Note For Noether's Theorem

Ting-Kai Hsu

December 28, 2023

Contents

1 Noether Theorem in Classical Mechanics **1**
1.1 Lagrangian Formalism 1
1.2 Examples in Classical Mechanics 4
1.2.1 Space Translation 4
1.2.2 Time Translation 5

2 Noether Theorem in Field Theory **5**

1 Noether Theorem in Classical Mechanics

1.1 Lagrangian Formalism

In Lagrangian formalism, we've defined the action by lagrangian,

$$S = \int_{t_1}^{t_2} L\left[x(t), \dot{x}(t)\right]\, dt \tag{1.1}$$

If action follows the *least action principle*, we would have the lagrangian be associated with the correct *equation of motion* of the system given by eq(1.1).

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{1.2}$$

Noether's theorem states that if given system has continuous symmetry, then there would be a conserved quantity associated with this symmetry.

What is continuous symmetry? One that leaves the action invariant even when the dynamical equations (equations of motion) are *not* satisfied, we call it **infinitesimal symmetry transformation**. We denote the infinitesimal symmetry transformation as δ_{S} to separate it from the variation δ.

We're interested in the case when the action doesn't change under a symmetry, and this implies there is special property behind the dynamics of the system.

If we would like to discuss the transformation on time, we should use another parameter,

$$\begin{aligned} t &= t(\tau) \\ q(t(\tau)) &= \mathcal{Q}(\tau) \end{aligned} \tag{1.3}$$

Generally, the transformation would be

$$\begin{aligned} t \to t' &= \tau'(t') \\ q \to q'(t') &= \mathcal{Q}(\tau') \end{aligned} \tag{1.4}$$

and the variation of position and time[1],

$$\begin{aligned} \delta_{\mathrm{S}} q = q'(t) - q(t) &= \mathcal{Q}'(\tau) - \mathcal{Q}(\tau) = \delta_{\mathrm{S}}\mathcal{Q} \\ \delta_{\mathrm{S}} t = t' - t &= \tau'(t) - \tau(t) = \delta_{\mathrm{S}}\tau \end{aligned} \tag{1.5}$$

Rewrite the original action,

$$S = \int_{t_1}^{t_2} dt\, L[q(t), \dot{q}(t)] = \int_{\tau_1}^{\tau_2} d\tau\, \left(\frac{dt}{d\tau}\right) L[\mathcal{Q}, \frac{d}{d\tau}\mathcal{Q}, t] \tag{1.6}$$

Redefine the new lagrangain,

$$\mathbb{L}\left[\mathcal{Q}, \frac{d}{d\tau}\mathcal{Q}, t, \frac{dt}{d\tau}\right] = \frac{dt}{d\tau} L \tag{1.7}$$

Now consider the infinitesimal symmetry transformation, and the new action would become,

$$S' = \int d\tau\, \mathbb{L}\left[\mathcal{Q}', \frac{d\mathcal{Q}'}{d\tau}, t', \frac{dt}{d\tau}'\right] + \int d\tau \frac{dK}{d\tau} \tag{1.8}$$

Adding a term of total derivative for general consideration, and we know it wouldn't affect the equation of motion. Therefore we can do the variation of action and lagrangian,

$$\delta S = S' - S = 0 \tag{1.9}$$

[1]The parameter should be the same.

The variation of action should vanish because we assume it is symmetry infinitesimal transformation. Let's consider the equation of motion corresponding to the new lagrangian $\mathbb{L}\left[\mathcal{Q}, \frac{d\mathcal{Q}}{d\tau}, t, \frac{dt}{d\tau}\right]$.

$$\begin{aligned} \frac{\partial \mathbb{L}}{\partial \mathcal{Q}} - \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial d_\tau \mathcal{Q}} = 0 \\ \frac{\partial \mathbb{L}}{\partial t} - \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial d_\tau t} = 0 \end{aligned} \tag{1.10}$$

The variation of lagrangian would become,

$$\delta \mathbb{L} = \mathbb{L}\left[\mathcal{Q}', \frac{d\mathcal{Q}}{d\tau}, t', \frac{dt'}{d\tau}\right] - \mathbb{L}\left[\mathcal{Q}', \frac{d\mathcal{Q}}{d\tau}, t', \frac{dt'}{d\tau}\right] \tag{1.11}$$

$$= \frac{\partial \mathbb{L}}{\partial \mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial \mathbb{L}}{\partial d_\tau \mathcal{Q}}\delta_{\mathrm{S}} d_\tau \mathcal{Q} + \frac{\partial \mathbb{L}}{\partial t}\delta_{\mathrm{S}} t + \frac{\partial \mathbb{L}}{\partial d_\tau t}\delta_{\mathrm{S}} d_\tau t$$

Plug in the equation of motion, we then got,

$$\delta \mathbb{L} = \frac{d}{d\tau}\left(\frac{\partial \mathbb{L}}{\partial d_\tau \mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial \mathbb{L}}{\partial d_\tau t}\delta_{\mathrm{S}} t\right)$$

To conclude, the variation of action would become,

$$\delta S = \int dt \, \frac{d}{d\tau}\left(\frac{\partial \mathbb{L}}{\partial d_\tau \mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial \mathbb{L}}{\partial d_\tau t}\delta_{\mathrm{S}} t + K\right) = 0 \tag{1.12}$$

Let turn $\mathcal{Q}$ and $\mathbb{L}$ back to original lagrangian and position,

$$\frac{\partial \mathbb{L}}{\partial d_\tau \mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial \mathbb{L}}{\partial d_\tau t}\delta_{\mathrm{S}} t = \frac{\partial L \frac{dt}{d\tau}}{\partial d_\tau \mathcal{Q}}\delta_{\mathrm{S}}\mathcal{Q} + \frac{\partial L \frac{dt}{d\tau}}{\partial d_\tau t}\delta_{\mathrm{S}} t$$

$$= \frac{\partial L}{\partial d_\tau \mathcal{Q}}\frac{dt}{d\tau}\delta_{\mathrm{S}}\mathcal{Q} + \left(\frac{\partial L}{\partial d_\tau t}\frac{dt}{d\tau} + L\right)\delta_{\mathrm{S}} t$$

$$= \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial d_\tau \mathcal{Q}}\frac{dt}{d\tau}\delta_{\mathrm{S}} q + \left(\frac{\partial L\left[q, \frac{dq}{d\tau} d_\tau t^{-1}\right]}{\partial d_\tau t}\frac{dt}{d\tau} + L\right)\delta_{\mathrm{S}} t$$

$$= \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial d_\tau \mathcal{Q}}\frac{dt}{d\tau}\delta_{\mathrm{S}} q + \left(\frac{\partial L\left[q, \frac{dq}{d\tau} d_\tau t^{-1}\right]}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial d_\tau t}\frac{dt}{d\tau} + L\right)\delta_{\mathrm{S}} t \tag{1.13}$$

with $d_\tau q \equiv \frac{dq}{d\tau}$. Consider,

$$\frac{\partial \dot{q}}{\partial d_\tau t}\frac{dt}{d\tau} = \frac{\partial\left(d_\tau q \cdot (d_\tau t)^{-1}\right)}{\partial d_\tau t} d_\tau t = -\frac{d_\tau q}{(d_\tau t)^2} d_\tau t = -\frac{d_\tau q}{d_\tau t} = \dot{q}$$

$$\frac{\partial \dot{q}}{\partial d_\tau \mathcal{Q}}\frac{dt}{d\tau} = 1$$

Therefore eq(1.13) would become,

$$= \frac{\partial L}{\partial \dot{q}}\delta_{\mathrm{S}} q + \left(-\frac{\partial L}{\partial \dot{q}}\dot{q} + L\right)\delta_{\mathrm{S}} t$$

We have the following quantity is conserved,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta_{\mathrm{S}} q + \left(-\frac{\partial L}{\partial \dot{q}}\dot{q} + L\right)\delta_{\mathrm{S}} t + K\right) = 0 \tag{1.14}$$

We call it **conserved quantity** or **conserved charge**.

1.2 Examples in Classical Mechanics

1.2.1 Space Translation

Consider the following infinitesimal symmetry transformation,

$$\begin{aligned} t \rightarrow t' &= t \\ q(t) \rightarrow q'(t') &= q(t) + \epsilon \end{aligned} \tag{1.15}$$

where ϵ is an arbitrary constant. Thus we have,

$$\begin{aligned} \delta_{\mathrm{S}} t &= 0 \\ \delta_{\mathrm{S}} q &= \epsilon \end{aligned} \tag{1.16}$$

The corresponding conserved quantity would be[2],

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\epsilon\right) = 0 \tag{1.17}$$

It is **canonical momentum**,

$$p \equiv \frac{\partial L}{\partial \dot{q}} \tag{1.18}$$

[2] K is zero in this case.

1.2.2 Time Translation

Consider the following infinitesimal symmetry transformation,

$$\begin{aligned} t &\to t' = t + \epsilon \\ q(t) &\to q'(t') = q(t+\epsilon) \end{aligned} \tag{1.19}$$

where ϵ is an arbitrary constant. Thus we have,

$$\begin{aligned} \delta_{\mathrm{S}} t &= \epsilon \\ \delta_{\mathrm{S}} q = q'(t') - q(t') = q(t+\epsilon) - q(t+\epsilon) &= 0 \end{aligned} \tag{1.20}$$

Thus the corresponding conserved quantity become,

$$\frac{d}{dt}\left[\left(-\frac{\partial L}{\partial \dot{q}}\dot{q} + L\right)\epsilon\right] = 0 \tag{1.21}$$

Which is hamiltonian,

$$H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L \tag{1.22}$$

2 Noether Theorem in Field Theory