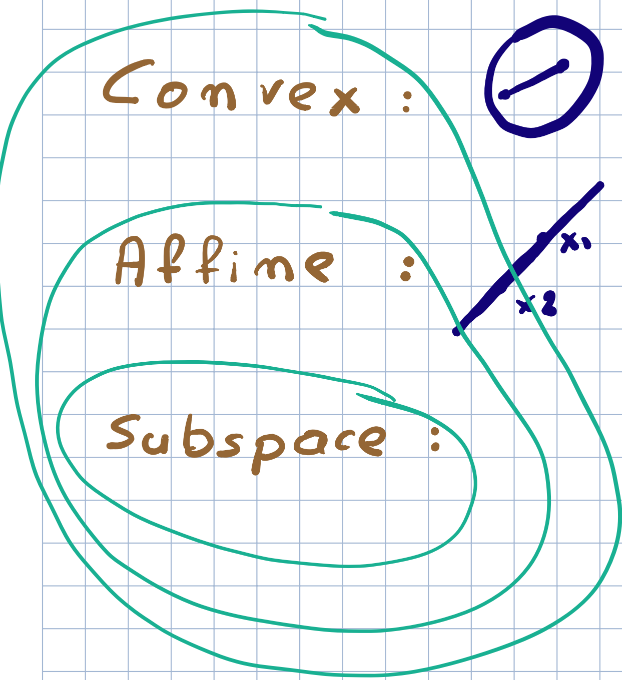
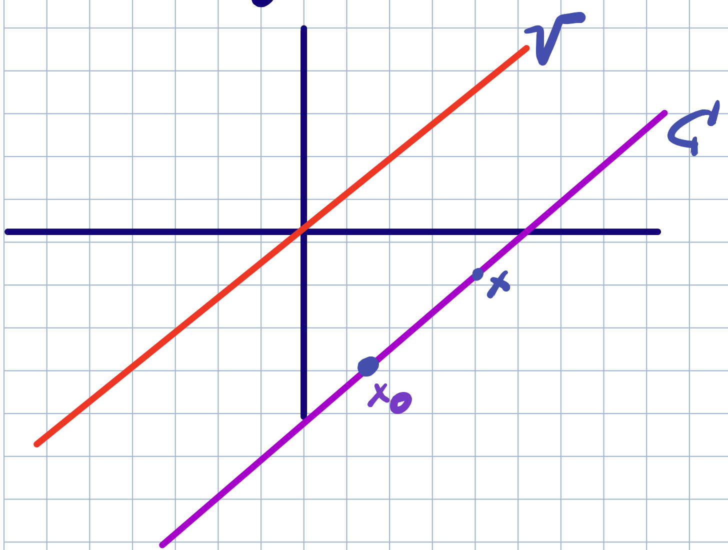
Lecture 6

In the last lecture:



Convex :	line segments	convex comb $\theta_1 x_1 + \theta_2 x_2$, $\Sigma\theta_i = 1$, $\theta_i \geq 0$	convex hull
Affine :	contains line	$\theta_1 x_1 + \theta_2 x_2$, $\Sigma\theta_i = 1$	affine hull
Subspace :	contains all linear combinations	$\theta_1 x_1 + \theta_2 x_2$	

For every affine set we have a parallel subspace



$V = \{ v \mid x - x_0 \in C,\ x_0 \in C \}$

$x_0 + V \rightarrow$ affine set

"Dimension" of affine set $\rightarrow$ dimension of parallel subsp.

$v_1, v_2, \dots, v_k \in C$ "affinely independent" iff $v_1 - v_k$, $v_2 - v_k$, $\vdots$, $v_{k-1} - v_k$ are linearly independ.

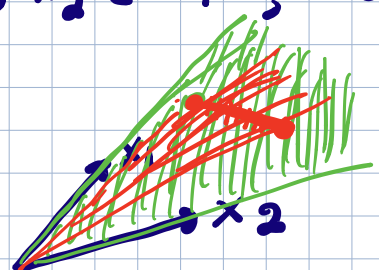
Solution set of equations $\quad C = \{ x \mid Ax = b \} \rightarrow$ affine set

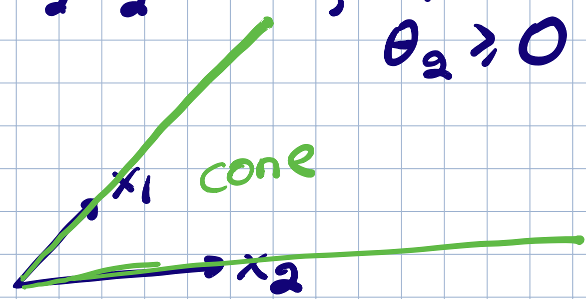
parallel subspace $\quad N(A)$

$$x = \underset{\in C}{x_0} + \underset{N(A)}{z}$$

Convex Cone

- A set C is called a cone, if for all $x \in C$, & $\theta \geq 0$
  $\theta x \in C$
- A set C is called a convex cone, if for $x_1, x_2 \in C$, $\theta_1 x_1 + \theta_2 x_2 \in C$, $\theta_1 \geq 0$, $\theta_2 \geq 0$





Problem 1

1) $S = \{ x \mid x = \underset{m \times k}{A} y + c, \; y \in \mathbb{R}^k, \; c \text{ constant} \}$

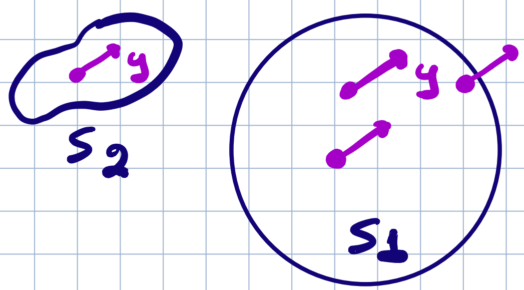
$$\theta_1 x_1 + \theta_2 x_2 = \theta_1 (A y_1 + c) + \theta_2 (A y_2 + c) =$$

$\theta_1 + \theta_2 = 1$

$$= A \underbrace{(\theta_1 y_1 + \theta_2 y_2)}_{y_3} + \overset{1}{\cancel{(\theta_1 + \theta_2)}}\, c \in S$$

2) The set $S = \{ x \mid x + S_2 \subseteq S_1 \}$, $S_1, S_2 \subseteq \mathbb{R}^n$, S_1 convex

We want $x + y \in S_1$, for all $y \in S_2$



$$S = \bigcap_{y \in S_2} \{ x \mid \underset{\text{fixed } y}{y} + x \in S_1 \} = \bigcap_{y \in S_2} \underbrace{(S_1 - y)}_{\text{shifted convex set = convex}}$$

• intersection of (infinite) convex sets

Polyhedron

$$P = \{ x \mid a_i^T x \leq b_i \underset{i=1,\dots,m}{}, \; c_i^T x = d_i \underset{i=1,\dots,k}{} \}$$

$$P = \{ x \mid \underset{m\times n}{A} x \leq b, \; \underset{k\times n}{C} x = d \}$$

Linearity Space

$$\mathcal{L} = \mathcal{N}\begin{pmatrix} A \\ C \end{pmatrix}$$

- if $x \in P$ and $v \in \mathcal{L}$ $\Rightarrow x + v \in P$

Pointed Polyhedron

$$\mathcal{L} = \{0\}$$

Face of a polyhedron:

a set of points $\in P$, where a subset of the inequalities are met with equality

$$P = \{ x \mid a_i^T x \leq b_i \underset{i=1,\dots,m}{}, \; c_i^T x = d_i \underset{i=1,\dots,k}{} \}$$

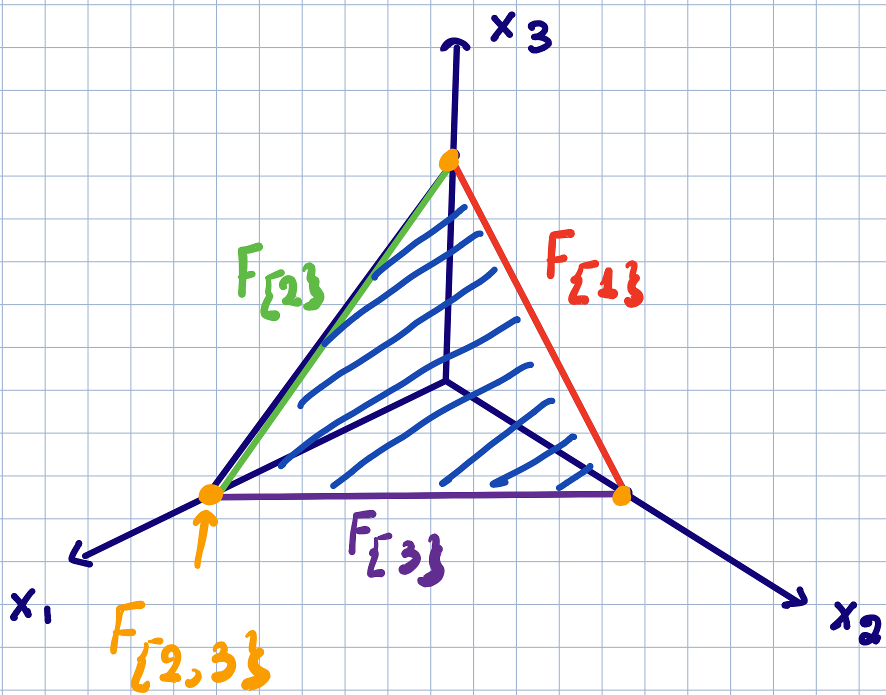
$$J \subseteq \{1, \dots, m\}$$

$$F_J = \{ x \in P \mid a_i^T x = b_i, \; i \in J \}$$

if this set is not empty it is called a face of P.

Example probability simplex

$$\{ x \in \mathbb{R}^3 \mid x_i \geq 0, \sum_{i=1,2,3} x_i = 1 \}$$



$F_0 = P$

$F_{\{1\}} = \{ x \in P \mid x_1 = 0 \}$

$F_{\{3\}} = \{ x \in P \mid x_3 = 0 \}$

$F_{\{2,3\}} = \{ x \in P \mid x_2 = x_3 = 0 \}$

$\cancel{F_{\{1,2,3\}} = \{ x \in P \mid x_1 = x_2 = x_3 = 0 \}}$

Minimal face : a face is minimal iff it does not contain another face.

Extreme point (or vertex) : minimal face of a pointed polyhedron

Properties

1) All faces of a polyhedron are also polyhedra with the same linearity space.

$$P = \{ x \mid Ax \leq b,\ Cx = d \} \qquad \mathcal{L} = \mathcal{N}\begin{pmatrix} A \\ C \end{pmatrix}$$

$$F_J = \{ x \mid \underbrace{A_J}_{\text{keep rows corresponding to indices in } J} x = b_J,\quad A_{\bar{J}}\, x \leq b_{\bar{J}},\ Cx = d \} \qquad \mathcal{L} = \mathcal{N}\begin{pmatrix} A \\ C \end{pmatrix}$$

2) A face is minimal iff it is an affine set and can be expressed as a set of solutions of equations

(proof ommitted for now)

3) Assume P is a pointed polyhedron, $\hat{x} \in P$ is a vertex if and only if

$$\operatorname{rank}\left(\begin{bmatrix} A_{J(\hat{x})} \\ C \end{bmatrix} \right) = n$$

active constraints.

where $J(\hat{x}) \rightarrow$ indices of constraints that $\hat{x}$ satisfies with equality

Given $\hat{x}$, to check if it is a vertex

$\rightsquigarrow$ find $J(\hat{x})$, say $J(\hat{x}) = \{1, \dots, k\}$

$$A_{J(\hat{x})} = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_k^T & - \end{bmatrix} \qquad \operatorname{rank}\begin{pmatrix} A_J \\ C \end{pmatrix} \overset{?}{=} n$$

Proof

1) if the rank condition is satisfied, $\hat{x}$ is the unique solution of a set of linear equations

$$\begin{pmatrix} A_{J(\hat{x})} \\ C \end{pmatrix} x = \begin{pmatrix} b_{J(\hat{x})} \\ d \end{pmatrix}$$

cannot contain another face.

2) if the rank condition does not hold, then there exists $v \in \mathcal{N}\begin{pmatrix} A_{J(\hat{x})} \\ C \end{pmatrix}$, with $v \neq 0$

Consider the point

$$\hat{x}' = \hat{x} + t\, v, \quad t \in \mathbb{R}, \quad v \in \mathcal{N}(\ ).$$

$$\begin{pmatrix} A_{J(\hat{x})} \\ C \end{pmatrix} (\hat{x} + t v) = \begin{pmatrix} b_{J(\hat{x})} \\ d \end{pmatrix} \checkmark$$

$$A_{\cancel{J}(\hat{x})} \hat{x} < b_{\cancel{J}(\hat{x})}$$

$$A_{J(\hat{x})} (\hat{x} + t v) \leq b_{J(\hat{x})}$$

we can select a t arbitrarily small so that inequalities hold

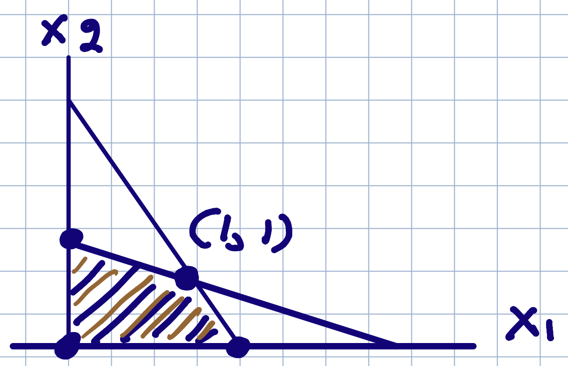
Break

Problem 2

1) Given $P = \{ x \in \mathbb{R}^2 \mid x_1 \geq 0,\ 2x_1 + x_2 \leq 3,\ x_2 \geq 0,\ x_1 + 2x_2 \leq 3 \}$

Is the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ a vertex?

→ pointed polyhedron → has vertices.



→ active constraints:

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

rank = 2. ⇒ it is a vertex

2) $P = \{ x \mid x \geq 0,\ \underset{\substack{\downarrow \\ m \times n}}{C} x = d \}$ $\quad x \in \mathbb{R}^n$, $m < n$

vertices: have at least $n-m$ zero entries

→ Pointed polyhedron ✓

→ vertices satisfy n active constraints (at least)

m → from matrix C

$n-m$ values have to be zero.

3) How many vertices can the polyhedron

$$P = \{ x \mid \underset{m \times n}{A} x \le b \} \quad \text{have?}$$

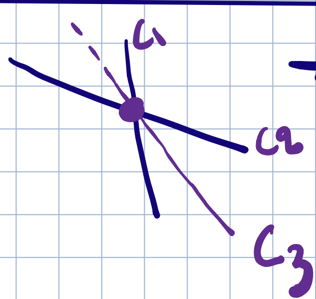
It can have $\binom{m}{n}$ vertices.

4) How many vertices does the polyhedron

$$S = \{ x \mid 0 \le x \le 1 \} \quad \text{have?} \quad x \in \mathbb{R}^n.$$

→ pointed.

→ 2^n vertices $[0 \ \ 0 \ \ 1 \ \ 1]$



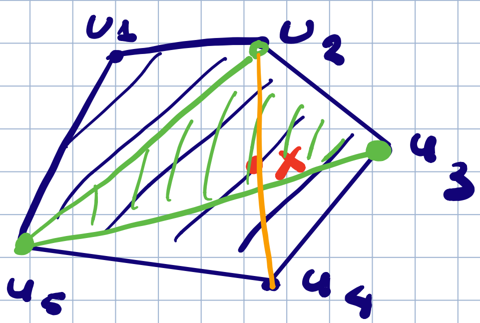
→ 2 dimensions $(n=2)$ vertex belongs in 2 lines

→ degenerate vertices → satisfy more than n constraints.

Caratheodory's theorem

Consider conv(S) convex hull of points $S = \{u_1, \dots, u_m\}$

$u_i \in \mathbb{R}^n$, $m >> n$



$$\text{conv}(S) = \{x \mid x = \theta_1 u_1 + \dots + \theta_m u_m, \ \theta_i \geq 0, \ \theta_1 + \theta_2 + \dots + \theta_m = 1\}$$

Any point $x \in$ conv(S) can be expressed as a convex combination of at most $k = n+1$ points u_i.

Consider a specific $x_0 \in$ conv(S)

Let P be the set of all vector $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{pmatrix} \in \mathbb{R}^m$ that can create x_0

$$P = \{\theta \mid \underbrace{\mathbf{1}^T \theta = 1}_{1 \text{ equality}}, \ \theta \geq 0, \ \underbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix} \theta = x_0}_{n \text{ equalities}} \}$$

- P is a polyhedron in $\mathbb{R}^m$
- pointed polyhedron
- consider a vertex → has to satisfy at least m equalities

at least $m - (n+1)$ constraints have to be equality
zero values

⇝ at most $n+1$ nonzero values.

Problem 3

Consider the set G of matrices $\mathbb{P} \in \mathbb{R}^{K\times K}$ with elements $p_{ij} \geq 0$, and $\sum_{j=1}^{K} p_{ij} = 1$ (sum of elements in each row equals one)

That is

$$G = \left\{ \mathbb{P} \in \mathbb{R}^{K\times K} \mid p_{ij} \geq 0, \sum_{j=1}^{K} p_{ij} = 1 \right\}$$

(i) is G a convex set? (why?)

(ii) can every $\mathbb{P}$ in G be expressed as a convex combination of matrices with exactly one 1 per row?

Solution

(i) $P_1, P_2 \in G$, $\theta_1 P_1 + \theta_2 P_2$, $\theta_1, \theta_2 \geq 0$, $\theta_1 + \theta_2 = 1$

$$\theta_1 \sum p_{ij}^{1} + \theta_2 \sum p_{ij}^{2} = \theta_1 + \theta_2 = 1$$

(ii) $P = \begin{pmatrix} - p_1 - \\ - p_2 - \\ - p_K - \end{pmatrix}$ $\rightsquigarrow$ create a vector $p = (-p_1 - \; - p_2 - \; \cdots - p_K -)$

$p \in \mathbb{R}^{K^2}$

$$G = \left\{ p \in \mathbb{R}^{K^2} \mid p \geq 0, \underbrace{K \text{ constraints } \mathbb{1}^T p_i = 1}_{\text{sum of rows equals } 1} \right\}$$

$\sim$ pointed polyhedron each vertex satisfies K^2 equalities

$\rightarrow$ K constraints from $\sum p_{ij} = 1$

$\rightarrow$ $K^2 - K$ of ≥ 0 become $=$ $\rightsquigarrow$ $K^2 - K$ $p_{ij} = 0$ $\rightarrow$

at most K nonzero elements

$$\begin{pmatrix} 0 & * & & \\ & 0 & * & \\ * & & & \\ & * & & \end{pmatrix}$$

k^2

k nonzero

To satisfy $\sum p_{ij} = 1$ there has to be exactly one 1 per row.