Scattering Theory

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This article is designed for leading the reader from old scattering theory of quantum mechanics to nowadays scattering theory of quantum field theory.

1 Potential Scattering

In this chapter we would study the theory of scattering in a simple but important case, the elastic scattering of a non-relativistic particle in a local potential, but using modern techniques that could easily be extended to more general problems.

1.1 In-States

Consider a non-relativistic particle of mass μ in a potential $V(\mathbf{x})$, whose Hamiltonian is,

$$H = H_0 + V(\mathbf{x}) \tag{1.1}$$

Here $H_0 = \mathbf{p}^2/2\mu$ is the kinetic energy operator. The potential is assumed to be function of position operator, and tends to zero as $|\mathbf{x}| = r \to \infty$. Then it is concerned here with a positive-energy particle[1]. The particle comes into facing the

[1] Actually, there could possibly be negative energy if it is in bound state, that is, the energy is less than potential at boundaries. But now, we are considering scattering state, which is opposite with bound state.

potential from great distances regarded to have no interaction with momentum $\hbar\mathbf{k}$, and is scattered, going out again to infinity, generally along a different direction.

In Heisenberg picture, for a particle with momentum $\hbar\mathbf{k}$ far from the scattering center if the measurement of the particle is made at very early times, and this situation is represented by a time-independent state vector $|\Psi_{\mathbf{k}}^{\text{in}}\rangle$. As mentioned, at very early times the particle is at a location that is far from the scattering center so that the potential is negligible there, so the state has an energy $\hbar^2\mathbf{k}^2/2\mu$, with the following relation,

$$H\,|\Psi_{\mathbf{k}}^{\text{in}}\rangle = \frac{\hbar^2\mathbf{k}^2}{2\mu}\,|\Psi_{\mathbf{k}}^{\text{in}}\rangle \tag{1.2}$$

Now switch to Schrödinger picture, the state would evolve as time goes on. As mentioned, the scattering state would be continuous *superposition*[2] of states with a spread of energies,

$$|\Psi_g(t)\rangle = \int d^3k/(2\pi)^3\, g(\mathbf{k}) \exp\left(-i\frac{\hbar t\mathbf{k}^2}{2\mu}\right) |\Psi_{\mathbf{k}}^{\text{in}}\rangle \tag{1.3}$$

where $g(\mathbf{k})$ is a smooth function that is peaked around some wave number $\mathbf{k}_0$. The eigenstate $|\Psi_{\mathbf{k}}^{\text{in}}\rangle$ satisfies the further condition that for any sufficiently smooth function $g(\mathbf{k})$, in the limit $t \to -\infty$,

$$|\Psi_g(t)\rangle \to \int d^3k/(2\pi)^3\, g(\mathbf{k}) \exp\left(-i\frac{\hbar t\mathbf{k}^2}{2\mu}\right) |\Phi_{\mathbf{k}}\rangle \tag{1.4}$$

where $|\Phi_{\mathbf{k}}\rangle$ are orthonormal eigenstates of the momentum operator $\mathbf{P}$ with eigenvalue $\hbar\mathbf{k}$,

$$\begin{aligned} \mathbf{P}\,|\Phi_{\mathbf{k}}\rangle &= \hbar\mathbf{k}\,|\Phi_{\mathbf{k}}\rangle \\ \langle\Phi_{\mathbf{k}}|\Phi_{\mathbf{k}'}\rangle &= (2\pi)^3\delta^{(3)}(\hbar\mathbf{k} - \hbar\mathbf{k}') \end{aligned} \tag{1.5}$$

and hence eigenstates of H_0 (not H), with eigenvalue $E(|\mathbf{k}|) = \hbar^2\mathbf{k}^2/2\mu$. Note that $|\Psi_{\mathbf{k}}^{\text{in}}\rangle$ and $|\Phi_{\mathbf{k}}\rangle$ belong to different Hilbert spaces. The incident wave packet $|\Psi_g(t)\rangle$ should satisfy the normalization condition $\langle\Psi_g(t)|\Psi_g(t)\rangle = 1$, which is equivalent to the condition at time limit,

$$\int\int d^3kd^3k'/(2\pi)^3\, g^*(\mathbf{k})g(\mathbf{k}') \exp\left(-i\frac{\hbar t}{2\mu}(\mathbf{k}'^2 - \mathbf{k}^2)\right)\delta^{(3)}(\hbar\mathbf{k} - \hbar\mathbf{k}')$$

[2]It would be discrete sum for a bound state.

$$= \hbar^{-3} \int d^3k/(2\pi)^3 \, |g(\mathbf{k})|^2 = 1 \tag{1.6}$$

Rewrite the equation (1.2) as,

$$(E(|\mathbf{k}|) - H_0) \, |\Psi_{\mathbf{k}}^{\text{in}}\rangle = V \, |\Psi_{\mathbf{k}}^{\text{in}}\rangle \tag{1.7}$$

This has a formal solution,

$$|\Psi_{\mathbf{k}}^{\text{in}}\rangle = \, |\Phi_{\mathbf{k}}\rangle + (E(|\mathbf{k}|) - H_0 + i\epsilon)^{-1} \, |\Psi_{\mathbf{k}}^{\text{in}}\rangle \tag{1.8}$$

The first term on RHS could always be added up because it is the *homogeneous* solution of equation (1.7), and ϵ is a positive infinitesimal quantity, which is inserted to give meaning to the operator $(E(|\mathbf{k}|) - H_0 + i\epsilon)^{-1}$ when we integrate over the eigenvalues of H_0[3]. It is known as the *Lippmann-Schwinger equation*. The special feature of the particular 'solution' is that it also satisfies the additional initial condition.

To see this, we could expand $V\,|\Psi_{\mathbf{k}}^{\text{in}}\rangle$ in the orthonormal free-particle states $|\Phi_{\mathbf{q}}\rangle$:

$$V\,|\Psi_{\mathbf{k}}^{\text{in}}\rangle = \int d^3q/(2\pi)^3 \, |\Phi_{\mathbf{q}}\rangle \, \langle\Phi_{\mathbf{q}}| \, V \, |\Psi_{\mathbf{k}}^{\text{in}}\rangle \tag{1.9}$$

Thus equation (1.8) becomes,

$$|\Psi_{\mathbf{k}}^{\text{in}}\rangle = \, |\Phi_{\mathbf{k}}\rangle + \hbar^3 \int d^3q/(2\pi)^3 \, (E(|\mathbf{k}|) - H_0 + i\epsilon)^{-1} \, |\Phi_{\mathbf{q}}\rangle \, \langle\Phi_{\mathbf{q}}| \, V \, |\Psi_{\mathbf{k}}^{\text{in}}\rangle$$

$$= \, |\Phi_{\mathbf{k}}\rangle + \hbar^3 \int d^3q/(2\pi)^3 \, (E(|\mathbf{k}|) - E(|\mathbf{q}|) + i\epsilon)^{-1} \, |\Phi_{\mathbf{q}}\rangle \, \langle\Phi_{\mathbf{q}}| \, V \, |\Psi_{\mathbf{k}}^{\text{in}}\rangle \tag{1.10}$$

In calculating the integral over $\mathbf{k}$ in equation (1.3),

$$\int d^3k/(2\pi)^3 \, g(\mathbf{k}) \frac{\exp(-i\hbar t\mathbf{k}^2/2\mu)}{E(|\mathbf{k}|) - E(q) + i\epsilon} \, \langle\Phi_{\mathbf{q}}| \, V \, |\Psi_{\mathbf{k}}^{\text{in}}\rangle$$

$$= \int d\Omega \int_0^\infty dk/(2\pi)^3 \, k^2 g(\mathbf{k}) \frac{\exp(-i\hbar t\mathbf{k}^2/2\mu)}{E(k) - E(q) + i\epsilon} \, \langle\Phi_{\mathbf{q}}| \, V \, |\Psi_{\mathbf{k}}^{\text{in}}\rangle$$

where $d\Omega = \sin\theta \, d\theta d\phi$. We could convert the integral over k to an integral over the kinetic energy, using $dk = \mu dE/k\hbar^2$. Now, when $t \to -\infty$, the exponential

[3] It is more reasonable in path integral formalism that the additional term must be added to prevent the integral diverges at large value of wave function.

term oscillates very rapidly, so that the only value of E that contribute are *those very near* $E(q)$, where the denominator also varies very rapidly.

$$= \int d\Omega/(2\pi)^3 \int_0^\infty dE\, \frac{\mu k}{\hbar^2} g(\mathbf{k}) \frac{\exp(-iEt/\hbar)}{E - E(q) + i\epsilon} \langle \Phi_{\mathbf{q}} | V | \Psi_{\mathbf{k}}^{\text{in}} \rangle$$

Hence, for the time limit $t \to -\infty$, we could set $k = q$ everywhere except in the rapidly varying exponential and denominator,

$$= \int d\Omega/(2\pi)^3\, g(\Omega) \frac{\mu q}{\hbar^2} \langle \Phi_{\mathbf{q}} | V | \Psi_{\mathbf{q}}^{\text{in}} \rangle \int_{\text{around E(q)}} dE\, \frac{\exp(-iEt/\hbar)}{E - E(q) + i\epsilon}$$

We could extend the integration range to the whole real axis, which is permissible because the integral receives no appreciable contributions anywhere that is far from $E(q)$,

$$\propto \int_{-\infty}^{\infty} dE\, \frac{\exp(-iEt/\hbar)}{E - E(q) + i\epsilon} \tag{1.11}$$

For $t \to -\infty$, we can close the contour of the integration with a very large semi-circle in the upper half of the complex plane, on which the integration is negligible because, for $\mathrm{Im}(E) > 0$ and $t \to -\infty$, the numerator $\exp(-iEt/\hbar)$ is exponentially small. The only singularity of the integration is a pole at $E = E(q) - i\epsilon$, which is always in the lower half plane because ϵ is positive infinitesimal parameter. Hence, the integral vanishes for $t \to -\infty$. Only the first term of equation (1.10) is left, which gives the correct condition for $t \to -\infty$

1.2 Scattering Amplitudes

In the previous section, we defined a state that at early times has the appearance of a particle traveling toward a collision with a scattering center. Now we must consider what this state looks like after the collision.

For this purpose, we consider the coordinate-space wave function of the state $|\Psi_{\mathbf{k}}^{\text{in}}\rangle$[4].

$$V |\Psi_{\mathbf{k}}^{\text{in}}\rangle = \int d^3x\, |\Phi_{\mathbf{x}}\rangle \langle \Phi_{\mathbf{x}} | V | \Psi_{\mathbf{k}}^{\text{in}} \rangle = \int d^3x\, |\Phi_{\mathbf{x}}\rangle V(\mathbf{x}) \psi_{\mathbf{k}}(\mathbf{x}) \tag{1.12}$$

where $\psi_{\mathbf{k}}(\mathbf{x})$ is the coordinate-space wave function of the in-state,

$$\psi_{\mathbf{k}}(\mathbf{x}) = \langle \Phi_{\mathbf{x}} | \Psi_{\mathbf{k}}^{\text{in}} \rangle \tag{1.13}$$

[4] By this way, it would give us a physically vivid picture about scattering.

Then, by taking the scalar product of the Lippmann-Schwinger equation (1.8), and using the fact that the scalar product of state of definite momentum and state of definite position would be plane-wave function,

$$\langle \Phi_{\mathbf{x}} | \Phi_{\mathbf{k}} \rangle = e^{i\mathbf{k}\cdot\mathbf{x}}$$

we have,

$$\psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \langle \Phi_{\mathbf{x}} | \left(E(|\mathbf{k}|) - H_0 + i\epsilon \right)^{-1} V \, |\Psi^{\text{in}}_{\mathbf{k}}\rangle$$

$$\psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} + \int d^3y \, \langle \Phi_{\mathbf{x}} | \left[E(|\mathbf{k}|) - H_0 + i\epsilon \right]^{-1} |\Phi_{\mathbf{y}}\rangle \, V(y) \psi_{\mathbf{k}}(\mathbf{y}) \tag{1.14}$$

where we define the Green function and then evaluate it by Fourier transform,

$$\begin{aligned}
G_k(\mathbf{x}-\mathbf{y}) &= \langle \Phi_{\mathbf{x}} | \left[E(|\mathbf{k}|) - H_0 + i\epsilon \right]^{-1} |\Phi_{\mathbf{y}}\rangle \\
&= \int d^3q/(2\pi)^3 \, \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{E(|\mathbf{k}|) - E(|\mathbf{q}|) + i\epsilon} \\
&= \int -d\cos(\theta) d\phi \int q^2 dq/(2\pi)^3 \, \frac{e^{iq|\mathbf{x}-\mathbf{y}|\cos(\theta)}}{\hbar^2 k^2/2\mu - \hbar^2 q^2/2\mu + i\epsilon} \\
&= \frac{2\pi}{(2\pi)^3} \int_0^\infty q^2 dq \, \frac{2i \sin(q|\mathbf{x}-\mathbf{y}|)}{q|\mathbf{x}-\mathbf{y}|} \frac{2\mu/\hbar^2}{k^2 - q^2 + i\epsilon} \\
&= \frac{2\mu}{\hbar^2} \frac{1}{4\pi^2 |\mathbf{x}-\mathbf{y}|} \int_{-\infty}^{\infty} dq \, \frac{q e^{iq|\mathbf{x}-\mathbf{y}|}}{k^2 - q^2 + i\epsilon} \\
&= -\frac{2\mu}{\hbar^2} \frac{1}{4\pi^2 |\mathbf{x}-\mathbf{y}|} e^{ik|\mathbf{x}-\mathbf{y}|}
\end{aligned} \tag{1.15}$$