Homework 3

Ting He
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Problem 1.

Problem Statement: given an $O(nlgk)$-time algorithm to merge k sorted lists into one sorted list, where n is. the total number of elements in all the input lists. (Hint: use a heap for k-way merging)
Assumptions:
none
Computations:
Think about the scenario that we compare each the smallest element from every sorted list from k lists, so every time we will have k elements to compare. Then we construct the min-heap data structure to find the smallest among these k elements. To construct the min-heap data structure, we need $O(lgk)$ time complexity. And it will return the min value of this structure, by returning the root, which cost $O(1)$. Once we return this root and append this element a new list. We will then add one more element from the previous array (which the element in new list comes from) to generate a new min-heap data structure, which represents that we need to construct the min-heap n times. In total, it takes $O(n * (lgk + 1))$ time complexity. Then we prove that it can achieve in $O(nlgk)$)
Conclusions: By constructing the min-heap structure, we can proof that it can achieve the problem by $O(nlgk)$

Problem 2.

Problem Statement: twenty-one card trick.
(a) prove that the top card in the chosen pile advances two positions in the next pile
(b) prove that the second and third positions advance by one.
(c) prove that onc ethe chosen card appears at position 11 in the stack, it stays at position 11.
(d) prove that the chosen card appearing in positions 5,6,or,7 within its pile will make it to the middle position after three deals.
(e) why this? "it seems that about half the time, you only need to go through two rounds of deals-not three-and the trick still works"
Assumptions:

none, but define the chosen card as my card in the following solution
Computations:
(a) the top card in the chosen pile will be in the position of $\lceil \frac{7+1}{3} \rceil$ in the next pile since we added the pile of my card into middle of other piles. The result of this computation is 3, which prove that it will move to next 2 positions than before.
(b) similarly, recorded the card we want to investigate is at the x position from our chosen pile. After first time of phasing, we will have $\lceil \frac{7+x}{3} \rceil$ and which give us 3 given $x = 2$ and 4 given $x = 3$
(c) After three times of phasing, we have calculation of $\lceil 7 + \lceil \frac{7+\lceil \frac{7+x}{3} \rceil}{3} \rceil = 10 + \lceil \frac{x+1}{9} \rceil$. Since $1 < x < 7$, we always get 11 from the calculation.
(d) given $x = 5, 6, 7$, we then have result of 11, which shows it will be the position 4 in the chosen pile.
Conclusions: the results shown in computation process

Problem 3.

Problem Statement: Peer-to-peer networks tend to grow through the arrival of new participants who join by linking into the existing structure.
(a) given the random process described above, what is the expected number of incoming links to node v_j in the resulting network?
(b) part a make precise a sense in which the nodes that arrive early carry an 'unfair' share of connections in the network. Another way to quantify the imbalance is to observed that, in a run of this random process, we expect many nodes to end up with no incoming links. Given a formula for the expected number of nodes with no incoming links in a network grown randomly according this model.
Assumptions:
Considering that will have n nodes in total, and we aim to evaluate whether node j will have incoming links.
Computations:
(a)

$$
\begin{aligned}
E(\sum_{i=1}^{n-j} prob(incoming - links - to - node - v_j)) &= \sum_{i=1}^{n-j} prob(incoming - links - to - node - v_j) \\
&= \frac{1}{j} + \frac{1}{j+1} + \cdots + \frac{1}{n-1}
\end{aligned}
$$

Given harmonic number of $\sum_{k=1}^{n} (\frac{1}{k}) = lnn + O(1)$, we can get the original equation is the same as $ln(n-1) - ln(j) = ln\frac{n-1}{j}$

(b)

$$
\begin{aligned}
A_i &= 1, if - no - incoming - link - for - i \\
&= 0, otherwise \\
\sum E(A_i) &= \frac{i-1}{i}\frac{i}{i+1}\frac{i+1}{i+2}\cdots\frac{n-2}{n-1} \\
&= \sum_{i=1}^{n}\frac{i-1}{n-1} \\
&= \frac{(n-1)n}{2(n-1)} \\
&= \frac{n}{2}
\end{aligned}
$$

Conclusions: (a) $ln\frac{n-1}{j}$ (b) $\frac{n}{2}$