Note For Noether's Theorem

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1 Noether Theorem in Classical Mechanics

1.1 Lagrangian Formalism

In Lagrangian formalism, we've defined the action by lagrangian,

$$S = \int_{t_1}^{t_2} L\left[x(t), \dot{x}(t)\right]\, dt \tag{1.1}$$

If action follows the *least action principle*, we would have the lagrangian be associated with the correct *equation of motion* of the system given by eq(1.1).

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{1.2}$$

Noether's theorem states that if given system has continuous symmetry, then there would be a conserved quantity associated with this symmetry.

What is continuous symmetry? One that leaves the action invariant even when the dynamical equations (equations of motion) are *not* satisfied, we call it **infinitesimal symmetry transformation**. We denote the infinitesimal symmetry transformation as δ_{S} to separate it from the variation δ.

We're interested in the case when the action doesn't change under a symmetry, and this implies there is special property behind the dynamics of the system. Consider the general infinitesimal transformation,

$$\begin{aligned} t \to t' &= t + \epsilon \frac{dt}{d\epsilon} + \cdots \\ x(t) \to x'(t') &= \mathcal{F}(x(t)) = x(t) + \frac{d\mathcal{F}}{d\epsilon}\epsilon \end{aligned} \tag{1.3}$$

Where ϵ is infinitesimal parameter.

Define the variation corresponding to the symmetry, and it should be expressed as infinitesimal transformation at the same point (or time, or generally coordinates).

$$\delta_{\mathrm{S}} x = x'(t) - x(t) \tag{1.4}$$

Let's find out how would the new action look,

$$S' = \int_{t_2}^{t_1} dt\, L\left[x'(t), \frac{d}{dt}x'(t)\right] \tag{1.5}$$

Note that we don't change the time in integral, this is because time t is a "dummy variable" that can be changed at will in the integral *without affecting action*. Importantly, symmetries will always be deformations of the fields (position), not the coordinates (time)[1]. So we can change the integral parameter in eq(1.5)

$$S' = \int_{t_1'}^{t_2'} dt'\, L\left[x'(t'), \frac{dx'(t')}{dt'}\right]$$

$$= \int_{t_1'}^{t_2'} dt \left(1 + \frac{d}{dt}\left(\epsilon \frac{dt}{d\epsilon}\right)\right) L\left[x(t) + \frac{d\mathcal{F}}{d\epsilon}\epsilon, \frac{dt}{dt'}\frac{dx'(t')}{dt}\right]$$

$$= \int_{t_1'}^{t_2'} dt \left(1 + \frac{d}{dt}\left(\epsilon \frac{dt}{d\epsilon}\right)\right) L\left[x(t) + \frac{d\mathcal{F}}{d\epsilon}\epsilon, \left(1 - \frac{d}{dt}\left(\epsilon \frac{dt}{d\epsilon}\right)\right)\frac{d\left(x(t) + \frac{d\mathcal{F}}{d\epsilon}\epsilon\right)}{dt}\right]$$

Then we expand the lagrangian in first order of ϵ,

$$\int dt \left(1 + \frac{d}{dt}\left(\frac{dt}{d\epsilon}\epsilon\right)\right)\left(L + \epsilon\frac{\partial L}{\partial x}\frac{d\mathcal{F}}{d\epsilon} + \frac{\partial L}{\partial \dot{x}}\left(\frac{d}{dt}(\epsilon\frac{d\mathcal{F}}{d\epsilon}) - \dot{x}\frac{d}{dt}(\epsilon\frac{dt}{d\epsilon})\right)\right)$$

[1]Please don't mix the transformation of parameters of the integral variable with true symmetries.

$$
\begin{aligned}
&= S + \int dt\, \epsilon \left(\frac{\partial L}{\partial x} \frac{d\mathcal{F}}{\partial \epsilon} + \frac{\partial L}{\partial \dot{x}} \frac{d}{dt} \frac{d\mathcal{F}}{d\epsilon} - \dot{x} \frac{\partial L}{\partial \dot{x}} \frac{d}{dt} \left(\frac{dt}{d\epsilon} \right) + L \frac{d}{dt} \frac{dt}{d\epsilon} \right) \\
&\quad + \int dt\, \frac{d\epsilon}{dt} \left(\frac{\partial L}{\partial \dot{x}} \frac{d\mathcal{F}}{d\epsilon} - \dot{x} \frac{\partial L}{\partial \dot{x}} \frac{dt}{d\epsilon} + L \frac{dt}{d\epsilon} \right)
\end{aligned}
$$

If the infinitesimal transformation is a symmetry, we would have the

1.2 More Friendly Way

Infinitesimal Transformation:
On-shell Variation: