$w_{ij} \overset{iid}{\sim} P(w)$ *   ◊ $\underline{W} \sim P(\underline{W})$

$\underline{W} \to \mathbb{R}^{N\times N}$ $P(\underline{W}) \to N^2$ parameter if gaussian

$$* \quad P(w_{ij}, w_{kl}) \neq P(w_{ij})\,P(w_{kl})$$

$$\Updownarrow$$

$i = k$	$i = l$	$j = k$	$j = l$	$i = l \quad j = k$

$\langle W_{ij} W_{il} \rangle$	$\langle W_{ij} W_{kl} \rangle$	$\langle W_{ij} W_{jk} \rangle$	$\langle W_{ij} W_{kj} \rangle$	$\underbrace{\langle W_{ij} W_{ji} \rangle}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
μ^{conv}	μ_2^{ch}	μ_2^{ch}	μ^{div}	μ^{R}

if $w_{ij} \overset{iid}{\sim} P(w) \Rightarrow \mu^{conv} = \mu^{div} = \mu_2^{ch} = \mu^{R} = (\mu_1)^2 = \langle W_{ij} \rangle$

$\kappa^{R} = \mu^{R} - (\mu_1)^2 = Cov(w_{ij}, w_{ji})$ | $\tau^{R} = \text{Corr Coef}\,(w_{ij}, w_{ji})$

$\kappa_2^{ch} =$

$\kappa^{div} =$

$\kappa^{conv} =$

2b

e.g.

◇ $\mu_3^{ch} = \langle w_{ik} w_{kl} w_{lj} \rangle_{ijkl}$

$$\mu_n^{ch} = \langle (W^n)_{ij} \rangle_{ij} \cdot \frac{1}{N^{n-1}}$$

cumulant ~~implies~~ combinatorial relationship:

$$\mu_n^{ch} = \sum_{\{n_1, \dots, n_t\} \in C(n)} \left(\prod_{i=1}^{t} \kappa_{n_i} \right)$$

e.g. $n=3$

$$\mu_3^{ch} = \kappa_3^{ch} + 2\kappa_2\kappa_1 + \kappa_1^3$$

(~10')

$$\frac{d\vec{r}}{dt} = -\vec{r} + \underline{\underline{W}}\,\phi(\vec{r}) + \vec{h}$$

linearize around $\vec{r}^*$

$$\frac{d\vec{r}}{dt} \approx -\vec{r}^* + \underline{\underline{\tilde{W}}} \cdot \vec{r} + \vec{h} \quad , \quad \tilde{W}_{ij} = W_{ij}\,\phi'(r_j)$$

fixed point

$$\vec{r}^* = \left(\underline{\underline{I}} - \underline{\underline{\tilde{W}}}\right)^{-1} \cdot \vec{h}$$

$\tilde{W} = W \quad , \quad \underline{\underline{\chi}} := \left(\underline{\underline{I}} - \underline{\underline{\tilde{W}}}\right)^{-1}$

χ_{ij} : linear response of neuron i to a pert. of neuron j (with filter)

$$\boxed{\left(\underline{\underline{I}} - \underline{\underline{W}}\right)^{-1} = \sum_{n=0}^{\infty} W^n}$$

$$\Rightarrow \chi_{ij} = \sum_{n=0}^{\infty} (W^n)_{ij} \Rightarrow \langle \chi_{ij} \rangle = \sum_{n=0}^{\infty} \langle (W^n)_{ij} \rangle_{ij} = \sum_{n=0}^{\infty} N^{n-1} \mu_n^{ch}$$

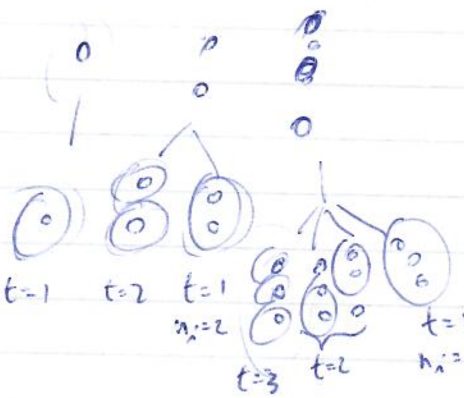
~7'

(4b)

Hu, ..., Shea-Brown

Resumming the cumulants:

$$\langle X_{ij} \rangle = \frac{1}{N} + \frac{1}{N}\sum_{n=1}^{\infty} \underbrace{\sum_{\{n_1,\dots,n_t\} \in C(n)} \left(\prod_{i=1}^{t} (N)^{n_i} \kappa_{n_i} \right)}_{\text{All cumulants} \to \text{one moment}}$$



$$\langle X_{ij} \rangle = \frac{1}{N} + \frac{1}{N}\sum_{t=1}^{\infty} \sum_{n_1,\dots,n_t=1}^{\infty} \underbrace{\left(\prod_{i=1}^{t} (N)^{n_i} \kappa_{n_i} \right)}_{\sum_{n_1,\dots,n_t} (n_1 \dots n_t)}$$

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \left(\prod_{\alpha \in \{i,j\}} X_\alpha \right)$$

$$= \sum_{\substack{i=1 \\ j=1}}^{2} x_i x_j = x_1 x_1 + x_1 x_2 + x_2 x_1 + x_2 x_2 = \prod_{\alpha=1}^{2} (x_1 + x_2) = \prod_{\alpha \in \{i,j\}} \left(\sum_{\alpha=1}^{2} X_\alpha \right)$$

$$\Rightarrow \langle X_{ij} \rangle = \frac{1}{N} + \frac{1}{N}\sum_{t=1}^{\infty} \prod_{i=1}^{t} \underbrace{\left(\sum_{n_i=1}^{\infty} (N)^{n_i} \kappa_{n_i} \right)}_{\text{does not depend on } t}$$

$$\langle \chi_{ij} \rangle = \frac{1}{N} + \frac{1}{N} \sum_{t=1}^{\infty} \left(\sum_{n=1}^{\infty} (N)^n K_n \right)^t$$

$$\sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

$$\langle \chi_{ij} \rangle = \frac{1}{N} + \frac{1}{N} \cdot \frac{\sum_{n=1}^{\infty} (N)^n K_n}{1 - \sum_{n=1}^{\infty} (N)^n K_n}$$

$$= \frac{1/N}{1 - \sum_{n=1}^{\infty} (N)^n K_n}$$

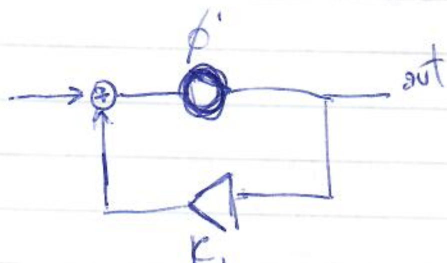
Only chain motif cumulants are relevant!

Applications

Networks where $\kappa_n = 0$ for $n > 1$

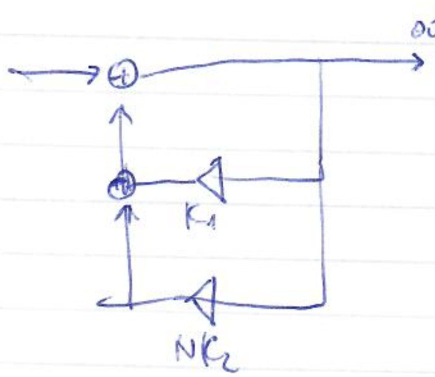
e.g. uniform degree, Erdos-Renyi ← If time allows, show how kumulants and degree relate.

$$\langle X_{ij} \rangle = \frac{1}{1-\kappa_1} = \frac{1}{1-\langle w_{ij} \rangle}$$



If $\kappa_2 \neq 0$ and $\kappa_n = 0$ for $n \geq 3$

$$\langle X_{ij} \rangle = \frac{1}{1-\kappa_1 - N\kappa_2}$$



Shao, ..., Ostojic, 2024

Go back to

$$X_0 = \left(\underline{\underline{I}} - \underline{\underline{W}}\right)^{-1}$$

Low rank approx. of $\underline{\underline{W}} \approx \underline{\underline{J}} = \frac{1}{N} \sum_{r=1}^{R} \vec{m}^{(r)} \vec{n}^{(r)T} = \frac{1}{N} \underline{\underline{M}}\, \underline{\underline{N}}^T$

To pick $\vec{m}, \vec{n}$, use ordered eigenvalues of W

$\Rightarrow \vec{m}^{(r)} = \sqrt{N}\, \vec{R}_r$, $\vec{n}^{(r)} = \lambda_r \sqrt{N}\, \vec{L}_r$ $\quad \left(\vec{L}_r^T \vec{R}_{r'} = \delta_{rr'}\right)$

right eig. $\qquad$ left eig.

$$\Rightarrow \underline{\underline{J}} = \sum_{r=1}^{R} \lambda_r \vec{R}_r \cdot \vec{L}_r^T$$

$$\Rightarrow X = \left(I - \frac{1}{N} \underline{M}\, \underline{N}^T\right)^{-1} \xrightarrow{\text{Woodbury}} = I + \underline{\underline{M}} \left(I - \underline{\underline{\Lambda}}\right)^{-1} \underline{\underline{N}}^T \cdot \frac{1}{N}$$

Λ: Diag$(\lambda_1, \ldots, \lambda_R)$

Woodbury

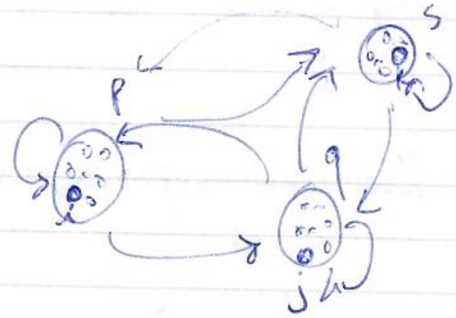
$$\cancel{(A + UV^T)^{-1} = A^{-1} - A^{-1} U (C^{-1} + V A^{-1} U)^{-1} V A^{-1}}$$

$$(A + UCV)^{-1} = A^{-1} - A^{-1} U \left(C^{-1} + V A^{-1} U\right)^{-1} V A^{-1}$$

(86)

Do motifs affect leading eigenvalues?

$$\underline{W} = \underbrace{\underline{W}^0}_{\text{deterministic}} + \underline{Z}$$



Assume pop. structure ($\underline{P}$ pops)

$$\mathbb{P}(W_{ij} = w) = f^{pq}(w)$$

$$\tau_2^{ch} = \frac{\langle W_{ij} W_{jk} \rangle - \langle W_{ij} \rangle \langle W_{jk} \rangle}{\mathrm{std}(w_{ij})\,\mathrm{std}(w_{jk})} = \rho(w_{ij}, w_{jk})$$

$$\tau_2^{ch} = \tau_{pqs}^{ch}$$

$$\det(W - \lambda I) = 0 \Rightarrow \det(W^0 + Z - \lambda I) = 0$$

$W^0 = \frac{1}{N} M_0 N_0^T$ (as before for J)

Use $\boxed{\det(\underbrace{UV^T}_{\text{rank } R} + A) = \det(I_R + V^T A^{-1} U)\det(A)}$ Matrix determinant lemma

9b

Q:
effect of $Z - N_0$ interaction

Non zero by assumption

$$\det(W - \lambda I) = \det(Z - \lambda I)\,\frac{1}{\lambda^N}\,\det\Big(\lambda I_{R_0} - \underbrace{\frac{1}{N} N_0^{+}\Big(1 - \frac{Z}{\lambda}\Big)^{-1} M_0}_{\text{interaction}}\Big)$$

expand as series
→ Motifs of Z!

$\langle Q \rangle = \frac{1}{N} N_0^{T} (1) -$ ⟹ Only chain motif matter.

Furthermore $\mu_1^Z = 0$, so only $\mu_2^Z \rightarrow \tau_{pq}^{ch}$ will matter.

The expression is different because τ_{pq}^{ch} are not homogeneous.

$$\langle (Z^2)_{ij} \rangle = \begin{cases} \frac{N}{P} \sum_{q=1}^{P} \kappa_{pqs}^{c} & \text{if } i \neq j \\ N/P \sum_{q=1}^{P} \kappa_{pqs}^{r} & \text{if } i = j \end{cases}$$

Block matrix in $P \times P$.

$$\left(I - \frac{Z}{\lambda}\right)^{-1} = \left(I - \frac{\langle Z^2 \rangle}{\lambda^2}\right)^{-1}$$

requires some work

Example: Fully-connected E-I network

with $J^0_{EE} = J^0_{IE} = J^0$, $J^0_{EI} = J^0_{II} = -g J^0$

J^0 is rank-1

$$\lambda_0 = \underbrace{(\alpha_E - g\alpha_I)}_{<0 \text{ (I-dominated)}} J^0 N$$

$$\sigma^2_{pq} = \frac{\sigma^2}{N}$$

$$\Rightarrow \lambda_{1,2} = \frac{\lambda_0 \mp \sqrt{\lambda_0^2 + 4\Delta^2}}{2}, \text{ with}$$

$$\Delta^2 = \sigma^2 \tau^c (N+1) + \sigma^2 \tau^r$$

Are τ^c, τ^r the same across populations?