Note For Noether's Theorem

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Contents

1 Noether Theorem in Classical Mechanics

1.1 Lagrangian Formalism

In Lagrangian formalism, we've defined the action by lagrangian,

$$S = \int_{t_1}^{t_2} L\left[x(t), \dot{x}(t)\right]\, dt \tag{1.1}$$

If action follows the *least action principle*, we would have the lagrangian be associated with the correct *equation of motion* of the system given by eq(1.1).

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{1.2}$$

Noether's theorem states that if given system has continuous symmetry, then there would be a conserved quantity associated with this symmetry.

What is continuous symmetry? One that leaves the action invariant even when the dynamical equations (equations of motion) are *not* satisfied, we call it **infinitesimal symmetry transformation**. We denote the infinitesimal symmetry transformation as δ_{S} to separate it from the variation δ.

We're interested in the case when the action doesn't change under a symmetry, and this implies there is special property behind the dynamics of the system.

If we would like to discuss the transformation on time, we should use another parameter,

$$\begin{aligned} t &= t(\tau) \\ q(t(\tau)) &= \mathcal{Q}(\tau) \end{aligned} \tag{1.3}$$

Generally, the transformation would be

$$\begin{aligned} t \to t' &= \tau'(t') \\ q \to q'(t') &= \mathcal{Q}(\tau') \end{aligned} \tag{1.4}$$

and the variation of position and time[1],

$$\begin{aligned} \delta_{\mathrm{S}} q = q'(t) - q(t) &= \mathcal{Q}'(\tau) - \mathcal{Q}(\tau) = \delta_{\mathrm{S}}\mathcal{Q} \\ \delta_{\mathrm{S}} t = t' - t &= \tau'(t) - \tau(t) = \delta_{\mathrm{S}}\tau \end{aligned} \tag{1.5}$$

Rewrite the original action,

$$S = \int_{t_1}^{t_2} dt\, L[q(t), \dot{q}(t)] = \int_{\tau_1}^{\tau_2} d\tau \, \left(\frac{dt}{d\tau}\right) L[\mathcal{Q}, \frac{d}{d\tau}\mathcal{Q}, t] \tag{1.6}$$

Redefine the new lagrangain,

$$\mathbb{L}\left[\mathcal{Q}, \frac{d}{d\tau}\mathcal{Q}, t, \frac{dt}{d\tau}\right] = \frac{dt}{d\tau} L \tag{1.7}$$

Now consider the infinitesimal symmetry transformation, and the new action would become,

$$S' = \int d\tau \, \mathbb{L}\left[\mathcal{Q}', \frac{d\mathcal{Q}'}{d\tau}, t', \frac{dt}{d\tau}'\right] + \int d\tau \frac{dK}{d\tau} \tag{1.8}$$

Adding a term of total derivative for general consideration, and we know it wouldn't affect the equation of motion. Therefore we can do the variation of action and lagrangian,

$$\delta S = S' - S = 0 \tag{1.9}$$

The variation of action should vanish because we assume it is symmetry infinitesimal transformation. Let's consider the equation of motion corresponding to the new lagrangian $\mathbb{L}\left[\mathcal{Q}, \frac{d\mathcal{Q}}{d\tau}, t, \frac{dt}{d\tau}\right]$.

$$\begin{aligned} \frac{\partial \mathbb{L}}{\partial \mathcal{Q}} - \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial d_\tau \mathcal{Q}} &= 0 \\ \frac{\partial \mathbb{L}}{\partial t} - \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial d_\tau t} &= 0 \end{aligned} \tag{1.10}$$

[1] The parameter should be the same.

The variation of lagrangian would become,

$$\delta\mathbb{L} = \mathbb{L}\left[\mathcal{Q}', \frac{d\mathcal{Q}}{d\tau}, t', \frac{dt'}{d\tau}\right] - \mathbb{L}\left[\mathcal{Q}', \frac{d\mathcal{Q}}{d\tau}, t', \frac{dt'}{d\tau}\right] \tag{1.11}$$

$$= \frac{\partial\mathbb{L}}{\partial\mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial d_\tau\mathcal{Q}}\delta_{\mathrm{S}}d_\tau\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial t}\delta_{\mathrm{S}}t + \frac{\partial\mathbb{L}}{\partial d_\tau t}\delta_{\mathrm{S}}d_\tau t$$

Plug in the equation of motion, we then got,

$$\delta\mathbb{L} = \frac{d}{d\tau}\left(\frac{\partial\mathbb{L}}{\partial d_\tau\mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial d_\tau t}\delta_{\mathrm{S}}t\right)$$

To conclude, the variation of action would become,

$$\delta S = \int dt\,\frac{d}{d\tau}\left(\frac{\partial\mathbb{L}}{\partial d_\tau\mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial d_\tau t}\delta_{\mathrm{S}}t + K\right) = 0 \tag{1.12}$$

Let turn $\mathcal{Q}$ and $\mathbb{L}$ back to original lagrangian and position,

$$\frac{\partial\mathbb{L}}{\partial d_\tau\mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial\mathbb{L}}{\partial d_\tau t}\delta_{\mathrm{S}}t = \frac{\partial L\frac{dt}{d\tau}}{\partial d_\tau\mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial L\frac{dt}{d\tau}}{\partial d_\tau t}\delta_{\mathrm{S}}t$$

$$= \frac{\partial L}{\partial d_\tau\mathcal{Q}}\frac{dt}{d\tau}\delta_{\mathrm{S}}\mathcal{Q} + \left(\frac{\partial L}{\partial d_\tau t}\frac{dt}{d\tau} + L\right)\delta_{\mathrm{S}}t$$

$$= \frac{\partial L}{\partial\dot{q}}\frac{\partial\dot{q}}{\partial d_\tau\mathcal{Q}}\frac{dt}{d\tau}\delta_{\mathrm{S}}q + \left(\frac{\partial L\left[q, \frac{dq}{d\tau}d_\tau t^{-1}\right]}{\partial d_\tau t}\frac{dt}{d\tau} + L\right)\delta_{\mathrm{S}}t$$

$$= \frac{\partial L}{\partial\dot{q}}\frac{\partial\dot{q}}{\partial d_\tau\mathcal{Q}}\frac{dt}{d\tau}\delta_{\mathrm{S}}q + \left(\frac{\partial L\left[q, \frac{dq}{d\tau}d_\tau t^{-1}\right]}{\partial\dot{q}}\frac{\partial\dot{q}}{\partial d_\tau t}\frac{dt}{d\tau} + L\right)\delta_{\mathrm{S}}t \tag{1.13}$$

with $d_\tau q \equiv \frac{dq}{d\tau}$. Consider,

$$\frac{\partial\dot{q}}{\partial d_\tau t}\frac{dt}{d\tau} = \frac{\partial\left(d_\tau q\cdot(d_\tau t)^{-1}\right)}{\partial d_\tau t}d_\tau t = -\frac{d_\tau q}{(d_\tau t)^2}d_\tau t = -\frac{d_\tau q}{d_\tau t} = \dot{q}$$

$$\frac{\partial\dot{q}}{\partial d_\tau\mathcal{Q}}\frac{dt}{d\tau} = 1$$

Therefore eq(1.13) would become,

$$= \frac{\partial L}{\partial\dot{q}}\delta_{\mathrm{S}}q + \left(-\frac{\partial L}{\partial\dot{q}}\dot{q} + L\right)\delta_{\mathrm{S}}t$$

We have the following quantity is conserved,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\delta_{\mathrm{S}}q + \left(-\frac{\partial L}{\partial\dot{q}}\dot{q} + L\right)\delta_{\mathrm{S}}t + K\right) = 0 \tag{1.14}$$