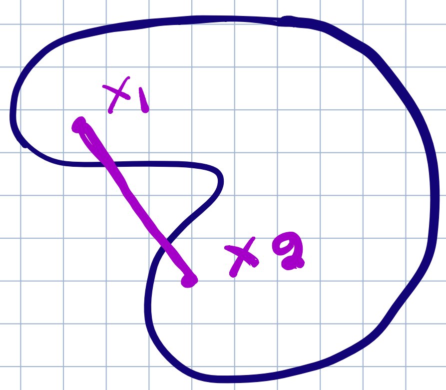
Lecture 5

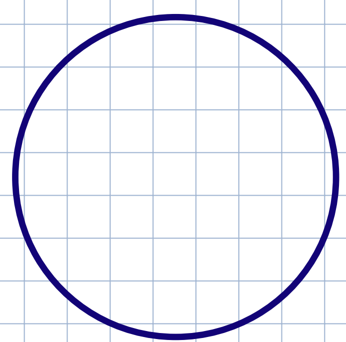
Geometry of LPs

Convex set $C \subseteq \mathbb{R}^n$

A set is convex iff for any $x_1, x_2 \in C$ the line that connects them also belongs in C

① 

not convex

② 

convex

$\forall$ for all $x_1, x_2 \in C$

$$\underbrace{\theta}_{\theta_1} x_1 + \underbrace{(1-\theta)}_{\theta_2} x_2 \in C$$

$$0 \leq \theta \leq 1$$

This definition extends recursively

$$\forall\, x_1, x_2, \dots, x_k \in C, \quad \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k \in C$$

$$\theta_i \geq 0, \quad \sum \theta_i = 1$$

Example → try to prove this holds

$k = 3 \qquad \theta_i \geq 0$

$\theta_1 + \theta_2 + \theta_3 = 1 \quad , \; x_1, x_2, x_3 \in C$

$$\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 = \theta_1 x_1 + \theta_2 x_2 + (1 - \theta_1 - \theta_2) x_3$$

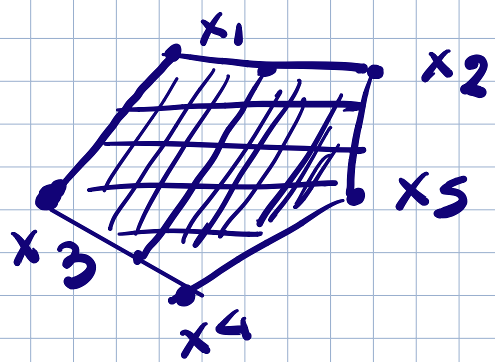
$$= (\theta_1 + \theta_2)\Bigg(\underbrace{\frac{\theta_1}{\theta_1+\theta_2} x_1 + \frac{\theta_2}{\theta_1+\theta_2} x_2}_{\substack{\theta x_1 + (1-\theta) x_2 \\ \text{some point} \\ x_4 \in C}}\Bigg) + (1 - \theta_1 - \theta_2) x_3$$

$$\underbrace{(\theta_1 + \theta_2)}_{\theta'} x_4 + (1 - \underbrace{\theta_1 - \theta_2}_{\theta'}) x_3 \in C$$

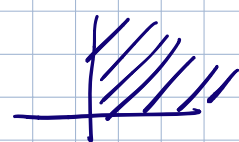
Definition

Given N points $x_1, x_2, \dots, x_N$, we define the convex hull of these points to be the set:

$$S = \left\{ y = \sum_{i=1}^{N} \theta_i x_i \;\middle|\; \theta_i \geq 0, \ \sum \theta_i = 1 \right\}$$



smallest convex set that contains $x_1, \dots, x_N$



Examples: are the following sets convex?

1) $\mathbb{R}^+ = \{ x \in \mathbb{R}^n \mid x \geq 0 \}$ non negative orthant.

convex

2) Sphere $S = \{ x \mid \|x - x_c\| \leq r \}$

To prove, consider $x_1, x_2 \in S$, we want to show $\theta x_1 + (1-\theta) x_2$, $\theta \geq 0$ $\in S$

$$\|\theta x_1 + (1-\theta) x_2 - x_c\| = \|\theta x_1 + (1-\theta) x_2 - \theta x_c - (1-\theta) x_c\|$$

$$= \|\theta (x_1 - x_c) + (1-\theta)(x_2 - x_c)\| \leq$$

$$\leq \theta \|x_1 - x_c\| + (1-\theta) \|x_2 - x_c\| \leq \theta r + (1-\theta) r = r$$

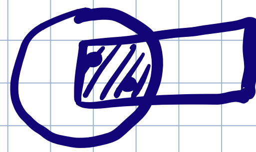
convex

3) Halfspace $\{x \mid a^T x \leq b\}$ convex

Hyperplane $\{x \mid a^T x = b\}$ convex.

4) Intersection of convex sets

S_1 is convex
S_2 is convex

$S_1 \cap S_2$  convex

$S_1 \cup S_2$ not convex

5) Polyhedron:

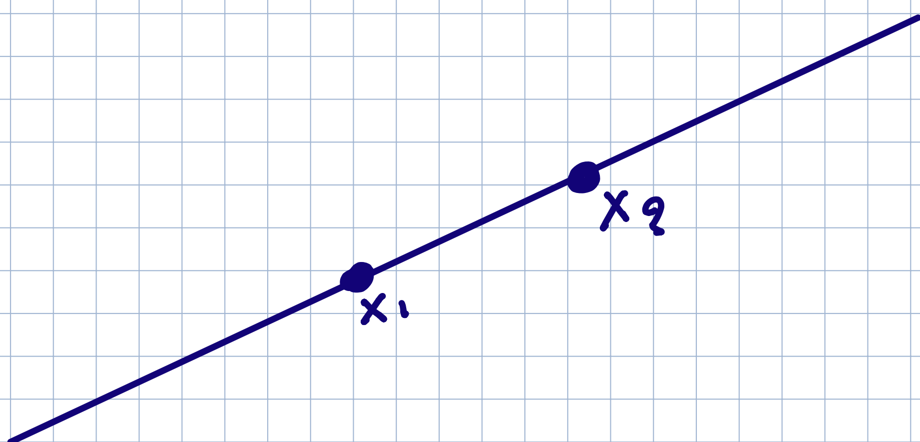
$P = \{x \mid a_i^T x \leq b_i, \; c_i^T x = d_i\}$

convex because intersection of convex sets.

Affine Set S

A subset $S \subseteq \mathbb{R}^n$ is an affine set iff the line that connects any 2 points $x_1, x_2 \in S$ also belongs in S

$\forall \; x_1, x_2 \in S, \quad \theta x_1 + (1-\theta) x_2 \in S$



Extends recursively

S is an affine set iff $x_1, \dots, x_k \in S$

$a_1x_1 + a_2x_2 + \dots + a_kx_k \in S$ affine combination

$a_1 + a_2 + \dots + a_k = 1$

Affine hull

$$\{a_1x_1 + \dots + a_kx_k \mid \Sigma a_i = 1\}$$

Example 1) hyperplane $\{x \mid a^Tx = b\}$ ✓

2) halfspace $\{x \mid a^Tx \le b\}$

hyperplane:

$x = \theta x_1 + (1-\theta)x_2$

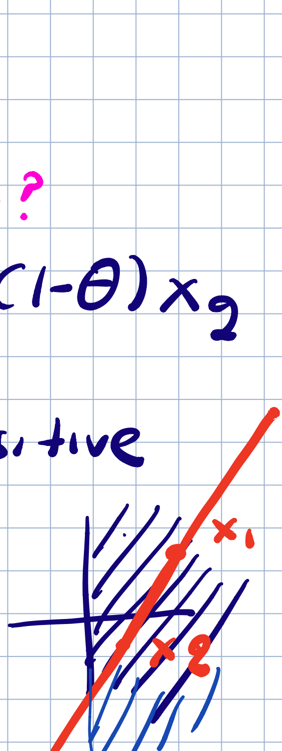
$$a^T(\theta x_1 + (1-\theta)x_2) = \theta \underbrace{a^Tx_1}_{b} + (1-\theta)\underbrace{a^Tx_2}_{b} = b$$

halfspace?

$x = \theta x_1 + (1-\theta)x_2$

$$a^T(\theta x_1 + (1-\theta)x_2) = \theta \underbrace{a^Tx_1}_{\le b} + (1-\theta)\underbrace{a^Tx_2}_{\le b}$$

θ is not positive



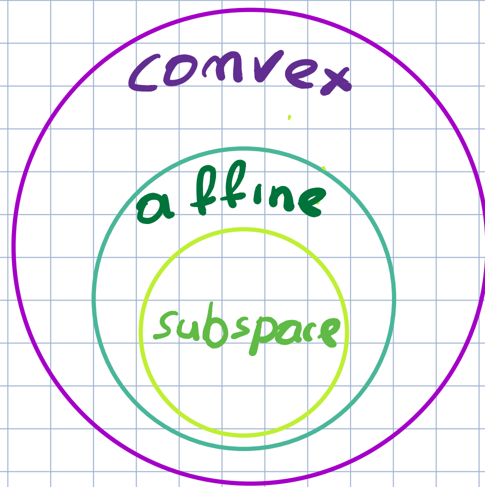
$a^Tx_1 \le b$

$\boxed{\theta < 0 \quad \theta a^Tx_1 \ge \theta b}$

Subspace $\mathcal{V}$: a set $V \subseteq \mathbb{R}^n$ is a subspace if $\forall\, x_1, x_2 \in \mathcal{V}$, $\alpha x_1 + \beta x_2 \in \mathcal{V}$

subspace always contains the origin

$x = 0$



Example: Let A be an $m \times n$ matrix.

1) $\mathcal{R}(A) = \{ x \mid x = Ay,\ y \in \mathbb{R}^n \}$ subspace ✓

$x_1, x_2 \in \mathcal{R}(A)$, $\alpha x_1 + \beta x_2 = \alpha A y_1 + \beta A y_2$

$$= A\underbrace{(\alpha y_1 + \beta y_2)}_{y'}$$

2) $\mathcal{N}(A) = \{ x \mid Ax = 0 \}$

$x_1, x_2 \in \mathcal{N}(A)$ $\quad A(\alpha x_1 + \beta x_2) =$

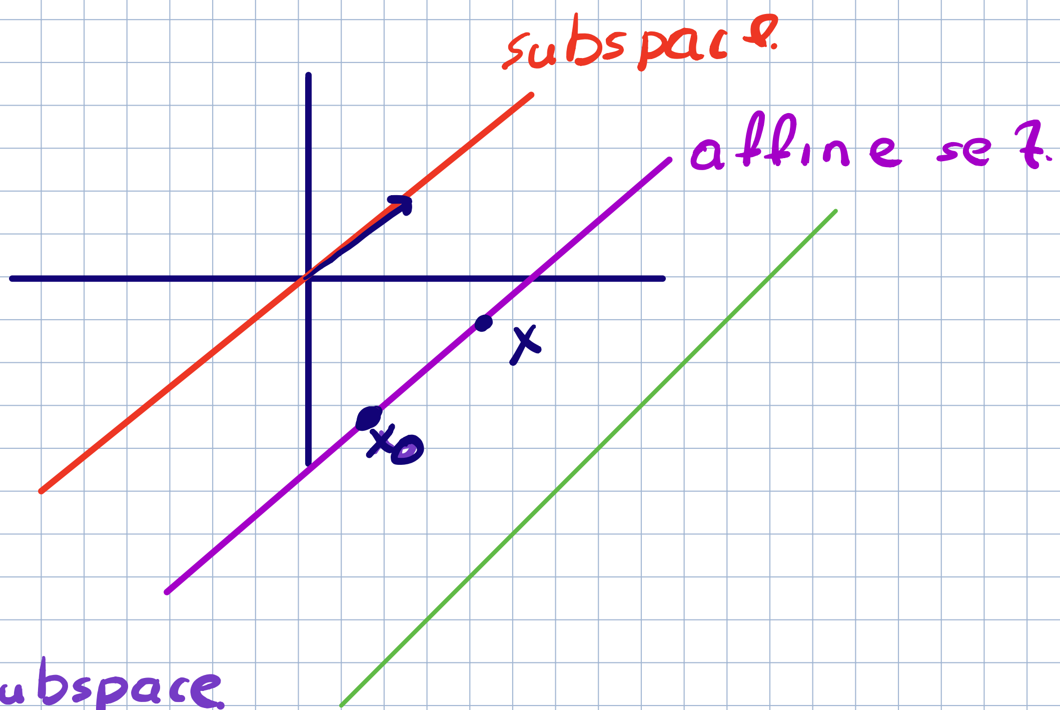
$$\alpha \underbrace{A x_1}_{0} + \beta \underbrace{A x_2}_{0} = 0$$

3) Set of solutions of linear equations

$\mathcal{C} = \{ x \mid Ax = b \}$ affine set (not a subspace)

$$A(ax_1 + bx_2) = aAx_1 + bAx_2 = \underbrace{(a+b)}_{1} b$$

Why are we interested in affine sets?
allow us to deal with sets independently of the coordinate system.



Parallel subspace

Let C be an affine set and let $x_0 \in C$ } the set $V = \{v = x - x_0 \mid x, x_0 \in C\}$ is a subspace called the parallel subspace.

Proof (that V is a subspace)
Assume $v_1, v_2 \in V$, we need to show that

$$av_1 + bv_2 \in V, \quad \forall a, b.$$

$v_1 = x_1 - x_0 \rightsquigarrow x_1 = v_1 + x_0$ $\quad x_1, x_2 \in C$

$v_2 = x_2 - x_0 \rightsquigarrow x_2 = v_2 + x_0$

$v_3 = a v_1 + b v_2 \rightsquigarrow x_3 - x_0$

equivalently, it is sufficient to show that $\exists\, x_3 \in C : x_3 = a v_1 + b v_2 + x_0$

$$= a \underbrace{(v_1 + x_0)}_{x_1 \in C} + b \underbrace{(v_2 + x_0)}_{x_2 \in C} + (1 - a - b) \underbrace{x_0}_{\in C}$$

affine combination $\Rightarrow \in C$

Summary

For every affine set C } parallel subspace $V = \{ x - x_0 ,\ x, x_0 \in C \}$

If C is affine, $C - x_0$ is a subspace.

If V is a subspace, $V + x_0$ is an affine set.

Set of solutions of linear equations

$$C = \{ x \mid \underset{m \times n}{A} x = b \} \quad \text{affine set}$$

① C is an affine set

② Consider $x_0 \in C$

$$V = \{ \underbrace{x - x_0}_{y} \mid \underset{Ax=b}{x \in C} \} = N(A)$$

$$A(x - x_0) = Ax - Ax_0 = b - b = 0$$

if x_0 is any solution ($\in C$) then any other solution can be expressed as $x_0 + z$, $z \in N(A)$.

$$A(x_0 + z) = Ax_0 + \cancel{Az}^{\,0} = b$$

③ Every affine set can be expressed as a set of solutions of linear equations.

<u>Procedure</u> Given an affine set C

i) construct the parallel subspace V

ii) find a matrix A, $N(A) = V$

iii) Select any $x_0 \in C$, and set

$$b = Ax_0$$

$$C = \{ x \mid Ax = b \}$$

Polyhedra

Polyhedra

$$P = \{ x \mid a_i^T x \leq b_i \ , \ c_i^T x = d_i \}$$
$i = 1, \dots, m$ $\qquad$ $i = 1, \dots, k$

intersection of a finite number of subspaces and hyperplanes

$$P = \{ x \mid \underset{m \times n}{A} x \leq b, \ \underset{k \times n}{C} x = d \}$$ convex

Linearity Space

We define the linearity space of P

$$\mathcal{L} = \text{nullspace} \underset{(m+k) \times n}{\begin{pmatrix} A \\ C \end{pmatrix}} \rightsquigarrow \text{subspace of } \mathbb{R}^n.$$

If $x \in P$, then $x + v \in P$, for all $v \in \mathcal{L}$

$$A(x+v) = Ax + \cancel{Av}\,0 = Ax \leq b$$
$$C(x+v) = Cx + \cancel{Cv}\,0 = Cx = d.$$

The opposite does not hold, we cannot express every point in P as $x_0 + v$, $v \in \mathcal{L}$, $x_0 \in P$.
the linearity space may be "empty" $\{0\}$
but the polyhedron is not empty.

Pointed Polyhedron: linearity space is $\{0\}$

it does not contain an entire line.

1) $\{ x \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0 \}$

$$Ax \leq b \Rightarrow \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

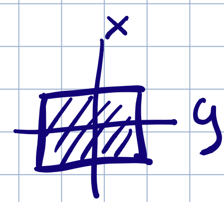
$\mathcal{L}(A) = \{0\}$, pointed polyhedron

2) Halfspace $\{ x \mid a^T x \leq b \}$

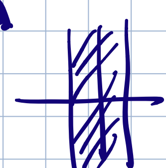
$\mathcal{L} = \{ x \in \mathbb{R}^n \mid a^T x = 0 \}$ for $n \geq 2$, not a pointed polyhedron.

3) $\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \underbrace{|x| \leq 1}_{\substack{x \leq 1 \\ -x \leq 1}},\ \underbrace{|y| \leq 1}_{\substack{y \leq 1 \\ -y \leq 1}},\ |z| \leq 1 \}$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$



$N(A) = \{0\}$, pointed polyhedron



4) $\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid |x| \leq 1,\ |y| \leq 1 \}$

not pointed contains $\{ (0, 0, z) \mid z \in \mathbb{R} \}$