

Note For Noether's Theorem

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1 Noether Theorem in Classical Mechanics

1.1 Lagrangian Formalism

In Lagrangian formalism, we've defined the action by lagrangian,

$$S = \int_{t_1}^{t_2} L[x(t), \dot{x}(t)] dt \quad (1.1)$$

If action follows the *least action principle*, we would have the lagrangian be associated with the correct *equation of motion* of the system given by eq(1.1).

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0 \quad (1.2)$$

Noether's theorem states that if given system has continuous symmetry, then there would be a conserved quantity associated with this symmetry.

What is continuous symmetry? One that leaves the action invariant even when the dynamical equations (equations of motion) are *not* satisfied, we call it **infinitesimal symmetry transformation**. We denote the infinitesimal symmetry transformation as δ_S to separate it from the variation δ .

We're interested in the case when the action doesn't change under a symmetry, and this implies there is special property behind the dynamics of the system.

If we would like to discuss the transformation on time, we should use another parameter,

$$\begin{aligned} t &= t(\tau) \\ q(t(\tau)) &= \mathcal{Q}(\tau) \end{aligned} \quad (1.3)$$

Generally, the transformation would be

$$\begin{aligned} t &\rightarrow t' = \tau'(t') \\ q &\rightarrow q'(t') = \mathcal{Q}(\tau') \end{aligned} \quad (1.4)$$

and the variation of position and time¹,

$$\begin{aligned} \delta_S q &= q'(t) - q(t) = \mathcal{Q}'(\tau) - \mathcal{Q}(\tau) = \delta_S \mathcal{Q} \\ \delta_S t &= t' - t = \tau'(t) - \tau(t) = \delta_S \tau \end{aligned} \quad (1.5)$$

Rewrite the original action,

$$S = \int_{t_1}^{t_2} dt L[q(t), \dot{q}(t)] = \int_{\tau_1}^{\tau_2} d\tau \left(\frac{dt}{d\tau} \right) L[\mathcal{Q}, \frac{d}{d\tau} \mathcal{Q}, t] \quad (1.6)$$

Redefine the new lagrangain,

$$\mathbb{L} \left[\mathcal{Q}, \frac{d}{d\tau} \mathcal{Q}, t, \frac{dt}{d\tau} \right] = \frac{dt}{d\tau} L \quad (1.7)$$

Now consider the infinitesimal symmetry transformation, and the new action would become,

$$S' = \int d\tau \mathbb{L} \left[\mathcal{Q}', \frac{d\mathcal{Q}'}{d\tau}, t', \frac{dt'}{d\tau} \right] + \int d\tau \frac{dK}{d\tau} \quad (1.8)$$

Adding a term of total derivative for general consideration, and we know it wouldn't affect the equation of motion. Therefore we can do the variation of action and lagrangian,

$$\delta S = S' - S = 0 \quad (1.9)$$

The variation of action should vanish because we assume it is symmetry infinitesimal transformation. Let's consider the equation of motion corresponding to the new lagrangian $\mathbb{L} \left[\mathcal{Q}, \frac{d\mathcal{Q}}{d\tau}, t, \frac{dt}{d\tau} \right]$.

$$\begin{aligned} \frac{\partial \mathbb{L}}{\partial \mathcal{Q}} - \frac{d}{d\tau} \frac{\partial \mathbb{L}}{\partial \frac{d\mathcal{Q}}{d\tau}} &= 0 \\ \frac{\partial \mathbb{L}}{\partial t} - \frac{d}{d\tau} \frac{\partial \mathbb{L}}{\partial \frac{dt}{d\tau}} &= 0 \end{aligned} \quad (1.10)$$

¹The parameter should be the same.

The variation of lagrangian would become,

$$\begin{aligned}\delta\mathbb{L} &= \mathbb{L} \left[\mathcal{Q}', \frac{d\mathcal{Q}}{d\tau}, t', \frac{dt'}{d\tau} \right] - \mathbb{L} \left[\mathcal{Q}', \frac{d\mathcal{Q}}{d\tau}, t', \frac{dt'}{d\tau} \right] \\ &= \frac{\partial\mathbb{L}}{\partial\mathcal{Q}}\delta_s\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial d_\tau\mathcal{Q}}\delta_s d_\tau\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial t}\delta_st + \frac{\partial\mathbb{L}}{\partial d_\tau t}\delta_s d_\tau t\end{aligned}\quad (1.11)$$

Plug in the equation of motion, we then got,

$$\delta\mathbb{L} = \frac{d}{d\tau} \left(\frac{\partial\mathbb{L}}{\partial d_\tau\mathcal{Q}}\delta_s\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial d_\tau t}\delta_st \right)$$

To conclude, the variation of action would become,

$$\delta S = \int dt \frac{d}{d\tau} \left(\frac{\partial\mathbb{L}}{\partial d_\tau\mathcal{Q}}\delta_s\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial d_\tau t}\delta_st + K \right) = 0 \quad (1.12)$$

Let turn \mathcal{Q} and \mathbb{L} back to original lagrangian and position,

$$\begin{aligned}\frac{\partial\mathbb{L}}{\partial d_\tau\mathcal{Q}}\delta_s\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial d_\tau t}\delta_st &= \frac{\partial L \frac{dt}{d\tau}}{\partial d_\tau\mathcal{Q}}\delta_s\mathcal{Q} + \frac{\partial L \frac{dt}{d\tau}}{\partial d_\tau t}\delta_st \\ &= \frac{\partial L}{\partial d_\tau\mathcal{Q}} \frac{dt}{d\tau} \delta_s\mathcal{Q} + \left(\frac{\partial L}{\partial d_\tau t} \frac{dt}{d\tau} + L \right) \delta_st \\ &= \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial d_\tau\mathcal{Q}} \frac{dt}{d\tau} \delta_sq + \left(\frac{\partial L [q, \frac{dq}{d\tau} d_\tau t^{-1}]}{\partial d_\tau t} \frac{dt}{d\tau} + L \right) \delta_st \\ &= \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial d_\tau\mathcal{Q}} \frac{dt}{d\tau} \delta_sq + \left(\frac{\partial L [q, \frac{dq}{d\tau} d_\tau t^{-1}]}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial d_\tau t} \frac{dt}{d\tau} + L \right) \delta_st\end{aligned}$$

Consider,

$$\frac{\partial \dot{q}}{\partial d_\tau t} \frac{dt}{d\tau} = \frac{\partial (d_\tau q \cdot (d_\tau t)^{-1})}{\partial d_\tau t}$$