Crane-Load Dynamics and Control
Working Note

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1 Introduction

2 Dynamics

2.1 Simple pendulum

A frame is define with horizontal xy plane, and z point in vertically upwards. Consider a point mass m attached to a fixed point with a mass-less rope of length L. The position of the attachment point is on the horizontal x axis with position $(x_0, 0, 0)$. Suppose that the position of the mass is $(x, 0, z)$.

Because of the constraint $(x - x_0)^2 + z^2 = L^2$, the mass has one degree of freedom, and the motion can be described with the generalized coordinate θ, which is the rotation of the pendulum about the y axis. The angle θ which satisfies $\sin\theta = (x - x_0)/L$ and $\cos\theta = -z/L$. The moment of the mass about the attachment point is $mg(x - x_0)$ in the y direction. This gives the equation of motion from a moment balance as

$$mL^2\ddot{\theta} = mg(x - x_0) \tag{1}$$

The sine of the angle θ is given by $\sin\theta = -(x - x_0)/L$, which gives the usual pendulum model

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{2}$$

which can be written

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0, \quad \omega_0 = \sqrt{g/L} \tag{3}$$

2.2 Lagrange formulation of simple pendulum

The equation of motion can also be found from the Lagrange formulation. The holonomic constraint is $x = x_0 - L\sin\theta$ and $z = -L\cos\theta$. The velocity components are $\dot{x} = -L\dot{\theta}\cos\theta$

and $\dot{z} = L\dot{\theta}\sin\theta$. The kinetic energy K and the potential energy U are given by

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}mL^2\dot{\theta}^2 \tag{4}$$

$$U = mg(L+z) = mgL(1-\cos\theta) \tag{5}$$

while the Lagrangian is

$$\mathcal{L} = K - U = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(\cos\theta - 1) \tag{6}$$

The equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{7}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(mL^2\dot{\theta}\right) \tag{8}$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta} = -mgL\sin\theta \tag{9}$$

This gives

$$mL^2\ddot{\theta} + mgL\sin\theta = 0 \tag{10}$$

2.3 Pendulum with moving attachment point

If the point of attachment is moving, then the velocity components are $\dot{x} = \dot{x}_0 - L\dot{\theta}\cos\theta$ and $\dot{z} = L\dot{\theta}\sin\theta$. The kinetic energy is

$$\begin{aligned} K &= \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) && (11)\\ &= \frac{1}{2}m\left((\dot{x}_0 - L\dot{\theta}\cos\theta)^2 + (L\dot{\theta}\sin\theta)^2\right) && (12)\\ &= \frac{1}{2}mL^2\dot{\theta}^2 - mL\dot{x}_0\dot{\theta}\cos\theta + \frac{1}{2}m\dot{x}_0^2 && (13)\end{aligned}$$

while the potential energy is $U = mgL(1-\cos\theta)$. Then the Lagrangian is

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 - mL\dot{x}_0\dot{\theta}\cos\theta + \frac{1}{2}m\dot{x}_0^2 + mgL(\cos\theta - 1) \tag{14}$$

equation of motion is found from

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mL^2\dot{\theta} - mL\dot{x}_0\cos\theta\right) && (15)\\ &= mL^2\ddot{\theta} - mL\ddot{x}_0\cos\theta + mL\dot{x}_0\dot{\theta}\sin\theta && (16)\end{aligned}$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta} = mL\dot{x}_0\dot{\theta}\sin\theta - mgL\sin\theta \tag{17}$$

This gives

$$mL^2\ddot{\theta} - mL\ddot{x}_0\cos\theta + mgL\sin\theta = 0 \tag{18}$$

which can be written

$$\ddot{\theta} + \omega_0^2\sin\theta = \frac{1}{L}\ddot{x}_0\cos\theta \tag{19}$$

2.4 Damping by feedback control

A damper can be made with the feedback

$$\ddot{x}_0 = -2L\zeta\omega_0\dot{\theta} \tag{20}$$

which gives

$$\ddot{\theta} + 2\zeta\omega_0\dot{\theta}\cos\theta + \omega_0^2\sin\theta = 0 \tag{21}$$

The linearized dynamics is found by with the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, which are valid for small θ. This gives the model

$$\ddot{\theta} + \omega_0^2\theta = \frac{1}{L}\ddot{x}_0 \tag{22}$$

$$\ddot{\theta} + 2\zeta\omega_0\dot{\theta} + \omega_0^2\theta = 0 \tag{23}$$

which is a linear damped oscillator with undamped natural frequency ω_0 and relative damping ζ. Note that the feedback can be implemented with $\dot{x}_0 = -2L\zeta\omega_0\theta$.

2.5 Control of suspension point in an outer loop

In this section the control of both the pendulum motion and the motion of the suspension point is studied. Suppose that the motion of the suspension point can be modeled with the double integrator $\ddot{x}_0 = u$, where u is the control input. Then the model is

$$\ddot{\theta} = -\omega_0^2\sin\theta + \frac{\cos\theta}{L}u \tag{24}$$

$$\ddot{x}_0 = u \tag{25}$$

The linearized model is

$$\ddot{\theta} = -\omega_0^2\theta + \frac{1}{L}u \tag{26}$$

$$\ddot{x}_0 = u \tag{27}$$

The control input is written $u = u_p + u_s$ where u_p is used to control the pendulum and u_s is used to control the suspension point. This gives

$$\ddot{x}_0 = u_p + u_s \tag{28}$$

The damping controller $u_p = -2L\zeta\omega_0\dot{\theta}$ is used to control the pendulum. This gives

$$\ddot{x}_0 = -2L\zeta\omega_0\dot{\theta} + u_s \tag{29}$$

where u_s is the control variable for the control of the suspension point. The closed loop pendulum dynamics is

$$\ddot{\theta} + 2\zeta\omega_0\dot{\theta}\cos\theta + \omega_0^2\sin\theta = \frac{u_s}{L} \tag{30}$$

This is linearized around *theta* $= 0$ and $\dot{\theta} = 0$ by setting $\sin\theta = \theta$ and $\cos\theta = 1$, which gives the linearized closed loop dynamics

$$\ddot{\theta} + 2\zeta\omega_0\dot{\theta} + \omega_0^2\theta = \frac{u_s}{L} \tag{31}$$

The Laplace transformation gives

$$(s^2 + 2\zeta\omega_0 + \omega_0^2)\theta(s) = \frac{u_s(s)}{L} \tag{32}$$

which gives

$$\theta(s) = \frac{1}{L(s^2 + 2\zeta\omega_0 + \omega_0^2)} u_s(s) \tag{33}$$

Insertion into the Laplace transformation of (29) gives

$$s^2 x_0(s) = -2L\zeta\omega_0 s\theta(s) + u_s(s) = \left(-\frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 + \omega_0^2} + 1\right) u_s(s) \tag{34}$$

which leads to

$$s^2 x_0(s) = \frac{s^2 + \omega_0^2}{s^2 + 2\zeta\omega_0 + \omega_0^2} u_s(s) \tag{35}$$

which can also be written

$$x_0(s) = \frac{1}{s^2} \frac{1 + \left(\frac{s}{\omega_0}\right)^2}{1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} u_s(s) \tag{36}$$

For frequencies $\omega \ll \omega_0$, this can be approximated by

$$x_0(s) = \frac{1}{s^2} u_s(s) \tag{37}$$

The position of the suspension point can then be controlled with a PD controller

$$u_s = k_p(x_d - x_0) + k_d(\dot{x}_d - \dot{x}_0) \tag{38}$$

where x_d is the desired position of the suspension point. The gains can be selected as $k_p = \omega_s^2$ and $k_d = 2\zeta_s\omega_s$ where $\omega_s \ll \omega_0$ and ζ_s can be selected in the range $[0.7, 1]$. The condition $\omega_s \ll \omega_0$ should be sufficiently well satisfied if $\omega_0 \geq 5\omega_s$.

2.6 Implementation of outer loop with velocity input

In industrial application it may not be possible to control the acceleration $a_0 = \ddot{x}_0$, while it may be possible to command the velocity $v_0 = \dot{x}_0$. The reason is that the input to an industrial system will typically be the velocity command v_{0d}. To model this it is assumed that the industrial system has a velocity loop with the desired velocity v_{0d}, and that there is a velocity loop

$$\dot{v}_0 = \frac{1}{T_v}(v_{0d} - v_0) \tag{39}$$

where T_v is the time constant of the velocity loop, which will normally be significantly faster than the dynamics of the pendulum, which means that $1/T_v \gg \omega_0$. To be more specific, it can be assumed that $1/T_v$ is one or two decades greater than ω_0.

Then if the commanded acceleration is u, the corresponding velocity command v_{0d} will be the integral of u so that $\dot{v}_{0d} = u$. The dynamics of the suspension point with velocity loop will be

$$\dot{v}_{0d} = u \tag{40}$$

$$\dot{v}_0 = \frac{1}{T_v}(v_{0d} - v_0) \tag{41}$$

$$\dot{x}_0 = v_0 \tag{42}$$

When $1/T_v$ is sufficiently large, that $v_0 \approx v_{0d}$ and it follows that $a_0 \approx u$. The dynamics of the pendulum will be

$$\ddot{\theta} + \omega_0^2 \sin\theta = \frac{1}{L}\ddot{x}_0 \cos\theta \tag{43}$$

where the acceleration is calculated from $\ddot{x}_0 = \dot{v}_0 = \frac{1}{T_v}(v_{0d} - v_0)$.

2.7 Lyapunov stability analysis

Consider the nonnegative function

$$V = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta) \tag{44}$$

The time derivative is

$$\dot{V} = \dot{\theta}(mL^2\ddot{\theta} + mgL\sin\theta) \tag{45}$$

The time derivative along the solutions of (18) is then

$$\dot{V} = \dot{\theta}(mL\ddot{x}_0\cos\theta) \tag{46}$$

If $\ddot{x}_0$ is selected as in (20), then

$$\dot{V} = -2\zeta mL^2\omega_0\dot{\theta}^2\cos\theta \leq 0 \quad \text{when} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \tag{47}$$

which means that when $-\pi/2 < \theta < \pi/2$, then V is decreasing whenever $\dot{\theta} \neq 0$. The implies stability of $\theta = 0$, $\dot{\theta} = 0$ due to LaSalle's theorem.

2.8 Pendulum with varying rope length

If the point of attachment is moving and the rope length is varying, then the velocity components are $\dot{x} = \dot{x}_0 - L\dot{\theta}\cos\theta - \dot{L}\sin\theta$ and $\dot{z} = L\dot{\theta}\sin\theta - \dot{L}\cos\theta$. The kinetic energy is

$$\begin{aligned}
K =& \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) && (48)\\
=& \frac{1}{2}m\left((\dot{x}_0 - L\dot{\theta}\cos\theta - \dot{L}\sin\theta)^2 + (L\dot{\theta}\sin\theta - \dot{L}\cos\theta)^2\right) && (49)\\
=& \frac{1}{2}m(\dot{x}_0^2 + L^2\dot{\theta}^2\cos^2\theta + \dot{L}^2\sin^2\theta) - m\dot{x}_0 L\dot{\theta}\cos\theta - m\dot{x}_0\dot{L}\sin\theta + mL\dot{L}\dot{\theta}\cos\theta\sin\theta && (50)\\
&+ \frac{1}{2}m(L^2\dot{\theta}^2\sin^2\theta + \dot{L}^2\cos^2\theta) - mL\dot{L}\dot{\theta}\cos\theta\sin\theta && (51)\\
=& \frac{1}{2}mL^2\dot{\theta}^2 - mL\dot{x}_0\dot{\theta}\cos\theta - m\dot{x}_0\dot{L}\sin\theta + \frac{1}{2}m\dot{x}_0^2 + \frac{1}{2}m\dot{L}^2 && (52)
\end{aligned}$$

while the potential energy is $U = mgL(1 - \cos\theta)$. The resulting Lagrangian is

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 - mL\dot{x}_0\dot{\theta}\cos\theta - m\dot{x}_0\dot{L}\sin\theta + \frac{1}{2}m\dot{x}_0^2 + \frac{1}{2}m\dot{L}^2 + mgL(\cos\theta - 1) \quad (53)$$

Then the Lagrangian equation of motion is found from

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mL^2\dot{\theta} - mL\dot{x}_0\cos\theta\right) && (54)\\
&= mL^2\ddot{\theta} + 2mL\dot{L}\dot{\theta} - mL\ddot{x}_0\cos\theta - m\dot{L}\dot{x}_0\cos\theta + mL\dot{x}_0\dot{\theta}\sin\theta && (55)
\end{aligned}$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta} = mL\dot{x}_0\dot{\theta}\sin\theta - m\dot{x}_0\dot{L}\cos\theta - mgL\sin\theta \quad (56)$$

which gives

$$mL^2\ddot{\theta} - mL\ddot{x}_0\cos\theta + 2mL\dot{L}\dot{\theta} + mgL\sin\theta = 0 \quad (57)$$

$$\ddot{\theta} - \frac{1}{L}\ddot{x}_0\cos\theta + \frac{g}{L}\sin\theta = -\frac{2}{L}\dot{L}\dot{\theta} \quad (58)$$

3 Spherical pendulum with Euler angles

3.1 Simple spherical pendulum model with Euler angles

Let the rotation from frame n to frame b be given in terms of the Euler angles ϕ_x about the x axis of the n frame followed by a rotation ϕ_y about the resulting y axis. The rotation matrix is then $\boldsymbol{R}_b^n = \boldsymbol{R}_x(\phi_x)\boldsymbol{R}(\phi_y)$. This gives

$$\boldsymbol{R}_b^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_x & -s_x \\ 0 & s_x & c_x \end{bmatrix}\begin{bmatrix} c_y & 0 & s_y \\ 0 & 1 & 0 \\ -s_y & 0 & c_y \end{bmatrix} = \begin{bmatrix} c_y & 0 & s_y \\ s_x s_y & c_x & -s_x c_y \\ -c_x s_y & s_x & c_x c_y \end{bmatrix} \quad (59)$$

Then the position of the mass relative to the attachment point is

$$\vec{r}_r = -L\vec{b}_3 = -\sin\phi_y L\vec{n}_1 + \sin\phi_x\cos\phi_y L\vec{n}_2 - \cos\phi_x\cos\phi_y L\vec{n}_3 \tag{60}$$

The coordinate vector form in frame n is $\boldsymbol{r}_r^n = L\boldsymbol{r}_3$ where $\boldsymbol{r}_3 = [s_y, -s_x c_y, c_x c_y]^{\mathrm{T}}$ is the last column of the rotation matrix $\boldsymbol{R}_b^n$. The velocity of the mass relative to the attachment point is

$$\begin{aligned}
\vec{v}_r = \frac{{}^n\mathrm{d}}{\mathrm{d}t}\vec{r}_r = & -\dot{\phi}_y\cos\phi_y L\vec{n}_1 && (61)\\
& + (\dot{\phi}_x\cos\phi_x\cos\phi_y - \dot{\phi}_y\sin\phi_x\sin\phi_y)L\vec{n}_2 && (62)\\
& + (\dot{\phi}_x\sin\phi_x\cos\phi_y + \dot{\phi}_y\cos\phi_x\sin\phi_y)L\vec{n}_3 && (63)
\end{aligned}$$

The kinetic energy K is given by $K = (1/2)m\vec{v}_r\cdot\vec{v}_r$, which gives

$$\begin{aligned}
\frac{2K}{mL^2} &= \dot{\phi}_y^2\cos^2\phi_y + \dot{\phi}_x^2(\sin^2\phi_x + \cos^2\phi_x)\cos^2\phi_y + \dot{\phi}_y^2(\sin^2\phi_x + \cos^2\phi_x)\sin^2\phi_y && (64)\\
&\quad - 2\dot{\phi}_x\dot{\phi}_y(\cos\phi_x\sin\phi_x\cos\phi_y\sin\phi_y) + 2\dot{\phi}_x\dot{\phi}_y(\cos\phi_x\sin\phi_x\cos\phi_y\sin\phi_y) && (65)\\
&= \dot{\phi}_x^2\cos^2\phi_y + \dot{\phi}_y^2 && (66)
\end{aligned}$$

The potential energy is $U = -mgL(\cos\phi_x\cos\phi_y - 1)$. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}mL^2\left(\dot{\phi}_x^2\cos^2\phi_y + \dot{\phi}_y^2\right) + mgL(\cos\phi_x\cos\phi_y - 1) \tag{67}$$

This gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_x}\right) &= \frac{1}{2}mL^2\frac{\mathrm{d}}{\mathrm{d}t}\left(2\dot{\phi}_x\cos^2\phi_y\right) = mL^2\ddot{\phi}_x\cos^2\phi_y - 2mL^2\dot{\phi}_x\dot{\phi}_y\sin\phi_y\cos\phi_y && (68)\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_y}\right) &= \frac{1}{2}mL^2\frac{\mathrm{d}}{\mathrm{d}t}\left(2\dot{\phi}_y\right) = mL^2\ddot{\phi}_y && (69)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi_x} &= \left(\frac{1}{2}mL^2\left(\dot{\phi}_x^2\cos^2\phi_y + \dot{\phi}_y^2\right) + mgL(\cos\phi_x\cos\phi_y - 1)\right) = -mgl\sin\phi_x\cos\phi_y && (70)\\
\frac{\partial\mathcal{L}}{\partial\phi_y} &= \left(\frac{1}{2}mL^2\left(\dot{\phi}_x^2\cos^2\phi_y + \dot{\phi}_y^2\right) + mgL(\cos\phi_x\cos\phi_y - 1)\right) = -mgl\cos\phi_x\sin\phi_y && (71)\\
&\qquad - mL^2\dot{\phi}_x^2\sin\phi_y\cos\phi_y && (72)
\end{aligned}$$

The resulting equations of motion are then

$$\begin{aligned}
mL^2\ddot{\phi}_x\cos^2\phi_y - 2mL^2\dot{\phi}_x\dot{\phi}_y\sin\phi_y\cos\phi_y &= -mgL\sin\phi_x\cos\phi_y && (73)\\
mL^2\ddot{\phi}_y + mL^2\dot{\phi}_x^2\sin\phi_y\cos\phi_y &= -mgL\cos\phi_x\sin\phi_y && (74)
\end{aligned}$$

which can also be written in the form

$$\begin{aligned}
\ddot{\phi}_x\cos\phi_y + \frac{g}{L}\sin\phi_x &= 2\dot{\phi}_x\dot{\phi}_y\sin\phi_y && (75)\\
\ddot{\phi}_y + \frac{g}{L}\cos\phi_x\sin\phi_y &= -\dot{\phi}_x^2\sin\phi_y\cos\phi_y && (76)
\end{aligned}$$

3.2 Spherical pendulum model with moving point of attachment

In the case of a moving point of attachment the position of the mass is

$$\vec{r}_r = (x_0 - \sin\phi_y L)\,\vec{n}_1 + (y_0 + \sin\phi_x \cos\phi_y L)\,\vec{n}_2 + (z_0 - \cos\phi_x \cos\phi_y L)\,\vec{n}_3 \tag{77}$$

The velocity of the mass is

$$\vec{v}_r = \frac{{}^n\mathrm{d}}{\mathrm{d}t}\vec{r}_r = \left(\dot{x}_0 - \dot{\phi}_y \cos\phi_y L\right)\vec{n}_1 \tag{78}$$

$$+ \left(\dot{y}_0 + \dot{\phi}_x \cos\phi_x \cos\phi_y L - \dot{\phi}_y \sin\phi_x \sin\phi_y L\right)\vec{n}_2 \tag{79}$$

$$+ \left(\dot{z}_0 + \dot{\phi}_x \sin\phi_x \cos\phi_y L + \dot{\phi}_y \cos\phi_x \sin\phi_y L\right)\vec{n}_3 \tag{80}$$

The kinetic energy is then

$$K = \frac{1}{2}mL^2\left(\dot{\phi}_x^2 \cos^2\phi_y + \dot{\phi}_y^2\right) + \frac{1}{2}m\left(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2\right) \tag{81}$$

$$- mL\dot{x}_0\dot{\phi}_y \cos\phi_y \tag{82}$$

$$+ mL\dot{y}_0\left(\dot{\phi}_x \cos\phi_x \cos\phi_y - \dot{\phi}_y \sin\phi_x \sin\phi_y\right) \tag{83}$$

$$+ mL\dot{z}_0\left(\dot{\phi}_x \sin\phi_x \cos\phi_y + \dot{\phi}_y \cos\phi_x \sin\phi_y\right) \tag{84}$$

The potential energy is $U = -mgL(\cos\phi_x \cos\phi_y - 1)$. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}mL^2\left(\dot{\phi}_x^2 \cos^2\phi_y + \dot{\phi}_y^2\right) + \frac{1}{2}m\left(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2\right) \tag{85}$$

$$- mL\dot{x}_0\dot{\phi}_y \cos\phi_y \tag{86}$$

$$+ mL\dot{y}_0\left(\dot{\phi}_x \cos\phi_x \cos\phi_y - \dot{\phi}_y \sin\phi_x \sin\phi_y\right) \tag{87}$$

$$+ mL\dot{z}_0\left(\dot{\phi}_x \sin\phi_x \cos\phi_y + \dot{\phi}_y \cos\phi_x \sin\phi_y\right) \tag{88}$$

$$+ mgL(\cos\phi_x \cos\phi_y - 1) \tag{89}$$

This gives the following equations for the ϕ_x equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_x}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(mL^2\dot{\phi}_x \cos^2\phi_y + mL\dot{y}_0 \cos\phi_x \cos\phi_y + mL\dot{z}_0 \sin\phi_x \cos\phi_y\right) \tag{90}$$

$$= mL^2\ddot{\phi}_x \cos^2\phi_y + mL\ddot{y}_0 \cos\phi_x \cos\phi_y + mL\ddot{z}_0 \sin\phi_x \cos\phi_y \tag{91}$$

$$- 2mL^2\dot{\phi}_x\dot{\phi}_y \sin\phi_y \cos\phi_y \tag{92}$$

$$- mL\dot{y}_0\dot{\phi}_x \sin\phi_x \cos\phi_y - mL\dot{y}_0\dot{\phi}_y \cos\phi_x \sin\phi_y \tag{93}$$

$$+ mL\dot{z}_0\dot{\phi}_x \cos\phi_x \cos\phi_y - mL\dot{z}_0\dot{\phi}_y \sin\phi_x \sin\phi_y \tag{94}$$

and

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi_x} &= -mgl \sin\phi_x \cos\phi_y && (95)\\
&- mL\dot{y}_0 \left(\dot{\phi}_x \sin\phi_x \cos\phi_y + \dot{\phi}_y \cos\phi_x \sin\phi_y\right) && (96)\\
&+ mL\dot{z}_0 \left(\dot{\phi}_x \cos\phi_x \cos\phi_y - \dot{\phi}_y \sin\phi_x \sin\phi_y\right) && (97)\\
& && (98)
\end{aligned}
$$

The equation for the ϕ_y is found from

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_y}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mL^2\dot{\phi}_y - mL\dot{x}_0\cos\phi_y - mL\dot{y}_0 \sin\phi_x \sin\phi_y + mL\dot{z}_0 \cos\phi_x \sin\phi_y\right) && (99)\\
&= mL^2\ddot{\phi}_y - mL\ddot{x}_0 \cos\phi_y - mL\ddot{y}_0 \sin\phi_x \sin\phi_y + mL\ddot{z}_0 \cos\phi_x \sin\phi_y && (100)\\
&+ mL\dot{x}_0\dot{\phi}_y \sin\phi_y && (101)\\
&- mL\dot{y}_0\dot{\phi}_x \cos\phi_x \sin\phi_y - mL\dot{y}_0\dot{\phi}_y \sin\phi_x \cos\phi_y && (102)\\
&+ mL\dot{z}_0\dot{\phi}_x \sin\phi_x \sin\phi_y - mL\dot{z}_0\dot{\phi}_y \cos\phi_x \cos\phi_y && (103)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi_y} &= -mgl \cos\phi_x \sin\phi_y && (104)\\
&- mL^2\dot{\phi}_x^2 \sin\phi_y \cos\phi_y && (105)\\
&+ mL\dot{x}_0\dot{\phi}_y \sin\phi_y && (106)\\
&- mL\dot{y}_0 \left(\dot{\phi}_x \cos\phi_x \sin\phi_y + \dot{\phi}_y \sin\phi_x \cos\phi_y\right) && (107)\\
&+ mL\dot{z}_0 \left(\dot{\phi}_x \sin\phi_x \sin\phi_y - \dot{\phi}_y \cos\phi_x \cos\phi_y\right) && (108)
\end{aligned}
$$

The resulting equations of motion are

$$
\begin{aligned}
mL^2\ddot{\phi}_x \cos^2\phi_y &= -mL\ddot{y}_0 \cos\phi_x \cos\phi_y - mL\ddot{z}_0 \sin\phi_x \cos\phi_y && (109)\\
&+ 2mL^2\dot{\phi}_x\dot{\phi}_y \sin\phi_y \cos\phi_y - mgL \sin\phi_x \cos\phi_y && (110)
\end{aligned}
$$

$$
\begin{aligned}
mL^2\ddot{\phi}_y &= mL\ddot{x}_0 \cos\phi_y + mL\ddot{y}_0 \sin\phi_x \sin\phi_y - mL\ddot{z}_0 \cos\phi_x \sin\phi_y && (111)\\
&- mL^2\dot{\phi}_x^2 \sin\phi_y \cos\phi_y - mgL \cos\phi_x \sin\phi_y && (112)
\end{aligned}
$$

which can also be written in the form

$$
\begin{aligned}
\ddot{\phi}_x \cos\phi_y + \frac{g}{L}\sin\phi_x &= 2\dot{\phi}_x\dot{\phi}_y \sin\phi_y - \frac{1}{L}\ddot{y}_0 \cos\phi_x - \frac{1}{L}\ddot{z}_0 \sin\phi_x && (113)\\
\ddot{\phi}_y + \frac{g}{L}\cos\phi_x \sin\phi_y &= -\dot{\phi}_x^2 \sin\phi_y \cos\phi_y && (114)\\
&+ \frac{1}{L}\ddot{x}_0 \cos\phi_y + \frac{1}{L}\ddot{y}_0 \sin\phi_x \sin\phi_y - \frac{1}{L}\ddot{z}_0 \cos\phi_x \sin\phi_y && (115)
\end{aligned}
$$

3.3 Damping by feedback control 1

Suppose that the attachment point is moving in the horizontal plane so that $\ddot{z}_0 = 0$. Let

$$\ddot{y}_0 = 2\zeta\omega_0 L\dot{\phi}_x \tag{116}$$

$$\ddot{x}_0 = -2\zeta\omega_0 L\dot{\phi}_y - \frac{1}{\cos\phi_y}\ddot{y}_0 \sin\phi_x \sin\phi_y \tag{117}$$

Then

$$\ddot{\phi}_x \cos\phi_y + 2\zeta\omega_0\dot{\phi}_x \cos\phi_x + \omega_0^2 \sin\phi_x = 2\dot{\phi}_x\dot{\phi}_y \sin\phi_y \tag{118}$$

$$\ddot{\phi}_y + 2\zeta\omega_0\dot{\phi}_y \cos\phi_y + \frac{g}{L}\cos\phi_x \sin\phi_y = -\dot{\phi}_x^2 \sin\phi_y \cos\phi_y \tag{119}$$

Linearization about $\phi_x = 0$ and $\phi_y = 0$ gives the two damped harmonic oscillators

$$\ddot{\phi}_x + 2\zeta\omega_0\dot{\phi}_x + \omega_0^2\phi_x = 0 \tag{120}$$

$$\ddot{\phi}_y + 2\zeta\omega_0\dot{\phi}_y + \omega_0^2\phi_y = 0 \tag{121}$$

3.4 Energy function 1

Consider the spherical pendulum when the motion of the attachment point is treated as an exogenous variable. Then the energy of the spherical pendulum is $V = K + U$ which gives

$$V = \frac{1}{2}mL^2\left(\dot{\phi}_x^2\cos^2\phi_y + \dot{\phi}_y^2\right) + mgL(1 - \cos\phi_x\cos\phi_y) \tag{122}$$

The time derivative of the energy function V is

$$\dot{V} = \dot{\phi}_x(mL^2\ddot{\phi}_x\cos^2\phi_y) + \dot{\phi}_y(mL^2\ddot{\phi}_y) \tag{123}$$

$$- mL^2\dot{\phi}_x^2\dot{\phi}_y\cos\phi_y\sin\phi_y \tag{124}$$

$$+ mgL\left(\dot{\phi}_y\cos\phi_x\sin\phi_y + \dot{\phi}_x\sin\phi_x\cos\phi_y\right) \tag{125}$$

and it follows that the time derivative along the the solutions of the system (113,115) is

$$\dot{V} = \dot{\phi}_x(-mL\ddot{y}_0\cos\phi_x\cos\phi_y - mL\ddot{z}_0\sin\phi_x\cos\phi_y \tag{126}$$

$$+ 2mL^2\dot{\phi}_x\dot{\phi}_y\sin\phi_y\cos\phi_y - mgL\sin\phi_x\cos\phi_y) \tag{127}$$

$$+ \dot{\phi}_y(mL\ddot{x}_0\cos\phi_y + mL\ddot{y}_0\sin\phi_x\sin\phi_y - mL\ddot{z}_0\cos\phi_x\sin\phi_y \tag{128}$$

$$- mL^2\dot{\phi}_x^2\sin\phi_y\cos\phi_y - mgL\cos\phi_x\sin\phi_y) \tag{129}$$

$$- mL^2\dot{\phi}_x^2\dot{\phi}_y\cos\phi_y\sin\phi_y \tag{130}$$

$$+ mgL\dot{\phi}_y\cos\phi_x\sin\phi_y + mgL\dot{\phi}_x\sin\phi_x\cos\phi_y \tag{131}$$

This gives the result

$$\dot{V} = -mL\dot{\phi}_x\cos\phi_y\left(\ddot{y}_0\cos\phi_x + \ddot{z}_0\sin\phi_x\right) \tag{132}$$

$$+ mL\dot{\phi}_y\left(\ddot{x}_0\cos\phi_y + \ddot{y}_0\sin\phi_x\sin\phi_y - \ddot{z}_0\cos\phi_x\sin\phi_y\right) \tag{133}$$

With $\ddot{z}_0 = 0$ and with the damping controllers this gives

$$\dot{V} = -2\zeta\omega_0 mL^2 \left(\dot{\phi}_x^2 \cos\phi_x \cos\phi_y + \dot{\phi}_y^2 \cos\phi_y\right) \tag{134}$$

which means that when $-\pi/2 < \phi_x < \pi/2$ and $-\pi/2 < \phi_y < \pi/2$, then V is decreasing whenever $\dot{\phi}_x$ or $\dot{\phi}_y$ are nonzero.

3.5 Damping by feedback control 2

Suppose that the attachment point is moving in the horizontal plane so that $\ddot{z}_0 = 0$. Let

$$\ddot{y}_0 = L(k_d\dot{\phi}_x + k_p\phi_x) \tag{135}$$

$$\ddot{x}_0 = -L(k_d\dot{\phi}_y + k_p\phi_y) - \frac{\ddot{y}_0 s_x s_y}{c_y} \tag{136}$$

Then the closed lop dynamics are given by

$$\ddot{\phi}_x c_y + k_d\dot{\phi}_x c_x + k_p\phi_x + \omega_0^2 s_x = 2\dot{\phi}_x\dot{\phi}_y s_y \tag{137}$$

$$\ddot{\phi}_y + k_d\dot{\phi}_y c_y + k_p\phi_y + \omega_0^2 c_x s_y = -L\dot{\phi}_x^2 s_y c_y \tag{138}$$

Linearization about $\phi_x = 0$ and $\phi_y = 0$ gives the two stable second order systems

$$\ddot{\phi}_x + k_d\dot{\phi}_x + (k_p + \omega_0^2)\phi_x = 0 \tag{139}$$

$$\ddot{\phi}_y + k_d\dot{\phi}_y + (k_p + \omega_0^2)\phi_y = 0 \tag{140}$$

4 Lyapunov analysis with cross terms

Consider a mass m with position $x_2 = x$ and velocity $x_1 = \dot{x}$. The equation of motion is

$$\dot{x}_1 = x_2 \tag{141}$$

$$m\dot{x}_2 = u \tag{142}$$

where the control variable u is the input force. The state vector is $\boldsymbol{x} = [x_1, x_2]^{\mathrm{T}}$.

Suppose that the position of the mass is controlled with the PD controller $u = -k_p x_1 - k_d x_2$. The closed-loop dynamics are given by

$$\dot{x}_1 = x_2 \tag{143}$$

$$m\dot{x}_2 = -k_p x_1 - k_d x_2 \tag{144}$$

The stability of the system can be established with the Lyapunov function candidate

$$V_1 = \frac{1}{2}k_p x_1^2 + \frac{1}{2}m x_2^2 = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x} \tag{145}$$

where

$$\boldsymbol{P}_1 = \begin{bmatrix} k_p & 0 \\ 0 & m \end{bmatrix} \tag{146}$$

Note that this function is the kinetic energy of the mass plus the potential energy of a spring with stiffness k_p which connects the mass point to the point at $x = 0$. The time derivative of V_1 along the trajectories of the system (143, 144) is

$$\dot{V}_1 = k_p x_1 \dot{x}_1 + x_2 m \dot{x}_2 = k_p x_1 x_2 + x_2(-k_p x_1 - k_d x_2) = -k_d x_2^2 \leq 0 \tag{147}$$

It is seen that $\dot{V}_1 = -\boldsymbol{x}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{x}$ where $\boldsymbol{Q}_1 = \mathrm{diag}(0, k_d)$. The matrix $\boldsymbol{Q}$ is positive semidefinite, and $\dot{V}_1$ is therefore negative semidefinite. This means that Lyapunov stability cannot be shown for this Lyapunov function candidate, because this would only be the case if $\boldsymbol{Q}$ is negative definite. Instead LaSalle's principle can be used to show that the origin is asymptotically stable.

It would be advantageous for the analysis if a Lyapunov function candidate V could be found so that the toime derivative $\dot{V}$ along the trajectories of (143, 144) was negative definite, as this could be used to establish exponential stability. This can be done by introducing cross terms in the Lyapunov function candidate, using the Lyapunov function candidate proposed in [1], where

$$V = V_1 + c k_d x_1^2 + c m x_1 x_2 \tag{148}$$

was used where the positive constant c of the cross term satisfies

$$k_d - cm = k_c > 0 \tag{149}$$

This Lyapunov function candidate can be written

$$V = \frac{1}{2}(k_p + c k_d) x_1^2 + c m x_1 x_2 + \frac{1}{2} m x_2^2 = \frac{1}{2} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x} \tag{150}$$

where

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} k_p + c k_d & cm \\ cm & m \end{bmatrix} \tag{151}$$

The matrix $\boldsymbol{P}$ is positive definite, which is verified by showing that the leading subdiagonals of the matrix have a positive determinant, which implies that $\boldsymbol{P}$ is positive definite. The first leading subdiagonal is $\boldsymbol{p}_{11} > 0$. The second leading subdiagonal has determinant

$$\begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} = (k_p + c k_d) m - c^2 m^2 = k_p m + c m k_c > 0 \tag{152}$$

where (149) is used. It follows that $\boldsymbol{P}$ is positive definite. The time derivative of V along the solutions of (143, 144) is then

$$\begin{aligned} \dot{V} &= (k_p + c k_d) x_1 \dot{x}_1 + x_2 m \dot{x}_2 + c m x_1 \dot{x}_2 + c m \dot{x}_1 x_2 & (153) \\ &= (k_p + c k_d) x_1 x_2 + x_2(-k_p x_1 - k_d x_2) + c x_1(-k_p x_1 - k_d x_2) + c m x_2^2 & (154) \\ &= -c k_p x_1^2 - k_c x_2^2 & (155) \\ &= - - \boldsymbol{x}^{\mathrm{T}} \boldsymbol{Q} \boldsymbol{x} & (156) \end{aligned}$$

where $\boldsymbol{Q} = \mathrm{diag}(c k_p, k_c)$ is positive definite. This means that $\dot{V}$ is negative semidefinite, and it follows that the system is exponentially stable.

References

[1] J. T. Wen and D. Bayard. A new class of control laws for robotic manipulators - Part I: Non-adaptive case. *International Journal of Control*, 47(5):1361–1385, 1988.