Quantum Field Theory Problems

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1 T P Symmetries

In S.Weinberg famous textbook about quantum field theory[4] section 2.6, he discusses parity$\mathcal{P}$ and time inversion$\mathcal{T}$. [1]

$$\begin{aligned}\mathcal{P}^{\mu}{}_{\nu} &= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\\ \mathcal{T}^{\mu}{}_{\nu} &= \begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\end{aligned} \tag{1.0.1}$$

[1] I use different notation with Weinberg.

The operators of $\mathcal{P}$ and $\mathcal{T}$ are believed to be

$$\begin{aligned}\mathrm{P} &\equiv U(\mathcal{P},0)\\ \mathrm{T} &\equiv U(\mathcal{T},0)\end{aligned} \tag{1.0.2}$$

He seems to define a new notation that corresponds to not only Lorentz transformation and translation but also parity and time inversion. The operators of Poincaré algebra would transform according to the following law,

$$\begin{aligned}\mathrm{P}U(\Lambda,a)\mathrm{P}^{-1} &= U(\mathcal{P}\Lambda\mathcal{P}^{-1},\mathcal{P}a)\\ \mathrm{T}U(\Lambda,a)\mathrm{T}^{-1} &= U(\mathcal{T}\Lambda\mathcal{T}^{-1},\mathcal{T}a)\end{aligned} \tag{1.0.3}$$

I'm confused about what he means,

> These transformation rules incorporate most of what is meant when we say that P or T are 'conserved'.

Later, he points out that the above equations of P and T are merely approximation. These are provided by T. D. Lee, C. N. Yang and others works[2][1]. Before reading the references, I think that the problem arises from that equation (1.0.3) that physicists originally define is wrong, or merely an approximation. Still, I would like to know why and why it is regarded as approximation.

2 Lippmann-Schwinger equation

$$\Psi_\alpha^+ = \Phi_\alpha + \left(E(|\mathbf{k}| - H_0 + i\epsilon)\right)^{-1} V\Psi_\alpha^+ \tag{2.0.1}$$

It seems Weinberg's reason[6] of adding the positive infinitesimal parameter ϵ is weird, and it is unclear and unintuitive for giving meaning to operator. Although the sentence of giving meaning to operator is unclear, it is also mentioned in Weinberg's another textbook for quantum field theory[5] that this is because of operator $(E_\alpha - H_0)$ isn't invertible, and this is because E_α is the eigenvalue of H_0; however, restricting the infinitesimal parameter to be positive is still confusing. Possible solution is to re-derive the Lippmann-Schwinger equation[3] in path integral formalism.

3 Meaning and Function of Potential Operator V

When we expand the second term of equation (2.0.1) with free-interaction state Φ_β, we get

$$\Psi_\alpha^+ = \Phi_\alpha + \int d\beta \, \frac{(\Phi_\beta, V\Psi_\alpha^+)}{(E(|\mathbf{k}|) - E_\beta + i\epsilon)} \Phi_\beta \tag{3.0.1}$$

and define the T matrix, which later will be seen in S matrix,

$$T_{\beta\alpha}^+ \equiv (\Phi_\beta, V\Psi_\alpha^+) \tag{3.0.2}$$

Now this is weird to me because it is unclear about the inner product between two different vectors from two different Hilbert spaces[2].

4 Notes for Lippmann-Schwinger Equation Paper

4.1 In Out States

In this section, I would derive Lippmann-Schwinger equation in similar way as S.Weinberg [6], but would first derive the formula of S-matrix and T-matrix. We define 'in' and 'out' states Ψ_α^+ and Ψ_β^- as eigenstates of the total Hamiltonian

$$H\Psi_\alpha^\pm = E_\alpha \Psi_\alpha^\pm \tag{4.1.1}$$

that both states look like an eigenstate Φ_α of the free-particle Hamiltonian

$$H_0\Phi_\alpha = E_\alpha \Phi_\alpha \tag{4.1.2}$$

Note that we describe these states in Schrödiger picture, and this means we must use wave packet to describe the behavior of particle because if not we only got a phase with same state when acting time-evolution operator on state $\exp(-iHt)\Psi_\alpha^\pm = \exp(-iE_\alpha t)\Psi_\alpha^\pm$

$$\Psi_g^\pm(t) = \int d\alpha \, g(\alpha) \exp(-iE_\alpha t)\Psi_\alpha^+ \tag{4.1.3}$$

[2]That is, one is one of the eigenvectors of free-interaction H_0 and the other is one of the eigenvectors of total Hamiltonian H.

where amplitude $g(\alpha)$ is a smooth-varying function when time evolves. We further require the 'in' and 'out' states to satisfy the condition

$$\Psi_g^{\pm}(t) \to \int d\alpha\, g(\alpha) \exp(-iE_\alpha t)\Phi_\alpha \tag{4.1.4}$$

when $t \to -\infty$ for 'in' state and $t \to \infty$ for 'out' state. Written equation (4.1.3) and (4.1.4) in

$$\begin{aligned} \Psi_g^{\pm}(t) &= \exp(-iHt) \int d\alpha\, g(\alpha)\Psi_\alpha^{\pm} \\ \Phi_g(t) &= \exp(-iH_0 t) \int d\alpha\, g(\alpha)\Phi_\alpha \end{aligned} \tag{4.1.5}$$

With condition (4.1.4) we could find out the relation

$$\Psi_\alpha^{\pm} = \Omega(\mp\infty)\Phi_\alpha \tag{4.1.6}$$

where $\Omega(t) = \exp(+iHt)\exp(-iH_0 t)$

4.2 S-matrix

Equation (4.1.1) implies the 'in' and 'out' states should inhabit same Hilbert space, so we can express the 'out' state as linear combination of 'in' states, and the coefficients are defined as S-matrix. S-matrix will completely contain the information of scattering

$$S_{\beta\alpha} = \left(\Psi_\beta^-, \Psi_\alpha^+\right) \tag{4.2.1}$$

we could further express S-matrix as by equation (4.1.6)

$$S_{\beta\alpha} = (\Phi_\beta, U(+\infty, -\infty)\Phi_\alpha) \tag{4.2.2}$$

where $U(t, t_0) = \Omega^\dagger(t)\Omega(t_0) = \exp(iH_0 t)\exp(-iH(t-t_0))\exp(-iH_0 t_0)$. and define the collision operator such that

$$(\Phi_\beta, S\Phi_\alpha) \equiv S_{\beta\alpha} \tag{4.2.3}$$

Then collision operator would be

$$S = U(+\infty, -\infty) \tag{4.2.4}$$

Now focus on the time-evolution operator $U(t, -\infty)$, it satisfies the initial condition and differential equation

$$\begin{aligned} U(-\infty, -\infty) &= 1 \\ i\frac{\partial}{\partial t}U(t, -\infty) &= V(t)U(t, -\infty) \end{aligned} \tag{4.2.5}$$

with $v(t) = \exp(iH_0 t)V\exp(-iH_0 t)$ The solution would be

$$U(t, -\infty) = 1 - i\int_{-\infty}^{t} d\tau\, V(\tau)\, U(\tau, -\infty) \tag{4.2.6}$$

References

[1] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay. Evidence for the 2π decay of the k_2^0 meson. *Phys. Rev. Lett.*, 13:138–140, Jul 1964.

[2] T. D. Lee and C. N. Yang. Question of parity conservation in weak interactions. *Phys. Rev.*, 104:254–258, Oct 1956.

[3] B. A. Lippmann and Julian Schwinger. Variational principles for scattering processes. i. *Phys. Rev.*, 79:469–480, Aug 1950.

[4] Steven Weinberg. *RELATIVISTIC QUANTUM MECHANICS*, page 49–106. Cambridge University Press, 1995.

[5] Steven Weinberg. *SCATTERING THEORY*, page 107–168. Cambridge University Press, 1995.

[6] Steven Weinberg. *POTENTIAL SCATTERING*, page 247–281. Cambridge University Press, 2015.