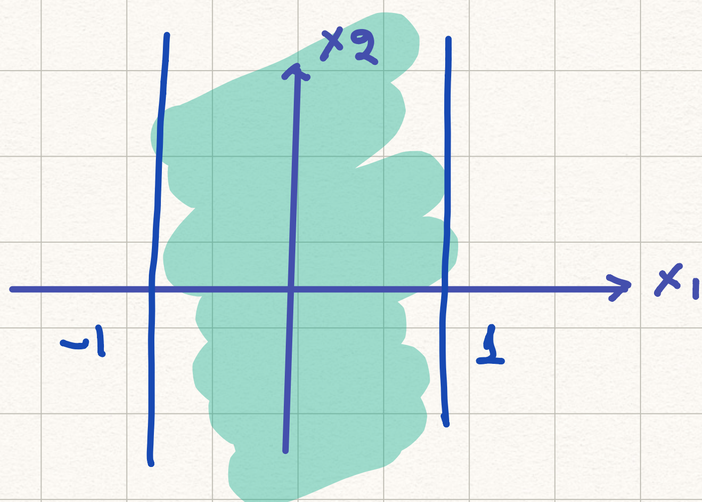
$$i = 1, \dots, m \qquad j = 1, \dots p$$

Polytope: bounded polyhedron (does not contain ∞)

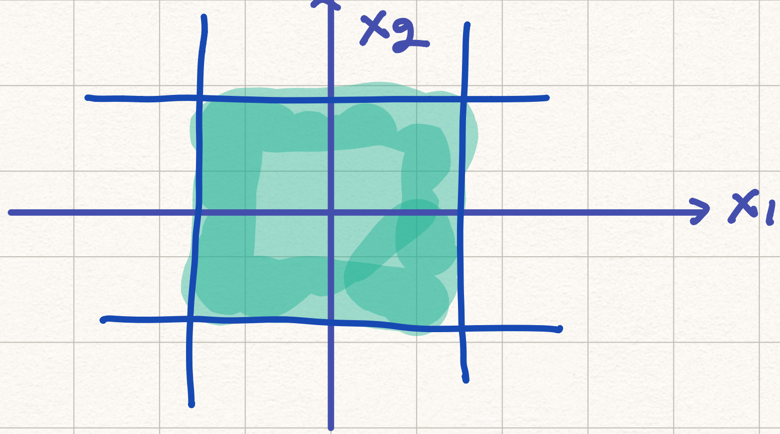
Examples

1) $P = \{ x \in \mathbb{R}^2 \mid |x_1| \leq 1 \}$

$-1 \leq x_1 \leq 1$



2) $P = \{ x \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1, \; -1 \leq x_2 \leq 1 \}$



Linear Program in:

inner product form:

$$\min \; c^T x$$
s.t.
$$a_i^T x \le b_i \,, \quad i = 1, \dots, m$$
$$d_i^T x = f_i \quad i = 1, \dots, p$$

matrix notation:

$$\min \; c^T x$$
s.t.
$$A x \le b$$
$$D x = f$$

$$\underset{m \times n}{A} = \begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_m^T - \end{bmatrix}$$

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = b \in \mathbb{R}^{m \times 1}$$

x feasible $\rightarrow$ satisfies all constraints.

Feasible set $\rightarrow$ polyhedron that contains feasible points

x^* optimal iff $c^T x^* \leq c^T x$, for all feasible x

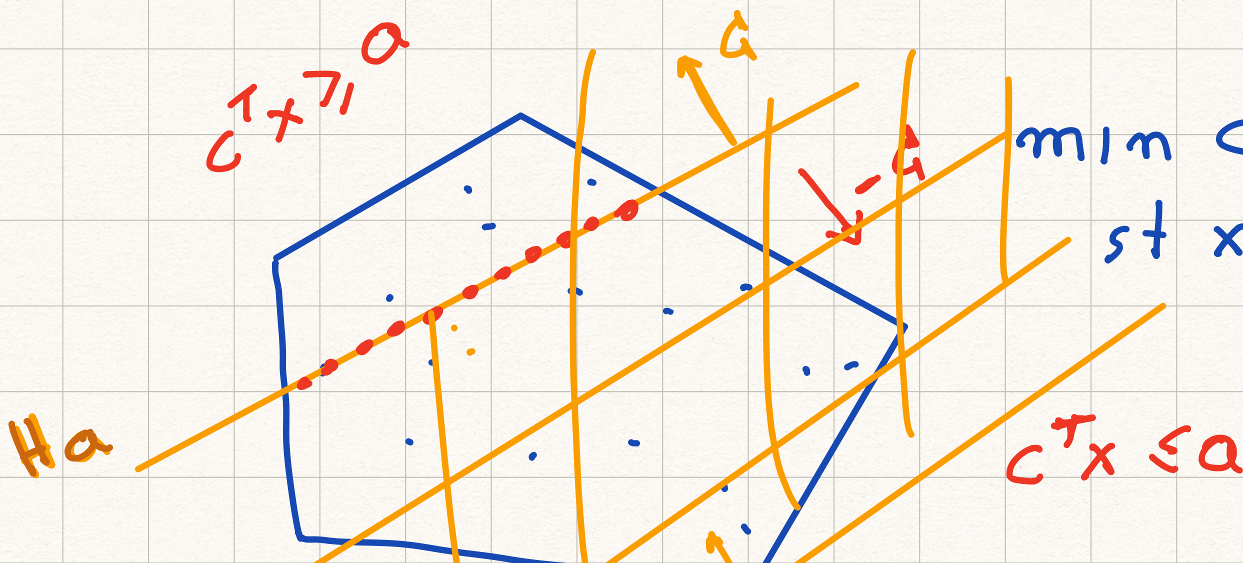
$p^* = c^T x^*$ optimal value

$\min c^T x$
st

unbounded $p^* = -\infty$

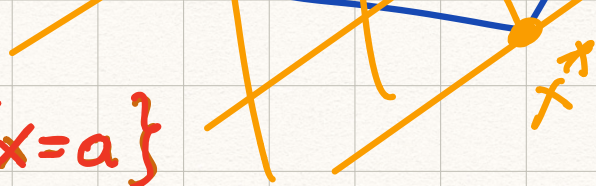
unfeasible $p^* = +\infty$
convention

Geometric interpretation



$\min c^T x$
st $x \in P$

$H_a = \{x \mid c^T x = a\}$



$c^T x^* \le c^T x$

More in the direction of $-c$ and go as far as possible while still having feasible points on the hyperplane $c^T x = a$