QFTII Final Project

Conformal Bootstrap

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Abstract

In conformal symmetry, we can write down its correlators. Among these correlators, the 4-point correlator is not trivial and can be calculated using the Operator Product Expansion (OPE) through different channels. These different channels should give the same answer, which imposes constraints on certain variables. This consistency condition allows us to use the bootstrap method to filter the possible solutions.

The conformal bootstrap is a powerful approach to studying conformal field theories (CFTs). By imposing the constraints of conformal symmetry and crossing symmetry on correlation functions, we can derive nontrivial results about the structure of the theory. In particular, the bootstrap provides restrictions on the scaling dimensions and operator coefficients that appear in the OPE, enabling us to solve or constrain the CFT data.

We will also introduce some background knowledge about the conformal bootstrap to facilitate understanding.

Contents

1 Introduction to Conformal Bootstrap and Its Language

This section provides a brief overview of the bootstrapping philosophy and the foundational language and conventions used in this field.

1.1 Bootstrap Philosophy

The critical universality of many microscopic theories allows us to focus on the CFT.

1.2 Path Integral Quantization

Path integral formalism chooses a specific time direction, and spacetime is divided into successive slices. Each slice is associated with a Hilbert space of states in the Heisenberg picture, and the progression from one slice to the next is governed by time evolution. The general correlator is defined as follows:

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle 0 | \,\mathrm{T} \left\{ \hat{\mathcal{O}}_1(t_1, \mathbf{x}_1) \cdots \hat{\mathcal{O}}_n(t_n, \mathbf{x}_n) \right\} | 0 \rangle \tag{1.1}$$

Here, note that the L.H.S represents a product of spacetime functions. At the same time, the R.H.S. is a product of operators in the Heisenberg picture, defined in distinct Hilbert spaces corresponding to each slice. This is the familiar form of the correlator within the path integral framework.

1.3 Stress Energy and Topological Surface Operator

Consider a quantum field theory (QFT) coupled to a background metric g, with correlators defined within the path integral formalism in Euclidean signature.

$$\langle 0 | \,\mathrm{T} \left\{ \hat{\mathcal{O}}_1(t_1, \mathbf{x}_1) \cdots \hat{\mathcal{O}}_n(t_n, \mathbf{x}_n) \right\} | 0 \rangle_g = \int \prod_y d\phi(y)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S[g,\phi]} \tag{1.2}$$

The stress-energy tensor is the conserved current associated with the diffeomorphism invariance of the action $S[g, \phi]$ in the flat spacetime limit $g \to \eta$. Consequently, we obtain the familiar Ward-Takahashi identity:

$$\partial_\mu \langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = -\sum_i \delta^4(x - x_i)\partial_i^\nu \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \quad (1.3)$$

Define **topological surface operator**:

$$P^\nu(\Sigma) \equiv -\int_\Sigma dS_\mu T^{\mu\nu}(x) \quad (1.4)$$

The integral is taken over a closed hypersurface within the spacetime manifold, which can be considered the boundary of a ball B. The presence of delta functions in the Ward-Takahashi identity allows for the deformation of this closed hypersurface, provided that the deformation does not intersect additional spacetime points. Let's consider a ball B that encloses a spacetime point x where an operator $\mathcal{O}(x)$ insertion occurs. In this case, the volume integral over the ball transforms into a surface integral over the boundary of B on the L.H.S:

$$\begin{aligned}\langle P^\nu(\Sigma)\mathcal{O}(x)\cdots\rangle &= \int_B d^4y\delta^4(y - x)\partial_x^\nu \langle \mathcal{O}(x)\cdots\rangle \\ &= \partial^\nu \langle \mathcal{O}(x)\cdots\rangle\end{aligned} \quad (1.5)$$

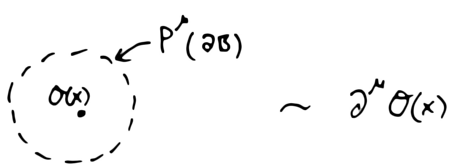


Figure 1: Illustration of the topological operator loop.

As long as no additional spacetime points with operator insertions are crossed, the surface Σ can be freely deformed from one slice to another, maintaining the invariance of $P^\mu(\Sigma)$. This invariance implies that the momentum $P^\mu(\Sigma)$ is topological in the path integral formulation, which in turn indicates that it is conserved. Now, consider two spatial slices at times t_1 and t_2 within the spacetime manifold, with $t_1 < t_2$.

Suppose there is an operator insertion $\mathcal{O}(t, \mathbf{x})$ at a time t such that $t_1 < t < t_2$. If we "sandwich" this operator between the slices at t_1 and t_2, then:

$$\lim_{t_1 \to t_2^-} \langle P^\mu(\Sigma_2)\mathcal{O}(x)\cdots\rangle - \langle P^\mu(\Sigma_1)\mathcal{O}(x)\cdots\rangle = \langle 0|\,\mathrm{T}\left\{\left[\hat{P}^\mu, \hat{\mathcal{O}}(x)\right]\cdots\right\}|0\rangle \quad (1.6)$$

The appearance of the commutator on the R.H.S arises due to the time ordering in the correlation function. Since the momentum $P^\mu(\Sigma)$ is topological, we can deform $\Sigma_2 - \Sigma_1$ to a sphere S that encloses the local point x:

$$\lim_{t_1 \to t_2^-} \langle P^\mu(\Sigma_2)\mathcal{O}(x)\cdots\rangle - \langle P^\mu(\Sigma_1)\mathcal{O}(x)\cdots\rangle = \langle P^\mu(S)\mathcal{O}(x)\cdots\rangle$$

$$= \partial_x^\mu\langle\mathcal{O}(x)\cdots\rangle$$

This is the familiar commutation relation for the momentum operator in quantum

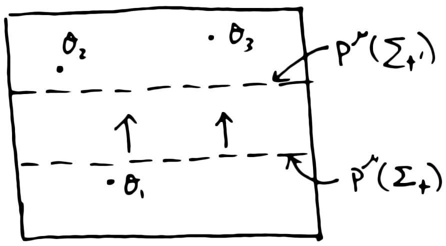


Figure 2: Illustration of the shrinking region in path integrals.

mechanics:

$$\left[\hat{P}^\mu, \hat{\mathcal{O}}(x)\right] = \partial^\mu\hat{\mathcal{O}}(x) \quad (1.7)$$

At first glance, the L.H.S of Eq(1.7) appears non-local because the topological momentum operator is expressed as a surface integral over a finite spacetime volume. However, by progressively deforming the boundary Σ to concentrate around the enclosed singularity increasingly, we reveal the locality of the transformation, as shown on the R.H.S of Eq(1.7).

1.4 Conformal Symmetry

Next, we can consider more symmetries, especially symmetries of general coordinate transform, infinitesimally,

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\nu(x) \tag{1.8}$$

the corresponding conserved charge[1] as an infinitesimal transformation generator:

$$Q_\epsilon(\Sigma) = -\int_\Sigma dS_\mu\, \epsilon_\nu(x) T^{\mu\nu}(x) \tag{1.9}$$

Stress energy $T^{\mu\nu}(x)$ is assumed to be conserved, symmetry, and *traceless*[2], then the conservation of Q implies:

$$\partial_\mu(\epsilon_\nu T^{\mu\nu}(x)) = 0$$

$$= \frac{1}{2}\left(\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)\right) \cdot T^{\mu\nu}(x)$$

Then, traceless $T^{\mu\nu}(x)$ leads to conformal Killing equation:

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \eta_{\mu\nu} c(x) \tag{1.10}$$

Constant spacetime translation satisfies the equation and the corresponding conserved charge Q_ϵ is momentum, and we'll discuss more symmetries that are solutions to the conformal Killing equation in the appendix. In terms of $\epsilon(x)$, $c(x) = 2/d\, \partial \cdot \epsilon(x)$. To see how metric transforms, first consider,

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu \epsilon^\mu(x)$$

$$\approx \left(1 + \frac{c(x)}{2}\right)\left(\delta^\mu_\nu + \frac{1}{2}(\partial_\nu \epsilon^\mu - \partial^\mu \epsilon_\nu)\right) \tag{1.11}$$

[1] One might argue why local symmetry can have a conserved charge, however, here we should view them as a special case.

[2] Tracelessness of stress energy is related to conformal symmetry, as we're about to show.

where we've used conformal Killing equation and ignore $\mathcal{O}(\epsilon^2)$ terms. The R.H.S of Eq(1.11) is an infinitesimal rescaling times an infinitesimal rotation. Equivalently, the transformation $x \to x'$ governed by the conformal Killing equation and stress energy rescales the metric by a scalar factor. Such transformations are called *conformal*[3].

2 Conformal Algebra

The conformal symmetry of spacetime is expressed by an extension of the Poincaré group, known as the conformal group. Conformal symmetry encompasses special conformal transformations and dilations and has 15 degrees of freedom in $3+1$ dimensions.

The Lie algebra of the conformal group has the following representation:

$$M_{\mu\nu} \equiv x_\mu \partial_\nu - x_\nu \partial_\mu \tag{2.1}$$

$$P_\mu \equiv \partial_\mu \tag{2.2}$$

$$D \equiv x^\mu \partial_\mu \tag{2.3}$$

$$K_\mu = 2x_\mu (x^\nu \partial_\nu) - x^2 \partial_\mu \tag{2.4}$$

Where $M_{\mu\nu}$ are the Lorentz generators, P_μ generates translations, which are also the generators in the Poincaré group. Also, D generates scaling transformations (also known as dilatations or dilations) and K_μ generates the special conformal transformations. It can be shown that these generators can be determined by the traceless stress tensor $T^{\mu\nu}$. A more rigorous discussion about conformal symmetry, conformal transformation, and how to solve its killing equations will be left in the appendix.

From the definition of generators (2.1)-(2.4), we can then derive their commutation

[3]By conformal, we mean the angles between two curves' intersections are invariant under the transformations, as seen from the metric transformation.

relations, as (2.5)-(2.10), and all other commutators are zero.

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= [(x_\mu \partial_\nu - x_\nu \partial_\mu), \partial_\rho] \\
&= [x_\mu \partial_\nu, \partial_\rho] - [x_\nu \partial_\mu, \partial_\rho] \\
&= -\delta_{\rho\mu} P_\nu + \delta_{\rho\nu} P_\mu.
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
[M_{\mu\nu}, K_\rho] &= [(x_\mu \partial_\nu - x_\nu \partial_\mu), 2x_\rho(x^\sigma \partial_\sigma) - x^2 \partial_\rho] \\
&= 2\big(x_\mu \delta_{\nu\rho} x^\sigma \partial_\sigma - x_\nu \delta_{\mu\rho} x^\sigma \partial_\sigma\big) + \big(x^2 \delta_{\rho\mu} \partial_\nu - x^2 \delta_{\rho\nu} \partial_\mu\big) \\
&= \delta_{\nu\rho}\big(2x_\mu (x \cdot \partial) - x^2 \partial_\mu\big) - \delta_{\mu\rho}\big(2x_\nu (x \cdot \partial) - x^2 \partial_\nu\big) \\
&= \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu.
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = [(x_\mu \partial_\nu - x_\nu \partial_\mu), (x_\rho \partial_\sigma - x_\sigma \partial_\rho)] \\
&= [x_\mu \partial_\nu, x_\rho \partial_\sigma] - [x_\nu \partial_\mu, x_\rho \partial_\sigma] - [x_\mu \partial_\nu, x_\sigma \partial_\rho] + [x_\nu \partial_\mu, x_\sigma \partial_\rho] \\
&= (x_\mu \delta_{\rho\nu} \partial_\sigma - x_\rho \delta_{\sigma\mu} \partial_\nu) - (x_\nu \delta_{\mu\rho} \partial_\sigma - x_\rho \delta_{\sigma\nu} \partial_\mu) \\
&\quad - (x_\mu \delta_{\sigma\nu} \partial_\rho - x_\sigma \delta_{\rho\mu} \partial_\nu) + (x_\nu \delta_{\sigma\mu} \partial_\rho - x_\sigma \delta_{\rho\nu} \partial_\mu) \\
&= \delta_{\rho\nu}(x_\mu \partial_\sigma - x_\sigma \partial_\mu) - \delta_{\mu\rho}(x_\sigma \partial_\nu - x_\nu \partial_\sigma) \\
&\quad + \delta_{\nu\sigma}(x_\rho \partial_\mu - x_\mu \partial_\rho) - \delta_{\mu\sigma}(x_\rho \partial_\nu - x_\nu \partial_\rho) \\
&= \delta_{\rho\nu} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\rho\mu} - \delta_{\mu\sigma} M_{\rho\nu}.
\end{aligned} \tag{2.7}$$

$$[D, P_\mu] = [x_\sigma \partial^\sigma, \partial_\mu] = -\delta_{\sigma\mu} \partial^\sigma = -\partial_\mu = -P_\mu. \tag{2.8}$$

$$
\begin{aligned}
[D, K_\mu] &= [x_\sigma \partial^\sigma, 2x_\mu(x^\nu \partial_\nu) - x^2 \partial_\mu] \\
&= 2[x_\sigma \eta^\sigma_\mu \partial^\sigma x^\nu \partial_\nu + x_\sigma x_\mu \delta^\sigma_\nu \partial^\nu + x_\sigma x_\mu x^\nu \partial^\sigma \partial_\nu \\
&\quad - (x_\mu x^\nu \delta_{\sigma\nu} \partial^\sigma + x_\mu x^\nu x_\sigma \partial^\sigma \partial_\nu)] \\
&\quad - [(2x_\sigma x^\sigma \partial_\mu + x_\sigma x^2 \partial^\sigma \partial_\mu) - (x^2 \delta_{\mu\sigma} \partial^\sigma + x^2 x_\sigma \partial_\mu \partial^\sigma)] \\
&= 2x_\sigma x_\mu \delta^\sigma_\nu \partial^\nu - x^2 \partial_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu = K_\mu.
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
[K_\mu, P_\nu] &= [2x_\mu(x^\sigma \partial_\sigma) - x^2 \partial_\mu, \partial_\nu] \\
&= 2(x_\mu x^\sigma \partial_\sigma \partial_\nu - (x_\mu x^\sigma \partial_\sigma \partial_\nu + \delta_{\mu\nu} x^\sigma \partial_\sigma + x_\mu \partial_\nu)) \\
&\quad - (x^2 \partial_\mu \partial_\nu - (x^2 \partial_\mu \partial_\nu + 2x_\nu \partial_\mu)) \\
&= -2\delta_{\mu\nu} x^\sigma \partial_\sigma - 2x_\mu \partial_\nu + 2x_\nu \partial_\mu = -2\delta_{\mu\nu} D - 2M_{\mu\nu}.
\end{aligned} \tag{2.10}
$$

Note that (2.5)-(2.7) show that P_μ and K_μ transform as vectors under rotation $SO(d)$ and $M_{\mu\nu}$ generates this algebra. (2.8)-(2.10) illustrate that P_μ and K_μ are lowering and raising operators of D, which is similar to $SU(2)$ algebra introduced in the quantum mechanics course.

Formally, we can define $L_{ab} = -L_{ba}$, where indices a, b run from -1 to dimension d. If we define L_{ab} in the way (2.11)-(2.15), then L_{ab} will satisfy the commutation relations of $SO(d+1, 1)$.

$$L_{\mu\nu} = M_{\mu\nu}, \tag{2.11}$$

$$L_{-1,0} = D, \tag{2.12}$$

$$L_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu), \tag{2.13}$$

$$L_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu). \tag{2.14}$$

Therefore, it is better to treat the algebra in space $\mathbb{R}^{1,d+1}$ than $\mathbb{R}^d$, which is the idea of embedding space formalism. In this formalism, we would define another vector X_M in $\mathbb{R}^{1,d+1}$, compared to x_μ in $\mathbb{R}^d$, and X_M should have restrictions (2.15)-(2.16):

$$X^2 = 0, \tag{2.15}$$

$$X^M \mapsto \rho X^M \quad \text{are equivalent to } x^\mu. \tag{2.16}$$

Since (2.15)-(2.16) give two restrictions to X_M, roughly speaking, the space described by X_M has the same dimension as x_μ, and x_μ is "embedded" in X_M.

3 Primary Operator and Correlation Function

Now, we want to act conformal generators onto operators $\mathcal{O}(x)$. We first have (3.1) in Chapter 2:

$$[P^\mu, \mathcal{O}(x)] = \partial^\mu \mathcal{O}(x) \tag{3.1}$$

In a rotationally-invariant QFT, local operators at the origin transform in irreducible representations of the rotation group $SO(d)$ as (3.2):

$$[M_{\mu\nu}, \mathcal{O}(0)] = S_{\mu\nu}\mathcal{O}(0) \tag{3.2}$$

where $S_{\mu\nu}$ are matrices sharing the same algebra as $M_{\mu\nu}$. Also, in a scale-invariant theory, acting D on local operators at the origin will be:

$$[D, \mathcal{O}(0)] = \Delta\mathcal{O}(0) \tag{3.3}$$

where Δ is the dimension of $\mathcal{O}(0)$.

And now, we can discuss the case where these operators act on local operators away from the origin, noting that $\mathcal{O}(x) = e^{x\cdot P}\mathcal{O}(0)$. Also, in our language, the shorthand notation $Q\mathcal{O}$ always stands for $[Q, \mathcal{O}]$, since it just means how topological surface operators act on $\mathcal{O}$.

$$\begin{aligned}
M_{\mu\nu}\mathcal{O}(x) &= M_{\mu\nu}e^{x\cdot P}\mathcal{O}(0) \\
&= e^{x\cdot P}\left(e^{-x\cdot P}M_{\mu\nu}e^{x\cdot P}\right)\mathcal{O}(0) \\
&= e^{x\cdot P}\left(-x_\mu P_\nu + x_\nu P_\mu + M_{\mu\nu}\right)\mathcal{O}(0) \\
&= \left(x_\nu\partial_\mu - x_\mu\partial_\nu + S_{\mu\nu}\right)e^{x\cdot P}\mathcal{O}(0) \\
&= \left(m_{\mu\nu} + S_{\mu\nu}\right)e^{x\cdot P}\mathcal{O}(x).
\end{aligned} \tag{3.4}$$

Here, $S_{\mu\nu}$ is similar to spin, while $m_{\mu\nu}$ is similar to orbital angular momentum in QFT language.

Then, let us discuss K_μ. We know K_μ is the lowering operator of D. From (2.9):

$$\begin{aligned}
DK_\mu\mathcal{O}(0) &= \left([D, K_\mu] + K_\mu D\right)\mathcal{O}(0) \\
&= (\Delta - 1)K_\mu\mathcal{O}(0).
\end{aligned} \tag{3.5}$$

We can repeatedly act K_μ onto $\mathcal{O}(0)$ to lower the dimension, but the dimension should not be unbounded, as the theory is physically sensible. Therefore, there exists $O(0)$ such that:

$$[K_\mu, \mathcal{O}(0)] = 0 \tag{3.6}$$

This is what we call a *primary operator*. Reversely, we can repeatedly act P_μ onto primary operators to get a series of operators $(P_\mu)^n\mathcal{O}(0)$, called *descendant operators*. Building these operators is similar to how we build states in $SU(2)$.

Similar to (3.4), when D acts on operators away from the origin, it becomes:

$$\begin{aligned}[D, \mathcal{O}(x)] &= De^{x \cdot P}\mathcal{O}(0) \\ &= e^{x \cdot P}\left(e^{-x \cdot P} D e^{x \cdot P}\right)\mathcal{O}(0) \\ &= e^{x \cdot P}\left(D + x \cdot [D, P]\right)\mathcal{O}(0) \\ &= e^{x \cdot P}\left(\Delta + x^\mu P_\mu\right)\mathcal{O}(0) \\ &= (\Delta + x^\mu \partial_\mu) e^{x \cdot P}\mathcal{O}(0) \\ &= (x^\mu \partial_\mu + \Delta)\mathcal{O}(x).\end{aligned} \tag{3.7}$$

Also, when K_μ acts on operators away from the origin, it becomes:

$$\begin{aligned}[K_\mu, \mathcal{O}(x)] &= K_\mu e^{x \cdot P}\mathcal{O}(0) \\ &= e^{x \cdot P}\left(e^{-x \cdot P} K_\mu e^{x \cdot P}\right)\mathcal{O}(0) \\ &= e^{x \cdot P}\left(K_\mu + x^\nu [K_\mu, P_\nu]\right)\mathcal{O}(0) \\ &= e^{x \cdot P}\left(K_\mu + 2x^\nu \delta_{\mu\nu} D - 2x^\nu M_{\mu\nu}\right)\mathcal{O}(0) \\ &= \left(2x_\mu \left(x \cdot \partial\right) - x^2 \partial_\mu + 2\Delta x_\mu - 2x^\nu S_{\mu\nu}\right) e^{x \cdot P}\mathcal{O}(0) \\ &= \left(2x_\mu \left(x \cdot \partial\right) - x^2 \partial_\mu + 2\Delta x_\mu - 2x^\nu S_{\mu\nu}\right)\mathcal{O}(x).\end{aligned} \tag{3.8}$$

Now, we could use what we have to determine the correlator. First, here comes the 2-correlator $\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle$. From translation and rotation invariance, we should have:

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = f(|x - y|) \tag{3.9}$$

Note that we are considering scale-invariant boundary conditions, which means:

$$0 = \langle 0|[D, \mathcal{O}_1(x)\mathcal{O}_2(y)]|0\rangle = \langle 0|[D, \mathcal{O}_1(x)]\mathcal{O}_2(y) + \mathcal{O}_1(x)[D, \mathcal{O}_2(y)]|0\rangle \tag{3.10}$$

Using (3.7), we get:

$$0 = \left(x^\mu \partial_\mu + \Delta_1 + y^\mu \partial_\mu + \Delta_2\right) f(|x - y|) \tag{3.11}$$

The solution of (3.11) is:

$$f(|x-y|) = \frac{C}{|x-y|^{\Delta_1+\Delta_2}} \tag{3.12}$$

Where C is a constant.

For the 3-correlator $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$, it should have the form:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{f_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{3.13}$$

where f_{ijk} is a constant.

For the 4-correlator $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$, something nontrivial will emerge. To write it down, we need conformal cross-ratios μ and ν:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \tag{3.14}$$

Notice they share the same index in their numerator and denominator, which is characteristic of cross-ratios.

For future use, we define z and $\bar{z}$ using μ and ν. First, we use conformal transformation to set:

$$x_1 = 0, \quad x_2 = z, \quad x_3 = 1, \quad x_4 \to \infty \tag{3.15}$$

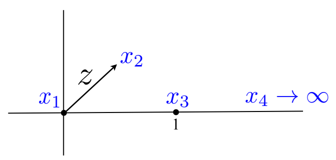


Figure 3: Using conformal transformations in $d = 2$ spacetime, any four points can be placed on a complex plane in the configuration shown above (from [4]).

Where z is the only degree of freedom. Then, we find:

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}) \tag{3.16}$$

Since the 4-correlator is nontrivial in terms of cross-ratios, we here only give the 4-correlator under scalar operators with dimension Δ_ϕ:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \tag{3.17}$$

Although we did not construct 3- and 4-correlators directly, we can check their correctness through the Ward identity.

4 Radial Quantization and Cylindrical Quantization

In a scale-invariant theory, it's natural to foliate spacetime with spheres around the origin and consider evolving states from smaller spheres to larger spheres using the dilation operator D, since this respects the symmetry, and this is called "radial quantization."

In usual quantization, we've learned that the evolution operator U is [6]:

$$U = e^{P^0 \Delta t} \tag{4.1}$$

The states are classified by operators and their momenta and energies:

$$P^\mu |k\rangle = k^\mu |k\rangle \tag{4.2}$$

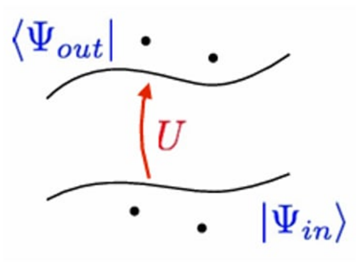


Figure 4: state evolution of normal quantization

But in radial quantization, the evolution operator U will change.

$$U = e^{D\Delta(\log r)} \tag{4.3}$$

While the states are classified by operators D:

$$D|\Delta\rangle = \Delta|\Delta\rangle \tag{4.4}$$

In the course last semester, we've learned about the correlator function in usual quantization:

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \langle 0|T\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\}|0\rangle \tag{4.5}$$

In radial quantization, the correlation function becomes the radially ordered product instead of the time-ordered product:

$$\begin{aligned} \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle &= \langle 0|R\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\}|0\rangle \\ = \theta(|x_n| - |x_{n-1}|)\cdots\theta(|x_2|-|x_1|)\langle 0|&\mathcal{O}(x_n)\cdots\mathcal{O}(x_1)|0\rangle + \text{permutations}. \end{aligned} \tag{4.6}$$

We can also define states through operators. First, let's construct a null state by performing the path integral over the sphere, with no operators inserted inside the sphere. Every operator acting on it becomes zero because all operators can be shrunk to zero within this sphere. Then we can insert an operator $\mathcal{O}$ into the sphere, perform the path

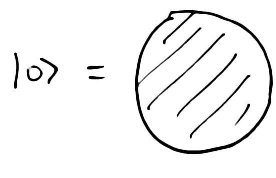


Figure 5: The vacuum in radial quantization is given by the path integral over the interior of the sphere, with no operator insertions.

integral, and define a state $\mathcal{O}|0\rangle$ spanned by field eigenstates:

$$\langle \phi_b|\mathcal{O}(x)|0\rangle = \int D\phi\, \mathcal{O}(x)e^{-S[\phi]}. \tag{4.7}$$

Therefore, roughly speaking, we could identify states as operators in QFT. This is called "state-operator correspondence."

Another way to formulate radial quantization is cylindrical quantization. Let's change our chart from $\mathbb{R}^d$ to $\mathbb{R} \times \mathbb{S}^{d-1}$ by changing $r = e^{\tau}$:

$$\begin{aligned} ds^2_{\mathbb{R}^d} &= dr^2 + r^2 ds^2_{\mathbb{S}^{d-1}} \\ &= r^2 \left(\frac{dr^2}{r^2} + ds^2_{\mathbb{S}^{d-1}} \right) \\ &= e^{2\tau} \left(d\tau^2 + ds^2_{\mathbb{S}^{d-1}} \right) \\ &= e^{2\tau} ds^2_{\mathbb{R}\times\mathbb{S}^{d-1}}. \end{aligned} \tag{4.8}$$

And the evolution operator U will be more natural:

$$U = e^{D\Delta\tau} \tag{4.9}$$

We can then define the operators in cylindrical quantization:

$$\mathcal{O}_{\text{cyl}}(\tau, \hat{n}) \equiv e^{\Delta\tau} \mathcal{O}_{\text{flat}}(x = e^{\tau}\hat{n}). \tag{4.10}$$

The correlators in cylindrical quantization are written as:

$$\begin{aligned} \langle \phi_{\text{cyl}}(\tau_1, \vec{n}_1)\phi_{\text{cyl}}(\tau_2, \vec{n}_2)\cdots\rangle &= f(\tau_i - \tau_j, \{\vec{n}_i\}) \\ &= r_1^{\Delta} r_2^{\Delta} \cdots \langle \phi_{\text{flat}}(r_1, \vec{n}_1)\phi_{\text{flat}}(r_2, \vec{n}_2)\cdots\rangle. \end{aligned} \tag{4.11}$$

which only depends on $\tau_i - \tau_j$ and $\vec{n}_i$.

There is an example for the 2-correlator:

$$\begin{aligned} \langle \mathcal{O}(\tau_1, \vec{n}_1)\mathcal{O}(\tau_2, \vec{n}_2)\rangle &= \frac{C|x_1|^{\Delta}|x_2|^{\Delta}}{|x_1 - x_2|^{2\Delta}} \\ &= \frac{Ce^{\Delta\tau_1}e^{\Delta\tau_2}}{|e^{\tau_1}\vec{n}_1 - e^{\tau_2}\vec{n}_2|^{2\Delta}} \\ &= \frac{Ce^{\Delta(\tau_1-\tau_2)}}{|e^{(\tau_1-\tau_2)}\vec{n}_1 - \vec{n}_2|^{2\Delta}}. \end{aligned} \tag{4.12}$$

Note that (4.12) can be expanded as a series of $e^{\Delta_n(\tau_1-\tau_2)}$, which comes from the exchange of states in the conformal multiple of $\mathcal{O}$.

5 Operator Product Expansion

Two operator products $\mathcal{O}_i(x)\mathcal{O}_j(0)$ can be covered inside a spacetime ball B that can separate them from other operator insertions. Every state on the boundary ∂B can be decomposed as a linear combination of primaries and descendants, then

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle \quad = \quad \sum_k C_{ijk}(x, \hat{P})\mathcal{O}_k(0)|0\rangle, \tag{5.1}$$

where k runs over primary operators and $C_{ijk}(x, \hat{P})$ is an operator, which becomes a coefficient depending on primary operators as the position is taken to be closed to origin $x \to 0$.

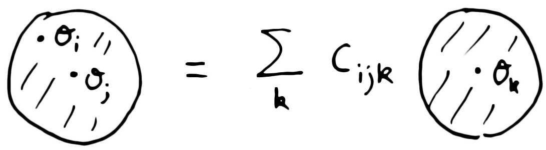


Figure 6: A state created by two operator insertions can be expanded as a sum of primary states with position-dependent coefficient.

Using the state-operator correspondence and promoting the momentum operator $\hat{P} \to \partial$

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \quad = \quad \sum_k C_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2), \qquad \text{(OPE)} \tag{5.2}$$

where it is understood that (5.2) is valid inside any n-point correlation function where the other operator insertions $\mathcal{O}_n(x_n)$ far away from them, $|x_{2n}| \geq |x_{12}|$. Eq. (5.2) is called the Operator Product Expansion (OPE). Operator product expansion can be thought of as an analog to Taylor expansion in calculus, and extract the divergence of infinite fluctuation due to small separation into coefficients, with the remaining part represented by a single operator.

We could alternatively perform an expansion around a different point x_3 that a sphere centered at x_3 encloses x_1 and x_2, giving

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \quad = \quad \sum_k C'_{ijk}(x_{13}, x_{23}, \partial_3)\mathcal{O}_k(x_3), \tag{5.3}$$

where $C'_{ijk}(x_{13}, x_{23}, \partial_3)$ is some other differential operator (figure 7). (5.3) shows that we can do the OPE whenever it's possible to draw any sphere that separates the two operators from all the other operator insertions in spacetime.

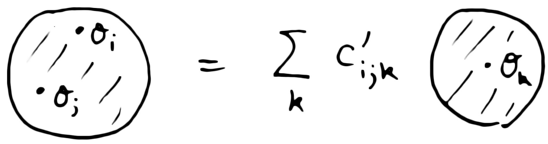


Figure 7: OPE doesn't have to be centered at the origin, and the expansion basis is located at some spacetime point as the dependence of the other operator's position is absorbed by the coefficients.

If the primaries carry spins, the OPE then becomes

$$\mathcal{O}_i^a(x_1)\mathcal{O}_j^b(x_2) \quad = \quad \sum_k C_{ijkc}^{ab}(x_{12}, \partial_2)\mathcal{O}_k^c(x_2), \tag{5.4}$$

where a, b, c are indices for (possibly different) representations of $\mathrm{SO}(d)$.

By acting on both sides of (5.1) with D,

$$\begin{aligned}
\text{RHS} &= (D\mathcal{O}_i(x))\,\mathcal{O}_j(0)|0\rangle - (\mathcal{O}_i(x)D)\,\mathcal{O}_j(0)|0\rangle + \mathcal{O}_i(x)\,(D\mathcal{O}_j(0))\,|0\rangle \\
&\qquad - \mathcal{O}_i(x)\,(\mathcal{O}_j(0)D)\,|0\rangle + \mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle \\
&= \left[D, e^{iP\cdot x}\mathcal{O}_i(0)\right]\mathcal{O}_j(0)|0\rangle + \Delta_j\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle + \mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \sum_k DC_{ijk}(x,\hat{P})\mathcal{O}_k(0)|0\rangle - \sum_k C_{ijk}(x,\hat{P})D\mathcal{O}_k(0)|0\rangle \\
&\qquad + \sum_k C_{ijk}(x,\hat{P})D\mathcal{O}_k(0)|0\rangle \\
&= \sum_k \left[D, C_{ijk}(x,\hat{P})\right]\mathcal{O}_k(0)|0\rangle + \sum_k \Delta_k C_{ijk}(x,\hat{P})\mathcal{O}_k(0)|0\rangle + \mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle
\end{aligned}$$

Then we have a differential equation of coefficient,

$$\begin{aligned}\left[D, C_{ijk}(x,\hat{P})\right] &= \\ (\Delta_i + \Delta_j - \Delta_k) C_{ijk}(x,\hat{P}) + x^\mu \cdot \partial_\mu C_{ijk}(x,\hat{P})\end{aligned} \tag{5.5}$$

The solution is given in order by order $\mathcal{O}(x \cdot \hat{P})$

$$C_{ijk}(x,\partial) \quad \propto \quad |x|^{\Delta_k - \Delta_i - \Delta_j} \left(1 + C_1 x^\mu \partial_\mu + C_2 (x\cdot\partial)^2 + C_3 x^2 \partial^2 + \dots\right) \tag{5.6}$$

5.1 Consistency with Conformal Invariance

By conformal invariance, such as dimensional analysis, rotational invariance, and non-trivially, and the special conformal transform, the form of the coefficient in OPE is strongly restricted.

We get a more interesting constraint by acting with SCT K_μ, for simplicity, suppose $\mathcal{O}_i$, $\mathcal{O}_j$, and $\mathcal{O}_k$ are scalars. Consistency with K_μ completely fixes C_{ijk} up to an overall coefficient so that the coefficients of each term in (5.6) are determined.

Take the correlation function of both sides of (5.2) with a third operator $\mathcal{O}_k(x_3)$ (Assume $|x_{23}| \geq |x_{12}|$, so that the OPE of $\mathcal{O}_i(x_1)$ and $\mathcal{O}_j(x_2)$ is valid),

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle \quad = \quad \sum_{k'} C_{ijk'}(x_{12},\partial_2)\langle \mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle. \tag{5.7}$$

The three-point function on the left-hand side is fixed by conformal invariance, as mentioned in the previous sections. Choosing **an orthonormal basis of primary operators** for the expansion, so that $\langle \mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle = \delta_{kk'} x_{23}^{-2\Delta_k}$. The sum then collapses to a single term, giving

$$\frac{f_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k} x_{23}^{\Delta_j+\Delta_k-\Delta_i} x_{31}^{\Delta_k+\Delta_i-\Delta_j}} \quad = \quad C_{ijk}(x_{12},\partial_2) x_{23}^{-2\Delta_k}. \tag{5.8}$$

This determines that C_{ijk} is proportional to f_{ijk}, times a differential operator that depends only on Δ_i' s. The coefficient operator can be obtained by matching both sides of (5.8) in small expansion $|x_{12}|/|x_{23}|$.

If $\Delta_i = \Delta_j = \Delta_\phi$, and $\Delta_k = \Delta$, then (5.6) becomes

$$C_{ijk}(x,\partial) \quad = \quad f_{ijk} x^{\Delta - 2\Delta_\phi} \left(1 + \frac{1}{2} x \cdot \partial + \alpha x^\mu x^\nu \partial_\mu \partial_\nu + \beta x^2 \partial^2 + \cdots \right) \tag{5.9}$$

5.2 Correlation Function with OPE

Equation (5.7) gives an example of using the OPE to reduce a three-point function to a sum of two-point functions. In general, we can use the OPE to reduce any n-point function to a sum of $n-1$-point functions by the assumption of locality,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle \quad = \quad \sum_k C_{12k}(x_{12},\partial_2)\langle \mathcal{O}_k(x_2)\cdots\mathcal{O}_n(x_n)\rangle . \tag{5.10}$$

Recursing, we can reduce everything to a sum of one-point functions, which are fixed by dimensional analysis,

$$\langle \mathcal{O}(x)\rangle \quad = \quad \begin{cases} 1 & \text{if } \mathcal{O} \text{ is the unit operator,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.11}$$

This gives an algorithm for computing any flat-space correlation function using the OPE. It shows that all these correlation functions are determined by dimensions Δ_i and three-point correlation function coefficients f_{ijk}.

6 Conformal Blocks

6.1 Using the OPE

We can use the OPE to compute a four-point correlation function of identical scalars. Recall that conformal invariance fixes the function up to some arbitrary function that is conformal-invariant

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \quad = \quad \frac{g(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}, \tag{6.1}$$

where the cross-ratios u, v are given previous section.

On the other hand, the OPE of two scalar fields takes the form

$$\phi(x_1)\phi(x_2) \quad = \quad \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} C(x_{12}, \partial_2)\mathcal{O}(x_2), \tag{6.2}$$

where we have extracted the factor f_{ijk} from (5.9) to define $C(x, \partial)$.

We can pair up the operator (12) (34) by assuming their relative positions are separated well to avoid operator insertions crossing, and then we can do the OPE in pairs,[4]

$$\begin{aligned}
&\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \\
&\qquad = \sum_{\mathcal{O},\mathcal{O}'} f_{\phi\phi\mathcal{O}} f_{\phi\phi\mathcal{O}'} C(x_{12}, \partial_2) C(x_{34}, \partial_4)\langle \mathcal{O}(x_2)\mathcal{O}'(x_4)\rangle \\
&\qquad = \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2 C(x_{12}, \partial_2) C(x_{34}, \partial_4)\frac{1}{x_{24}^{2\Delta_\mathcal{O}}} \\
&\qquad = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2 g_{\Delta_\mathcal{O}}(x_i),
\end{aligned} \tag{6.3}$$

In the second line, we use the fact that the correlation function vanishes unless the operators have the same eigenvalue, and

$$g_\Delta(x_{12}, x_{34}, x_{24}) \quad \equiv \quad x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} C(x_{12}, \partial_2) C(x_{34}, \partial_4)\frac{1}{x_{24}^{2\Delta}}. \tag{6.4}$$

Recall that we have chosen an orthonormal basis for the primary operators and used

$$\langle \mathcal{O}(x)\mathcal{O}'(0)\rangle \quad = \quad \frac{\delta_{\mathcal{O}\mathcal{O}'}}{x^{2\Delta_\mathcal{O}}}. \tag{6.5}$$

[4] Although this computation will look like we need $x_{3,4}$ to be sufficiently far from $x_{1,2}$, it in fact will be correct whenever we can draw any sphere separating x_1, x_2 from x_3, x_4, and this is because the beautiful scale invariance in CFT.

The functions $g_\Delta(x_{12}, x_{34})$ are called *conformal blocks*. They are functions of the conformal cross-ratios u, v alone, then the conformal block decomposition

$$g(u,v) \quad = \quad \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} g_{\Delta_{\mathcal{O}}}(u,v). \tag{6.6}$$

Using the differential operator (5.9), (6.4) becomes

$$\begin{aligned} g_\Delta(u,v) &= x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} x_{12}^{\Delta-2\Delta_\phi}\left(1+\cdots\right) x_{34}^{\Delta-2\Delta_\phi}\left(1+\cdots\right)\frac{1}{x_{24}^{2\Delta}} \\ &= \left(\frac{x_{12}x_{34}}{x_{13}x_{24}}\right)^{\Delta} \cdot \frac{x_{13}^{\Delta}}{x_{24}^{\Delta}}\left(1+\cdots\right) \\ &= \quad u^{\Delta/2}\left(1+\cdots\right). \end{aligned} \tag{6.7}$$

6.2 Conformal Block in Radial Quantization

As we mentioned, conformal invariance determines the form of the correlation function, but we also note that the simplest possible conformal invariants are the cross ratios u, v, so the four-point correlation function can be fixed up to a cross-ratio function. This is similar to the spherical symmetric function $f(r)$, in which we cannot tell the dependence on r simply by spherical symmetry.

From this point of view, it is obvious that the conformal block, as the simplest building block for a higher-point correlation function, should solely depend on cross-ratio, as it represents the conformal invariance of the four-point correlation function.

In radial quantization, suppose an origin is chosen such that $|x_{3,4}| \geq |x_{1,2}|$, then

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \quad = \quad \langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle. \tag{6.8}$$

For a primary operator $\mathcal{O}$, let $|\mathcal{O}|$ be the projector onto the conformal multiplet of $\mathcal{O}$,

$$|\mathcal{O}| \equiv \sum_{\alpha,\beta=\mathcal{O},P\mathcal{O},PP\mathcal{O},\ldots} \mathcal{N}^{-1}_{\alpha\beta}|\alpha\rangle\langle\beta| \tag{6.9}$$

where normalization factor $\mathcal{N}_{\alpha\beta} \equiv \langle\alpha|\beta\rangle$. Summing all projects of all primaries, we get the identity

$$\mathbf{1} \quad = \quad \sum_{\mathcal{O}} |\mathcal{O}|. \tag{6.10}$$

Inserting the identity into (6.8) gives

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \quad = \quad \sum_{\mathcal{O}} \langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle. \tag{6.11}$$

Then,

$$\begin{aligned} &\langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle = \\ \sum_{\mathcal{O}',\mathcal{O}''} &\langle 0|\mathcal{R}\{f_{\phi\phi\mathcal{O}'}C(x_{12},\partial_2)\mathcal{O}'(x_2)\}|\mathcal{O}|\mathcal{R}\{f_{\phi\phi\mathcal{O}''}C(x_{34},\partial_4)\mathcal{O}''(x_4)\}|0\rangle \\ &= \frac{f^2_{\phi\phi\mathcal{O}}}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} g_{\Delta_{\mathcal{O}}}(u,v). \end{aligned} \tag{6.12}$$

Each term in the sum is a conformal block times a squared OPE coefficient and some conventional powers of x_{ij}.

This expression makes it clear why g_Δ is a function of cross-ratio u and v: *the projector $|\mathcal{O}|$ commutes with all conformal generators (by construction); therefore, (6.12) above satisfies all the same Ward identities as a four-point function of primaries, and must take the form of a four-point correlation function*. In path integral language, $|\mathcal{O}|$ can be thought of as a surface operator, and we insert it on a sphere separating $x_{1,2}$ from $x_{3,4}$.

6.3 Methods of Conformal Casimir

In [3], the calculation of the conformal block can be done by the differential equation in embedding formalism mentioned previously. Recall that the conformal group is isomorphic to $SO(d+1,1)$, with generators L_{AB} given in the previous section. The usual quadratic Casimir operator $C = -\frac{1}{2}L^{AB}L_{AB}$ acts with the same eigenvalue on every state in an irreducible representation in $SO(d+1,1)$.

This eigenvalue is given by

$$\begin{aligned} C|\mathcal{O}\rangle &= \lambda_\Delta|\mathcal{O}\rangle, \\ \lambda_\Delta &\equiv \Delta(\Delta-d). \end{aligned} \tag{6.13}$$

Casimir operator C gives this same eigenvalue λ_Δ when acting on the projection operator $|\mathcal{O}|$ from left or right,

$$C|\mathcal{O}| = |\mathcal{O}|C = \lambda_\Delta|\mathcal{O}|. \tag{6.14}$$

We've defined X^M to generalize the form of a conformal generator, which can be written as a $SO(d+1,1)$ operator:

$$L_{AB} = X_A\frac{\partial}{\partial X^B} - X_B\frac{\partial}{\partial X^A} \tag{6.15}$$

In quantum mechanics, the commutation relation of momentum operator $\hat{p}$ and wavefunction $\psi(x)$ can be represented by the partial derivative on wavefunction:

$$[\hat{p},\psi(x)] = -i\frac{\partial\psi(x)}{\partial x} \tag{6.16}$$

Similarly, we can promote the conformal generators L_{AB} to some differential operator $\mathcal{L}_{AB,i}$ that gives differential equation of $\phi(x_i)$,

$$\begin{aligned} (\mathcal{L}_{AB,1}+\mathcal{L}_{AB,2})\phi(x_1)\phi(x_2) &= ([L_{AB},\phi(x_1)]\phi(x_2)+\phi(x_1)[L_{AB},\phi(x_2)]) \\ &= [L_{AB},\phi(x_1)\phi(x_2)]\,. \end{aligned} \tag{6.17}$$

Thus,

$$\begin{aligned} [C,\phi(x_1)\phi(x_2)] &= \mathcal{D}_{1,2}\phi(x_1)\phi(x_2), \\ \text{where} \qquad \mathcal{D}_{1,2} &\equiv -\frac{1}{2}(\mathcal{L}_1^{AB}+\mathcal{L}_2^{AB})(\mathcal{L}_{AB,1}+\mathcal{L}_{AB,2}). \end{aligned} \tag{6.18}$$

We then have

$$
\begin{aligned}
&\mathcal{D}_{1,2}\langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle \\
&\qquad = \langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|C\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle \\
&\qquad = \lambda_\Delta\langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle. \qquad (6.19)
\end{aligned}
$$

Plugging in (6.12), we find that g_Δ satisfies the differential equation

$$
\mathcal{D}_{1,2}\frac{f^2_{\phi\phi\mathcal{O}}}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}g_{\Delta_\mathcal{O}}(u,v) \quad = \quad \lambda_{\Delta_\mathcal{O}}\frac{f^2_{\phi\phi\mathcal{O}}}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}g_{\Delta_\mathcal{O}}(u,v) \qquad (6.20)
$$

By transformation of variables, the cross-ratio becomes

$$
\begin{aligned}
u \equiv \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} &= z\bar{z} \\
v \equiv \frac{x_{23}^2x_{14}^2}{x_{13}^2x_{24}^2} &= (1-z)(1-\bar{z})
\end{aligned} \qquad (6.21)
$$

where z is a complex number.

$$
\mathcal{D}g_\Delta(u,v) \quad = \quad \lambda_\Delta g_\Delta(u,v), \qquad (6.22)
$$

where the second-order differential operator $\mathcal{D}$ is given by $\mathcal{D}_{1,2}$

$$
\begin{aligned}
\mathcal{D} \quad = \quad & 2(z^2(1-z)\partial_z^2 - z^2\partial_z) + 2(\bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}) \\
& +2(d-2)\frac{z\bar{z}}{z-\bar{z}}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}). \qquad (6.23)
\end{aligned}
$$

We then determine the conformal block $g_\Delta(u,v)$ by solving the differential equation (6.22). Other methods like recursion relations can also solve the block.

6.4 Series Expansion of Conformal Blocks

As we have shown, we can perform coordinate transformation as long as the conformal invariance is maintained, which allows us to write the conformal block elegantly. In this section, we introduce radial coordinate formalism [4] which manifests the conformal invariance explicitly. Consider $d = 2$ conformal field theory and four spacetime point x_1, x_2, x_3, x_4, we can move x_4 to $(\infty, 0)$ by special conformal transformation (solving $1 - 2(a \cdot x_4) + a^2 x_4^2 = 0$ for a). Then we apply translations in $0-$ and $1-$ components so that x_1 sits in origin, and we can apply rotation transformation and dilatation transformation so that x_3 is located at $(1, 0)$, leaving $x_2 = z$ unfixed.

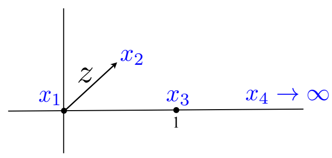


Figure 8: Using conformal transformations in $d = 2$ spacetime, any four points can be placed on a complex plane in the configuration shown above (from [4]).

We have recovered the formula (6.21) for cross-ratio in the previous section. Moreover, we can transform a line segment to a circle by conformal transformation, and one famous example is *fractional linear transformations*. If we consider mapping line segment x_{34} to a unit circle and line segment x_{12} to a circle of radius ρ. It turns out that the mapping is non-trivial than fractional linear transformation. If we consider placing x_i in the following configuration The mapping is

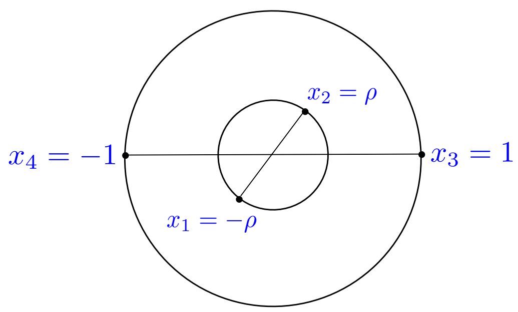


Figure 9: Any four points can be brought to the above configuration using conformal transformations. (from [4].)

$$w = \frac{z}{(1 + \sqrt{1 - z})^2}, \qquad z = \frac{4w}{(1 + w)^2} \tag{6.24}$$

Note that the non-trivial point of this mapping is that $x_1 = (0, 0)$ in Fig.(8) is mapped to origin at first sight, but in fact, we should forget Fig.(8) and view (6.24) as a transformation of z. u and v are invariant under this transformation, so it is conformal:

$$u = \frac{16w\bar{w}}{|w+1|^2| - w - 1|^2} = z\bar{z}$$

$$v = \frac{|w-1|^2| - w + 1|^2}{| - w - 1|^2|w+1|^2} = (1-z)(1-\bar{z})$$

Now we are ready to use this picture to derive the expansion of the conformal block, and now the radially ordered product of the 4-point correlation function is easily realized by demanding $|\rho| < 1$ in Fig.(9).

Since the spacetime points are now placed in a spherical symmetry way, Cylindrical quantization is a natural way to simplify the Euclidean space $\mathbb{R}^d$ to $(0, \infty) \times \mathbb{S}^{d-1}$ and promotes the radial distance as evolution "time" $\tau = -\ln(r)$. In cylindrical quantization, Fig(9) corresponds to placing cylinder operators $\mathcal{O}_{\text{cyl}}(\tau, \mathbf{n}) = \exp(-\tau\Delta) \cdot \mathcal{O}_{\text{rad}}(x = \exp(-\tau)\mathbf{n}$ at diametrically opposite points $\pm\mathbf{n}$ and $\pm\mathbf{n}'$ on $\mathbb{S}^{d-1}$, with $\cos\theta = \mathbf{n} \cdot \mathbf{n}'$, and with the pairs separated by time $\tau = -\ln r$ shown in Fig.(10). If we denote $\mathbf{n}'$ as $\mathcal{O}_1$ and $\mathcal{O}_2$ insertions, $\mathbf{n}$ as $\mathcal{O}_3$ and $\mathcal{O}_4$, and we also define the state with normalization factor from (6.12)

$$|\psi(\mathbf{n})\rangle \equiv \frac{4^{\Delta_\phi}}{f_{\phi\phi\mathcal{O}}} \phi_{\text{cyl}}(0, \mathbf{n}) \phi_{\text{cyl}}(0, -\mathbf{n}) |0\rangle, \tag{6.25}$$

where 4^{Δ_ϕ} is from $x_{12}^{-2\Delta_\phi} = (2|\rho|)^{-2\Delta_\phi}$ then the conformal block is

$$g_\Delta(u, v) = \langle \psi(\mathbf{n}) | |\mathcal{O}| e^{-\tau D} | \psi(\mathbf{n}') \rangle, \tag{6.26}$$

where $\tau = -\ln(1 - |\rho|)$.

The eigenvalue of D for a descendant $P^{\mu_1} \cdots P^{\mu_n} |\mathcal{O}\rangle$ is $\Delta + n$ on the cylinder. Within the n-th level, the $\mathrm{SO}(d)$ spins of a spinless primary are caused by the momentum operator:

$$j \in \{n, n-2, \ldots\}. \tag{6.27}$$

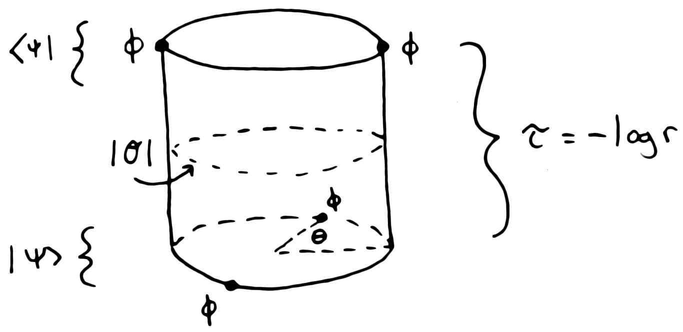


Figure 10: Configuration on the cylinder corresponding to (6.26).

Then we denote the set of descendant and primary states as $|n, j\rangle^{\mu_1\cdots\mu_j}$ with eigenvalue $\Delta+n$ and spin j. The projector $|\mathcal{O}|$ as a summation of primaries and descendants, and (6.26) becomes

$$r^{\Delta+n}\langle\psi(\mathbf{n})|n, j\rangle^{\mu_1\cdots\mu_j}{}_{\mu_1\cdots\mu_j}\langle n, j|\psi(\mathbf{n}')\rangle, \tag{6.28}$$

where $r^{\Delta+n} = (1-|\rho|)^{\Delta+n}$ comes from the operator $\exp\left(-\tau\hat{D}\right)$. By rotational invariance, Recall in the theory of hyperspherical harmonics, the Gegenbauer polynomial provides an orthogonal basis, so (6.28) can be expanded as

$$r^{\Delta+n}\langle\psi(\mathbf{n})|n, j\rangle^{\mu_1\cdots\mu_j}{}_{\mu_1\cdots\mu_j}\langle n, j|\psi(\mathbf{n}')\rangle \quad\propto\quad r^{\Delta+n}C_j^{\frac{d-2}{2}}(\cos\theta), \tag{6.29}$$

where $\cos(\theta) = \mathbf{n}'\cdot\mathbf{n}$

Summing over descendants, we find

$$g_\Delta(u, v) \quad=\quad \sum_{\substack{n=0,2,\ldots\\ j}} B_{n,j}r^{\Delta+n}C_j^{\frac{d-2}{2}}(\cos\theta), \tag{6.30}$$

where j goes over (6.27) and $B_{n,j}$ are constant coefficients. Notice a few properties in [8]:

- The leading term in the r expansion comes from the primary state $|\mathcal{O}\rangle$ with $n = 0$ and $j = 0$. This can be used as a boundary condition in the Casimir equation to determine the higher coefficients $B_{n,j}$.

- The $B_{n,j}$ are positive in a unitary theory because they are given by norms of projections of $|\psi\rangle$ onto dilatation and spin eigenstates.
- The $B_{n,j}$ are rational functions of Δ. This follows because the Casimir eigenvalue λ_Δ is polynomial in Δ, or from the fact that the differential operators $C_a(x,\partial)$ appearing in the OPE (6.2) have a series expansion in x with rational coefficients.

(6.30) shows that spinless blocks are invariant under $x_1 \leftrightarrow x_2$ or $x_3 \leftrightarrow x_4$, where $r = 1-|\rho|$ is invariant and $\cos(\theta) = \mathbf{n}'\cdot\mathbf{n} \to -\cos(\theta)$, but $C_j^{\frac{d-2}{2}}(\cos\theta)$ is symmetric function of $\cos\theta$, and $u \to \frac{u}{v}, v \to \frac{1}{v}$,

$$g_\Delta(u,v) = g_\Delta\left(\frac{u}{v},\frac{1}{v}\right) \tag{6.31}$$

7 Conformal Bootstrap

7.1 OPE Associativity and Crossing Symmetry

OPE determines n-point functions as sums of $(n-1)$-point functions,

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_k(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \sum_k C_{12k}(x_{12},\partial_2)\langle\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle, \tag{7.1}$$

and finally in terms of Δ_i, f_{ijk}, called ”CFT data” as the differential operators $C_{ijk}(x,\partial)$ are determined by conformal symmetry in terms of dimensions Δ_i, spins, and 3-point correlation function coefficients f_{ijk}.

The bootstrap program uses consistent conditions to find out if a provided ”CFT data” builds up a consistent CFT.

Doing the OPE between different pairs of operators in different orders (see Fig 11), we get different expressions for the same correlation function in terms of CFT data. There is no way to prevent them from being the same, it is the consistent conditions on

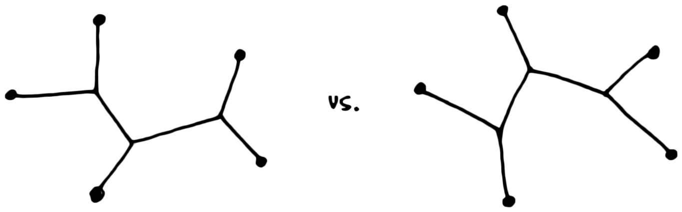


Figure 11: Two different pairings of evaluating a five-point function using the OPE. Dots represent operator insertions, and vertices represent the pairing order.

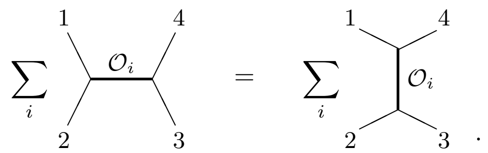
the OPE.

$$\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3 = \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3, \tag{7.2}$$

or more explicitly,

$$C_{12i}(x_{12},\partial_2)C_{i3j}(x_{23},\partial_3)\mathcal{O}_j(x_3) = C_{23i}(x_{23},\partial_3)C_{1ij}(x_{13},\partial_3)\mathcal{O}_j(x_3). \tag{7.3}$$

Consider additional $\mathcal{O}_4(x_4)$ gives the so-called *crossing symmetry equation*



(7.4)

The left-hand side is the conformal block expansion of $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$ in the $12 \leftrightarrow 34$ channel, while the right-hand side is the expansion in the $14 \leftrightarrow 23$ channel. This is just an analog to describe conformal blocks as we do in scattering amplitudes.

7.2 Identical Scalar

Consider the 4-point correlation function of the identical scalar field ϕ.

- We have the OPE

$$\phi(x_1)\phi(x_2) \quad = \quad \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} C(x_{12}, \partial_2)\mathcal{O}(x_2), \tag{7.5}$$

where Δ is the dimension of $\mathcal{O}$.

- An orthonormal basis of operators $\mathcal{O}$ are chosen as we did in (6.5). Unitarity of the operator implies that the three-point coefficients $f_{\phi\phi\mathcal{O}}$ are real.

- Each spinless operator $\mathcal{O}$ satisfies the unitarity bounds

$$\begin{aligned} \Delta &= 0 \text{ (unit operator), or} \\ \Delta &\geq \frac{d-2}{2} \end{aligned} \tag{7.6}$$

- The conformal block expansion for 4-point correlation functions, note the expansion is in $12 \leftrightarrow 34$ channel.

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \quad = \quad \frac{g(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \tag{7.7}$$

$$g(u,v) \quad = \quad \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2 g_\Delta(u,v), \tag{7.8}$$

where $g_\Delta(u,v)$ are conformal blocks, and the cross ratios are

$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = (1-z)(1-\bar{z}) = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \tag{7.9}$$

- Crossing symmetry is equivalent to the condition that the four-point function is invariant under the exchange $1 \leftrightarrow 3$ or $2 \leftrightarrow 4$ and $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$,

$$g(u,v) \quad = \quad \left(\frac{u}{v}\right)^{\Delta_\phi} g(v,u). \tag{7.10}$$

$$g(u,v) \quad = \quad g(\frac{u}{v}, \frac{1}{v}) \tag{7.11}$$

All other permutations can be generated from these.

7.3 An Infinite Number of Primaries

Consider a limit $x_2 \to x_1$, equivalent to $z \to 0$ with $z = \bar{z}$.

Recall that the blocks go like $g_\Delta(u,v) \sim u^{\Delta/2}(1+\cdots) \sim (z\bar{z})^{\Delta/2}$ in this limit, so the left-hand side of (7.10) is dominated by the unit operator, which is the smallest dimension ($\Delta = 0$) operator in the OPE:

$$\text{LHS} \quad : \quad 1 + \ldots \qquad (z \to 0). \tag{7.12}$$

The crossing symmetry $u \leftrightarrow v$ corresponds to $(z,\bar{z}) \to (1-z, 1-\bar{z})$. In the limit $z \to 0$, each conformal block $g_\Delta((1-z)(1-\bar{z}), z\bar{z}) \to \log z$.

Thus, each term on the right-hand side goes like

$$\text{each term, RHS} \quad : \quad |z|^{2\Delta_\phi} \ln|z| + \cdots \qquad (z \to 0). \tag{7.13}$$

As $z \to 0$, any finite sum of terms of the form (7.13) vanishes. Thus, for a sum of operators on the right-hand side to reproduce that on the left-hand side, **an infinite number of primary operators is needed**.

The general feature of the crossing equation (7.4) is that it cannot be satisfied block-by-block. Analyzing different limits of the crossing equation can give other consistent information about the distribution of Δ of primaries, which is called the "CFT spectrum".

7.4 Bounds on CFT Data

The dimension of primaries must be positive, but can we know more about the dimension spectrum of the primaries? The paper [5] derives the lower bound on the dimension of primaries by geometrically studying the crossing equation. If we rewrite the crossing equation (7.10)

$$\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} \underbrace{\left(v^{\Delta_\phi} g_\Delta(u,v) - u^{\Delta_\phi} g_\Delta(v,u)\right)}_{F_\Delta^{\Delta_\phi}(u,v)} \quad = \quad 0. \tag{7.14}$$

$F_\Delta^{\Delta_\phi}(u,v)$ can be viewed as vectors $\vec{F}_\Delta^{\Delta_\phi}$ in the infinite-dimensional vector space of u and v. Because the coefficients $f^2_{\phi\phi\mathcal{O}}$ are positive, so (7.14) is

$$\sum_{\Delta} p_\Delta \vec{F}_\Delta^{\Delta_\phi} \quad = \quad 0, \qquad p_\Delta \geq 0, \tag{7.15}$$

where Δ runs over the dimensions of primaries in the OPE.

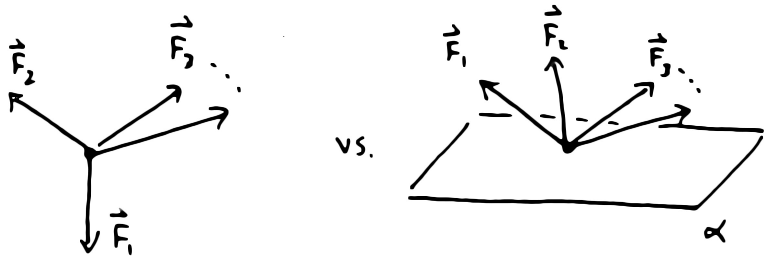


Figure 12: The L.H.S vectors can be summed to zero with positive coefficients. But the R.H.S vectors can't because there is a potential plane α that all vectors are on one of its sides.

Eq. (7.15) says that the sum of a bunch of vectors with positive coefficients vanishes. The L.H.S of Fig. 12 shows a case where it's possible, while the R.H.S shows a case where it's impossible. The situation of R.H.S is equivalent to having a *separating plane* α through the origin such that all the $\vec{F}_\Delta^{\Delta_\phi}$ lie on one side of α. If such α exists, then the $\vec{F}_\Delta^{\Delta_\phi}$ fail to satisfy the crossing equation (7.10), for any choice of positive coefficients $p_\Delta = f^2_{\phi\phi\mathcal{O}}$. The above argument can be straightforwardly implemented into the following algorithm for finding a possible bound for CFT:

---

1. Make a hypothesis for which dimensions Δ appear in the OPE of $\phi\phi$.

2. Search for a linear functional α that is nonnegative acting on all $\vec{F}_{\Delta}^{\Delta_\phi}$ satisfying the hypothesis,

$$\alpha(\vec{F}_{\Delta}^{\Delta_\phi}) \geq 0, \tag{7.16}$$

and strictly positive on at least one operator (usually taken to be the unit operator).

3. If α exists, the hypothesis is wrong. We see this by applying α to both sides of (7.15) and finding a contradiction.

7.5 A 2d CFT Example

Consider a 2d CFT with a real scalar field ϕ with dimension $\Delta_\phi = \frac{1}{8}$. Given (7.15), we have infinite choices of vector set depending on the cross-ratio pair (u, v), and we also need to sum over all dimensions, which are positive running from $0 \to \infty$, and even spin l [5]. We then choose three different vector sets with normalization, and consider their linear combination to form a two-component vector $\vec{v}$:

$$\begin{aligned}
\vec{v}(F) \quad &= \quad \left(H\left(\frac{1}{2},\frac{3}{5}\right) - H\left(\frac{1}{2},\frac{1}{3}\right), H\left(\frac{1}{2},\frac{3}{5}\right) - H\left(\frac{1}{3},\frac{1}{4}\right)\right) \in \mathbb{R}^2, \\
&\text{where} \quad H(z,\bar{z}) = \frac{F_{\Delta,\ell}^{\Delta_\phi}(u,v)}{u^{\Delta_\phi} - v^{\Delta_\phi}}, \\
&\qquad u = z\bar{z}, \\
&\qquad v = (1-z)(1-\bar{z}).
\end{aligned} \tag{7.17}$$

Because it is a linear combination of the three vector sets, the vectors $\vec{v}(F_{\Delta,\ell}^{\Delta_\phi})$ also sum $\sum_{\Delta,\ell}$ to zero with positive coefficients,

$$\sum_{\Delta,\ell} p_{\Delta,\ell}\,\vec{v}(F_{\Delta,\ell}^{\Delta_\phi}) \quad = \quad 0. \tag{7.18}$$

[5] Previously, we didn't discuss primaries with spin, please see [8].

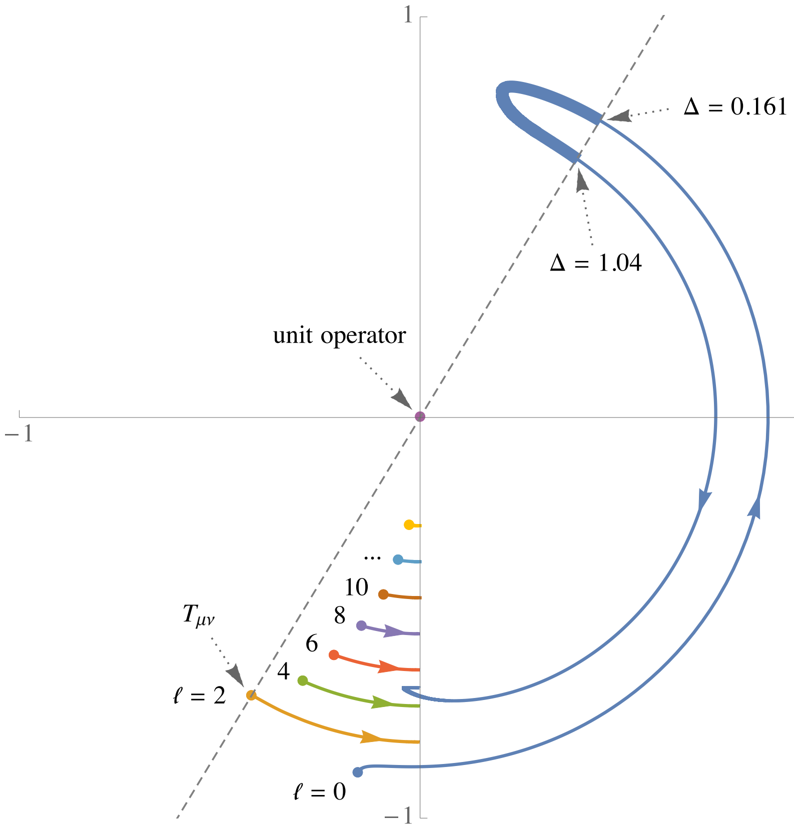


Figure 13: Vectors $\vec{v}(F^{\Delta_\phi}_{\Delta,\ell})$ for all values of Δ, ℓ satisfying the 2d unitarity bound $\Delta \geq \ell$. Dots represent vectors at the unitarity bound $\Delta = \ell$. As Δ varies, $\vec{v}(F^{\Delta_\phi}_{\Delta,\ell})$ the unitarity bound out a curve starting at the dot and approaching the negative y-axis as $\Delta \to \infty$. All vectors are normalized for visual simplicity, except for the unit operator $\vec{v}(F^{\Delta_\phi}_{0,0}) = \vec{0}$. The dashed line splits the figure into two half-spaces with the stress tensor $\vec{v}(F^{\Delta_\phi}_{2,2})$ on the boundary because it is the left-bottom-most.

We plot $\vec{v}(F^{\Delta_\phi}_{\Delta,\ell})$ for all Δ, ℓ satisfying the unitarity bounds (7.6), as shown in Fig. (13). The formula of 2d conformal blocks is given in [3] [6].

As Δ varies from the unitarity bound ℓ to ∞, $\vec{v}(F^{\Delta_\phi}_{\Delta,\ell})$ sweeps out a curve. The curves for higher spin primaries $\ell \geq 2$ are converging quickly at large Δ. The scalar curve circles counterclockwise partway around the origin[7] before coming back and converging as $\Delta \to \infty$.

[6]
$$g^{(2d)}_{\Delta,\ell}(u,v) = k_{\Delta+\ell}(z)k_{\Delta-\ell}(\bar{z}) + k_{\Delta-\ell}(z)k_{\Delta+\ell}(\bar{z})$$

[7] Where the unit operator sits.

We can draw a possible α by observing that the primaries of $l = 2, \Delta = 2$ [8] is the left, bottom-most operator, then we can draw a dashed line between it and the origin, where the unit operator sits. This dashed line is the possible α plane, if our theory doesn't have operators lying on both sides of α, then our theory cannot satisfy (7.15), and thus inconsistent.

The region $\Delta \in [0.161, 1.04]$ of the scalar ($\ell = 0$ curve lies on a different side of α from the other curves. To satisfy (7.18), we must include at least one vector from this region. Thus, we immediately conclude:

In a unitary 2d CFT with a scalar field ϕ of dimension $\Delta_\phi = \frac{1}{8}$, there must exist a scalar in the OPE of $\phi\phi$ with dimension $\Delta \in [0.161, 1.04]$.

7.6 Numerical Techniques

As mentioned in the last section, had we picked a different two-dimensional subspace (7.17), we would have gotten different numbers. Moreover, we can consider higher-dimensional subspaces and derive even stronger bounds. Although possible, it is time-consuming and inefficient to go one by one and see if we get a stronger bound. Instead, we can refine our technique of computation.

The hard part of Algorithm 7.4 is the middle step: finding a functional α such that

$$\alpha(\vec{F}^{\Delta_\phi}_{\Delta,\ell}) \geq 0, \qquad \text{for all } \Delta, \ell \text{ satisfying our hypothesis.} \tag{7.19}$$

We have two immediate difficulties:

1. The space of possible α's is infinite-dimensional.

2. There are an infinite number of positivity constraints (7.19) — one for each Δ, ℓ satisfying our hypothesis. ℓ ranges from 0 to ∞, and Δ varies continuously, besides some know dimensions are discrete.

For the first difficulty, we restrict to a finite-dimensional subspace of α. If we find α in our subspace that satisfies the positivity constraints, we can immediately rule out

[8] Which is stress-energy, see [8].

our hypothesis. If we can't find α, then we can't conclude anything about the spectrum: either no functional exists, or we just weren't searching a big enough subspace of α.

In the example from section 7.5, we restricted α to linear combinations of the components of $\vec{v}(F)$ in (7.17). For numerical computations, we take linear combinations of derivatives of vector sets around the crossing-symmetric point $z = \bar{z} = \frac{1}{2}$,

$$\alpha(F) \quad = \quad \sum_{m+n\leq\Lambda} a_{mn}\partial_z^m \partial_{\bar{z}}^n F(z,\bar{z})|_{z=\bar{z}=\frac{1}{2}}, \tag{7.20}$$

where Λ is cutoff that we restrict the space of α to be finite-dimensional. The functional α is now parameterized by a finite number of coefficients a_{mn}, which a computer can search over. For the second difficulty, we can do a similar thing by adding a cutoff to spins and dimensions, and more detailed techniques are listed in [8].

7.7 Improvment of Computation and Critical Bound

Let's compute the upper bound on the existing lowest-dimension scalar in the OPE of ϕ field [9]. The procedure is as follows

1. Assume a upper bound Δ_0 and that all scalars in the OPE of $\phi\phi$ have dimension $\Delta \geq \Delta_0$.

2. Search for a_{mn} with some cutoff:

$$\sum_{m+n\leq\Lambda} a_{mn}\partial_z^m \partial_{\bar{z}}^n F_{\Delta,\ell}^{\Delta_\phi}(z,\bar{z})|_{z=\bar{z}=\frac{1}{2}} \geq 0,$$

$$\text{for all} \quad \ell = 0, 2, \ldots, \ell_{\max}, \quad \Delta \geq \begin{cases} 0 & \text{(unit operator)}, \\ \Delta_0 & (\ell = 0), \\ \ell + d - 2 & (\ell > 0). \end{cases} \tag{7.21}$$

3. If (7.21) has a solution, then there must exist a scalar primary with dimension below Δ_0.

[9] a $\mathbb{Z}_2$ symmetry is assumed, and ϕ is odd under this symmetry so that ϕ doesn't appear in its own OPE.

The best upper bound of the lowest-dimensional primary is called the critical bound $\Delta_0^{\text{crit.}}$. Above the critical bound (7.21) has a solution and below it, there is no solution.

An implementation of this procedure is in [7]. Running the code for $\Lambda = 6, 8, 12, 16, 20, 28$ gives the bounds shown in figure 14.

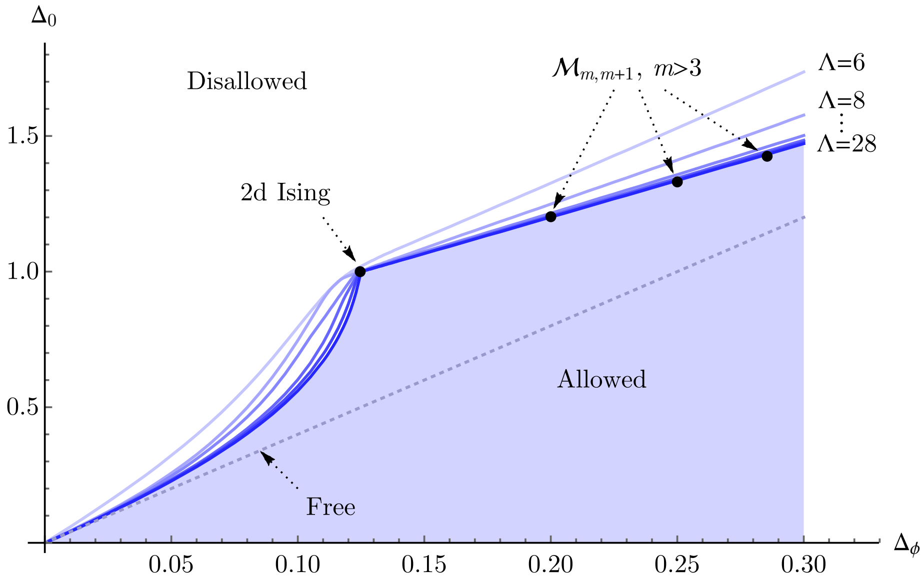


Figure 14: Upper bounds on the dimension Δ_0 of the lowest dimension scalar in the $\phi \times \phi$ OPE as a function of Δ_ϕ, for 2d CFTs with a $\mathbb{Z}_2$ symmetry. The bounds are computed for $\Lambda = 6, 8, 12, 16, 20, 28$, with the strongest bound (darkest blue curve) corresponding to $\Lambda = 28$. The black dots represent the unitary minimal models $\mathcal{M}_{m,m+1}$, and the 2d Ising model is one of them. The dashed line represents the lowest dimension scalar in an OPE of operators $\cos(k\phi)$ in the free boson theory.

As the cutoff Λ on the number of derivatives increases [10], the critical bounds $\Delta_0^{\text{crit.}}(\Delta_\phi)$ become stronger. Remarkably, the strongest bounds are nearly saturated by interesting physical theories. The most obvious feature of Fig. 14 is a *kink* near the location of the 2d Ising model with $(\Delta_\phi, \Delta_0) = (\frac{1}{8}, 1)$, and other exactly soluble unitary minimal models $\mathcal{M}_{m,m+1}$ also lie near the critical bound. The bounds for different Λ at the 2d Ising point $\Delta_\phi = \frac{1}{8}$ are given in table 1.

[10] That is, we increase the dimension of the space of functional α

Table 1: Upper bounds on Δ_ϵ in the 2d Ising model, computed with different cutoffs Λ on the number of derivatives.

Λ	6	8	12	16	20	28
$\Delta_0^{\mathrm{crit.}}(\Delta_\phi = \frac{1}{8})$	1.020	1.0027	1.00053	1.000043	1.0000070	~ 1.0000005

8 Conclusion

From crossing symmetry of 4-correlators and crossing equation, they have bounded the data of CFT. These boundaries of data are the upper bound of dimension that is allowed in $\phi \times \phi$ OPE operators under different spin. Here, they are more concerned about the scalar case. Furthermore, there exists a unitary CFT theory (2d Ising model and unitary minimal models) that saturates these upper bounds. In the 3d Ising case, using different kinds of correlators to calculate gives us a better bound of CFT data.

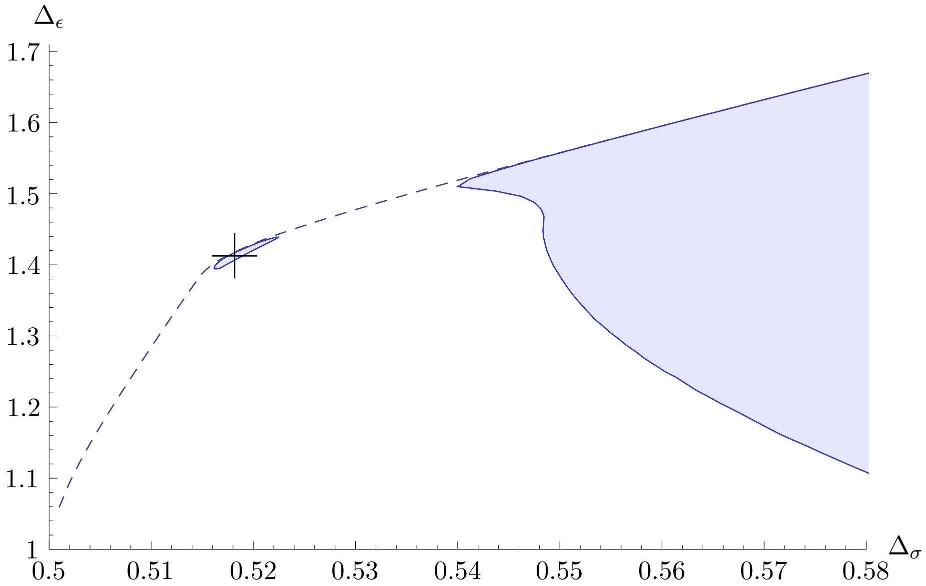


Figure 15: Bound on $(\Delta_\sigma, \Delta_\epsilon)$ in a unitary 3d CFT with a $\mathbb{Z}_2$ symmetry and two relevant scalars σ, ϵ with $\mathbb{Z}_2$ charges $-, +$. The bound comes from the crossing symmetry equations of the 4-point correlation function $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, and is computed with cutoff $\Lambda = 12$. The allowed region is now a small island near the 3d Ising point (black cross).

Appendix

Derivation of Ward-Takahashi Identity for Stress Energy

Recall the definition of stress-energy in QFT with a background metric g:

$$\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_g = \frac{2}{\sqrt{|g(x)|}}\frac{\delta}{\delta g_{\mu\nu}(x)}\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_g$$

For simplicity, we consider only a single operator insertion $\mathcal{O}_1(x_1)$ in Eq(1.3):

$$\frac{2}{\sqrt{|g|}}\frac{\delta}{\delta g_{\mu\nu}}\langle\mathcal{O}_1(x_1)\rangle_g = \frac{2}{\sqrt{|g|}}\frac{\delta}{\delta g_{\mu\nu}}\int\prod_y d\phi(y)\,\mathcal{O}_1(x_1)e^{S[g,\phi]}$$

$$= \frac{2}{\sqrt{|g|}}\int\prod_y d\phi(y)\,\left(\frac{\delta\mathcal{O}_1(x_1)}{\delta g_{\mu\nu}(x)}e^{-S[g,\phi]} - \mathcal{O}_1(x_1)e^{-S[g,\phi]}\frac{\delta S[g,\phi]}{\delta g_{\mu\nu}(x)}\right)$$

Rewrite the operator $\mathcal{O}_1(x_1) = \int d^4y\,\sqrt{|g|}\delta^4(y-x_1)\mathcal{O}_1(y)/\sqrt{|g|}$:

$$\frac{\delta\sqrt{|g|}}{\delta g_{\mu\nu}} = \frac{\delta\exp\left(\frac{1}{2}\operatorname{Tr}\ln(g)\right)}{\delta g_{\mu\nu}}$$

$$= \frac{\sqrt{|g|}}{2}g^{\mu\nu}$$

Then,

$$\frac{\delta\mathcal{O}_1(x_1)}{\delta g_{\mu\nu}(x)} = \int d^4y\,\sqrt{|g|}\delta^4(y-x_1)\mathcal{O}_1(y)\cdot(-\frac{1}{2\sqrt{|g|}}g^{\mu\nu})\delta^4(y-x)$$

$$= -\frac{g^{\mu\nu}(x)}{2}\delta^4(x-x_1)\mathcal{O}_1(x_1)$$

So the Ward-Takahashi identity of stress energy is Eq(1.3) under flat spacetime limit:

$$\partial_\mu\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\rangle = -\partial^\nu\langle\delta^4(x-x_1)\mathcal{O}_1(x_1)\rangle$$

$$= -\delta^4(x-x_1)\partial_1^\nu\langle\mathcal{O}_1(x_1)\rangle$$

We've used integration by parts and disregarded the total derivative term, as the Ward-

Takahashi identity has physical significance only when it is integrated over a spacetime volume, as in Eq(1.5).

Conformal Symmetry

In sec 1.4, solutions of the conformal Killing equation give us special symmetry that keeps metric conformal, and in this section, we'll briefly go over some solutions [2].

General Solutions

Recall conformal Killing equation Eq.(1.10) and we can express $c(x)$ in terms of $\epsilon(x)$, it is possible to further settle down the form of conformal symmetry infinitesimally. Use a similar trick solving the relation of Christoffel symbols and metrics [1], applying an extra derivative on the conformal Killing equation, permuting the indices, and taking a linear combination [2]:

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\mu\rho}\partial_\nu c + \eta_{\nu\rho}\partial_\mu c - \eta_{\mu\nu}\partial_\rho c$$

Contracting the indices μ and ν,

$$2\partial^2\epsilon_\mu = (2-d)\partial_\mu c$$

Additionally, applying ∂_ν on this equation:

$$2\partial_\nu\partial^2\epsilon_\mu = (2-d)\partial_\nu\partial_\mu c$$

compared with Eq.(1.10):

$$\partial_\mu\partial^2\epsilon_\nu + \partial_\nu\partial^2\epsilon_\mu = \eta_{\mu\nu}\partial^2 c$$

Now the L.H.S is $(2-d)\partial_\nu\partial_\mu c$, we have a differential equation of $c(x)$ along:

$$(d-1)\partial^2 c(x) = 0$$

We see that conformal symmetry is related to the dimension! In $d = 1$, there is no restriction on $c(x)$, which means any smooth transformation would be conformal in $1D$ because the angle is meaningless in $1D$. Now if we consider $d \geq 3$, then we must have:

$$\partial^2 c(x) = 0$$

thus function $c(x)$ is at most linear in coordinates:

$$c(x) = A + B_\mu x^\mu = \frac{2}{d}\partial \cdot \epsilon(x)$$

In general, the infinitesimal parameter $\epsilon(x)$ is:

$$\epsilon_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\rho\nu}x^\rho x^\nu$$

where a, b, c are constant, and $c_{\mu\rho\nu} = c_{\mu\nu\rho}$. The constant term a_μ doesn't have any restriction and represents the infinitesimal translation in coordinates. Plug the solution back into the conformal Killing equation, the first-order part gives:

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}b^\rho{}_\rho \eta_{\mu\nu}$$

Therefore, the symmetric part of the coefficient of linear term is proportional to flat spacetime metric (as it should):

$$b_{\mu\nu} = \frac{1}{d}b^\rho{}_\rho \eta_{\mu\nu} + m_{\mu\nu}$$

The symmetric part represents an infinitesimal scale transformation, and the anti-symmetric part is an infinitesimal rotation. Plug in the solution into a second-order differential equation, and the coefficient of the quadratic term can be expressed as:

$$c_{\mu\nu\rho} = \eta_{\mu\rho}\beta_\nu + \eta_{\nu\mu}\beta_\rho - \eta_{\nu\rho}\beta_\mu$$

where $\beta_\mu = c^\sigma{}_{\sigma\mu}/d$, and the corresponding infinitesimal transformation is so-called **special conformal transformation** (SCT).

Dilatation Transformation

We now focus on the dilatation transformation, characterized by:

$$\epsilon_\mu(x) = \frac{b^\rho{}_\rho}{d} x_\mu$$

The corresponding conserved charge, derived from the stress-energy tensor, is given by:

$$Q_\epsilon(\Sigma) = -\int_\Sigma dS_\mu \frac{b^\rho{}_\rho}{d} x_\nu T^{\mu\nu}(x)$$

This suggests a form for the generator that relates to the momentum operator, specifically:

$$d = x^\mu p_\mu$$

Here, the momentum operator typically generates constant translations, but in this context, it acquires an explicit linear dependence on x. Thus, in our formulation, the dilatation transformation is expected to generate:

$$x^\mu \to x'^\mu = x^\mu + \epsilon x^\mu$$

However, this description is not fully rigorous, particularly when considering fields $\phi(x)$ that possess a scaling dimension, as outlined in [2]. Despite this limitation, the relation between the conformal transformation (as defined in Eq. (1.9)) and the momentum operator (associated with spacetime translations) offers valuable intuition, even though a formal derivation would require additional considerations.

Special Conformal Transformations

The generator of special conformal transformations is given by

$$k_\mu = 2x_\mu(x \cdot \partial) - x^2 \partial_\mu$$

To interpret its physical meaning and the transformations it generates, we consider the inversion:

$$I : x^\mu \to x'^\mu = \frac{x^\mu}{x^2}$$

Following our approach in the last section, we assume the generator of special conformal transformations to be related to the momentum operator:

$$k^\mu = -Ip^\mu I$$

so that

$$\begin{aligned} -Ip_\mu I &= \frac{\partial}{\partial x'^\mu} \\ &= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \\ &= -\left(x^2\partial_\mu - 2x_\mu(x\cdot\partial)\right) = k_\mu \end{aligned}$$

where

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{1}{x^4}\left(\delta^\mu_\nu x^2 - 2x^\mu x_\nu\right)$$

Considering the finite transformation $e^{a\cdot k}$ acting on x, the special conformal transformation becomes manifest as a translation preceded and followed by an inversion:

$$x^\mu \to x^\mu_{\text{I}} = \frac{x^\mu}{x^2}$$

$$x^\mu_{\text{I}} \to x^\mu_{\text{trans}} = \frac{x^\mu}{x^2} - a^\mu$$

$$\begin{aligned} x^\mu_{\text{trans}} \to x^\mu_{\text{SCT}} &= \frac{\frac{x^\mu}{x^2} - a^\mu}{\left(\frac{x^\mu}{x^2} - a^\mu\right)^2} \\ &= \frac{x^\mu - a^\mu x^2}{x^2 - 2(a\cdot x) + a^2} \end{aligned}$$

If a^μ is taken to be infinitesimal, this recovers the solution to the conformal Killing equation discussed in the previous section. Notably, the special conformal transformation (SCT) can be viewed as a transformation that shifts infinity while leaving the origin fixed.

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