Project ----- Week04
Tina Zhao

Problem 1

① Classical Brownian Motion:

$$P_t = P_{t-1} + r_t$$

$$\mathbb{E}(P_t) = P_{t-1} + \mathbb{E}(r_t) \qquad r_t \sim N(0, \sigma^2)$$

$$= P_{t-1}$$

$$Var(P_t) = Var(r_t)$$

$$= \sigma^2$$

② Arithmetic Return

$$P_t = P_{t-1}(1 + r_t)$$

$$\mathbb{E}(P_t) = P_{t-1} + P_{t-1}\mathbb{E}(r_t) \qquad r_t \sim N(0, \sigma^2)$$

$$= P_{t-1}$$

$$Var(P_t) = P_{t-1}^2 Var(r_t)$$

$$= P_{t-1}^2 \sigma^2$$

③ Log Return / Geometric Brownian Motion

$$P_t = P_{t-1} e^{r_t}$$

$$\mathbb{E}(P_t) = P_{t-1}\mathbb{E}(e^{r_t}) \qquad \ln(e^{r_t}) \sim N(0, \sigma^2)$$

$$= e^{\frac{\sigma^2}{2}} P_{t-1} > P_{t-1} \qquad \mathbb{E}(e^{r_t}) = e^{\frac{\sigma^2}{2}}$$

$$(\sigma^2 > 0) \qquad Var(e^{r_t}) = (e^{\sigma^2} - 1)e^{\sigma^2}$$

$$Var(P_t) = P_{t-1}^2 Var(e^{r_t})$$

$$= P_{t-1}^2 (e^{\sigma^2} - 1)e^{\sigma^2}$$

The first two methods should result in the same expectation (equals to P_{t-1}) while the geometric return will have a larger expectation and positively related to the size of sigma. The variance of P_t depends on sigma when assuming classical Brownian motion, and related to the value of P_{t-1} for other two methods.

Simulate the three methods, changing the value of sigma and P_{t-1} to verify the result.

```
rt ~ N( 0, 1 )
Pt-1 =  1
Classical Brownian Method:
 E(P) =  1.0035256192471083 Expected(Pt-1): 1
Var(P) =  0.9942590689266244 Expected(σ^2): 1
Arithmetic Return:
 E(P) =  1.0035256192471083 Expected(Pt-1): 1
Var(P) =  0.9942590689266244 Expected(Pt-1^2*σ^2): 1
Geometic Brownian Motion:
 E(P) =  1.6502507854553548 Expected(e^(σ^2/2) * Pt-1): 1.6487212707001282
Var(P) =  4.626707012527816 Expected(Pt-1^2 * (e^(σ^2)-1) ) * e^(σ^2)): 4.670774270471604
```

```
rt ~ N( 0, 1 )
Pt-1 =  3
Classical Brownian Method:
 E(P) =  3.0055686519128573 Expected(Pt-1): 3
Var(P) =  0.9940276415693544 Expected(σ^2): 1
Arithmetic Return:
 E(P) =  3.016765955738573 Expected(Pt-1): 3
Var(P) =  8.945526765809502 Expected(Pt-1^2*σ^2): 9
Geometic Brownian Motion:
 E(P) =  4.96597377171533 Expected(e^(σ^2/2) * Pt-1): 4.946163812100385
Var(P) =  41.90967028298915 Expected(Pt-1^2 * (e^(σ^2)-1) ) * e^(σ^2)): 42.036968434244436
```

```
rt ~ N( 0, 1 )
Pt-1 =  -1
Classical Brownian Method:
 E(P) =  -0.9948295922477342 Expected(Pt-1): -1
Var(P) =  1.0045670281323267 Expected(σ^2): 1
Arithmetic Return:
 E(P) =  -1.005150407752266 Expected(Pt-1): -1
Var(P) =  1.0045672345486363 Expected(Pt-1^2*σ^2): 1
Geometic Brownian Motion:
 E(P) =  -1.6591232757206302 Expected(e^(σ^2/2) * Pt-1): -1.6487212707001282
Var(P) =  4.6532396964212595 Expected(Pt-1^2 * (e^(σ^2)-1) ) * e^(σ^2)): 4.670774270471604
```

```
rt ~ N( 0, 2 )
Pt-1 =  1
Classical Brownian Method:
 E(P) =  0.9987160968410883 Expected(Pt-1): 1
Var(P) =  2.0030020362142107 Expected(σ^2): 2
Arithmetic Return:
 E(P) =  0.9987160968410883 Expected(Pt-1): 1
Var(P) =  2.0030020362142107 Expected(Pt-1^2*σ^2): 2
Geometic Brownian Motion:
 E(P) =  2.712746459474568 Expected(e^(σ^2/2) * Pt-1): 2.718281828459045
Var(P) =  48.70272583230862 Expected(Pt-1^2 * (e^(σ^2)-1) ) * e^(σ^2)): 47.20909393421359
```

The result shows consistency in simulated data and the thermotical calculations (ignoring deviations from the random draws).

Problem 2

Implement the return_calculate function which allow three kinds of return calculating method.

Use the Arithmetic return for the input data.

Calculate VaR under 4 different distribution assumptions.

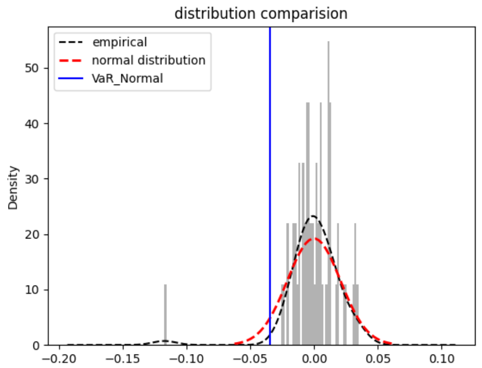
Calculating 5% VaR

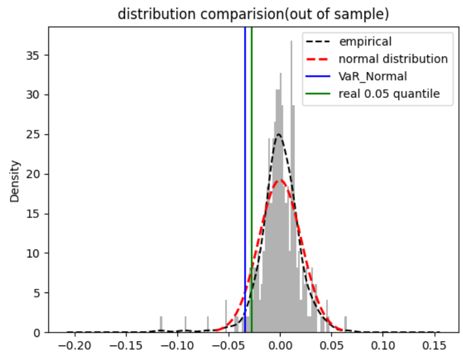
Following are the results

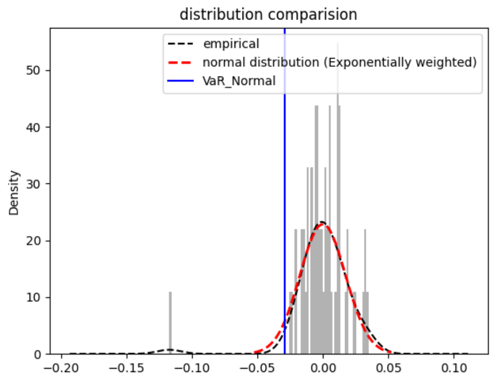
```
VaR calculations:
   Normal Distribution: 3.41% in dollars: 1.902
Exponentially Weighted: 2.87% in dollars: 1.600
         MLE fitted T: 2.73% in dollars: 1.519
   Historic Simulation: 2.50% in dollars: 1.391
```

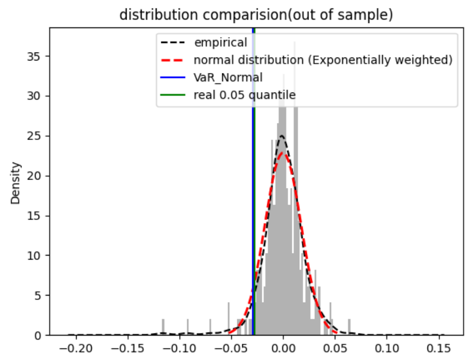
Plot the sample data and the assumed distribution for the for cases.

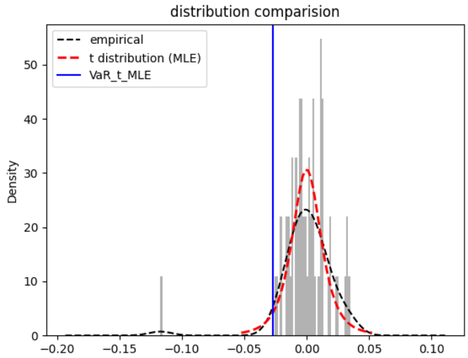
Also plot the downloaded out of sample data, with the in sample simulated distribution, and the estimated VaR.

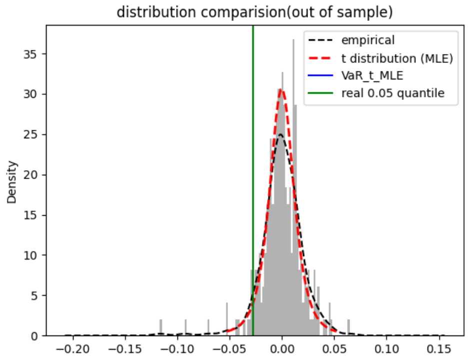


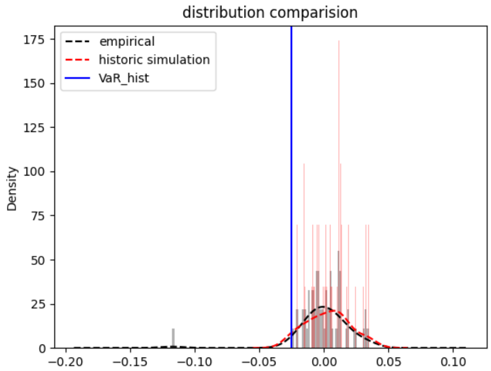


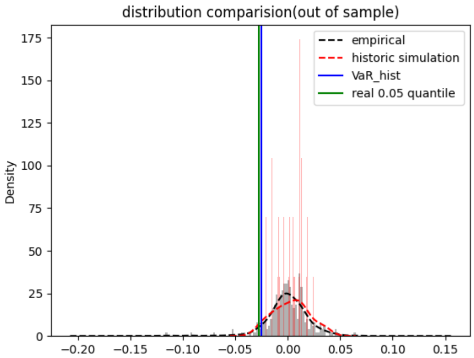












The main characteristics of the empirical data includes excess kurtosis, fat tail, and negatively skewed, which can be clearly observed from the first two plots. In theory, t distribution should be used to capture these properties. The third method (t distribution estimated with MLE) indeed result in a VaR that is closest to the out of sample data's real quantile.

From the graphs, it seems that the exponentially weighted normal distribution matches the shape of the data the best, and the simple normal distribution gives the largest VaR(largest expected loss in a 5% worst situation). However, the normal distribution is thin-tailed, we can deduct from the graph that if we are interested in a more extreme case, like choosing 1% VaR, both the normal distributions with and without a weighted adjustment will most likely underestimate the risk.

Extreme value is also an issue. The historic simulation method should be the one that has the best chance to avoid understate the impact of extreme value, but it depends on whether the extreme value will be picked during the simulation. When having a small data set, this is really volatile. Perhaps using KDE first can help.

Problem 3

I am using the t-distribution estimate by MLE to calculate VaR for the portfolios.

When choosing methods, I first excluded the historic simulation method. The data we have (60 returns) is kind of inadequate to get a good result. When working on Problem 2, I noticed that as there is a return datapoint much smaller than the rest of the series, if I do a large number of times of simulations, the 0.05 quantile almost always land on that extreme value. However, if I reduce the times of simulation, it's hard to get a stable result. Which make the VaR from historic simulation method less informative.

Also from Problem2, the exponentially weighted normal distribution has the most desired shape of distribution, while the t-distribution appears to be exaggerating on the kurtosis. That being said, we are most concerned about the tail distribution, which makes the t-distribution more suitable for calculating VaR as it will give a more robust estimate.

Using t distribution, following is the estimated risk for the portfolios:

```
----Portfolio 1----
VaR[Return] 1.65%
VaR[portfolio Value]: (a loss of)  $ 6002.954453962024

----Portfolio 2----
VaR[Return] 1.46%
VaR[portfolio Value]: (a loss of)  $ 4775.207786430735

----Portfolio 3----
VaR[Return] 1.10%
VaR[portfolio Value]: (a loss of)  $ 3591.438028104183

----Total----
VaR[Return] 1.38%
VaR[portfolio Value]: (a loss of)  $ 14011.428774479446
```

All the calculated portfolio VaR are smaller than what we see in the single stock's VaR in Problem2. As VaR is a measure of risk, this can be a demonstration of the value of portfolio diversification.