Lecture 10

Today

Theorem of alternatives } next lecture:
Proof of strong duality } integer LP.

Theorem of alternatives

Farkas lemma I For each A and b, exactly one of the following two statements is true:

① There exist x such that $Ax \leq b$ (feasible)

② or there exist z, with

$$z \geq 0, \quad A^T z = 0, \quad b^T z < 0$$

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① and ② cannot be true at the same time:

$$0 = z^T A x \leq z^T b < 0 \qquad \text{contradiction.}$$

We will skip proof that at least one is true.

Farkas lemma II

For given $\underset{n\times m}{A}$ and b, either

① there exist x, with $Ax = b$ and $x \geq 0$

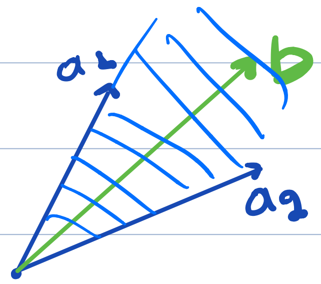
or

② there exists y with $A^T y \geq 0$, $b^T y < 0$

but not both

Geometric interpretation

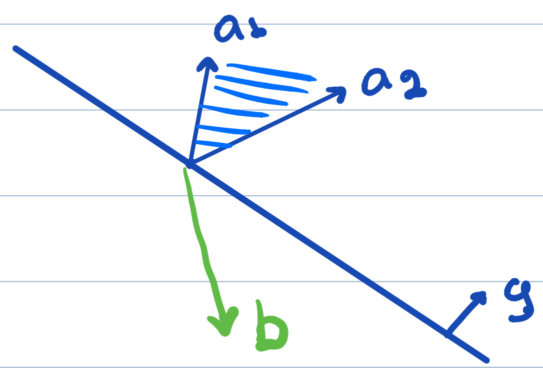
$$Ax = b \Rightarrow \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} x_1 + \begin{bmatrix} | \\ a_2 \\ | \end{bmatrix} x_2 + \dots + \begin{bmatrix} | \\ a_m \\ | \end{bmatrix} x_m = \begin{bmatrix} | \\ b \\ | \end{bmatrix}$$



$x_i \geq 0$

b belongs in the cone generated by the columns of A.

② there exists a hyperplane with normal vector y that strictly separates the a's & b



$$- a_i^T - \; y \geq 0$$

$$b^T y < 0$$

Proof : Apply Farkas lemma I to

$Ax \leq b$, $Ax \geq b$

$$\left.\begin{matrix} Ax = b \\ x \geq 0 \end{matrix}\right\} \Rightarrow \underbrace{\begin{pmatrix} A \\ -A \\ -I \end{pmatrix}}_{A_1} x \leq \underbrace{\begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}}_{b_1}$$

$$A_1 x \leq b_1$$

either there exists x that satisfies

$A_1 x \leq b_1$, or.

there exists $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, such that

$$z \geq 0, \quad A_1^T z = 0, \quad b_1^T z < 0$$

- $z \geqslant 0 \rightsquigarrow u, v, w \geqslant 0$

$$\left.\begin{array}{l} \bullet \ (A^T \ \ -A^T \ \ -I)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0 \Rightarrow A^T\overbrace{(u-v)}^{\text{let } y = u-v} = w \\ \bullet \ (b^T \ \ -b^T \ \ 0)\begin{pmatrix} u \\ v \\ w \end{pmatrix} < 0 \Rightarrow b^T(u-v) < 0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} A^T y = w \\ b^T y < 0 \\ w \geqslant 0 \end{array}\right\} \quad \boxed{\begin{array}{l} A^T y \geqslant 0 \\ b^T y < 0 \end{array}}$$

Mixed inequalities & equalities

Either

① there exists x: $\begin{array}{l} Ax \leq b \\ Cx = d \end{array}$, or

② there exist y and z, $\begin{array}{l} z \geqslant 0 \\ A^T z + C^T y = 0 \\ b^T z + d^T y < 0 \end{array}$

$n \times m$

Example 1 Let P be a matrix with elements p_{ij}, such that $p_{ij} \geq 0$ and the columns of P sum to 1, namely: $\sum_{i=1}^{n} p_{ij} = 1$, for $j = 1, \ldots, n$

Show that there exists $y \in \mathbb{R}^n$ such that
$$Py = y, \quad y \geq 0, \quad \sum_{i=1}^{n} y_i = 1$$

We will use theorem of alternatives.
Farkas lemma II.

① $Ax = b$ and $x \geq 0$

② $A^T y \geq 0$, $b^T y < 0$

alternative ① for us:
$$\begin{matrix} (P - I)\, y = 0 \\ 1^T \quad y = 1 \end{matrix} \Rightarrow \underbrace{\begin{bmatrix} P - I \\ 1^T \end{bmatrix}}_{A_1} y = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_1}, \; y \geq 0$$

Assume ① does not hold, then ② must be true:

Let $y = \begin{pmatrix} z \\ \omega \end{pmatrix}$, $z \in \mathbb{R}^n$, $\omega \in \mathbb{R}$

$$A_1^T y \geq 0 \Rightarrow (P - I)^T z + \omega 1 \geq 0, \; \omega < 0$$

$$P^T z - z + \omega \geq 0 \Rightarrow P^T z \geq z \underbrace{- \underbrace{\omega}_{<0}}_{>0} \Rightarrow$$

$$\Rightarrow \quad P^T z > z$$

this is a contradiction.

$$\begin{pmatrix} — & p_1^T & — \\ — & p_2^T & — \\ & \vdots & \end{pmatrix} \begin{pmatrix} | \\ z \\ | \end{pmatrix} > \begin{pmatrix} | \\ z \\ | \end{pmatrix}$$

$i^* = \arg\max_i z_i$

$$\sum_{j=1}^{n} P_{ij} z_j > z_i$$

z $$\sum P_{ij} z_j \leq \sum P_{ij} z_{i^*} = z_{i^*} \cdot \cancelto{1}{\sum P_{ij}}$$

contradicts our assumption $z_{i^*} < \sum P_{ij} z_i$

Strong duality proof

$$\min \; c^T x \qquad\qquad \max \; -\lambda^T b$$

$$\text{st} \quad \underset{m\times n}{A} x \le b \qquad\qquad \text{st} \quad c + A^T\lambda = 0$$

$x \in \mathbb{R}^n \qquad\qquad \lambda \ge 0 \qquad \lambda \in \mathbb{R}^m$

We will prove that strong duality holds when both the primal and dual are feasible.

$$\begin{cases} A x^* \le b \\ c^T x^* + b^T \lambda^* \le 0 \\ \lambda^* \ge 0 \\ c + A^T \lambda^* = 0 \end{cases}$$

$\leftarrow$ it is sufficient to show ≤ 0 since we already proved the other direction from weak duality $c^T x^* \ge -\lambda^{*T} b$

Use mixed-equalities-inequalities format.

$$\underbrace{\begin{array}{c} \\ m \\ m \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} n & m \end{array} \\ \begin{pmatrix} A & 0 \\ 0 & -I \\ c^T & b^T \end{pmatrix} \end{array}}_{A_1} \overbrace{\begin{pmatrix} x^* \\ \lambda^* \end{pmatrix}}^{x'} \le \underbrace{\begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}}_{b_1}$$

$1 \times (n+m)$

$$A_1 x' \le b_1$$

1

$$\underbrace{\begin{pmatrix} 0 & -A^T \end{pmatrix}}_{C_1} \begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \underbrace{c}_{d_1}$$

$$C_1 x' = d_1$$

Lets create the alternative

$$z \geq 0$$
$$A_1^T z + C_1^T y = 0$$
$$b_1^T z + d_1^T y < 0$$

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \geq 0$$

$$\begin{pmatrix} A^T & 0 & c \\ 0 & -I & b \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -A \end{pmatrix} y = 0$$

$$(b^T \ 0 \ 0) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + c^T y < 0$$

$$A^T z_1 + c z_3 = 0$$
$$\left.\begin{array}{l} -z_2 + b z_3 - A y = 0 \end{array}\right\} \left.\begin{array}{l} b z_3 - A y = z_2 \\ z_2 \geq 0 \end{array}\right\} A y \leq b z_3$$
$$b^T z_1 + c^T y < 0$$

Let $z_1 = u \quad 1 \times m$

$y = \omega \quad 1 \times n$

$z_3 = t \quad 1 \times 1$

$$\boxed{\begin{array}{c} A^T u + t c = 0 \\ A \omega \leq t b \\ b^T u + c^T \omega < 0 \\ u \geq 0, \quad t \geq 0 \end{array}}$$

We want to show that there do not exist $u \in \mathbb{R}^m$, $t \in \mathbb{R}$ and $w \in \mathbb{R}^n$ so that the above holds

① $t > 0 \Rightarrow$ $A^T \frac{u}{t} + c = 0$ $\qquad \boxed{b^T \frac{u}{t} + c^T \frac{w}{t} < 0}$

$A \frac{w}{t} \leq b$ $\qquad \frac{u}{t} \geq 0$

feasible solution for the primal $\qquad$ feasible solution for dual

contradicts weak duality $\qquad c^T \frac{w}{t} < -b^T \frac{u}{t}$ not possible.

$$\min c^T x \qquad\qquad \max -\lambda^T b$$

$$\text{st} \quad \underset{m \times n}{A} x \leq b \qquad\qquad \text{st} \quad c + A^T \lambda = 0$$

$$\lambda \geq 0 \qquad \lambda \in \mathbb{R}$$

② $t = 0 \Rightarrow$ $A^T u = 0$

$A w \leq 0$

$\left.\begin{array}{l} b^T u + c^T w < 0 \end{array}\right\}$ either $b^T u < 0$ or $c^T w < 0$

$u \geq 0$

Assume $b^T u < 0$

$$\left.\begin{array}{l} A^T u = 0 \\ b^T u < 0 \\ u \geq 0 \end{array}\right\}$$ from Farkas lemma I

then $Ax \leq b$ not feasible $\Rightarrow$ primal is not feasible (contradiction)

Assume $c^T w < 0$

$$\begin{array}{l} Aw \leq 0 \\ c^T w < 0 \end{array}$$ contradicts feasibility of the dual problem.

$$\underbrace{\lambda^T A w}_{\leq 0} + \underbrace{c^T w}_{< 0} = 0$$ not possible.

This concludes the proof of strong duality theorem.