

QIC Final Project: Anisotropic Transmission of quantum information through quantum fields

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ABSTRACT: In this letter, we briefly review the possible way to transmit the quantum information via quantum fields [1], and then we discuss

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1 Quantum Channel: Via Quantum Mechanics

In quantum information theory, the information is represented by a qubit, and it can be transformed, projected, and transmitted based on basic quantum mechanics posulates. In this letter, we focus on the transmission of a qubit from a spacetime emitter Alice A to a receiver Bob B .

There are various ways to transmit a qubit without contacting, which are based on the *resources* Alice and Bob share. For instance, if an entangled state is shared, they can transmit the qubit by Alice performing the Bell measurement and then send the result (a classical cbit) to Bob, which is the well-known *quantum teleportation*. Here, we simply consider transmission by a third quantum bit C , $\hat{\rho}_{C,0}$. Denote Alice's qubit as $\hat{\rho}_{A,0}$ and Bob's qubit $\hat{\rho}_{B,0}$; the transmission is done by performing SWAP between A and C , and then between C and B . The whole process is unitary and does not violate the non-cloning process because Alice's qubit becomes $\hat{\rho}_{C,0}$.

The SWAP operator can be derived by assuming $\hat{\rho}_{C,0} = |0\rangle\langle 0|$ and $\hat{\rho}_{A,0} = |a\rangle\langle a|$ with $\langle a|0\rangle \neq 0$:

$$U \rho_{A,0} \otimes \rho_{C,0} U^\dagger = \rho_{C,0} \otimes \rho_{A,0} \quad (1.1)$$

Remark: The transmission of qubit described above is rather trivial; however, it is based on an important fact that the dimension of the Hilbert space of C is the same as those of the Hilbert space of A and B , so there is an isomorphism between the Hilbert spaces. As we will see in the next section, the Hilbert space (or more precisely, the Fock space) of quantum fields is infinite-dimensional, and therefore there is no isomorphism like SWAP gate in the quantum mechanic case.

2 Quantum Channel: Via Quantum Fields

2.1 Brief Review on Quantum Field Theory

2.2 Unruh-DeWitt model

3 Non-isotropic Smearing Function

3.1 Problem Setup

The smearing function used in the paper is isotropic (i.e., $F_A = \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{|\mathbf{x}|^2}{\sigma^2}\right)$), which leads to isotropic propagation of information. In our study, we aim to investigate whether employing a non-isotropic smearing function could lead to different behavior.

As a first step, we consider a smearing function F_A with axial symmetry, i.e., $F_A = R(r)\Theta(\theta)$. And do following simplification,

1. $R(r)$ is localized at $r = 0$, i.e., $R(r \rightarrow \infty) = 0$
2. $\Theta(\theta) = \frac{\delta(\theta)}{\sin \theta}$,

where the $\frac{1}{\sin \theta}$ factor arises from the solid angle element in spherical coordinates. The reason of first simplification is that Alice should not be able to encode the information with the infinite range. And the reason of second simplification is the delta function should provide a well non-isotropic condition.

Under this construction, the smearing function F_{B1} becomes

$$F_{B1}(\mathbf{x}) = - \int dr d\theta d\phi r^2 \sin \theta R(r) \frac{\delta(\theta)}{\sin \theta} \frac{\delta(|\mathbf{r} - \mathbf{x}| - \Delta)}{4\pi |\mathbf{r} - \mathbf{x}|} \quad (3.1)$$

To evaluate the delta function, we use the following identity:

$$\int dx g(x) \delta(f(x)) = \int df \frac{g}{|f'|} \delta(f) = \sum_i \frac{g(x_{i0})}{|f'(x_{i0})|}$$

where x_{i0} are all x within the domain of integration which satisfy $f(x_{i0}) = 0$. Since in our case, we only integrate over positive real values of r , we only consider the solutions $r \in \mathbb{R}^+$ in the following calculation.

Before proceeding, we need to express $|\mathbf{r} - \mathbf{x}|$ in terms of (r, θ, ϕ) , so that we can perform the integration explicitly. Given that the smearing function is axially symmetric, we align \mathbf{x} along the $\phi = 0$ plane, i.e., we choose coordinates such that $\mathbf{x} = (x, \theta_x, 0)$, since any $\mathbf{x} = (x, \theta_x, \phi)$ yields the same value due to the symmetry. Then, the distance between \mathbf{r} and \mathbf{x} can be written as

$$|\mathbf{r} - \mathbf{x}| = \sqrt{r^2 + x^2 - 2rx \cos \gamma},$$

where γ is the angle between \mathbf{r} and \mathbf{x} . Using the spherical coordinate expression for the angle between two vectors, we have

$$\cos \gamma = \cos \theta \cos \theta_x + \sin \theta \sin \theta_x \cos \phi.$$

The whole function becomes

$$F_{B1}(\mathbf{x}) = - \int d\theta d\phi dr r^2 R(r) \delta(\theta) \frac{\delta \left(\sqrt{r^2 + x^2 - 2rx \cos \gamma} - \Delta \right)}{4\pi \sqrt{r^2 + x^2 - 2rx \cos \gamma}} \quad (3.2)$$

The solutions of $\sqrt{r^2 + x^2 - 2rx \cos \gamma} - \Delta = 0$ are

$$r_{\pm} = x \cos \gamma \pm \sqrt{x^2 \cos^2 \gamma - (x^2 - \Delta^2)}$$

We mention before, we only need the positive real solutions. So now we discuss when will r_{\pm} be positive, when will be negative, or even be complex.

$\cos \gamma > 0$	$\Delta^2 > x^2$	$ \sin \gamma < \frac{\Delta}{x}$	remain solutions	reason of unavalible
True	True	-	+	\mathbb{R}^-
True	False	True	\pm	-
True	False	False	0	$\mathbb{C} - \mathbb{R}$
False	True	-	+	\mathbb{R}^-
False	False	-	0	\mathbb{R}^-

Then we do another simplification, that is setting $\theta_x = 0, \pi$, i.e., align \mathbf{x} on the axis of \mathbf{r} . The reason we do this simplification is that we wonder how should Bob measure the field on the axis where also should has the most significant non-isotropic phenonenon.

3.2 Result

Since we set $\theta_x = 0, \pi$ and the delta function makes the smearing function only distribute in $\theta = 0$, we can rewrite $\cos \gamma$ as 1, -1, and r_{\pm} as $x \pm \Delta, -x \pm \Delta$. We now

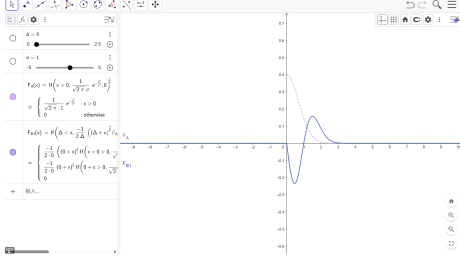
able to derive the F_{B1} explicitly.

$$\begin{aligned}
F_{B1}(\mathbf{x}) &= - \int d\theta \delta(\theta) \int d\phi \int dr r^2 R(r) \frac{\delta\left(\sqrt{r^2 + x^2 - 2rx \cos \gamma} - \Delta\right)}{4\pi \sqrt{r^2 + x^2 - 2rx \cos \gamma}} \\
&= -\frac{1}{2} \sum \frac{r_{\pm}^2 R(r_{\pm})}{\sqrt{r_{\pm}^2 + x^2 - 2r_{\pm}x \cos \gamma}} \frac{1}{\frac{r_{\pm} - x \cos \gamma}{\sqrt{r_{\pm}^2 + x^2 - 2r_{\pm}x \cos \gamma}}} \\
&= \begin{cases} -\frac{1}{2} \sum \frac{r_{\pm}^2}{r_{\pm} - x} R(r_{\pm}) & \text{if } \theta_x = 0 \\ -\frac{1}{2} \sum \frac{r_{\pm}^2}{r_{\pm} + x} R(r_{\pm}) & \text{if } \theta_x = \pi \end{cases} \\
&= \begin{cases} -\frac{1}{2} \frac{(x+\Delta)^2}{(x+\Delta)-x} R(x+\Delta) & \text{if } \theta_x = 0 \wedge \Delta^2 > x^2 \\ -\frac{1}{2} \left(\frac{(x+\Delta)^2}{(x+\Delta)-x} R(x+\Delta) + \frac{(x-\Delta)^2}{(x-\Delta)-x} R(x-\Delta) \right) & \text{if } \theta_x = 0 \wedge \Delta^2 < x^2 \\ -\frac{1}{2} \frac{(\Delta-x)^2}{(\Delta-x)+x} R((\Delta-x)) & \text{if } \theta_x = \pi \wedge \Delta^2 > x^2 \\ 0 & \text{if } \theta_x = \pi \wedge \Delta^2 < x^2 \end{cases} \\
&= \begin{cases} -\frac{1}{2\Delta} (x+\Delta)^2 R(x+\Delta) & \text{if } \theta_x = 0 \wedge \Delta^2 > x^2 \\ -\frac{1}{2\Delta} ((x+\Delta)^2 R(x+\Delta) - (x-\Delta)^2 R(x-\Delta)) & \text{if } \theta_x = 0 \wedge \Delta^2 < x^2 \\ -\frac{1}{2\Delta} (\Delta-x)^2 R((\Delta-x)) & \text{if } \theta_x = \pi \wedge \Delta^2 > x^2 \\ 0 & \text{if } \theta_x = \pi \wedge \Delta^2 < x^2 \end{cases} \tag{3.3}
\end{aligned}$$

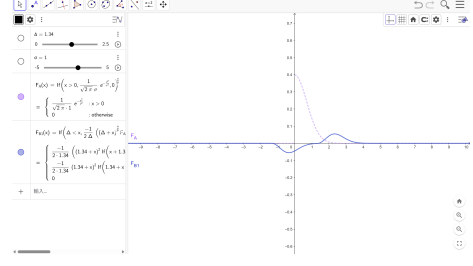
Note that when $\theta_x = \pi$, the smearing function of Bob is located directly behind that of Alice along the same axis. Therefore, if we are only interested in the behavior of the smearing function along this axis, we can equivalently set $x \rightarrow -x$ while taking $\theta_x = \pi$. So the smearing function becomes

$$F_{B1}(\mathbf{x}) = \begin{cases} -\frac{1}{2\Delta} ((x+\Delta)^2 R(x+\Delta) - (x-\Delta)^2 R(x-\Delta)) & \text{if } \Delta < x \\ -\frac{1}{2\Delta} (x+\Delta)^2 R(x+\Delta) & \text{if } -\Delta < x < \Delta \\ 0 & \text{if } x < -\Delta \end{cases}$$

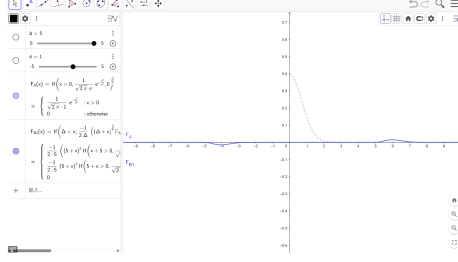
We can try to plug in the Gaussian function to $R(r)$ first. And get following figure,



(a) $t \rightarrow 0$



(b) $t = 1.34$



(c) $t = 5$

From the figures above, we can observe that the propagation of F_{B1} does not exhibit any non-isotropic behavior. The distribution remains symmetric along the axis, indicating that the chosen smearing function, despite being axially symmetric, does not lead to anisotropic propagation in this setup.

to calculate F_{B2}, F_{B3} , we could explicitly write out what they are,

$$F_{B2}(\mathbf{x}) = - \int d\theta \delta(\theta) \int d\phi \int dr r^2 R(r) \frac{\delta' \left(\sqrt{r^2 + x^2 - 2rx \cos \gamma} - \Delta \right)}{4\pi \sqrt{r^2 + x^2 - 2rx \cos \gamma}}$$

$$F_{B3}(\mathbf{x}) = - \int d\theta \delta(\theta) \int d\phi \int dr r^2 R(r) \frac{\delta'' \left(\sqrt{r^2 + x^2 - 2rx \cos \gamma} - \Delta \right)}{4\pi \sqrt{r^2 + x^2 - 2rx \cos \gamma}}$$

where $\delta'(x) = \frac{d\delta(x)}{dx}$ and $\delta''(x) = \frac{d^2\delta(x)}{dx^2}$. Note that is derivative is differential over x , in the original equation, it becomes differentiating over $\sqrt{r^2 + x^2 - 2rx \cos \gamma}$. Thus we need to do a change of variable, Chain rule, to change the variable to r . After these messy calculation, we obtain the following equations

$$F_{B2}(\mathbf{x}) = \int \frac{1}{\Delta} (\delta'(r - r_+) + \delta'(r - r_-)) g(r) r^2 R(r) dr \quad (3.4)$$

$$F_{B3}(\mathbf{x}) = \int \frac{1}{\Delta} (g(r)(\delta'(r - r_+) + \delta'(r - r_-))) g(r) r^2 R(r) dr \quad (3.5)$$

where $g(r) = \frac{dr}{d\sqrt{r^2+x^2-2rx\cos\gamma}}$. By using the following properties of Dirac delta function,

$$\int dx \delta'(x-a)f(x) = f'(a) \quad (3.6)$$

$$\int dx \delta''(x-a)f(x) = f''(a) \quad (3.7)$$

we obtain

$$F_{B2}(\mathbf{x}) = \frac{\Delta(\Delta-x)e^{-\frac{(x-\Delta)^2}{\sigma^2}}((x-\Delta)^2-\sigma^2)}{\sqrt{\Delta^2}\sigma^2} \text{sgn}(x-\Delta) \\ + \frac{\Delta(\Delta+x)e^{-\frac{(\Delta+x)^2}{\sigma^2}}((\Delta+x)^2-\sigma^2)}{\sqrt{\Delta^2}\sigma^2} \text{sgn}(x+\Delta)$$

$$F_{B3}(\mathbf{x}) = \frac{e^{-\frac{(x-\Delta)^2}{\sigma^2}}(-5\Delta^2\sigma^2+2\Delta^4+\sigma^4+x^2(12\Delta^2-5\sigma^2)-8\Delta x^3+2x^4+x(10\Delta\sigma^2-8\Delta^3))}{\sigma^4} \text{sgn}(x-\Delta) \\ + \frac{e^{-\frac{(\Delta+x)^2}{\sigma^2}}(-5\Delta^2\sigma^2+2\Delta^4+\sigma^4+x^2(12\Delta^2-5\sigma^2)+8\Delta x^3+2x^4+2x(4\Delta^3-5\Delta\sigma^2))}{\sigma^4} \text{sgn}(x+\Delta)$$

The figure of both is plotted below, we can see that it is indeed surviving in the $\theta = 0$ direction and $\theta = \pi$ direction. Thus we can't make a isotropic quantum teleportation via this setup.

Acknowledgments

References

- [1] Petar Simidzija, Aida Ahmadzadegan, Achim Kempf, and Eduardo Martín-Martínez. Transmission of quantum information through quantum fields. *Phys. Rev. D*, 101:036014, Feb 2020.