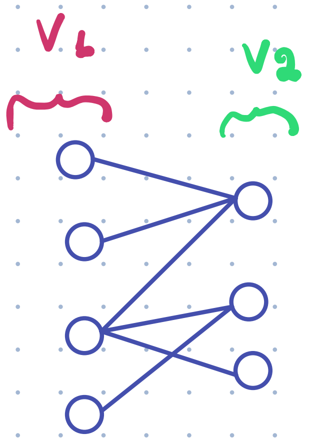
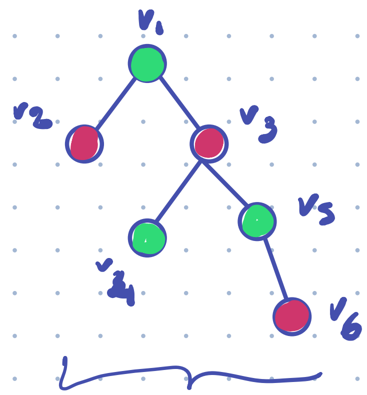
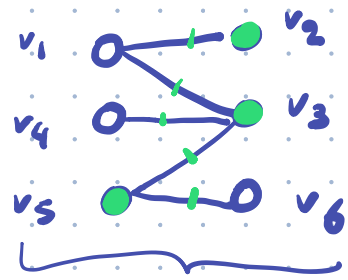
Lecture 13 Bipartite Graphs and matching

A graph $G=(V,E)$ is called bipartite if we can partition the vertices in two sets V_1 and V_2, $V=V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ so that each edge has one end at V_1 and one end at V_2





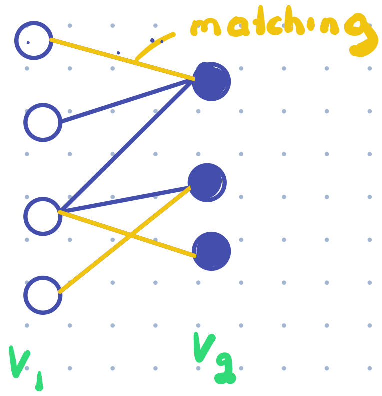
Trees are bipartite graphs



A vertex cover S is a subset of the vertices such that every edge has at least one endpoint incident to S

for each edge uv, either u or v belongs to S

A matching M is a subset of the edges so that no vertex in G is incident to more than one edges in M

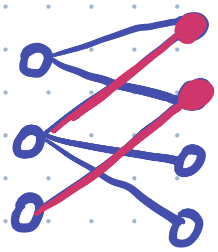


We are interested in finding a maximum matching

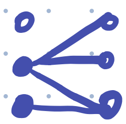
$|M| \leq \min\{|V_1|, |V_2|\}$

vertex cover → find the smallest

Claim: $|M| \leq |S|$ size of matching ≤ size of vertex cover



because all edges in M need to be covered, and all of them touch different vertices - not possible to cover 2 edges in M with less than 2 vertices

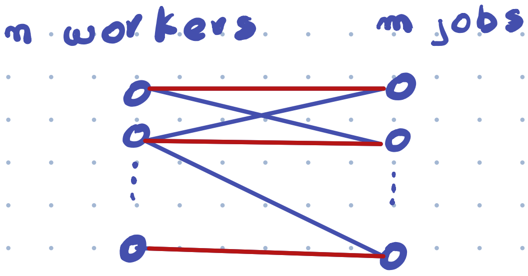


Königs theorem for bipartite graphs

max size of a matching = min size of a vertex cover

Applications

① Personnel assignment: in a company n workers are available for m jobs and each worker is qualified for one or more jobs. Can all workers be assigned, one person per job, to jobs they are qualified for?



edge ij exists if worker i is qualified for job j

We are looking to find a maximum matching.

- doctors to hospitals
- advertisements to people

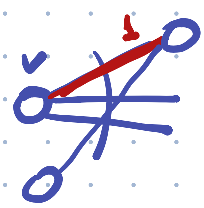
ILP for max matching

use an indicator variable $x(e) = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$

$$\max \sum_{e \in E} x(e)$$

s.t.

$$\sum_{e \in \delta(v)} x(e) \leq 1, \text{ for all vertices } v$$



$\delta(v)$ = set of edges incident to v.

$$x(e) \geq 0, \quad x(e) \in \mathbb{Z}$$

LP relaxation in a matrix format. Let $x = \begin{pmatrix} x(e_1) \\ x(e_2) \\ \vdots \end{pmatrix}$

$$\boxed{\begin{array}{ll} \max & \mathbb{1}^T x \\ \text{st} & M x \leq \mathbb{1} \\ & x \geq 0 \end{array}}$$

dual variables

$\rightarrow \lambda$

$\rightarrow v$

each column has exactly two ones, one in V_1 & another in V_2

$$\begin{array}{ccc} & & \begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array} \\ \begin{array}{c} V_1 \\ \\ \\ \\ \\ \\ V_2 \end{array} & \begin{array}{c} v_1\\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_n\end{array} & \underbrace{\begin{bmatrix} 1 & 0 & & \\ 0&1&&\\ 0&0&&\\ \vdots&\vdots&&\\ 1&0&&\\0&0&&\\0&1&&\end{bmatrix}}_{m} \end{array} \begin{pmatrix} x(e_1)\\x(e_2)\\ \vdots \\ x(e_n)\end{pmatrix} \le \begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$$

Derive the dual of the matching LP

$$\mathcal{L}(x,\lambda,\nu) = -\mathbb{1}^T x + \lambda^T(Mx - \mathbb{1}) + \nu^T(-x) =$$
$$= (-\mathbb{1}^T + \lambda^T M - \nu^T)x - \lambda^T \mathbb{1}$$

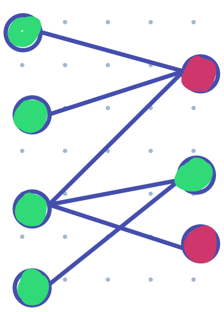
$$\left.\begin{array}{ll} \max & -\lambda^T \mathbb{1} \\ \text{st} & -\mathbb{1}^T + \lambda^T M + \nu^T = 0 \\ & \lambda, \nu \geq 0 \end{array}\right\} \quad \begin{array}{l} \min \ \mathbb{1}^T \lambda \\ M^T \lambda \geq \mathbb{1} \\ \lambda \geq 0 \end{array}$$

$$\min \sum_{u \in V} \lambda(u)$$ dual LP

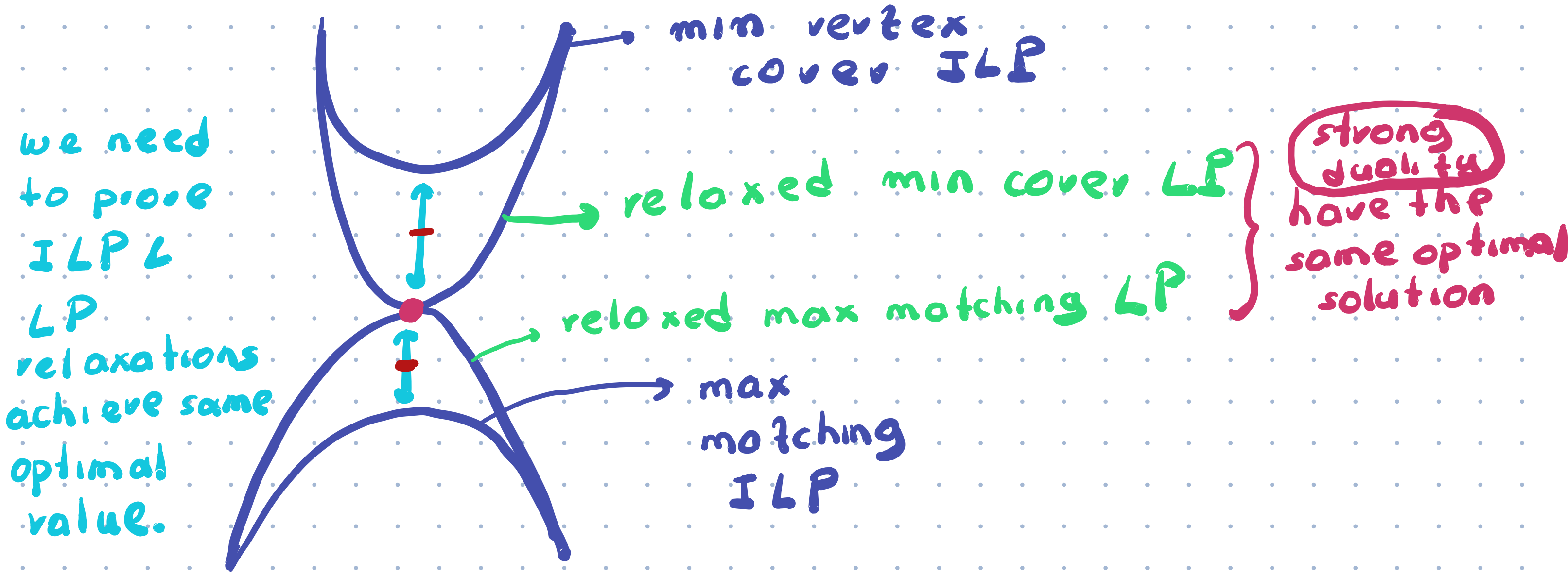
st $\lambda(u) + \lambda(v) \geq 1$ for every edge $e=uv$

$\lambda(u) \geq 0$

assume $\lambda(u) \in \{0, 1\}$ intege dual LP



this is the min-cover ILP



Sufficient to prove that M is TUM.

$$
M = \begin{array}{c} \\ V_1 \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ V_2 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array}
\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array} \\
\begin{bmatrix} 1 & 0 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ 1 & 0 & & \\ 0 & 0 & & \end{bmatrix} \end{array}
$$

Proof

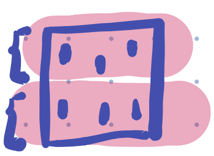
Consider any square submatrix of M.

Consider the columns of this submatrix and take cases.

(i) there exists an all zero column $\rightarrow$ det = 0

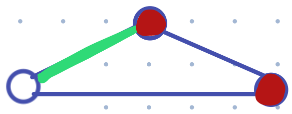
(ii) if there exists a column that has only one 1, expand the det along this column $\rightarrow$ consider a smaller submatrix

(iii) all columns have exactly 2 ones $\rightarrow$ det = 0



$\rightarrow$ if we sum the rows corresponding to vertices in V_1, and we sum the rows corresponding to vertices in V_2, we get the same row $\rightarrow$ rows linear dependent.

Why bipartite graphs



max matching ≠ min cover.

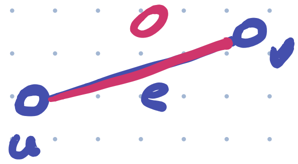
$\boxed{|M| \le |S|}$ — in general graphs Konigs theorem does not hold.

$$\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline v_1 & & & \\ v_2 & & & \\ v_3 & & & \end{array} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

M → not TUM matrix

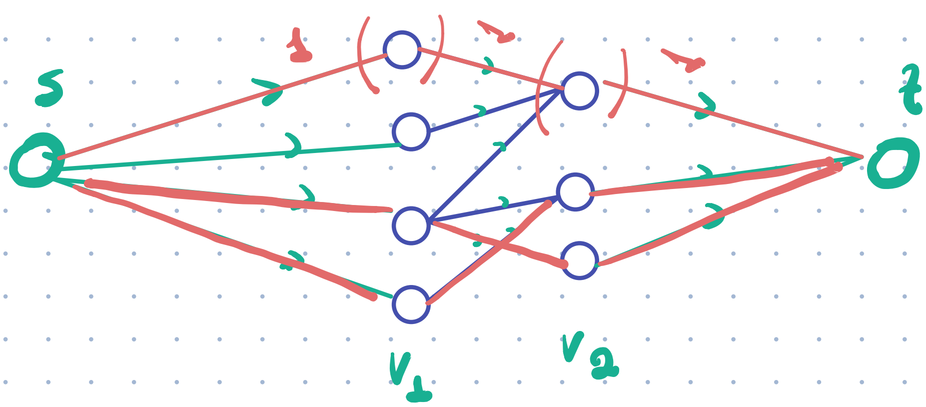
KKT conditions $\quad x(e)\,[\underbrace{\lambda(u) + \lambda(v) - 1}_{0}] = 0$, $\quad x(e) \to 1$

$e = uv$

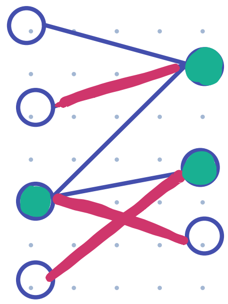


if $x(e) = 1$ belongs in matching ⇒ either u or v is in the cover.

We can solve the max matching problem by solving a max-flow problem.



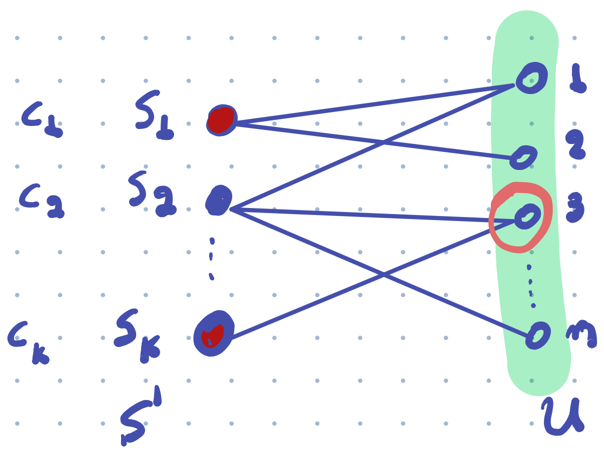
use unit capacity edges



Is matching optimal?
certificate of optimality
if there exists a vertex
cover of same size

Set Cover Problem

Given a universe $\mathcal{U}$ of n elements, a collection of subsets of $\mathcal{U}$ $S = \{S_1, S_2, \dots, S_k\}$ and a cost function $c: S \to \mathbb{R}^+$, find a minimum cost subcollection of S that covers all the elements in $\mathcal{U}$.



edges indicate which elements belong in each subset

ILP

$$x_i = \begin{cases} 1 & \text{select } S_i \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{x_i} \sum_{i=1}^{k} c_i x_i \qquad \text{min cost}$$

$$\text{s.t.} \quad \sum_{u \in S_i} x_i \geq 1 \leftarrow \text{for every element } u \text{ in } \mathcal{U}$$

cover every element

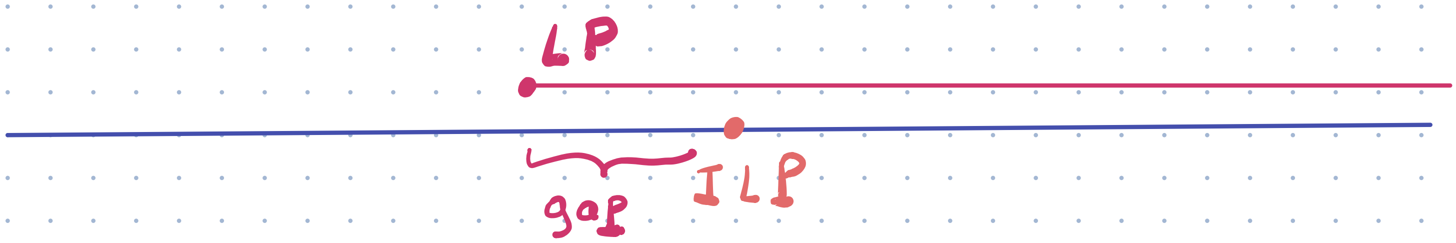
$$x_i \geq 0, \quad x_i \in \mathbb{Z}$$

LP relaxation

fractional set cover problem

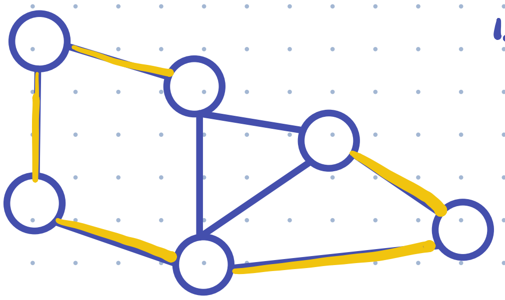
$$\min \sum c_i x_i$$
$$\text{s.t.} \sum x_i \geq 1 \quad \text{for all } u.$$
$$x_i \geq 0, \; x_i \in \mathbb{R}$$

$$M^{v_2} \quad \begin{array}{c|c} & \begin{array}{cccc} S_1 & \overset{v_1}{S_2} & \dots & S_k \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array} & \begin{pmatrix} 1 & 1 & & 1 \\ 1 & 1 & & 1 \\ & & & \\ & & & \end{pmatrix} \end{array} \rightarrow \text{not TUM matrix}$$



Use approximation algorithm to solve ILP.
(discuss one on Thursday)

Spanning tree ILP formulation



$G=(V,E)$, spanning tree $T=(V,E_1)$ is a subgraph that contains all the vertices, and is a tree $E_1 \subseteq E$

connected
no cycles.

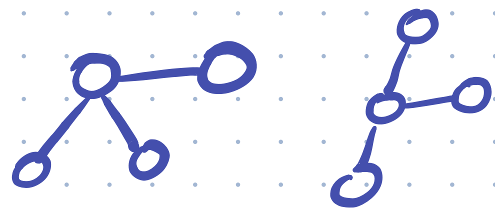
$|E_1| = |V|-1$

Assume cost c_{ij} with using edge ij in the tree. Find the spanning tree of minimum cost.

Use an indicator variable $x_{ij} = \begin{cases} 1 & \text{if edge } ij \text{ belongs in } T \\ 0 & \text{otherwise} \end{cases}$

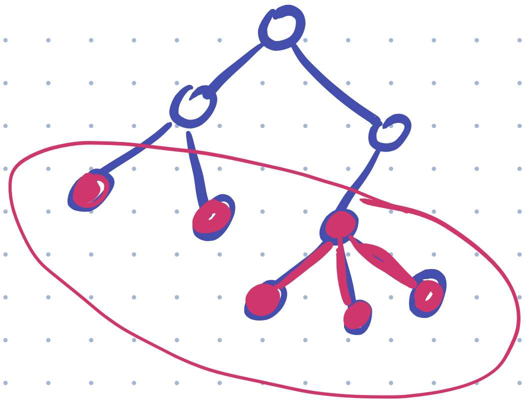
$$\min \sum_{ij \in E} c_{ij} x_{ij}$$

s.t.



main observation to derive constraints:

if T is a tree, every subgraph of T has to be either a tree or a forest.



subtour elimination constraints

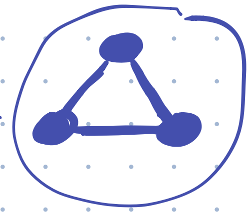
for the original tree

$$\sum_{ij \in E_1} x_{ij} = |V|-1$$

for every subset of vertices S,

$$\sum x_{ij} \le |S| - 1$$

edge ij belongs in the induced subgraph of vertices in S



$|S| = 3$

$\sum x_{ij} \le 2$

$$\min \sum c_{ij} x_{ij}$$

s.t. $\sum x_{ij} = |V| - 1$ → original tree

$\sum x_{ij} \le |S| - 1$ for every $S \subseteq V$

$x_{ij} \in \{0, 1\}$