Notes of Lectures on Quantum Mechanics by P.M.Dirac

Ting-Kai Hsu

January 26, 2024

Contents

1 Transformation That Doesn't Change State

If we start with given initial variables and solve for the equations of motion, whose solutions should contain arbitrary functions of time $v_a(t)$ as we discuss above. The initial variables we need would q and p but not the initial value of arbitrary functions $v_a(t)$, and this is quite important because *initial physical state* of the system is uniquely determined by q and p but not v, which is consistent with the system without constraints.

As time evolves, the initial state must determine the state at later time. However, q and p would not be uniquely determined by the initial state due to the arbitrary functions in solutions of equations of motion. That means *the state doesn't uniquely determine q and p but a given set of q and p would uniquely determine a state.* There must be several choices of q and p which correspond to the same physical state. To explicitly show this, consider a general dynamical variable g with initial value g_0 and a short time interval δt. The value of this dynamical variable at time δt would be given by the equations of motion,

$$g(\delta t) = g_0 + \dot{g}\delta t$$

$$= g_0 + [g, H_T]\,\delta t$$

$$\approx g_0 \delta t \left\{ [g, H'] + v_a [g, \phi_a] \right\} \tag{1.1}$$

The coefficients v are completely arbitrary thus we can consider different functions of time v', which therefore gives a different $g(\delta t)$. Define the difference between different coefficients,

$$\Delta g(\delta t) = \epsilon_a \left[g, \phi_a \right] \tag{1.2}$$

where

$$\epsilon_a = \delta t (v_a - v'_a) \tag{1.3}$$

which is a small arbitrary parameter, small because of short time interval δt and arbitrary because of v and v'. Consider changing all the Hamiltonian variables (like coordinates and canonical momentum) with the rule eq(1.2) yet would describe exactly the same state. We come to the conclusion that the ϕ_a which appear in the theory as primary, first-class constraints, have the meaning: *as generating functions of infinitesimal contact transformations, they lead to changes in coordinates and canonical momenta that don't affect the physical state*.

To push the theory further by considering successive contact infinitesimal transformations. First by $\epsilon_a \phi_a$ and then $\gamma_{a'} \phi_{a'}$, thus we have,

$$\begin{aligned} \Delta g &= \epsilon_a \left[g, \phi_a \right] \\ \Delta' g &= \gamma_{a'} \left[g + \Delta g, \phi_{a'} \right] \end{aligned} \tag{1.4}$$

Write them altogether,

$$g''_1(\delta t) = g(\delta t) + \epsilon_a \left[g, \phi_a \right] + \gamma_{a'} \left[g + \epsilon_a \left[g, \phi_a \right], \phi_{a'} \right] \tag{1.5}$$

However, if the transformations are applied in the reverse order,

$$g''_2(\delta t) = g(\delta t) + \gamma_{a'} \left[g, \phi_{a'} \right] + \epsilon_a \left[g + \gamma_{a'} \left[g, \phi_{a'} \right], \phi_a \right] \tag{1.6}$$

These two would correspond to different change in q and p, yet both of them wouldn't change the physical state. The difference between these two quantity would yield,

$$\Delta_{12} g \equiv g''_1 - g''_2 \approx \epsilon_a \gamma_{a'} \left\{ \left[\left[g, \phi_a \right], \phi_{a'} \right] - \left[\left[g, \phi_{a'} \right], \phi_a \right] \right\}$$

Further simplify this by Jacobi identity,

$$\left[\left[g, \phi_a \right], \phi_{a'} \right] + \left[\left[\phi_a, \phi_{a'} \right], g \right] + \left[\left[\phi_{a'}, g \right], \phi_a \right] = 0$$

Then the equation becomes,

$$\Delta_{12} g = \epsilon_a \gamma_{a'} \left[g, [\phi_a, \phi_{a'}] \right] \tag{1.7}$$

Therefore, we immediately see that $\Delta_{12} g$ represents a change in Hamiltonian variables yet does not involve any change in the physical state. We use,

$$[\phi_a, \phi_{a'}] \tag{1.8}$$

as a generating function of infinitesimal contact transformation that causes no harm to the physical state.

By the theorem previously proved, the Poisson brackets of first-class constraints would be first-class, too.

The final result is that those transformations of dynamical variables which do not change physical states are infinitesimal contact transformations in which the generating functions is a primary first-class constraint or possibly a secondary first-class constraint.