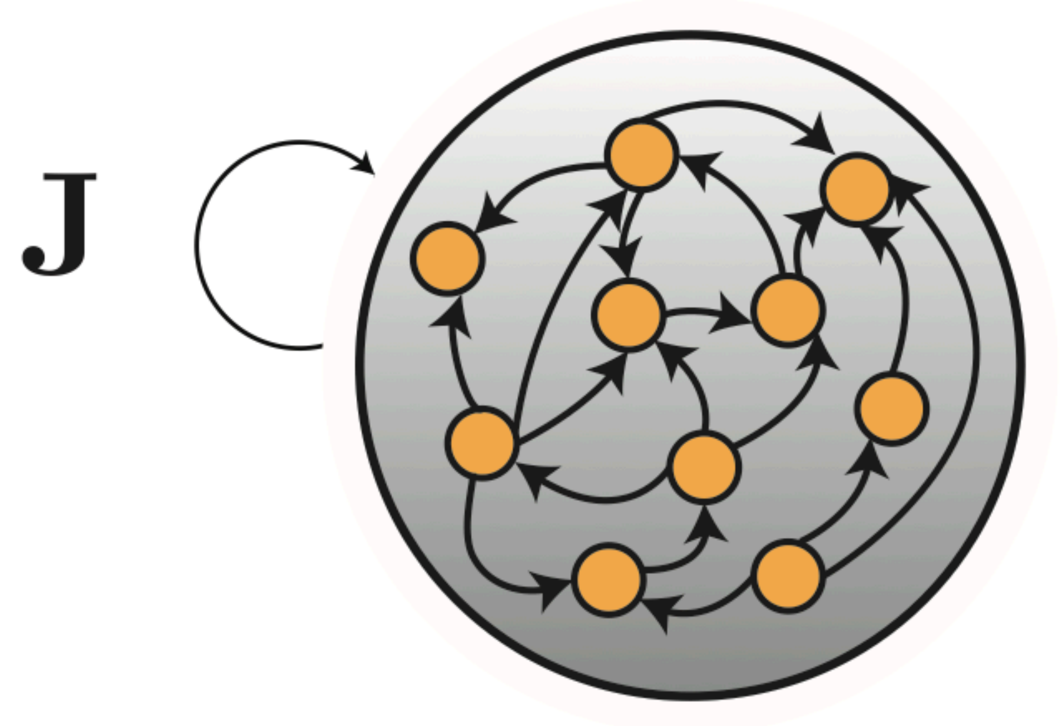
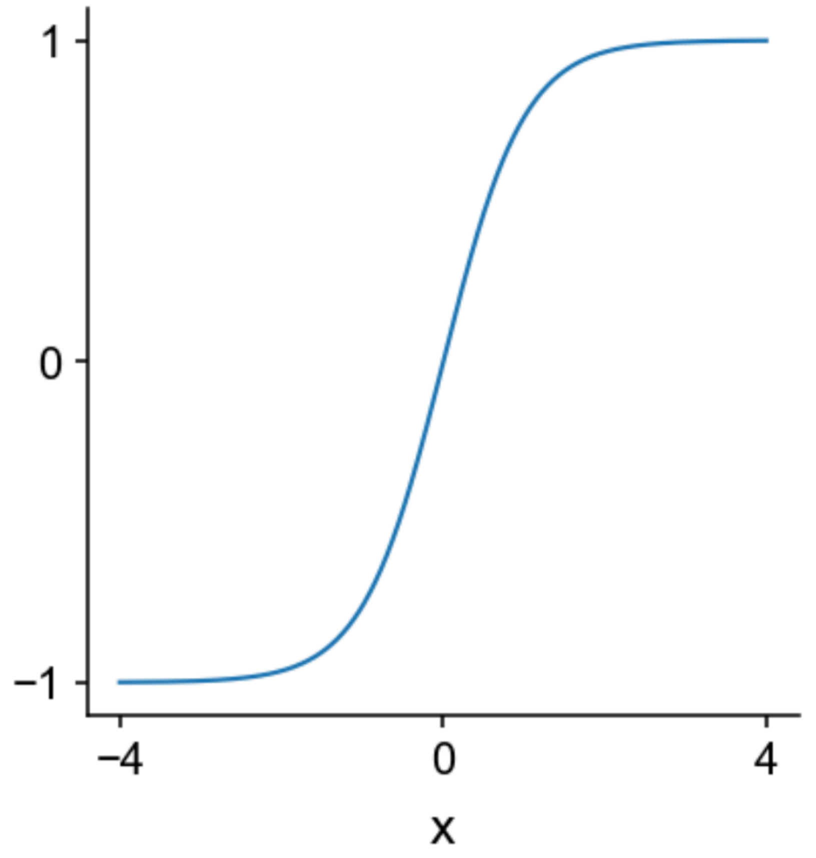
Classical chaotic RNN (Sompolinsky et al. 1988)

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} J_{ij}\phi(x_j(t))$$



$$J_{ij} \sim \mathcal{N}(0, g^2/N)$$

$$\phi(x) = \tanh(x)$$

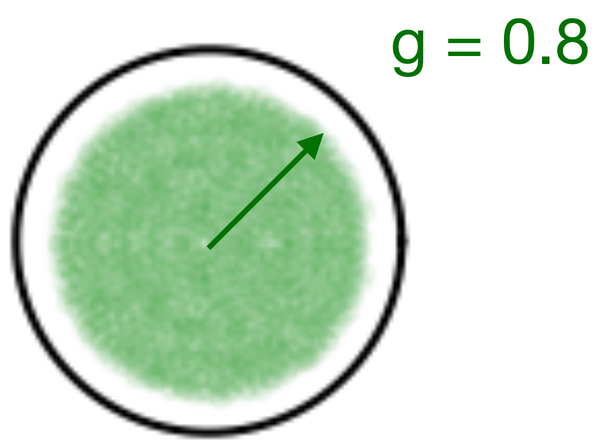


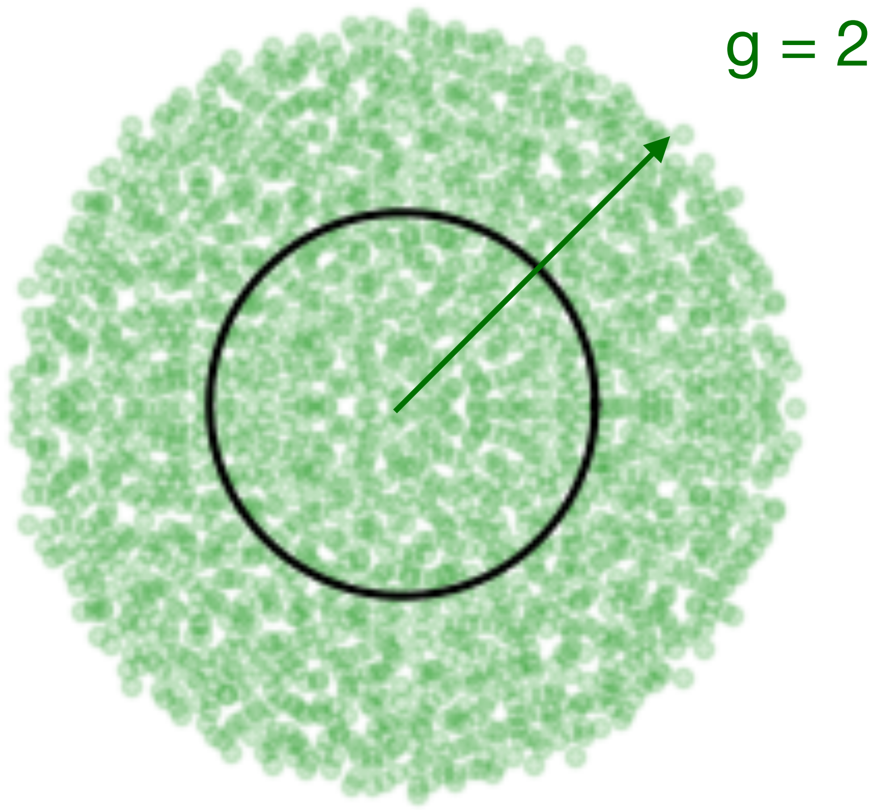
- The couplings J are initially chosen randomly, then are fixed
- Dynamics over the x variables are deterministic

Girko's Circular Law

- Consider a matrix whose entries are identically and independently distributed with 0 mean and variance 1/N
- The eigenvalues of this matrix are uniformly distributed over the unit circle in the complex plane

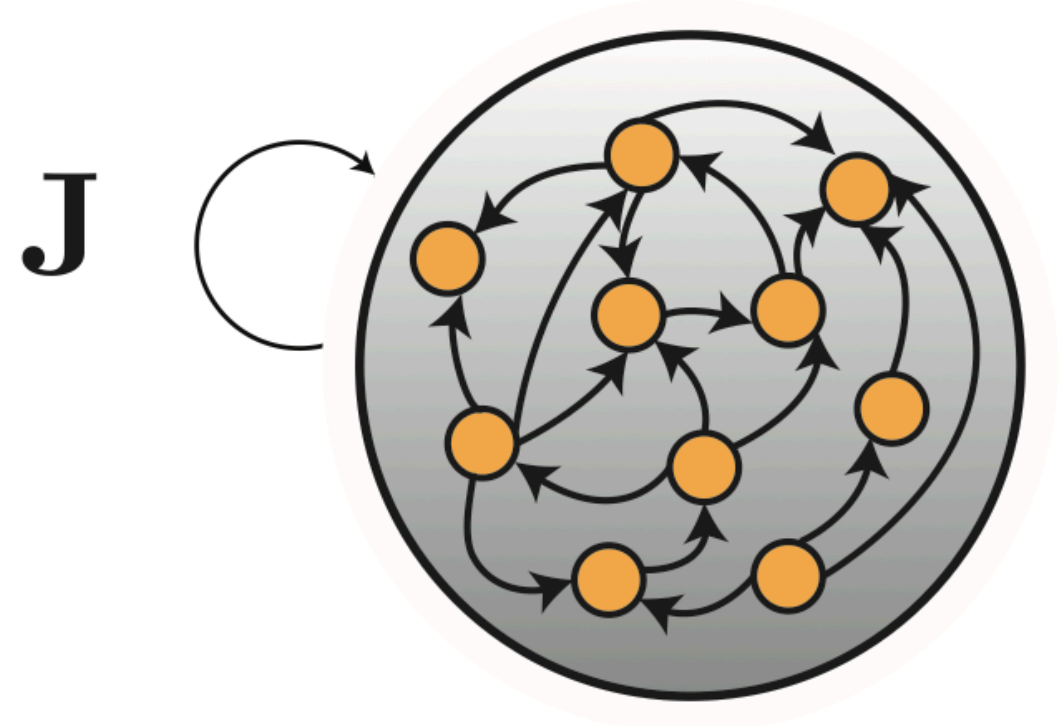
$$J_{ij} \sim \mathcal{N}(0, g^2/N)$$





Quiescent state

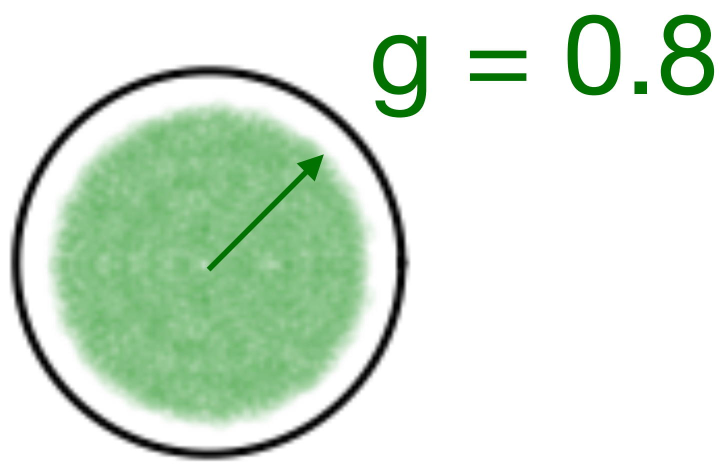
$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} J_{ij}\phi(x_j(t))$$



$$J_{ij} \sim \mathcal{N}(0, g^2/N)$$

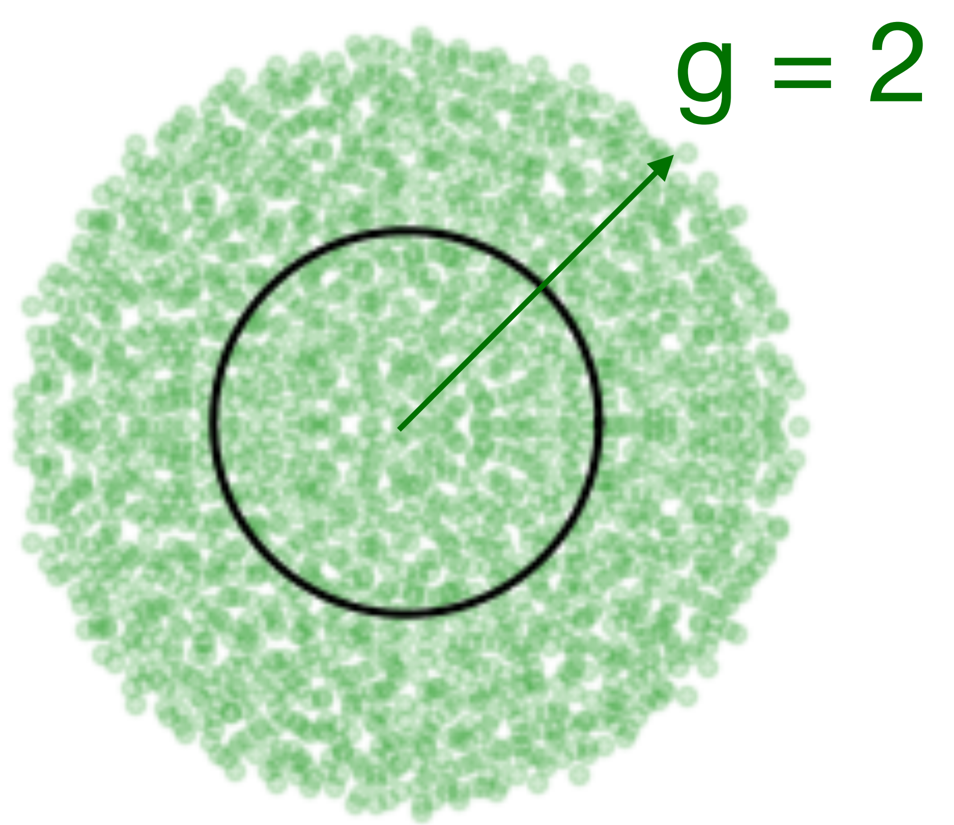
Origin is *always* a fixed point

$$x_j = 0 \text{ for all } i \implies \dot{x}_i(t) = 0 + \sum_j J_{ij}\phi(0) = 0$$

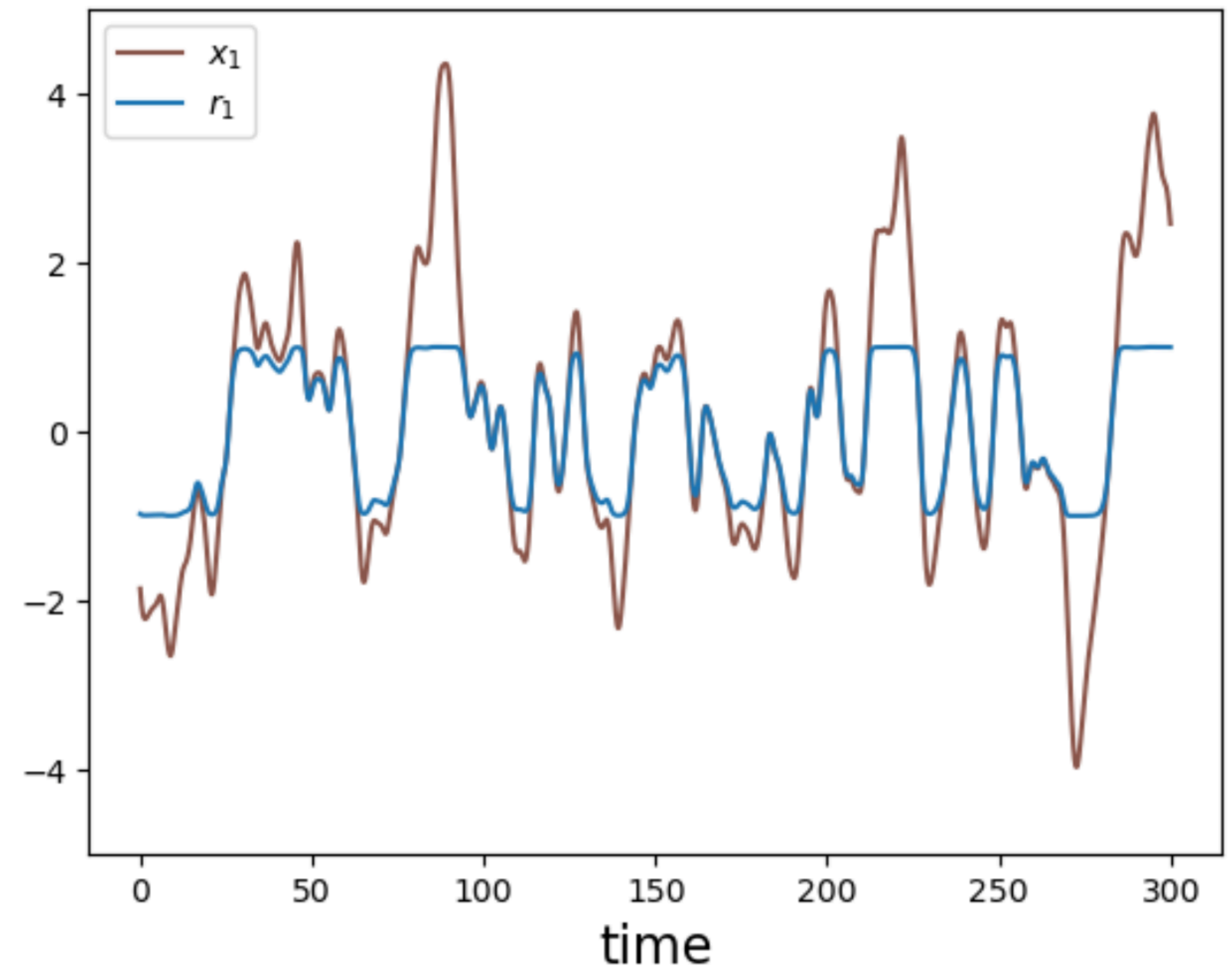


When g < 1, the origin is a *stable* fixed point and the state always gets pulled to the origin no matter where it starts.

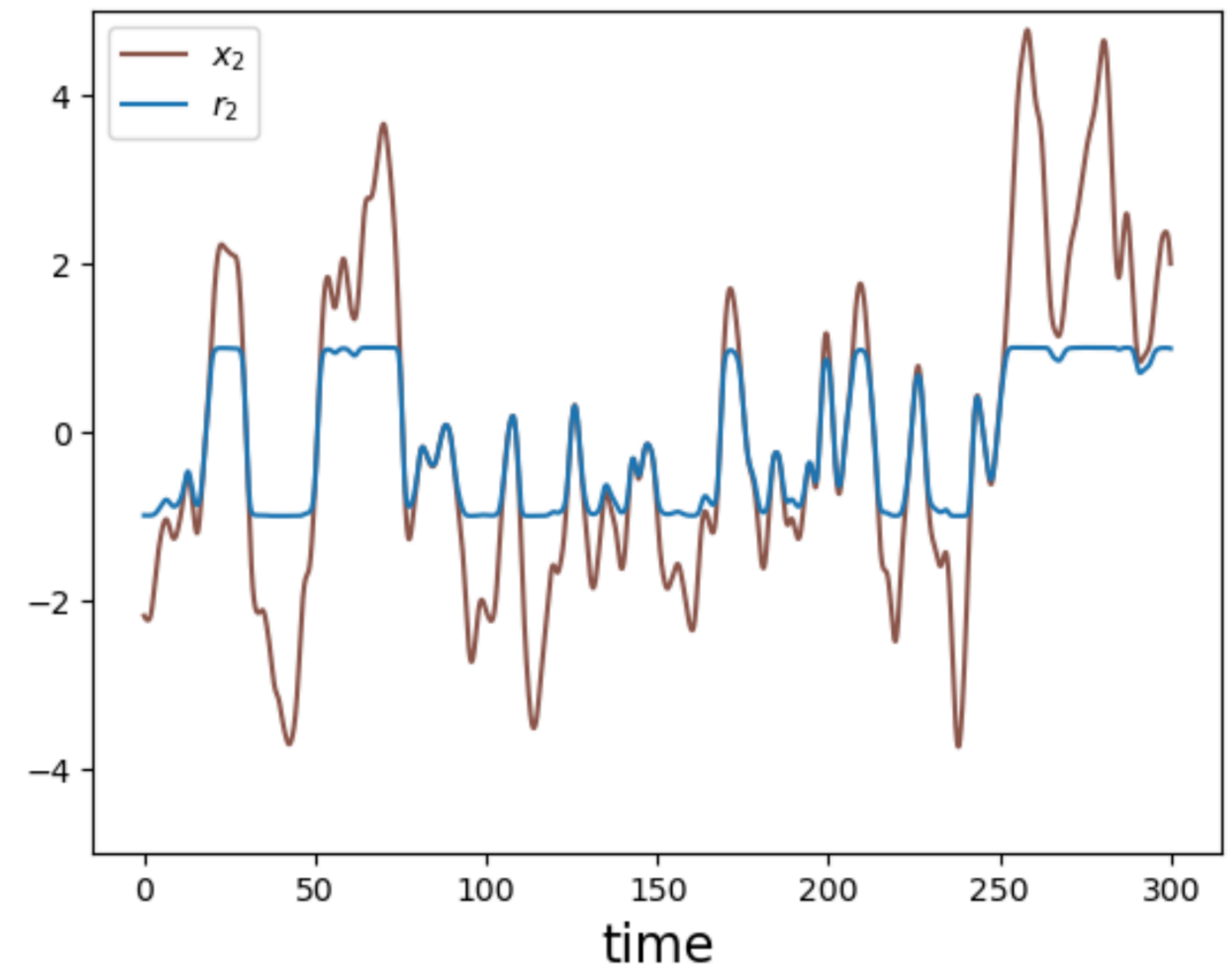
Chaotic state



Unit 1



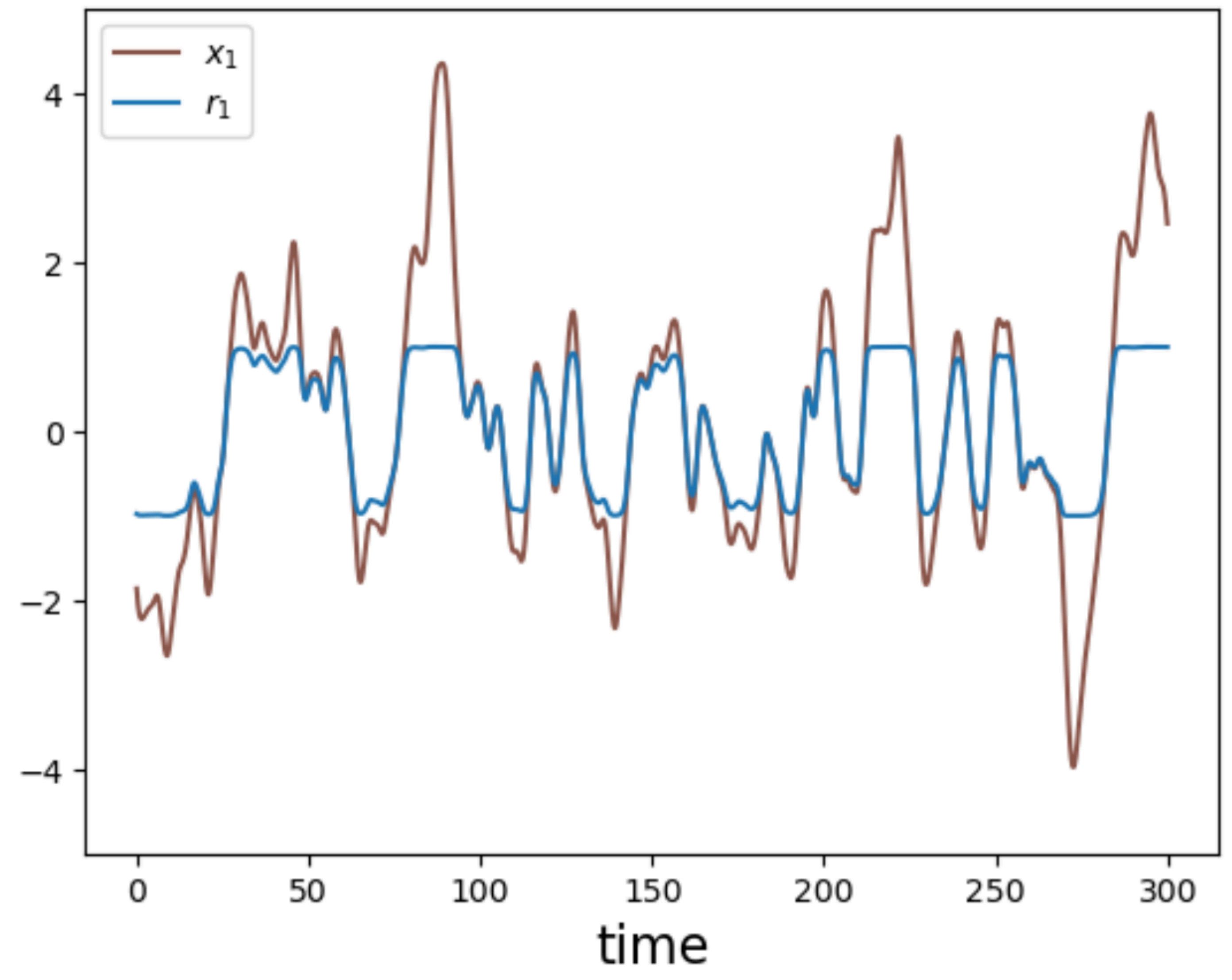
Unit 2



From eigenvalue perspective to unit perspective

Ansatz

- In the large N limit…
- Every unit's activity can be described by an identical and independent random process with **zero mean**
- Each process has a shared **autocovariance function** describing its temporal structure



$$\dot{x}_i(t) = -x_i(t) + \underbrace{\sum_{j=1}^{N} J_{ij}\phi(x_j(t))}_{\eta(t)}$$

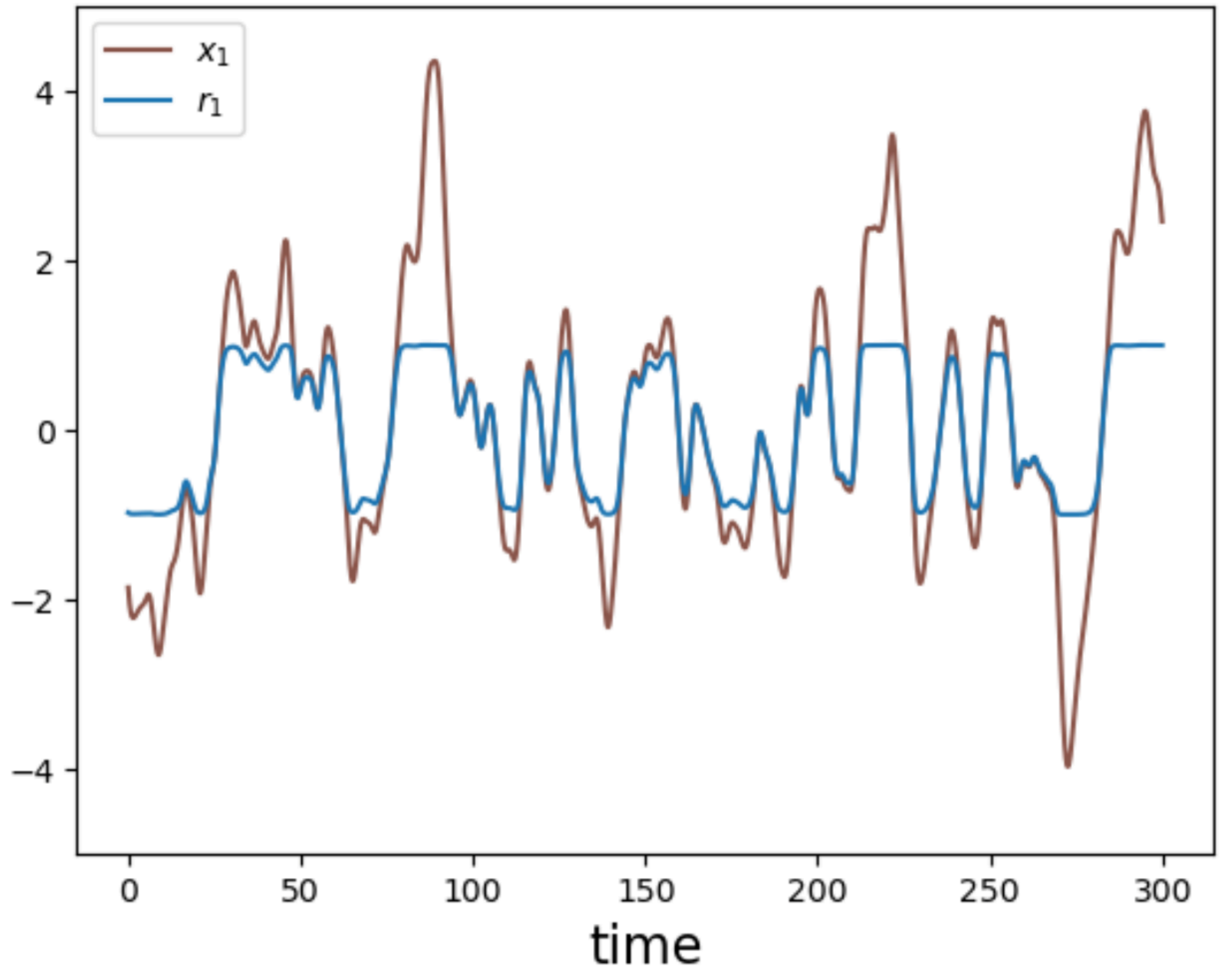
$\eta(t)$ **gaussian**

(by sloppy CLT-like argument)

From eigenvalue perspective to unit perspective

Ansatz

- In the large N limit…
- Every unit's activity can be described by an identical and independent random process with **zero mean**
- Each process has a shared **autocovariance function** describing its temporal structure



$$\dot{x}(t) = -x(t) + \eta(t)$$

$$\eta \sim \mathcal{GP}(0, C^{\eta}(\tau))$$

Reminder about Gaussian processes

Imagine a multivariate Gaussian
But there are infinitely many dimensions
Those dimensions are organized along a *continuum*

$$y_1, y_2, \cdots, y_N \sim \mathcal{N}(\vec{0}, \mathbf{\Sigma})$$

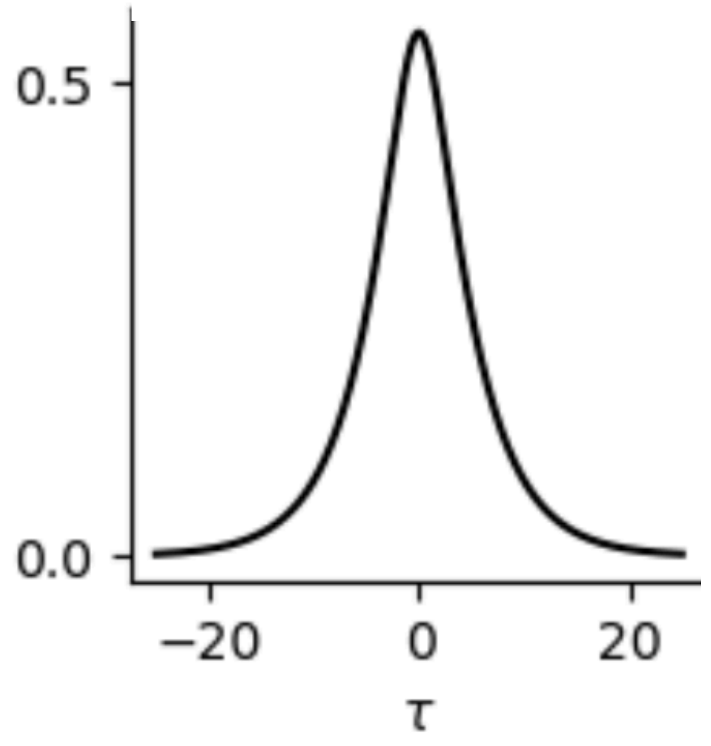
$$x(t=0), x(t=0.00001), x(t=0.00002), \cdots, x(t=T) \sim \mathcal{N}(0, C(t,t'))$$

A process is *stationary* if $C(t,t') = C(t-t') = C(\tau)$

Could have explicit functional form

$$C(t-t') = \sigma^2 \exp\left(-\frac{|t-t'|^2}{2L^2}\right)$$

Or not—just some set of values that defies closed-form description



“Measuring” the autocovariance function by averaging over time for stationary processes

$$C^x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt x(t) x(t-\tau)$$
$$= \langle x(t) x(t-\tau) \rangle_t$$

notational convenience

$$\int dt \approx \lim_{T\to\infty} \int_{-T/2}^{T/2} dt$$

Tempting to write this as a convolution

$$(f * g)(\tau) = \int dt f(t) g(\tau - t)$$

Exercise: prove that the autocovariance of x is equivalent to the convolution of x(t) with x(-t), its time-reversed self

$$(x * \tilde{x})(\tau) = \int dt x(t) \tilde{x}(\tau - t) = \int dt x(t) x(t-\tau) = \langle x(t) x(t-\tau) \rangle_t$$

$$C^x(\tau) = \langle x_i(t)x_i(t+\tau)\rangle_t$$

$$C^\phi(\tau) = \langle \phi_i(t)\phi_i(t+\tau)\rangle_t$$

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} J_{ij}\phi(x_j(t))$$

x_1

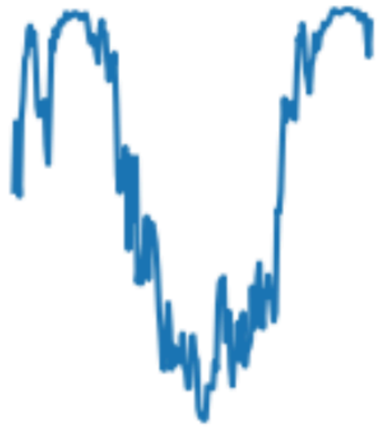
$= \mathrm{LPF}\Big($ $J_{11}\times$

ϕ_1

$+$ $J_{12}\times$

ϕ_2

$+$ $J_{13}\times$

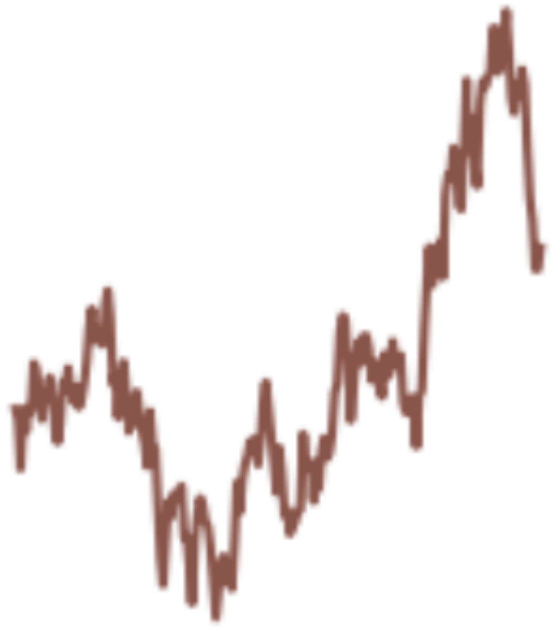


ϕ_3

$+ \cdots \Big)$

ϕ_1

$= \phi\Big($



x_1

$\Big)$

Key tricks in Fourier space

$$\phi(\omega) = \int dt e^{-i\omega t} \phi(t)$$

$$\frac{d}{dt} \sim -i\omega$$

Temporal derivative is the same as multiplying by -i omega in Fourier space (integrate by parts)

$$C^{\phi}(\omega) = \phi(\omega)\phi(\omega)^* = \|\phi(\omega)\|^2$$

Autocovariance function in Fourier space is the power spectrum

since $C^{\phi}(\tau) = (\phi * \tilde{\phi})(\tau)$

$$\delta(t-\tau) \to \int dt e^{-i\omega t} \delta(t-\tau) = e^{-i\omega\tau}$$

Delta function in Fourier space is just a standing wave

Deriving the self-consistency condition

Original equation

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} J_{ij}\phi_j(t)$$

Translate into Fourier domain

$$-i\omega x_i(\omega) = -x_i(\omega) + \sum_{j=1}^{N} J_{ij}\phi_j(\omega)$$

Solve for x_i

$$\implies x_i(\omega) = \frac{1}{1 - i\omega}\sum_{j=1}^{N} J_{ij}\phi_j(\omega)$$

$$C^x(\omega) = x_i(\omega)x_i(\omega)^* = \left(\frac{1}{1 - i\omega}\sum_{j=1}^{N} J_{ij}\phi_j(\omega)\right)\left(\frac{1}{1 + i\omega}\sum_{j'=1}^{N} J_{ij'}\phi_{j'}(\omega)^*\right)$$

$$C^x(\omega) = x_i(\omega)x_i(\omega)^* = \left(\frac{1}{1-i\omega}\sum_{j=1}^{N} J_{ij}\phi_j(\omega)\right)\left(\frac{1}{1+i\omega}\sum_{j'=1}^{N} J_{ij'}\phi_{j'}(\omega)^*\right)$$

$$= \frac{1}{1+\omega^2}\sum_{jj'} J_{ij}J_{ij'}\phi_j(\omega)\phi_{j'}(\omega)^*$$

$$= \frac{1}{1+\omega^2}\sum_{j} J_{ij}^2\phi_j(\omega)\phi_j(\omega)^*$$

assume 0 for j ~= j'

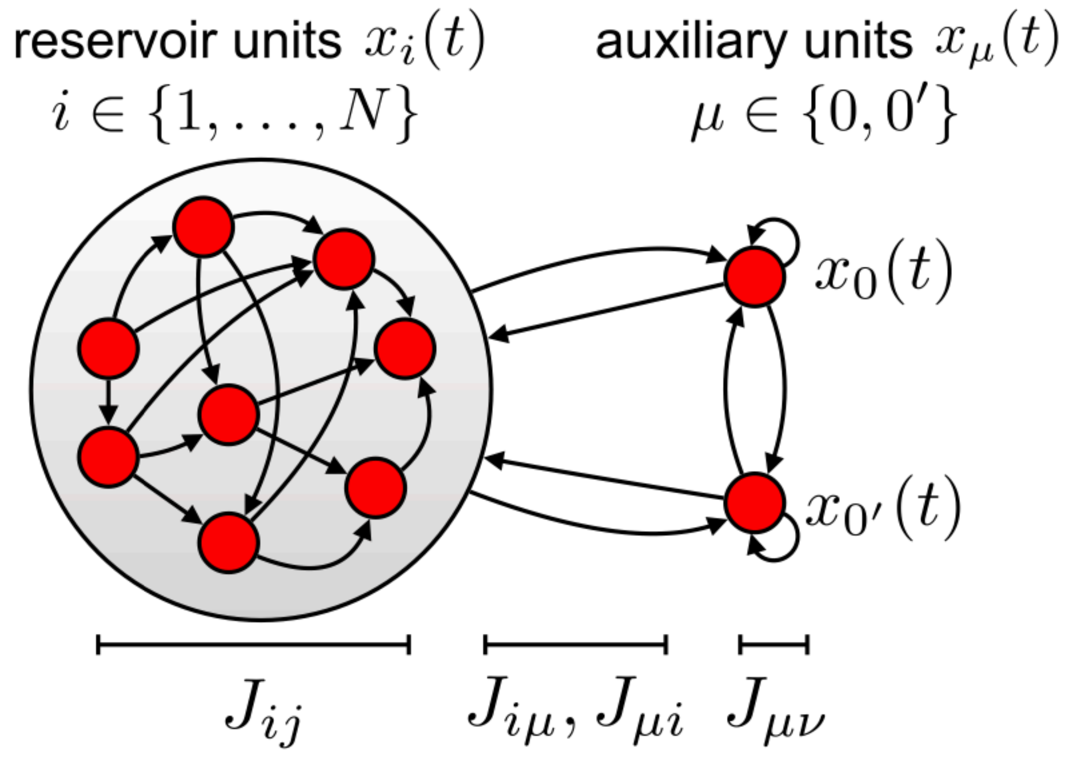
(not quite true! $\phi_j(\omega)\phi_{j'}(\omega)^* \sim \mathcal{O}(1/\sqrt{N})$
which has implications for
dimensionality of activity)

take J average

$$C^x(\omega) = \frac{1}{1+\omega^2}\frac{g^2}{N}\sum_{j} C^\phi(\omega)$$

$$= \frac{1}{1+\omega^2}g^2C^\phi(\omega)$$

David G. Clark



Back to the temporal domain

$$C^x(\omega) = \frac{1}{1+\omega^2} g^2 C^\phi(\omega) \qquad \implies (1+\omega^2) C^x(\omega) = g^2 C^\phi(\omega)$$

$$\implies \left(1 - \frac{d^2}{d\tau^2}\right) C^x(\tau) = g^2 C^\phi(\tau)$$

$$\ddot{C}^x(\tau) = C^x(\tau) - g^2 C^\phi(\tau)$$

$$\ddot{C}^x(\tau) = C^x(\tau) - g^2 C^\phi(\tau)$$

$C^\phi(\tau)$ is fully determined by $C^x(\tau)$ since $\phi = \tanh(x)$

$$\langle \phi(x(t))\phi(x(t+\tau))\rangle$$

$$x(t) = \sqrt{C^x(0) - C^x(\tau)} \times x_1 + \sqrt{C^x(\tau)}z$$
$$x(t+\tau) = \sqrt{C^x(0) - C^x(\tau)} \times x_2 + \sqrt{C^x(\tau)}z$$

$$x_1, x_2, z \sim \mathcal{N}(0,1)$$

$$\ddot{C}^x(\tau) = C^x(\tau) - g^2 C^\phi(\tau)$$

$$\begin{aligned} x(t) &= \sqrt{C^x(0) - C^x(\tau)} \times x_1 + \sqrt{C^x(\tau)}z \\ x(t+\tau) &= \sqrt{C^x(0) - C^x(\tau)} \times x_2 + \sqrt{C^x(\tau)}z \end{aligned} \qquad x_1, x_2, z \sim \mathcal{N}(0,1)$$

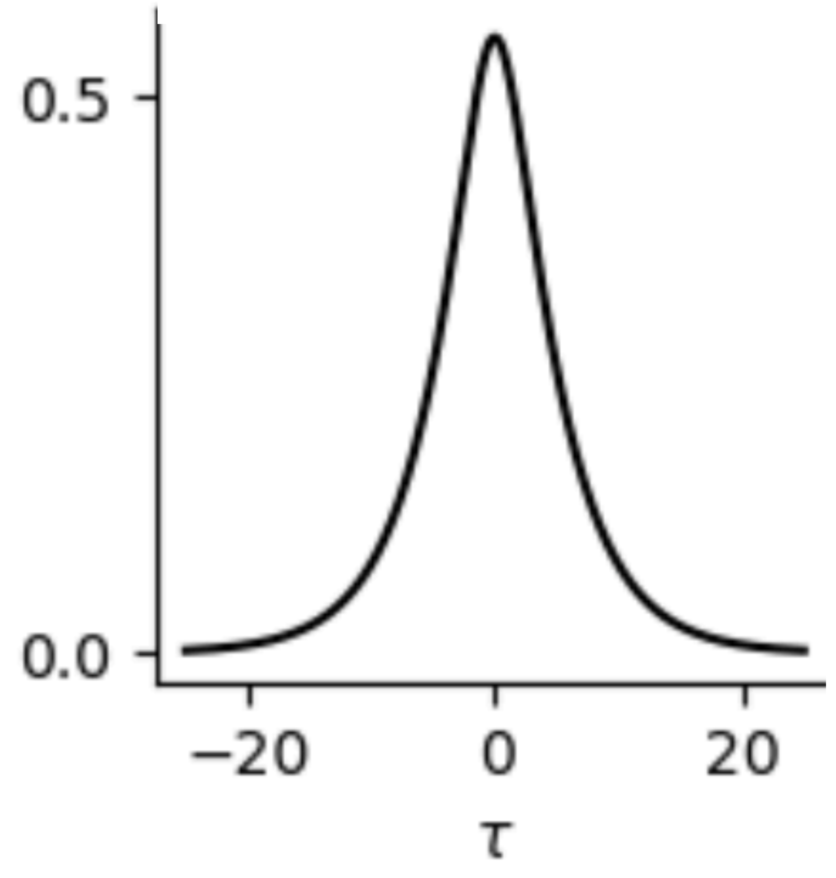
$$\begin{aligned} C^\phi(\tau) = \int dz \int dx_1 \int dx_2 \frac{1}{(2\pi)^{3/2}} e^{-z^2/2} e^{-x_1^2/2} e^{-x_2^2/2} \phi\left(\sqrt{C^x(0) - C^x(\tau)}x_1 + \sqrt{C^x(\tau)}z\right) \\ \times \phi\left(\sqrt{C^x(0) - C^x(\tau)}x_2 + \sqrt{C^x(\tau)}z\right) \end{aligned}$$

$$C^\phi(\tau) = \int Dz \left[\int Dx \phi\left(\sqrt{C^x(0) - C^x(\tau)}x + \sqrt{C^x(\tau)}z\right)\right]^2$$

$$\ddot{C}^x(\tau) = C^x(\tau) - g^2 C^\phi(\tau)$$

$$C^\phi(\tau) = \int Dz \left[\int Dx \phi \left(\sqrt{C^x(0) - C^x(\tau)} x + \sqrt{C^x(\tau)} z \right) \right]^2$$

TLDR: we derive a self-consistency condition that implies a second-order ODE, a particular solution of which is the autocorrelation kernel for the currents



How did we arrive at this Ansatz?

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} J_{ij}\phi(x_j(t))$$

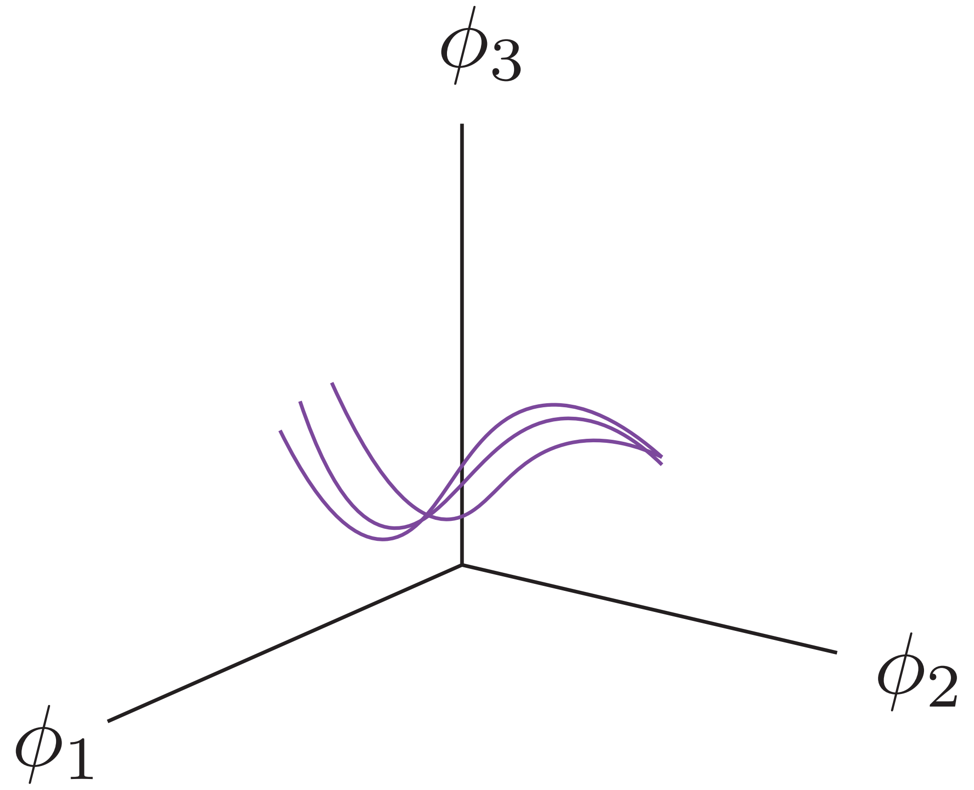
$$J_{ij} \sim \mathcal{N}(0, g^2/N)$$

$$\implies \dot{x} = -x + \eta$$

$$\eta \sim \mathcal{GP}(0, g^2 C^{\phi})$$

MSRJD path integral method (Crisanti 2018, Helias group tutorials)

Compact description of dynamics $T[x_i](t) = \sum_j J_{ij}\phi_j(t)$ $T[x](t) = (1+\partial_t)x(t)$



$$\delta\Big(T[x_i](t) - \sum_j J_{ij}\phi_j(t)\Big)$$

for all t, i

Delta-function enforcement of deterministic dynamics

$$Z[\mathbf{J}] = \int \mathcal{D}\mathbf{x}(t) \prod_{t,i} \overbrace{\delta\Big(T[x_i](t) - \sum_j J_{ij}\phi_j(t)\Big)}$$

$$\mathcal{D}\mathbf{x}(t) = dx_1(t=0)dx_1(t=0.0001)dx_1(t=0.0002)\cdots dx_1(t=T)dx_2(t=0)dx_2(t=0.0001)dx_2(t=0.0002)\cdots$$

Fourier representation of the Delta function

For every i and t, define a “conjugate variable” to represent the delta function in Fourier space

$$\delta\Big(T[x_i](t) - \sum_j J_{ij}\phi_j(t)\Big) = \frac{1}{2\pi}\int d[\hat{x}_i(t)] \exp\left[i\hat{x}_i(t)\Big(T[x_i](t) - \sum_j J_{ij}\phi_j(t)\Big)\right]$$

Integrate over all double-infinity of these conjugate variables

$$Z[\mathbf{J}] \propto \int \mathcal{D}\mathbf{x}(t) \int \mathcal{D}\hat{\mathbf{x}}(t) \boxed{\prod_{t,i}} \exp\left[\hat{x}_i(t)\Big(T[x_i](t) - \sum_j J_{ij}\phi_j(t)\Big)\right]$$

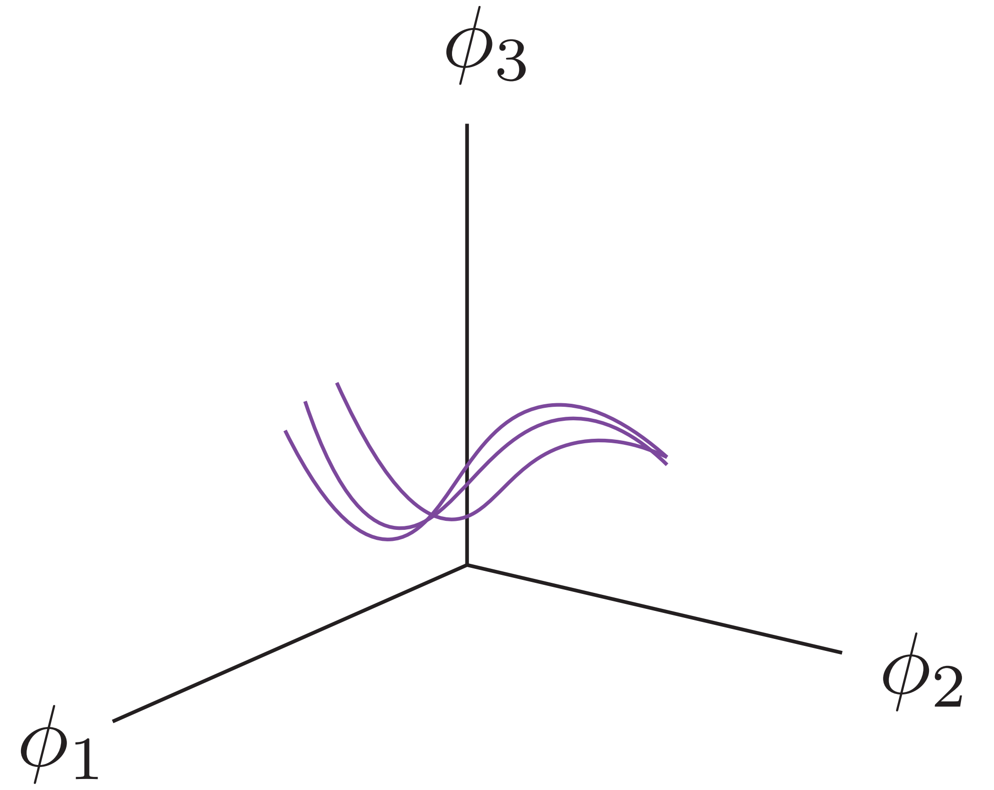
$$Z[\mathbf{J}] = \int \mathcal{D}\mathbf{x}(t) \int \mathcal{D}\hat{\mathbf{x}}(t) \exp\left[i \boxed{\sum_i \int dt} \hat{x}_i(t) T[x_i](t) - i \boxed{\sum_{ij} J_{ij} \int dt} \hat{x}_i(t)\phi_j(t)\right]$$

product of exponents turns into sum (and integral) into argument of exponent

Disorder average of the path integral

$$Z[\mathbf{J}] = \int \mathcal{D}\mathbf{x}(t) \int \mathcal{D}\hat{\mathbf{x}}(t) \exp\left[i \sum_i \int dt \hat{x}_i(t) T[x_i](t) - i \sum_{ij} J_{ij} \int dt \hat{x}_i(t) \phi_j(t) \right]$$

$$\begin{aligned} \langle Z \rangle_{\mathbf{J}} &= \int \prod_{ij} dJ_{ij} \exp\left(- \frac{N}{2g^2} \sum_{ij} J_{ij}^2 \right) \times Z[\mathbf{J}] \\ &= \int \mathcal{D}\mathbf{x}(t) \int \mathcal{D}\hat{\mathbf{x}}(t) \int \prod_{ij} dJ_{ij} \exp\left[\sum_{ij} \left(-\frac{N}{2g^2} J_{ij}^2 - i \hat{x}_i \phi_j J_{ij} \right) + i \sum_i \hat{x}_i T[x_i] \right] \end{aligned}$$



Completing the square

$$\langle Z \rangle_{\mathbf{J}} = \int \mathcal{D}\mathbf{x}(t) \int \mathcal{D}\hat{\mathbf{x}}(t) \int \prod_{ij} dJ_{ij} \exp\left[\sum_{ij}\left(-\frac{N}{2g^2}J_{ij}^2 - i\hat{x}_i\phi_j J_{ij}\right) + i\sum_i \hat{x}_i T[x_i]\right]$$

$$-\frac{N}{2g^2}J_{ij}^2 - i\hat{x}_i\phi_j J_{ij} = \cancel{-\frac{N}{2g^2}\left(J_{ij} - igN^{-1}\hat{x}_i\phi_j\right)^2} - \frac{g^2}{2N}(\hat{x}_i\phi_j)^2$$

$$\int dJ_{ij} \rightarrow \text{ const.}$$

$$-\frac{g^2}{2N}(\hat{x}_i\phi_j)^2 = -\frac{g^2}{2N}\int dt\hat{x}_i(t)\phi_j(t)\int dt'\hat{x}_i(t')\phi_j(t')$$

$$= -\frac{g^2}{2}\int dt dt'\hat{x}_i(t)\left(\frac{1}{N}\phi_j(t)\phi_j(t')\right)\hat{x}_i(t')$$

$$\langle Z\rangle_{\mathbf{J}} = \int \mathcal{D}\mathbf{x}(t) \int \mathcal{D}\hat{\mathbf{x}}(t) \exp\left[i \sum_i \hat{x}_i T[x_i] - \frac{g^2}{2} \sum_i \underbrace{\int dt dt' \hat{x}_i(t) \underbrace{\left(\frac{1}{N} \sum_j \phi_j(t)\phi_j(t') \right)}_{C^{\phi}(t,t')} \hat{x}_i(t')}_{\hat{x}_i C^{\phi} \hat{x}_i} \right]$$

$$\langle Z\rangle_{\mathbf{J}} = \prod_i \int \mathcal{D}x_i(t) \int \mathcal{D}\hat{x}_i(t) \exp\left[i\hat{x}_i T[x_i] - \frac{g^2}{2}\hat{x}_i C^{\phi} \hat{x}_i \right]$$

$$1 = \int \mathcal{D}C^{\phi}(t,t') \int \mathcal{D}\hat{C}^{\phi}(t,t') \exp\left[\frac{1}{2}\hat{C}^{\phi}\Big(NC^{\phi} - \sum_i \phi_i(t)\phi_i(t') \Big) \right]$$

$$\langle Z \rangle_{\mathbf{J}} = \int \mathcal{D}C^{\phi}(t,t') \int \mathcal{D}\hat{C}^{\phi}(t,t') \exp\left[\frac{1}{2}N\hat{C}^{\phi}C^{\phi}\right] \prod_i \underbrace{\int \mathcal{D}x_i(t) \int \mathcal{D}\hat{x}_i(t) \exp\left[i\hat{x}_i T[x_i] - \frac{g^2}{2}\hat{x}_i C^{\phi}\hat{x}_i - \phi_i \hat{C}^{\phi}\phi_i\right]}_{W_{\text{single-site}}(C^{\phi},\hat{C}^{\phi}) \quad \text{identical for every i}}$$

$$\begin{aligned}
\langle Z \rangle_{\mathbf{J}} &= \int \mathcal{D}C^{\phi}(t,t') \int \mathcal{D}\hat{C}^{\phi}(t,t') \exp\left[\frac{1}{2}N\hat{C}^{\phi}C^{\phi}\right] \prod_i W_{\text{single-site}}(C^{\phi},\hat{C}^{\phi}) \\
&= \int \mathcal{D}C^{\phi}(t,t') \int \mathcal{D}\hat{C}^{\phi}(t,t') \exp\left[\frac{1}{2}N\hat{C}^{\phi}C^{\phi}\right] W^{N}_{\text{single-site}}(C^{\phi},\hat{C}^{\phi}) \\
&= \int \mathcal{D}C^{\phi}(t,t') \int \mathcal{D}\hat{C}^{\phi}(t,t') \exp\left[\frac{1}{2}N\hat{C}^{\phi}C^{\phi}\right] \exp[N \log W_{\text{single-site}}(C^{\phi},\hat{C}^{\phi})] \\
&= \int \mathcal{D}C^{\phi}(t,t') \int \mathcal{D}\hat{C}^{\phi}(t,t') \exp\left[\frac{1}{2}N\hat{C}^{\phi}C^{\phi} + N \log W_{\text{single-site}}(C^{\phi},\hat{C}^{\phi})\right] \\
&= \int \mathcal{D}C^{\phi}(t,t') \int \mathcal{D}\hat{C}^{\phi}(t,t') \exp\left[- N\mathcal{S}[C^{\phi},\hat{C}^{\phi}]\right]
\end{aligned}$$

$$\int \mathcal{D}C^{\phi}(t,t') \int \mathcal{D}\hat{C}^{\phi}(t,t') \exp\Big[- N\mathcal{S}[C^{\phi}, \hat{C}^{\phi}]\Big]$$

N is huge, can use saddle-point approximation

$$\begin{aligned} e^{-Nf(x)} &= e^{-Nf(x^*)-Nf'(x^*)(x-x^*)-N\frac{1}{2}f''(x^*)(x-x^*)^2+\cdots} \\ &= e^{-Nf(x^*)+0-N\frac{1}{2}f''(x^*)(x-x^*)^2+\cdots} \\ &= e^{-Nf(x^*)} \times e^{-N\times \text{positive number}} \end{aligned}$$

$$\mathcal{S}[C^\phi, \hat{C}^\phi] = -\frac{1}{2}\hat{C}^\phi C^\phi - \log W_{\text{single-site}}(C^\phi, \hat{C}^\phi)$$

$$W_{\text{single-site}}(C^\phi, \hat{C}^\phi) = \int \mathcal{D}x(t)\mathcal{D}\hat{x}(t)\exp\left[i\hat{x}T[x] - \frac{g^2}{2}\hat{x}C^\phi\hat{x} - \phi\hat{C}^\phi\phi\right]$$

Exercise: find saddle point

$$0 = \frac{\delta\mathcal{S}}{\delta C^\phi}$$

$$0 = \frac{\delta\mathcal{S}}{\delta\hat{C}^\phi}$$