S-matrix

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January 1, 2024

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1 Interacting Field Theory

Let's try to figure out the *two-point function* or *two-point correlation Green's function*,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle \tag{1.1}$$

Note that Ω is ground state and x and y are spacetime four vectors with T denoting the time-ordering operator[1]. One must be careful that the ground state $|\Omega\rangle$ here is different with the ground state $|0\rangle$ in free theory[2]. The correlation function can be seen as the amplitude of the propagation for a particle between y and x[3].

1.1 Free Theory

The lagrangian,

$$\mathcal{L} = (\partial_\mu\phi)^2 - m^2\phi^2 \tag{1.2}$$

[1] I couldn't figure out why we need this operator here.

[2] Where I don't know why?

[3] Does this mean there is only spacetime distance difference in these states?

The two-point function of free field could be directly computed by the solution of the field,

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}\right) \tag{1.3}$$

This is the form in Schrödinger picture, and next we should consider the time evolution of this field operator, that is, in Heisenberg's picture.

$$\phi(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt} \tag{1.4}$$

Also note that the creator and annihilator would have the following relation with the hamiltonian of the system,

$$\begin{aligned} Ha_{\mathbf{p}} &= a_{\mathbf{p}}\left(H - E_{\mathbf{p}}\right) \\ Ha_{\mathbf{p}}^\dagger &= a_{\mathbf{p}}^\dagger\left(H + E_{\mathbf{p}}\right) \end{aligned} \tag{1.5}$$

These are the properties of the annihilator and creator, which are related to SHO in quantum mechanics. Thus we have,

$$\begin{aligned} e^{iHt}a_{\mathbf{p}}e^{-iHt} &= a_{\mathbf{p}}e^{i(H-E_{\mathbf{p}})t}e^{-iHt} = a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} \\ e^{iHt}a_{\mathbf{p}}^\dagger e^{-iHt} &= a_{\mathbf{p}}e^{i(H+E_{\mathbf{p}})t}e^{-iHt} = a_{\mathbf{p}}e^{iE_{\mathbf{p}}t} \end{aligned} \tag{1.6}$$

Then eq(1.3) becomes,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \tag{1.7}$$

with $p_0 = E_{\mathbf{p}}$

Remark:

2 In Out State

To study S-matrix, that is, the scattering process in quantum field theory, one must know what is "In" and "Out" state. These physical quantities are related to the interacting field and process, which is of importance. In interacting field theory, we have to use nonlinear term of hamiltonian and lagrangian, with different Fourier modes that represent the different particle which can occupy them respectively. In order to **preserve causality**, that is, to make sure the formalism obey the principle of relativity, we must have the products of fields at the **same spacetime**

point. We've already learnt the scattering theory in quantum mechanics, and we would review them in this section.

First we separate the total hamiltonian into free hamiltonian (or one may say asymptotic hamiltonian) and interacting hamiltonian.

$$H(t) = H_0(t) + H_{\text{int}}(t) \tag{2.1}$$

Suppose we're in Heisenberg's picture.