Notes of Lectures on Quantum Mechanics by P.M.Dirac

Ting-Kai Hsu

February 18, 2024

Contents

1 Transformation That Doesn't Change State

If we start with given initial variables and solve for the equations of motion, whose solutions should contain arbitrary functions of time $v_a(t)$ as we discuss above. The initial variables we need would q and p but not the initial value of arbitrary functions $v_a(t)$, and this is quite important because *initial physical state* of the system is uniquely determined by q and p but not v, which is consistent with the system without constraints.

As time evolves, the initial state must determine the state at later time. However, q and p would not be uniquely determined by the initial state due to the arbitrary functions in solutions of equations of motion. That means *the state doesn't uniquely determine q and p but a given set of q and p would uniquely determine a state.* There must be several choices of q and p which correspond to the same physical state. To explicitly show this, consider a general dynamical variable g with initial value g_0 and a short time interval δt. The value of this dynamical variable at time δt would be given by the equations of motion,

$$g(\delta t) = g_0 + \dot{g}\delta t$$

$$
\begin{aligned}
&= g_0 + [g, H_T]\,\delta t \\
&\approx g_0 \delta t \left\{[g, H'] + v_a [g, \phi_a]\right\}
\end{aligned} \tag{1.1}
$$

The coefficients v are completely arbitrary thus we can consider different functions of time v', which therefore gives a different $g(\delta t)$. Define the difference between different coefficients,

$$\Delta g(\delta t) = \epsilon_a \left[g, \phi_a\right] \tag{1.2}$$

where

$$\epsilon_a = \delta t (v_a - v'_a) \tag{1.3}$$

which is a small arbitrary parameter, small because of short time interval δt and arbitrary because of v and v'. Consider changing all the Hamiltonian variables (like coordinates and canonical momentum) with the rule eq(1.2) yet would describe exactly the same state. We come to the conclusion that the ϕ_a which appear in the theory as primary, first-class constraints, have the meaning: *as generating functions of infinitesimal contact transformations, they lead to changes in coordinates and canonical momenta that don't affect the physical state*.

To push the theory further by considering successive contact infinitesimal transformations. First by $\epsilon_a \phi_a$ and then $\gamma_{a'} \phi_{a'}$, thus we have,

$$
\begin{aligned}
\Delta g &= \epsilon_a \left[g, \phi_a\right] \\
\Delta' g &= \gamma_{a'} \left[g + \Delta g, \phi_{a'}\right]
\end{aligned} \tag{1.4}
$$

Write them altogether,

$$g''_1(\delta t) = g(\delta t) + \epsilon_a \left[g, \phi_a\right] + \gamma_{a'} \left[g + \epsilon_a \left[g, \phi_a\right], \phi_{a'}\right] \tag{1.5}$$

However, if the transformations are applied in the reverse order,

$$g''_2(\delta t) = g(\delta t) + \gamma_{a'} \left[g, \phi_{a'}\right] + \epsilon_a \left[g + \gamma_{a'} \left[g, \phi_{a'}\right], \phi_a\right] \tag{1.6}$$

These two would correspond to different change in q and p, yet both of them wouldn't change the physical state. The difference between these two quantity would yield,

$$\Delta_{12} g \equiv g''_1 - g''_2 \approx \epsilon_a \gamma_{a'} \left\{\left[\left[g, \phi_a\right], \phi_{a'}\right] - \left[\left[g, \phi_{a'}\right], \phi_a\right]\right\}$$

Further simplify this by Jacobi identity,

$$\left[\left[g, \phi_a\right], \phi_{a'}\right] + \left[\left[\phi_a, \phi_{a'}\right], g\right] + \left[\left[\phi_{a'}, g\right], \phi_a\right] = 0$$

Then the equation becomes,

$$\Delta_{12} g = \epsilon_a \gamma_{a'} \left[g, [\phi_a, \phi_{a'}]\right] \tag{1.7}$$

Therefore, we immediately see that $\Delta_{12} g$ represents a change in Hamiltonian variables yet does not involve any change in the physical state. We use,

$$[\phi_a, \phi_{a'}] \tag{1.8}$$

as a generating function of infinitesimal contact transformation that causes no harm to the physical state.

By the theorem previously proved, the Poisson brackets of first-class constraints would be first-class, too.

The final result is that those transformations of dynamical variables which do not change physical states are infinitesimal contact transformations in which the generating functions is a primary first-class constraint or possibly a secondary first-class constraint.

2 The Problem of Quantization

We were led to the idea that there are certain changes in the p's and q's that don't correspond to a change of state, and which have as generators first-class secondary constraints. This implies a more general equation of motion that allows variation of a dynamical variable g with the time contains *not only* any variation given by equation (1.1) $\dot{g} = [g, H_T]$, *but also* any variation which does not correspond to a change of current state.

$$\dot{g} = [g, H_E] \tag{2.1}$$

where H_E stands for the extended Hamiltonian, consisting of the previous total Hamiltonian H_T, plus all generators that don't change the current state, with arbitrary coefficients[1]:

$$H_E = H_T + v'_{a'} \phi_{a'} \tag{2.2}$$

Those additional generators $\phi_{a'}$, which haven't been included in H_T are the first-class secondary constraints.

[1]Recall that we mentioned in the last section that the arbitrariness of a system is determined by the number of generators that don't change the system.

The general Hamiltonian theory here applies to a finite number of degrees of freedom. We could easily extend N into infinity with the original indices denoting the degrees of freedom is $n = 1, \cdots, N$. Furthermore, the suffix could be taken from discrete to continuous, that is, we have q_n's and p_n's variables becomes q_x and p_x, where x is the suffix which could be any values in a continuous range. In continuous case, the sums over n in the previous work would become integrals.

However,