Lecture 7

Today: • solve examples
• duality

Problems from past exams

Prove that, for $x \in R^n$, if the function $f(x)$ is a convex function, then the set $C = \{x | f(x) \leq b\}$ is a convex set, with $b \in R$ a given constant.

$\theta_1 x_1 + \theta_2 x_2$
$\theta_1, \theta_2 \geq 0$
$\theta_1 + \theta_2 = 1$
$x_1, x_2 \in C$

$f(\theta_1 x_1 + \theta_2 x_2) \overset{?}{\leq} b$

$\leq \theta_1 f(x_1) + \theta_2 f(x_2) \leq b$ ✓

Can you find the solution to the following problem (call this P1), by solving an LP?

$$\begin{aligned} \text{minimize}_x \quad & ||x||_1^2 + 2||x||_1 \\ \text{subject to} \quad & Ax = b, \end{aligned} \tag{1}$$

where $x \in R^n$, A is an $m \times n$ matrix and $b \in R^m$. If yes, explain which LP you can solve, if not, explain why.

Objective function is increasing with $\|x\|_1$

$\min \; \|x\|_1^2 + 2\|x\|_1 \rightsquigarrow \min \|x\|_1 \;\; \text{s.t.} \;\; Ax = b \rightsquigarrow \min \sum z_i \;\; \vdots$

$z_i = |x_i|^2 + 2|x_i|$

$i = 1, \dots, m.$

$y_{1i} = x_i$

missing their relationship

$$\min_{z_i, y_i} \sum z_i + \|y\|_1$$

s.t. $Ay \leq b$

y, z

$$\|x\| = (|x_1|^2 + |x_2| + \dots + |x_n|)$$

$$z_i = |x_i|^2 + 2|x_i|$$

$|x_1 x_2|$ $+$

Consider the optimization problem

$$(1)\quad \begin{aligned} \min \quad & \frac{3x_1^2 + 5x_2^2 + x_3 + x_4 + 6}{4x_1^2 + x_2^2 + x_4 + 1} \\ \text{st} \quad & x_3 - x_4 \le 8 \\ & 0 \le x_i \le 10, \quad i \in \{1,2,3,4\} \end{aligned}$$

Show that it is equivalent to the LP

$$(2)\quad \begin{aligned} \min \quad & 3y_1 + 5y_2 + y_3 + y_4 + 6y_5 \\ \text{st} \quad & y_3 - y_4 \le 8y_5 \\ & y_i \le 100 y_5 \quad i \in \{1,2\} \\ & y_i \le 10 y_5 \quad i \in \{3,4\} \\ & y_i \ge 0 \quad i \in \{1,2,3,4,5\} \\ & 4y_1 + y_2 + y_4 + y_5 = 1 \end{aligned}$$

We will first argue that (1) is equivalent

$$(3)\quad \begin{aligned} \min \quad & \frac{3z_1 + 5z_2 + z_3 + z_4 + 6}{4z_1 + z_2 + z_4 + 1} \\ & z_3 - z_4 \le 8 \\ & z_i \ge 0, \quad i = 1, \dots, 4 \\ & z_i \le 100 \quad i = 1,2 \\ & z_i \le 10 \quad i = 3,4 \end{aligned}$$

use change of variables:

$z_1 = x_1^2$

$z_2 = x_2^2$

$z_3 = x_3$

$z_4 = x_4$

$x_1 = \sqrt{z_1}$

$x_2 = \sqrt{z_2}$

We will prove that (2) and (3) are equivalent

$$y_1 = \frac{z_1}{4z_1 + z_2 + z_4 + 1}, \quad y_2 = \frac{z_2}{4z_1 + z_2 + z_4 + 1}, \quad y_3 = \frac{z_3}{4z_1 + z_2 + z_4 + 1}$$

$$y_4 = \frac{z_4}{4z_1 + z_2 + z_4 + 1}, \quad y_5 = \frac{1}{4z_1 + z_2 + z_4 + 1}$$

$$y_i \geq 0$$

$$y_3 - y_4 \leq 8 y_5$$

$$(*) \quad 4y_1 + y_2 + y_4 + y_5 = 1$$

$$y_i \leq 100\, y_5 \quad i = 1, 2$$

$$y_i \leq 10\, y_5 \quad i = 3, 4$$

Proof of equivalence

i) from (3) to (2)

Assume z_i's are feasible in (3). We will show that the corresponding y's are also feasible.

Indeed: $y_i \geq 0$ ✓ $i = 1, \dots, 5$

since $\left.\begin{array}{l} z_i \leq 100 \\ y_i = z_i y_5 \end{array}\right\} \Rightarrow y_i \leq 100\, y_5$ ✓ $i = 1, 2$

$i = 3, 4$ $\left.\begin{array}{l} z_i \leq 10 \\ y_i = z_i y_5 \end{array}\right\} \Rightarrow y_i \leq 10\, y_5$ ✓ $i = 3, 4$

(ii) from (2) to (3)

Assume y's feasible, we can construct z that are feasible in (3)

- Observe that $y_5 > 0$

if $y_5 = 0 \Rightarrow y_1, y_2, y_3, y_4 = 0$

$$\left(z_1 = \frac{y_1}{y_5}, \; z_2 = \frac{y_2}{y_5}, \; z_3 = \frac{y_3}{y_5}, \; z_4 = \frac{y_4}{y_5} \right)$$

then the condition

$4y_1 + y_2 + y_4 + y_5 = 1$ cannot hold.

The two problems are equivalent because.

→ any feasible sol in (2) ⇝ feasible sol in (3) that achieves same obj. function value

$p^* \leq q^*$

→ any feasible sol in (3) ⇝ feasible sol in (2) that achieves same obj. value

$q^* \leq p^*$

$p^* = q^*$

Duality

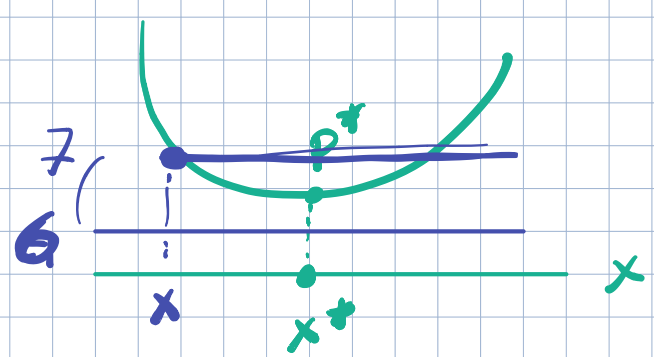
Assume we are given the LP:

$$\begin{aligned}
\min \quad & 2x_1 + 3x_2 \\
\text{st} \quad & x_1 + 2x_2 \geq 4 \rightarrow C_1 \qquad \lambda_1 \\
& x_1 \geq 2 \rightarrow C_2 \qquad \lambda_2 \\
& x_1 + 3x_2 \geq 3 \rightarrow C_3 \qquad \lambda_3 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Let x_1^*, x_2^* achieve optimal $p^* = 2x_1^* + 3x_2^*$

Upper bound on p^*:

any feasible solution gives an upper bound on p^*.



Eg, $x_1 = 2$, $x_2 = 1$, feasible, so

$$p^* \leq 7$$

Lower bound on p^*

$$2x_1 + 3x_2 \underset{\substack{\uparrow \\ \text{because} \\ x_1,\ x_2 \geq 0}}{\geq} \underbrace{x_1 + 2x_2}_{C_1} \geq 4$$

$$2x_1 + 3x_2 \geq \underbrace{x_1}_{c_2} + \underbrace{x_1 + 2x_2}_{c_1} \geq 2 + 4 = 6$$

In general:

$$2x_1 + 3x_2 \geq \lambda_1 (\underbrace{x_1 + 2x_2}_{c_1}) + \lambda_2 \underbrace{x_1}_{c_2} + \lambda_3 (\underbrace{x_1 + 3x_2}_{c_3})$$

$$\geq \lambda_1 \cdot 4 + \lambda_2 \cdot 2 + \lambda_3 \cdot 3$$

for the inequality to hold

$$(\underbrace{\lambda_1 + \lambda_2 + \lambda_3}_{\leq 2}) x_1 + (\underbrace{2\lambda_1 + 3\lambda_3}_{\leq 3}) x_2$$

$$\lambda_i \geq 0$$

dual program

$$\boxed{\begin{aligned} \max \quad & 4\lambda_1 + 2\lambda_2 + 3\lambda_3 \\ \text{s.t.} \quad & \lambda_1 + \lambda_2 + \lambda_3 \leq 2 \\ & 2\lambda_1 + 3\lambda_3 \leq 3 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}}$$

finds the best lower bound for our LP.

How do we find systematically the dual program.

Start primal

inequality form

$$\min\ c^T x$$
$$\text{s.t. } \underset{m \times n}{A} x \le b$$
$$x \in \mathbb{R}^n$$

primal

$x^* \in \mathbb{R}^n$

$$\max\ -b^T \lambda$$
$$\text{s.t. } A^T \lambda + c = 0$$
$$\lambda \ge 0$$

dual

$\lambda^* \in \mathbb{R}^m$

To find the dual:

1) form Lagrangian: augment the objective function with a weighted sum of constraints

$$\mathcal{L}(\underset{n\times 1}{x}, \underset{m\times 1}{\lambda}) = c^T x + \lambda^T (Ax - b)$$

$$= (c^T + \lambda^T A)x - \lambda^T b$$

2) form Lagrange dual function, defined as

$$g(\lambda) = \inf_x \mathcal{L}(x, \lambda)$$

$$= \inf_x \left([c^T + \lambda^T A] x - \lambda^T b \right)$$

Claim $g(\lambda) \leq p^*$, for any $\lambda \geq 0$

proof

$$g(\lambda) = \inf_x \left[(c^T + \lambda^T A)x - \lambda^T b \right] \leq \underbrace{c^T x^*}_{p^*} + \underbrace{\lambda^T (A x^* - b)}_{\leq 0}$$

inf over x is smaller or equal to the evaluation at a specific x^*

$c^T x^* = p^*$ $\qquad \leq p^*$

$\underset{\geq 0}{\lambda^T} (A x^* - b) \leq 0$

$$g(\lambda) = \inf_x \left(\underbrace{(c^T + \lambda^T A)}_{\neq 0} x - \lambda^T b \right) = \begin{cases} -\lambda^T b & \text{if } c^T + \lambda^T A = 0 \\ -\infty & \text{otherwise} \end{cases}$$

↳ not a very useful LB :)

To find the best lower bound,

$$\boxed{\begin{aligned} \max\ & -b^T \lambda \\ \text{st}\ & A^T \lambda + c = 0 \\ & \lambda \geq 0 \end{aligned}}$$

dual program

Apply formula to previous example:

$$\min \underbrace{\begin{pmatrix} 2 & 3 \end{pmatrix}}_{c^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{st} \quad \underbrace{\begin{pmatrix} -1 & -2 \\ -1 & 0 \\ -1 & -3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}}_{\substack{A \\ m \times n}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \underbrace{\begin{pmatrix} -4 \\ -9 \\ -3 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

$m = 5$

$\lambda \in \mathbb{R}^5.$

$$\max \quad (4\ 2\ 3\ 0\ 0)\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_5\end{pmatrix}$$

$$\text{st} \quad \begin{pmatrix}-1 & -1 & -1 & -1 & 0\\ -2 & 0 & -3 & 0 & -1\end{pmatrix}\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_5\end{pmatrix} + \begin{pmatrix}2\\ 3\end{pmatrix} = 0$$

$$\max \quad 4\lambda_1 + 2\lambda_2 + 3\lambda_3$$

$$\text{st} \quad \left.\begin{array}{l} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2 \\ 2\lambda_1 + 3\lambda_3 + \lambda_5 = 3 \\ \lambda_i \geq 0,\ i = 1, \ldots, 5 \end{array}\right] \longleftrightarrow \left[\begin{array}{l} \lambda_1 + \lambda_2 + \lambda_3 \leq 2 \\ 2\lambda_1 + 3\lambda_3 \leq 3 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}\right.$$

$\lambda_4, \lambda_5 \rightarrow$ slack variables

Standard form

primal

p^* optimal value, x^* optimal $\in \mathbb{R}^n$

$$\boxed{\begin{aligned} &\min\ c^T x \\ &\text{s.t. } \underset{m\times n}{A}\, x = b \\ &\qquad -x \leq 0 \end{aligned}}$$

$Ax = b \rightarrow \lambda_1 \in \mathbb{R}^m$

$-x \leq 0 \rightarrow \lambda_2 \in \mathbb{R}^n$

$$\mathcal{L}(x, \lambda_1, \lambda_2) = c^T x + \lambda_1^T (Ax - b) - \lambda_2^T x$$
$$= -b^T \lambda_1 + (c^T + \lambda_1^T A - \lambda_2^T) x$$

$$g(\lambda_1, \lambda_2) = \inf_x \mathcal{L}(x, \lambda_1, \lambda_2)$$
$$= \begin{cases} -b^T \lambda_1, & \underline{c^T + \lambda_1^T A - \lambda_2^T = 0} \\ -\infty, & \text{otherwise.} \end{cases}$$

Claim $\quad g(\lambda_1, \lambda_2) \leq p^*$ provided $\lambda_2 \geq 0$ and for all λ_1

Proof

$$g(\lambda_1, \lambda_2) = \inf_x \mathcal{L}(x, \lambda_1, \lambda_2)$$
$$\leq \mathcal{L}(x^*, \lambda_1, \lambda_2) =$$
$$= \underbrace{c^T x^*}_{p^*} + \underbrace{\lambda_2^T}_{\geq 0} \underbrace{(-x^*)}_{\leq 0} + \lambda_1^T \underbrace{(Ax^* - b)}_{0}$$

provided

$\lambda_2 \geq 0$ $\leq p^*$ ≤ 0

<u>Dual</u>

$$\boxed{\begin{aligned} \max \quad & -b^T\lambda_1 \\ \text{s.t.} \quad & A^T\lambda_1 + c = \lambda_2 \\ & \lambda_2 \geq 0 \end{aligned}}$$

General form LP

primal

$\min \; c^T x$

s.t. $Ax \leq b \rightsquigarrow \lambda$

$\quad\; Cx = d \rightsquigarrow \nu$

p^*, x^*

$$\mathcal{L}(x, \lambda, \nu) = c^T x + \lambda^T(Ax - b) + \nu^T(Cx - d)$$

$$g(\lambda, \nu) = \begin{cases} -b^T\lambda - d^T\nu, & \text{if } c^T + \lambda^T A + \nu^T C = 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

Claim

$g(\lambda, \nu) \leq p^*$ provided $\lambda \geq 0$

$$g(\lambda, \nu) = \inf_x \mathcal{L}(x, \lambda, \nu) \leq \mathcal{L}(x^*, \lambda, \nu)$$

$$= c^T x^* + \underbrace{\underbrace{\lambda^T}_{\geq 0} \underbrace{(Ax^* - b)}_{\leq 0}}_{\leq 0} + v^T \underbrace{(Cx^* - d)}_{0}$$

$$\leq c^T x^* = p^*$$

Dual:

$$\max \ -b^T \lambda - d^T v$$
$$\text{s.t.} \quad A^T \lambda + C^T v + c = 0$$
$$\lambda \geq 0$$

Another form we sometimes use:

$$\boxed{\begin{aligned} \min \ & \sum c_j x_j \\ \text{s.t.} \ & \sum_j a_{ij} x_j \geq b_i \\ & x_j \geq 0 \end{aligned}} \longleftrightarrow \boxed{\begin{aligned} \max \ & \sum b_i y_i \\ \text{s.t.} \ & \sum_i a_{ij} y_i \leq c_j \\ & y_i \geq 0 \end{aligned}}$$

p^* primal | dual q^*