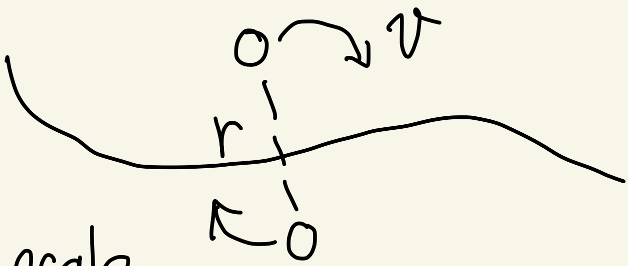
EFT method in binary inspiral

hep-th/049156

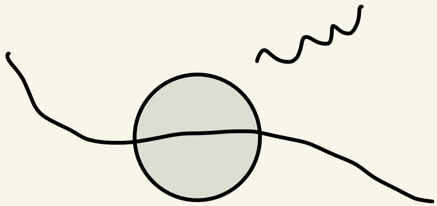
hep-th/1601.04914.

EFT setup



μ

$1/r$ orbital scale



v/r wavelength scale

Core: NR limit $|v| \ll 1$

DoFs:

Plank mass

1. $g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{M}\left(\bar{h}_{\mu\nu} + H_{\mu\nu}\right)$

   gauge symmetry

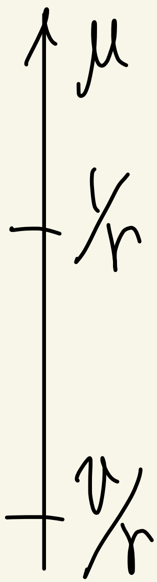
2. worldline cooridnate $x^\mu(\lambda)$

   RPI: $\lambda \to \lambda'$

we choose $\lambda = t$

3. spherical symmetry SO(3)

Spinless p.p.

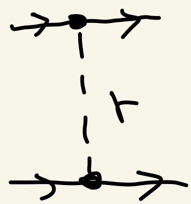


p.p + GR

$\Big\downarrow$ SDH

bound system + $\hbar$

Region of momentum.



$$H: \quad \partial_i H \sim \frac{1}{r} H$$

$$\partial_0 H \sim \frac{v}{r} H$$

NR limit!

$\bar{h}$: $\partial_\mu \bar{h} \sim \frac{v}{r} \bar{h}$ (on-shell)

$$H_{\mu\nu}(x) = \int_{\mathbb{k}} H_{\mu\nu\mathbb{k}}(x^0)\, e^{i\mathbb{k}\cdot\mathbb{x}}$$

$$\partial_i H_{\mu\nu} \sim \int_{\mathbb{k}} -i\mathbb{k}\, H_{\mu\nu\mathbb{k}}(x^0)\, e^{i\mathbb{k}\cdot\mathbb{x}}$$

$$\partial_0 H_{\mu\nu} \sim \int_{\mathbb{k}} \partial_0 H_{\mu\nu\mathbb{k}}(x^0)\, e^{i\mathbb{k}\cdot\mathbb{x}}$$

seperate the mixing power counting

$$S \equiv S_{pp}(x, \bar{h}, H) + S_{EH}(\bar{h} + H)$$

$$+ S_{GF}(\bar{h}, H) + \cancel{S_{ghost}}$$

no quantum effect.

$$G^{\mu} = \bar{D}_{\lambda} H^{\mu\lambda} - \frac{1}{2}\bar{D}^{\mu} H^{\lambda}{}_{\lambda} \quad \text{(harmonic gauge)}$$

$$S_{GF}(\bar{h}, H) = \int d^4x \sqrt{-\bar{g}}\; G^{\mu} G_{\mu}$$

$$\Rightarrow S_{EH}(\bar{h}+H) + S_{GF}(\bar{h}, H)$$

Contribute $\mathcal{L}_{H^2}$ $\qquad S_{H^2} = \int dt\, \mathcal{L}_{H^2}$

$$= -\frac{1}{2}\int_{\mathbb{k}} \Big(\mathbb{k}^2 H_{\mathbb{k},\mu\nu} H^{\mu\nu}_{-\mathbb{k}} - \frac{\mathbb{k}^2}{2} H_{\mathbb{k}} H_{-\mathbb{k}}$$

$$- \partial_0 H^{\mu\nu}_{\mathbb{k}}\, \partial_0 H_{-\mathbb{k}\,\mu\nu} + \frac{1}{2}\partial_0 H_{\mathbb{k}}\, \partial_0 H_{-\mathbb{k}} \Big)$$

Dynamic part viewed as interaction

$$\langle H_{\mathbb{k},\mu\nu}(t_1) H_{\mathbb{q},\alpha\beta}(t_2) \rangle$$ instataneous potential

$$= -(2\pi)^3 \delta^{(3)}(\mathbb{k}+\mathbb{q})\, \delta(t_1-t_2) \frac{i}{\mathbb{k}^2} P_{\mu\nu;\alpha\beta}$$

Polarization $P_{\mu\nu;\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)$

$$\Rightarrow [H_{\mathbb{k}}]^2 = \left(\frac{1}{r}\right)^{-3} \times \left(\frac{r}{v}\right)^{-1} \times \left(\frac{1}{r}\right)^{-2}$$

$$[H_{\mathbb{k}}] = r^2\sqrt{v}$$

$$\langle \bar{h}_{\mu\nu}(x)\, \bar{h}_{\alpha\beta}(y) \rangle$$

$$= \int_k e^{-ik\cdot(x-y)} \cdot \frac{i}{k^2+i\epsilon} P_{\mu\nu;\alpha\beta}$$

$$[\bar{h}]^2 = \left(\frac{v}{r}\right)^4 \times \left(\frac{v}{r}\right)^{-2}$$

$$[\bar{h}] = \frac{v}{r}$$

Perturbation Theory. (no external $\bar{h}$)

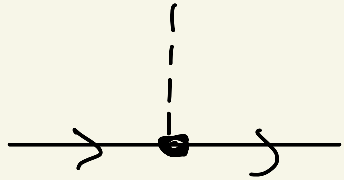
$$S_{pp} = -m\int d\tau \left((\bar{g}_{\mu\nu} + \frac{1}{M} H_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu \right)^{\frac{1}{2}}$$

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}$$

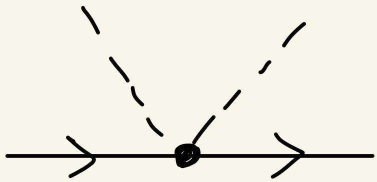
$$d\tau = dt(1-v^2)^{1/2}$$

$$= -m\int dt \cdot \frac{1}{2} v^2 + \frac{m}{M}\int dt \left[-\frac{1}{2} H_{00} + H_{0i} V_i \right.$$

$$-\frac{1}{4}H_{00}v^2 - \frac{1}{2}H_{ij}v^iv^j\Big]$$



$$+\frac{m}{8M^2}\int dt\, H_{00}^2 + \cdots$$



$$e^{iW_{NR}(x,\bar{h})} \equiv \int DH_k\, e^{iS}$$

expansion in v order.

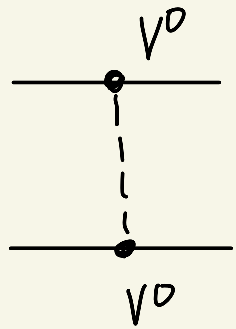
$O(L)$

$$W_{NR} = -\sum_a m_a \int dt\, \frac{1}{2}v_a^2 + \cdots$$

$$\int \mathcal{D}H_{\mathbf{k}} \left(1 + \frac{1}{2}\left(i\sum \frac{m}{M}\int dt \frac{-H_{00}}{2}\right)^2 + \cdots\right) e^{iS'}$$

$$\downarrow$$

$$-\frac{1}{4}\left(\sum \frac{m}{M}\int dt \int_{\mathbf{k}} H^{t}_{\mathbf{k}00}\, e^{i\mathbf{k}\cdot\mathbf{x}}\right)^2$$



$$\Rightarrow i\int dt \frac{G_N m_1 m_2}{|\mathbf{x}_1(t)-\mathbf{x}_2(t)|}$$ Newtonian Potential

$$G_N = \frac{1}{32\pi M^2}$$

$$(\sum \quad)(\sum \quad)$$

$$\frac{1}{r} \sim \int_{\mathbf{k}} \frac{i}{\mathbf{k}^2} e^{i\mathbf{k}\cdot\mathbf{r}}$$ in calculation

$$\xrightarrow{r\to 0} \int_{\mathbf{k}} \frac{i}{\mathbf{k}^2}$$ scaleless $= 0$.

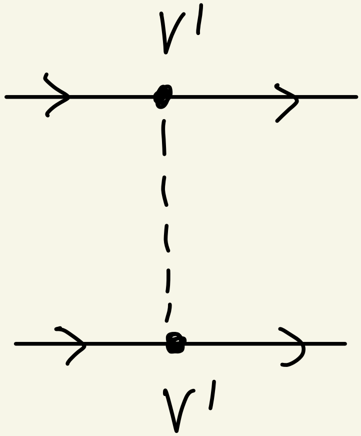
$$W_{NR} = -\sum_a m_a \int dt \, \overbrace{\tfrac{1}{2} v_a^2}^{O(L)} + \int dt \frac{G_N m_1 m_2}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|} \; O(L) + \ldots$$

$$\frac{r}{v} \times \left(\frac{m}{M}\right)^2 \times \frac{1}{r} = L$$

viral thm. (in NR limit) $\frac{G_N m^2}{r} \sim m v^2$

$$\frac{m^2}{M^2} \sim m r v^2 \sim L v$$

NLO : $L v^2$



$$\int DH_{\mathbf{k}}\left(1+\frac{1}{2}\left(i\sum \frac{m}{M}\int dt\, H_{0i}v^i\right)^2+\cdots\right)e^{iS}$$

$$\downarrow$$

$$\left(i\sum\frac{m}{M}\int dt\int_{\mathbf{k}} H^{t}_{0i\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, v^i\right)^2$$

$$= \underset{\substack{\uparrow \\ \text{over counting}}}{-2}\,\frac{m_1 m_2}{M^2}\int dt\ \mathbf{v}_1\cdot\mathbf{v}_2\,\frac{1}{4\pi|\mathbf{x}_1-\mathbf{x}_2|}\times \underset{\substack{\uparrow \\ \text{polarization}}}{\frac{1}{2}}$$

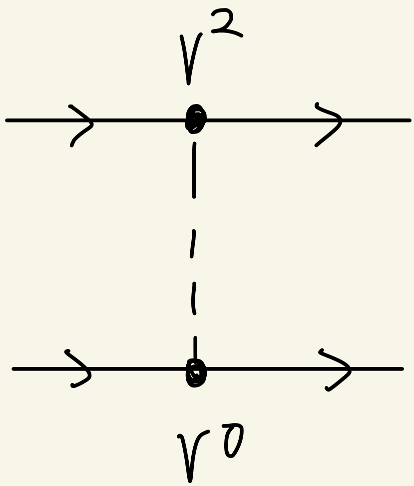
$$\Rightarrow -4i\int dt\,(\mathbf{v}_1\cdot\mathbf{v}_2)\,\frac{G_N m_1 m_2}{|\mathbf{x}_1-\mathbf{x}_2|}$$

$$\overset{\mathcal{O}(L)}{W_{NR} = -\sum_a m_a \int dt \, \frac{1}{2} v_a^2} +$$

$$\int dt \, \frac{G_N m_1 m_2}{|x_1(t) - x_2(t)|} \quad \mathcal{O}(L)$$

$$-4 \int dt \, (v_1 \cdot v_2) \frac{G_N m_1 m_2}{|x_1 - x_2|} \quad \mathcal{O}(Lv^2) \quad + \cdots$$

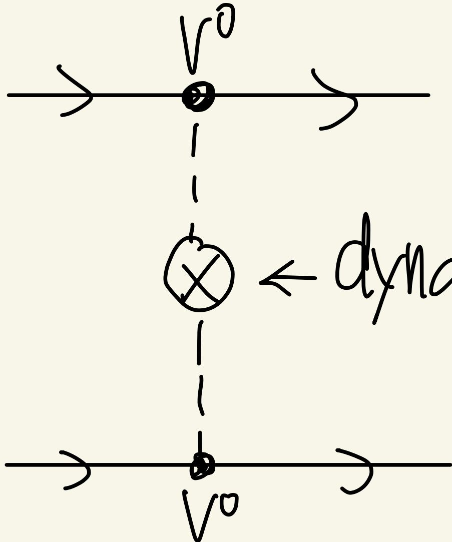
$$\frac{r}{v} \times v^2 \times Lv \times \frac{1}{r} = Lv^2$$



$$\int DH_{kk}\left(1+\left[i\sum\frac{m}{M}\int dt\,-\frac{H_{00}}{2}\right]\left[i\sum\frac{m}{M}\int dt'\left(-\frac{H_{00}}{4}v^2-\frac{H_{ij}}{2}v^iv^j\right)\right]+\cdots\right)e^{iS'}$$

$$\Rightarrow \frac{3i}{2}\int dt\,\frac{G_N m_1 m_2}{|x_1-x_2|}\left(v_1^2+v_2^2\right)$$

again $\mathcal{O}(Lv^2)$



$\leftarrow$ dynamic insertion from $\mathcal{L}_{H^2}$

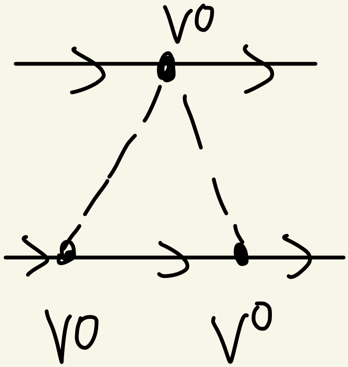
$$\int DH_{\mathbb{k}} \left(1 + \frac{1}{2} \left(i \sum \frac{m}{M} \int dt - \frac{H_{00}}{2} \right)^2 \cdot \right.$$

$$\left(-\frac{i}{2} \int_{\mathbb{k}} \int dt \left(-\partial_0 H^{\mu\nu}_{\mathbb{k}} \partial_0 H^{\mu\nu}_{-\mathbb{k}} + \frac{1}{2} \partial_0 H_{\mathbb{k}} \partial_0 H_{-\mathbb{k}} \right) \right)$$

$$\left. + \cdots \right) e^{iS'}$$

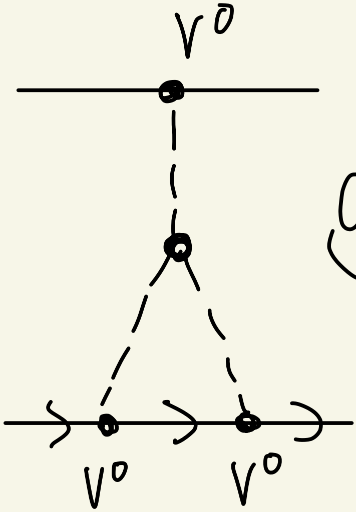
$$\Rightarrow \frac{i}{2} \int dt \frac{G_N m_1 m_2}{|\mathbb{x}_1 - \mathbb{x}_2|} \left(\mathbb{v}_1 \cdot \mathbb{v}_2 - \frac{(\mathbb{v}_1 \cdot \mathbb{x}_{12})(\mathbb{v}_2 \cdot \mathbb{x}_{12})}{|\mathbb{x}_1 - \mathbb{x}_2|^2} \right)$$

$$\mathbb{x}_{12} \equiv \mathbb{x}_1 - \mathbb{x}_2$$



$$\int DH_{kk}\left(1+\frac{1}{2}\left[i\sum\frac{m}{M}\int dt\,\frac{H_{00}}{2}\right]^2 \times \left[i\sum\frac{m}{8M^2}\int dt\,H_{00}^2\right]+\cdots\right)e^{iS'}$$

$$\Rightarrow \frac{i}{2}\int dt\,\frac{G_N^2\,m_1m_2(m_1+m_2)}{|\vec{x}_1-\vec{x}_2|^2}$$



graviton–graviton vertex from S_{EH}

$$\Rightarrow -i\int dt \, \frac{G_N^2 m_1 m_2}{|x_1 - x_2|^2}(m_1 + m_2)$$

$\Rightarrow \mathcal{O}(Lv^2)$ correction to binding energy

L_{EIH} calculated in Post Newtonian Approx.

Angular momentum as loop counting parameter.

1. physical intuition ($\hbar$)

2. rigorous proof $\left(\frac{\hbar}{L}\right)$

$$1 - \mathbb{L} = V - P$$

$$g = \eta + h$$

[p.p - graviton vertex $S_{p.p}$]

$\sim m$

[grav-grav vertex S_{EH}]

$\sim M^2$

[propagator of H] $\sim M^{-2}$

N_p : pp-grav

N_g : grav-grav

$\Rightarrow N_p + N_g = V.$

P : # propagator

[W_{int} w/o radiation] =

$$m^{N_p} \, M^{2N_g} \, M^{-2P} \times \gamma^{N_p + 2N_g - 2P}$$

$$= (mr)^{N_p+2N_g-2P} \times (m/M)^{-2N_g+2P}$$

$$= L^{N_p+2N_g-2P} \, L^{-N_g+P}$$

$$= L^{V-P}$$

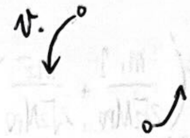
$$\mathbb{L}+V-P=1$$

$$= L^{1-\mathbb{L}}$$

3 scales 1. internal structure scale r_s $\quad \mu = 1/r_s$

2. orbital scale r. $\quad \mu = 1/r$

3. radiation wavelength. r/v $\quad \mu = v/r$



NRGR : DoFs 1. gravitational field. $g_{\mu\nu}(x)$.

2. particle worldline coordinates $x^\mu(\lambda)$

3. SO(3) for spherical symmetry.

Start at orbital scale. $S_{EH} = -2M_{Pl}^{-2} \int d^4x \sqrt{g}\, R(x)$

$$S_{pp} = \sum_{a=1}^{2} \left(m_a \int d\tau_a \right) + \cdots \text{ (deviation of geodesic)}$$

related to finite size eff.

internal structure of black hole

Toy model (scalar field)

$$S = \int d^4x \, \frac{1}{2}(\partial\phi)^2 - \sum_{a=1}^{2} m_a \int d\tau_a \left(1 + \frac{\phi(x_a)}{2\sqrt{2} M_{Pl}} \right)$$

$$= -\sum_a m_a \int d\tau_a + \int d^4x \left(\frac{1}{2}(\partial\phi)^2 + J[\phi]\phi \right), \quad J[x_a] = -\sum_a \int d\tau_a \frac{m_a}{2\sqrt{2} M_{Pl}} \delta^{(4)}(x - x_a)$$

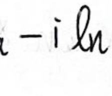
If define $e^{(iW[x])} = \int D\phi \, e^{iS[\phi,x]} = e^{-i(\sum_a m_a \int d\tau_a)} \times \underbrace{\int D\phi \, e^{iS'[\phi,x_a]}}_{Z[J]}$

$W[x] = -\sum_a m_a \int d\tau_a - i \ln(Z[J])$,

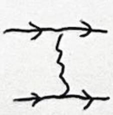
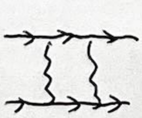
Gaussian integral $Z[J] = \int D\phi \, e^{i\int d^4x (\frac{1}{2}(\partial\phi)^2 + J(x_a)\phi)} \propto \exp\left(\frac{-1}{2} \int d^4x d^4y \, J(x) D(x-y) J(y) \right)$

vertex of graviton and point particle

$\int d^4x \, J(x) e^{-ip\cdot x}$



propagator of graviton $D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon} e^{ik(x-y)}$

$Z[J] =$  +  $+ \cdots = \exp($  $)$

$$W[x] = -\sum_a m_a \int d\tau_a + \frac{i}{2}\int d^4x\, d^4y\, J(x)\, D(x-y)\, J(y)$$

$$= \text{''} + \frac{i}{2}\times \sum_{\substack{a,b\\=1,2}} \frac{m_a m_b}{(2\sqrt{2})^2 M_{Pl}^2} \int d\tau_a\, d\tau_b\, D(x_a - x_b)$$ diverges when $x_a = x_b$.

Potential mode: $k^0 \sim \frac{v}{r}$, $|\mathbb{k}| \sim \frac{1}{r}$ (off-shell)

$\because$ slow varying mode. $k_0^2 = \omega^2 \ll |\mathbb{k}|^2$ corresponds to instantaneous potential.

$$D(x_a - x_b) = \int \frac{d\omega\, d^3\mathbb{k}}{(2\pi)^4} \frac{i}{\omega^2 - \mathbb{k}^2 + i\epsilon} e^{-i\omega(t_a - t_b) + i\mathbb{k}\cdot(\mathbb{x}_a - \mathbb{x}_b)}$$

$$= \int_{\omega,\mathbb{k}} \frac{-i}{\mathbb{k}^2}\left(1 + \frac{\omega^2}{\mathbb{k}^2} + \cdots\right) e^{-i\omega(t_a - t_b) + i\mathbb{k}\cdot(\mathbb{x}_a - \mathbb{x}_b)}$$

The relative scale fixed by $\frac{k^0}{|\mathbb{k}|} \sim E/p \sim v$ (?)

$$\mathrm{Re}(W[x]) = -\sum_a m_a \int dx^0 \left(1 - \frac{1}{2} V_a^2\right) + \frac{1}{2}\sum_{\substack{a,b\\=1,2}} \frac{m_1 m_2}{8 M_{Pl}^2} \int dx_a^0\, dx_b^0 \int_{\omega,\mathbb{k}} \frac{1}{\mathbb{k}^2} e^{-i\omega(t_a - t_b) + i\mathbb{k}\cdot(\mathbb{x}_a - \mathbb{x}_b)}$$

$$= \sum_a m_a \int dx^0 \frac{1}{2} V_a^2 - \frac{1}{2}\int dx^0 \sum_{a,b} \frac{G_N m_a m_b}{|x_a - x_b|} + \cdots$$

$d\tau_a = dx^0 (1 - V_a^2)^{1/2}$ or $d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

Self-interaction becomes scaleless $\int \frac{d^D\mathbb{k}}{(2\pi)^D} e^{-i\mathbb{k}\cdot\mathbb{x}} \frac{1}{\mathbb{k}^2} = 0$. when $\mathbb{x} \to 0$.

3 Real World. $S_{pp} = -m \int d\tau \left((\eta_{\mu\nu} + \frac{1}{M_{Pl}} h_{\mu\nu})\, \dot{x}^\mu \dot{x}^\nu \right)^{\frac{1}{2}}$, $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$

Take $d\tau = dt(1 - V^2)^{1/2}$, $V = \frac{d\mathbb{x}}{dt}$.

$$= -m \int dt \times \frac{1}{2} V^2 + \frac{m}{M_{Pl}} \int dt \left(-\frac{1}{2} H_{00} - \frac{1}{4} H_{00} V_a^2 - \frac{1}{2} H_{ij} V^i V^j + H_{0i} V_i^a \right) + \cdots$$

$$e^{i S_{pp}[x_a, \bar{h}, H] + i S_{EH}[\bar{h} + H] + i S_{GF}[\bar{h}, H] + i S_{ghost}} \equiv e^{iS}$$

$$e^{iW_{NR}[x_a,\bar{h}]} \equiv \int DH_{\mathbb{k},\mu\nu}\, e^{iS}$$

Where $H_{\mathbb{k},\mu\nu}(t)$ by $H_{\mu\nu}(x) \equiv \int \frac{d^3\mathbb{k}}{(2\pi)^3} H_{\mathbb{k},\mu\nu}(t)\, e^{-i\mathbb{k}\cdot\mathbb{x}}$

because $\partial_i H_{\mu\nu}(x) \sim \frac{1}{r} H_{\mu\nu}(x)$, $\partial_0 H_{\mu\nu}(x) \sim \frac{v}{r} H_{\mu\nu}(x)$

Seperate $\mathbb{k}$ and k^0 , $\mathbb{k} \sim \frac{v}{r}$ now explicitly.

$$S_{GF}[\bar{h},H] = \int d^4x \sqrt{-\bar{g}} \left(D_\lambda H^{\mu\lambda} - \frac{1}{2} D^\mu H^\lambda{}_\lambda\right)^2$$

Resulting H^2 term. $\mathcal{L}_{H^2} = -\frac{1}{2}\int_{\mathbb{k}} \left[\mathbb{k}^2 H_{\mathbb{k},\mu\nu} H^{\mu\nu}_{-\mathbb{k}} - \frac{\mathbb{k}^2}{2} H_{\mathbb{k}} H_{-\mathbb{k}} - \partial_0 H^{\mu\nu}_{\mathbb{k}}\, \partial_0 H^{\mu\nu}_{-\mathbb{k}} + \frac{1}{2}\partial_0 H_{\mathbb{k}} \partial_0 H_{-\mathbb{k}}\right]$

First term gives instantaneous propagator of H.

by computing $\langle H_{\mathbb{k},\mu\nu}(t_1) H_{\mathbb{q},\alpha\beta}(t_2)\rangle \equiv \left[\left(-i\frac{\delta}{\delta J^{\mu\nu}_{\mathbb{k}}(t_1)}\right)\left(-i\frac{\delta}{\delta J^{\alpha\beta}_{\mathbb{q}}(t_2)}\right) \int DH\, e^{iS + \int dt \int_{\mathbb{k}} J_{\mathbb{k}\mu\nu} H^{\mu\nu}_{\mathbb{k}}}\right]_{J=0.}$

$$= -(2\pi)^3 \delta^{(3)}(\mathbb{k}+\mathbb{q})\, \delta(t_1-t_2)\, \frac{i}{\mathbb{k}^2} \underbrace{P_{\mu\nu;\alpha\beta}}_{=\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta})} \Rightarrow [H_{\mathbb{k},\mu\nu}(t)] = r^2\sqrt{v}$$

§ Perturbation Theory (Leading order of v)

$$\int DH_{\mathbb{k},\mu\nu}\, e^{iS} = \int DH_{\mathbb{k},\mu\nu}\, e^{-i\sum_a m_a \int dt \frac{1}{2} v_a^2}\, e^{\overbrace{\sum i \frac{m_a}{M_{Pl}} \int dt \left(-\frac{1}{2}H_{00}+\cdots\right)}^{LO} + iS_{GF} + iS_{ghost} + iS_{EH} + S[\bar{h}]}$$

$$= e^{-i\sum_a m_a \int dt \frac{v_a^2}{2}} \int DH_{\mathbb{k},\mu\nu} \left(1 + \frac{1}{2}\left(i\sum_a \frac{m_a}{M_{Pl}} \int dt \frac{-H_{00}}{2}\right)^2 + \cdots\right) e^{iS'}$$

$$\Downarrow$$

$$e^{\sum \frac{im_a}{M_{Pl}} \int dt\, -\frac{1}{2} H_{00}}$$

$$\Rightarrow \int DH_{\mathbb{k}} \left(1 - \frac{1}{2} \left(\sum_a \frac{m_a}{M} \int dt \int_{\mathbb{k}} -\frac{1}{2} H^t_{\mathbb{k}00} e^{-i\mathbb{k}\cdot\mathbb{x}} \right)^2 + \cdots \right) e^{iS'}$$

$$\|$$

$$\frac{1}{4} \sum_a \frac{m_a}{M} \int dt_1 \int_{\mathbb{k}} H^{t_1}_{\mathbb{k}00} e^{-i\mathbb{k}\cdot\mathbb{x}_1} \cdot \sum_b \frac{m_b}{M} \int dt_2 \int_{\mathbb{q}} H^{t_2}_{\mathbb{q}00} e^{-i\mathbb{q}\cdot\mathbb{x}_2}$$

$$= \int DH_{\mathbb{k}} e^{iS'} - \frac{1}{8} \left(\sum_a \frac{m_a}{M} \sum_b \frac{m_b}{M} \int dt_1 dt_2 \int_{\mathbb{k},\mathbb{q}} e^{-i\mathbb{k}\cdot\mathbb{x}_1} e^{-i\mathbb{q}\cdot\mathbb{x}_2} \right) \underbrace{\int DH_{\mathbb{p}} H^{t_1}_{\mathbb{k}00} H^{t_2}_{\mathbb{q}00} e^{iS'}}_{} + \cdots$$

$$\langle H^{t_1}_{\mathbb{k}00} H^{t_2}_{\mathbb{q}00} \rangle$$

$$\|$$

$$-\frac{1}{2} (2\pi)^3 \delta^{(3)}(\mathbb{k}+\mathbb{q}) \delta(t_1 - t_2) \frac{i}{\mathbb{k}^2}$$

$$= \int DH_{\mathbb{k}} e^{iS'} + i \int dt \frac{G_N m_1 m_2}{|\mathbb{x}_1 - \mathbb{x}_2|} + \cdots$$

$$\Rightarrow e^{iW[\bar{h}, x]} = e^{-i \sum m_a \int dt \frac{1}{2} v_a^2} \left(1 + i \int dt \frac{G_N m_1 m_2}{|\mathbb{x}_1 - \mathbb{x}_2|} + \cdots \right)$$

$$= \exp\left(-i \sum_a m_a \int dt \frac{v_a^2}{2} + i \int dt \frac{G_N m_1 m_2}{|\mathbb{x}_1 - \mathbb{x}_2|} \right)$$

§ Power Counting and Angular momentum.

LO : kinematic term. $m \times \frac{r}{v} \times v^2 = mvr = L$

Newtonian potential $\left(\frac{m}{M}\right)^2 \times \left(\frac{r}{v}\right)^2 \times \left(\frac{1}{r}\right)^{3\times 2} \times r^3 \times \frac{v}{r} \times r^2 = Lv \times \frac{r^2}{v^2} \frac{1}{r^6} r^5 \times \frac{v}{r} = L.$

$\frac{m}{M}$ estimated by viral thm. $m v^2 \sim \frac{m^2}{M^2 r} \Rightarrow \frac{m}{M} \sim \sqrt{Lv}$

Absorb $1/M$ into $h \Rightarrow g = \eta + h.$

$S_{pp} = m \int d\tau = m\left(\int dt + \int dt \left(-\frac{1}{2} H + \cdots \right) + \cdots \right)$ every insertion of point particle

have m dimension.

Meanwhile $S_{EH} = M^{+2}\int d^4x\,\sqrt{-g}\,R$, any graviton insertion have M^{+2}

And graviton propagator have M^{-2}

$\Rightarrow$ N_m: insertion of point particle, for each term in perturbation

N_g: " of graviton.

P: " of propagator.

we have $m^{N_m} M^{+2N_g} M^{-2P} r^{N_m+2N_g-2P}$ for dimensionless S_{int}.

define $V = N_m + N_g$, $(mr)^{N_m+2N_g-2P}(m/M)^{-2N_g+2P} \sim L^{N_m+2N_g-2P} L^{-N_g+P} = L^{V-P} = L^{1-\mathbb{L}}$ *

The simplest connected diagram. $\Rightarrow V=2,\ P=1 \ \sim L$.

Adding add. vertex will add at least 1 propagator

$\Rightarrow$ Power counting of $L \sim L^{1-n}$ for $n \geq 0$.

Moreover, we've topological relation. $1-V+P = \mathbb{L}$, $\mathbb{L}$: # of loop.

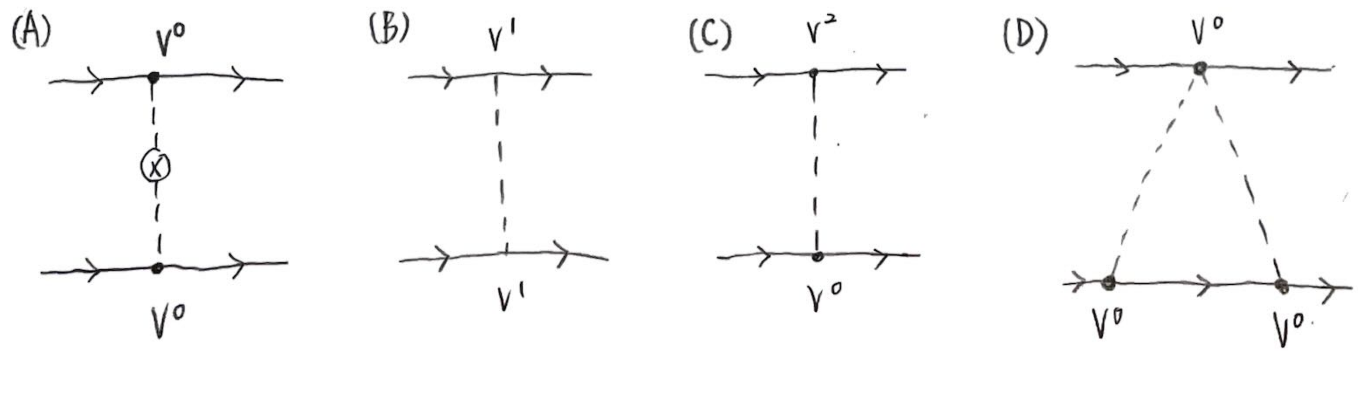
If we rewrite $\hbar$ back $Z[J] \equiv e^{\frac{i}{\hbar}W[J]}$, $W[J] = \sum_{\mathbb{L}} \frac{\hbar^{\mathbb{L}}}{(\mathbb{L})!} W_{\mathbb{L}}$

$$= e^{i\sum_{\mathbb{L}} \frac{1}{\mathbb{L}!}\left(\frac{\hbar}{L}\right)^{\mathbb{L}-1} W_{\mathbb{L}}}$$

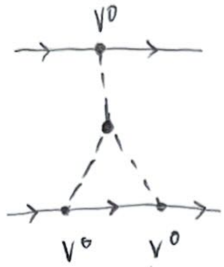
$\frac{\hbar}{L} \ll 1$ is Loop counting parameter.

§ Perturbation theory (NLO)

We can get terms in v^2 in $W[x_a, \hbar]$ by considering connected diagram in potential mode.

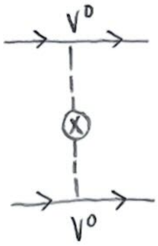


(E)



Note that there cannot be potential mode graviton put on-shell and thus no external potential graviton.

(A)



⊗ represents a kinematic insertion of potential mode

Recall H^2 term $\mathcal{L}_{H^2} = -\frac{1}{2}\int_{\mathbb{k}}\Big[\mathbb{k}^2 H_{\mathbb{k},\mu\nu}H^{\mu\nu}_{-\mathbb{k}} - \frac{\mathbb{k}^2}{2}H_{-\mathbb{k}}H_{\mathbb{k}} \underbrace{-\,\partial_0 H^{\mu\nu}_{\mathbb{k}}\partial_0 H^{\mu\nu}_{-\mathbb{k}} + \frac{1}{2}\partial_0 H_{\mathbb{k}}\partial_0 H_{-\mathbb{k}}}_{\text{kinematic term.}}\Big]$

$$e^{iW[x,\bar{h}]} = \int DH_{\mathbb{k},\mu\nu}\left(1+\frac{1}{2}\left(i\sum_a \frac{m_a}{M}\int dt\,\frac{-H_{00}}{2}\right)^2\left(-\frac{i}{2}\int\!\!\int_{\mathbb{k}} dt'\left(-\partial_0' H^{\mu\nu}_{\mathbb{k}}\partial_0' H^{\mu\nu}_{-\mathbb{k}}+\frac{1}{2}\partial_0' H_{\mathbb{k}}\partial_0' H_{-\mathbb{k}}\right)\right)+\cdots\right)e^{iS'}$$

$$\left(-i\sum_a \frac{m_a}{M}\int dt\int_{\mathbb{p}} \frac{H_{00\mathbb{p}}}{2}\underbrace{}_{e^{-i\mathbb{p}\cdot x_a}}\right)\left(-i\sum_b \frac{m_b}{M}\int dt_2\int_{\mathbb{q}} \frac{H_{00\mathbb{q}}}{2}\underbrace{}_{e^{-i\mathbb{q}\cdot x_b}}\right)\left(+\frac{i}{2}\int_{\mathbb{k}}\int dt'\, H^{\mu\nu}_{\mathbb{k}}P_{\mu\nu;\alpha\beta}\,\partial_0^2 H^{\alpha\beta}_{-\mathbb{k}}\right)$$

$$=+\frac{i}{8}\left(\sum_a \frac{m_a}{M}\int dt_1\int_{\mathbb{p}}\right)\left(\sum_b \frac{m_b}{M}\int dt_2\int_{\mathbb{q}}\right)\left(\int_{\mathbb{k}}\int dt'\, P_{\mu\nu;\alpha\beta}\right)\left(\int DH\left(H_{00\mathbb{p}}H_{00\mathbb{q}}(\partial_0^2 H^{\mu\nu}_{\mathbb{k}})H^{\alpha\beta}_{-\mathbb{k}}\right)e^{iS'}\right)$$

$$\|$$

$$-(2\pi)^3\delta^{(3)}(\mathbb{p}+\mathbb{k})\frac{d^2}{dt'^2}\delta(t_1-t')\frac{i}{\mathbb{k}^2}P_{00,\mu\nu}$$
$$-(2\pi)^3\delta^{(3)}(\mathbb{q}-\mathbb{k})\delta(t_2-t')\frac{i}{\mathbb{k}^2}P_{00,\alpha\beta} \qquad +(\mathbb{p}\leftrightarrow\mathbb{q}).$$

$$=+\frac{i}{8}\left(\sum_a \frac{m_a}{M}\int dt_1\right)\left(\sum_b \frac{m_b}{M}\int dt_2\right)\left(\int_{\mathbb{k}}\int dt'\, \underbrace{P_{\mu\nu;\alpha\beta}P_{00,\mu\nu}P_{00,\alpha\beta}}_{\|}\, e^{-i\mathbb{k}(x_b-x_a)}\frac{d^2}{dt'^2}\delta(t_1-t')\cdot\delta(t_2-t')\frac{1}{\mathbb{k}^4}\right)+(\mathbb{p}\leftrightarrow\mathbb{q})$$

$$=-\frac{i}{8}\left(\sum_a \frac{m_a}{M}\int dt_1\right)\left(\sum_b \frac{m_b}{M}\int dt_2\right)\left(\int_{\mathbb{k}}\int dt'\, e^{-i\mathbb{k}\cdot(x_1-x_2)}\frac{d^2}{dt'^2}\delta(t_1-t')\cdot\delta(t_2-t')\frac{1}{\mathbb{k}^4}\right)+(\mathbb{p}\leftrightarrow\mathbb{q})$$

$$\boxed{\begin{aligned}&\frac{1}{8}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}\right)\left(\eta_{0\mu}\eta_{0\nu}+\eta_{0\nu}\eta_{0\nu}-\eta_{00}\eta_{\mu\nu}\right)\left(\eta_{0\alpha}\eta_{0\beta}+\eta_{0\beta}\eta_{0\alpha}-\eta_{00}\eta_{\alpha\beta}\right)\\ &=\frac{1}{8}(2\cdot 4-16)=-1\end{aligned}}$$

$$=\frac{i}{2}\int dt\,\frac{G_N m_1 m_2}{|x_1-x_2|}\times\left(v_1\cdot v_2-\frac{(v_1\cdot x_{12})(v_2\cdot x_{12})}{|x_1-x_2|^2}\right)$$

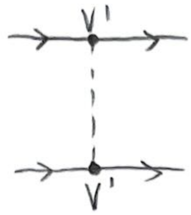
\#

Power counting: $\left(\sqrt{Lv}\times\frac{r}{v}\times\left(\frac{1}{r}\right)^3 r^2\sqrt{v}\right)^2\left(\left(\frac{1}{r}\right)^3\frac{r}{v}r^2\sqrt{v}\left(\frac{v}{r}\right)^2 r^2\sqrt{v}\right)$

$$=L\times\left(\frac{1}{r^3}\frac{r}{v}r^2\frac{v^2}{r^2}r^2 v\right)=Lv^2$$

\#

(B)



from $S_{pp} = -m_a\int dt \times \frac{1}{2} v^2 + \sum_a \frac{m_a}{M}\int dt\left(-\frac{1}{2} H_{00} - \frac{1}{4} H_{00} v_a^2 - \frac{1}{2} H_{ij} v^i v^j + \underline{H_{0i} v_i^a}\right)$

$$\int DH_{k,\mu\nu}\, e^{iS} = e^{-i\sum_a m_a \int dt \frac{v^2}{2} + i\int dt \frac{G_N m_1 m_2}{|x_1 - x_2|}} \int DH_{k,\mu\nu}\left(1+\frac{1}{2}\left(i\sum_a \frac{m_a}{M}\int dt\, H_{0i} v_i^a\right)^2+\cdots\right) e^{iS'}$$

$$= e^{iS_L + iS_A}\int DH_{k,\mu\nu}\left(1+\frac{1}{2}\left(i\sum_a \frac{m_a}{M}\int dt\, H_{0i} v_i^a\right)^2+\cdots\right) e^{iS'}$$

$$-\frac{1}{2}\left(\sum_a \frac{m_a}{M}\int dt_1\, v_i^a\right)\left(\sum_b \frac{m_b}{M}\int dt_2\, v_j^b\right)\left(\int_k\int_q e^{-ik\cdot x_a}\cdots\right)\underbrace{\int DH_{p,\mu\nu}\, H^{t_1}_{k,0i} H^{t_2}_{q,0j}\, e^{iS'}}_{= -(2\pi)^3\delta^{(3)}(k+q)\,\delta(t_1-t_2)\frac{i}{k^2}\underline{P_{0i;0j}}}$$

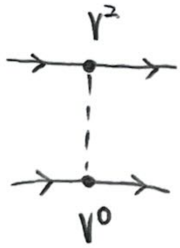
$$\underline{P_{0i;0j}} = \frac{1}{2}(-\delta_{ij})$$

$a=1,\ b=2$

$$= -\frac{1}{2}\times 2\,\frac{m_1 m_2}{M^2}\int dt_1 dt_2\, v_i^1 v_i^2 \left(\frac{1}{2}\right)\int_k e^{-ik(x_1-x_2)}\,\delta(t_1-t_2)\frac{i}{k^2}$$

$$= -\frac{i}{2}\frac{m_1 m_2}{M^2}\int dt\,(v_1\cdot v_2)\frac{1}{4\pi|x_1-x_2|} = -4i\int dt\,(v_1\cdot v_2)\frac{G_N m_1 m_2}{|x_1-x_2|}$$ #

(C)



from S_{pp}, $\sum_a \frac{m_a}{M}\int dt\left(-\frac{1}{4}H_{00} v_a^2\right)$ and $\sum_a \frac{m_a}{M}\int dt\left(-\frac{1}{2}H_{ij}v^i v^j\right)$

$$= e^{iS_L + iS_A + iS_B}\int DH_{k,\mu\nu}\left(1+\left[i\sum_a \frac{m_a}{M}\int dt_1 -\frac{1}{2}H_{00}\right]\left[i\sum_b \frac{m_b}{M}\int dt_2 -\frac{1}{4}H_{00} v_b^2\right]+\cdots\right) e^{iS'}$$

$$\Rightarrow -\frac{1}{8}\left(\sum_a \frac{m_a}{M}\int dt_1\right)\left(\sum_b \frac{m_b}{M}\int dt_2\, v_b^2\right)\left(\int_k\int_q e^{-ik\cdot x_a} e^{-iq\cdot x_b}\right)\underbrace{\int DH_{k,\mu\nu}\, H^{t_1}_{k,00} H^{t_2}_{q,00}\, e^{iS'}}_{-(2\pi)^3\delta^{(3)}(k+q)\,\delta(t_1-t_2)\frac{i}{k^2}P_{00;00}}$$

$$P_{00;00} = \frac{1}{2}$$

$$= +\frac{i}{16}\frac{m_1 m_2}{M^2}\int dt\,(v_1^2+v_2^2)\underbrace{\int_k e^{-ik\cdot(x_1-x_2)}\frac{1}{k^2}}_{\frac{1}{4\pi r}}$$

$$= \frac{i}{2}\int dt\,(v_1^2+v_2^2)\frac{G_N m_1 m_2}{|x_1-x_2|}$$

(C) part 2. $\int DH_{k,\mu\nu}\left(1+\left[i\sum_a \frac{m_a}{M}\int dt_1 -\frac{1}{2}H_{00}\right]\left[i\sum_b\frac{m_b}{M}\int dt_2 -\frac{1}{2} H_{ij}v^iv^j\right]+\cdots\right)e^{iS'}$

$$\Rightarrow -\frac{1}{4}\left(\sum_a\frac{m_a}{M}\int dt_1\right)\left(\sum_b\frac{m_b}{M}\int dt_2\, v_b^iv_b^j\right)\left(\int_k\int_q e^{-ik\cdot x_a}e^{-iq\cdot x_b}\right)\underbrace{\int DH_{p,\mu\nu}H^{t_1}_{k,00}H^{t_2}_{q,ij}\,e^{iS'}}_{-(2\pi)^3\delta^{(3)}(k+q)\,\delta(t_1-t_2)\,\frac{i}{k^2}P_{00;ij}}$$

$$P_{00;ij} = \frac{1}{2}\delta_{ij}$$

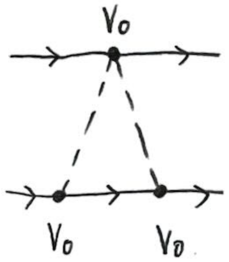
$$= +\frac{i}{4}\frac{m_1m_2}{M^2}\int dt\left(v_1^iv_1^j+v_2^iv_2^j\right)\int_k e^{-ik\cdot(x_1-x_2)}\frac{1}{k^2}\frac{1}{2}\delta_{ij}$$

$$= \frac{i}{4}\frac{m_1m_2}{M^2}\int dt\left(v_1^2+v_2^2\right)\frac{1}{4\pi|x_1-x_2|}$$

$$= i\int dt\,(v_1^2+v_2^2)\frac{G_Nm_1m_2}{|x_1-x_2|}$$

$$(C) = \frac{3i}{2}\int dt\,\frac{G_Nm_1m_2}{|x_1-x_2|}(v_1^2+v_2^2)\ \#$$

(D)



$$S_{pp} = -m\int d\tau\left((\eta_{\mu\nu}+\frac{1}{M}h_{\mu\nu})\dot{x}^\mu\dot{x}^\nu\right)^{\frac{1}{2}}$$

$$= -m\int dt \times\frac{1}{2}v^2 + \frac{m}{M}\int dt\left(-\frac{H_{00}}{2}-\frac{H_{00}}{4}v_a^2-\frac{1}{2}H_{ij}v^iv^j+H_{0i}v_i^a\right)$$

$$+\left(-m\int dt\,(1-v^2)^{1/2}\right)\left(-\frac{1}{8}\right)\frac{1}{M^2}\left(h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)^2+\cdots = +\frac{m}{8M^2}\int dt\,(1-v^2)^{-\frac{3}{2}}\left(h_{00}+h_{ij}v^iv^j-2h_{0i}v^i\right)^2$$

$$\dot{x} = \frac{dx}{dt}\frac{dt}{d\tau} = (1-v^2)^{-\frac{1}{2}}v \qquad = \frac{m}{8M^2}\int dt\left(1+\frac{3}{2}v^2\right)\left(h_{00}^2+\cdots\right)$$

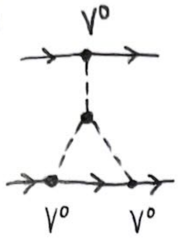
$$\int DH_{k,\mu\nu}e^{iS} = e^{iS_L+iS_A+iS_B+iS_C}\int DH_{k,\mu\nu}\left(1+\frac{1}{2}\left[i\sum_a\frac{m_a}{M}\int dt-\frac{H_{00}}{2}\right]^2\left[i\sum_b\frac{m_b}{8M^2}\int dt_2H_{00}^2\right]+\cdots\right)e^{iS'}$$

$$\Rightarrow -\frac{i}{8}\left(\sum_a\frac{m_a^{(1)}}{M}\int dt_1^{(1)}\right)^2\left(\sum_b\frac{m_b}{8M^2}\int dt_2\right)\left(\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}e^{-ik_1\cdot x_a}e^{-ik_2\cdot x_a'}e^{-i(k_3+k_4)\cdot x_b}\right)$$

$$\left(\int DH_{p,\mu\nu}H^{t_1}_{k_1 00}H^{t_1'}_{k_2 00}H^{t_2}_{k_3 00}H^{t_2}_{k_4 00}e^{iS'}\right)$$

$$= \frac{im_1m_2(m_1+m_2)}{128M^4}\int dt_1dt_1'dt_2\,\delta(t_1-t_2)\delta(t_1-t_1')\left[\int_k\frac{1}{k^2}e^{-ik\cdot(x_1-x_2)}\right]^2$$

$$= \frac{i}{2}\int dt\,\frac{G_N^2m_1m_2(m_1+m_2)}{|x_1-x_2|^2}\quad\#$$

(E) 

from Einstein-Hilbert action.

$$S_{EH} = \int d^4x \sqrt{-g}\, R(x)$$

$$(E) = i\frac{m_1 m_2 (m_1+m_2)}{16 M^3}\int dt_1\, dt_2\, dt_2' \int_{\mathbb{k}_1,\mathbb{k}_2,\mathbb{k}_3} e^{i\sum_i \mathbb{k}_i\cdot \mathbb{x}_i}\left(-\frac{i}{4M}\delta(t_1-t_2)\delta(t_1-t_2')(2\pi)^3\delta(\mathbb{k}_1+\mathbb{k}_2+\mathbb{k}_3)\prod_{r=1}^{3}\frac{i}{\mathbb{k}_r^2}\times\sum_{r=1}^{3}\mathbb{k}_r^2\right)$$

$$= -i\int dt\, \frac{G_N^2 m_1 m_2}{|\mathbb{x}_1-\mathbb{x}_2|^2}(m_1+m_2)$$

Correction of binding energy to $O(Lv^2)$

$$L_{Lv^2} = \underbrace{\frac{1}{8}\sum m_a v_a^4}_{S_{pp,\mathrm{free}}} + \frac{G_N m_1 m_2}{2|\mathbb{x}_1-\mathbb{x}_2|}\left[3(v_1^2+v_2^2) - 7(v_1\cdot v_2) - \frac{(v_1\cdot \mathbb{x}_{12})(v_2\cdot \mathbb{x}_{12})}{|\mathbb{x}_1-\mathbb{x}_2|^2}\right] - \frac{G_N^2 m_1 m_2(m_1+m_2)}{2|\mathbb{x}_1-\mathbb{x}_2|^2}.$$

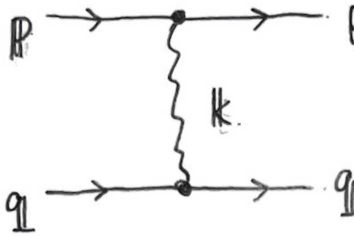
#.

Complement given by 張誌皓

1. Method of region in this model seperate "soft region" into

potential region and radiation region.

Consider two-body scattering (NR)



in soft region, $|\mathbf{k}| \ll |\mathbf{p}|, |\mathbf{q}|$

mom. cons. $\mathbf{p} + \mathbf{q} = \mathbf{p}' + \mathbf{q}'$

$p^{\mu} \sim (mv^2, mv)$

$\Rightarrow |\mathbf{k}| \sim m\delta v$.

enegy cons. $p^0 + q^0 = p'^0 + q'^0$

$k^0 \sim mv\delta v \sim |\mathbf{k}| v$.

2. gauge fixing hamonic gauge $g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = 0$.

$$\left(\eta_{\mu\nu} - \frac{1}{M} h_{\mu\nu}\right) \frac{1}{2}\left(\eta^{\alpha\beta} - \frac{1}{M} h^{\alpha\beta}\right) \frac{1}{M}\left(\frac{\partial h_{\nu\beta}}{\partial x^{\mu}} + \frac{\partial h_{\mu\beta}}{\partial x^{\nu}} - \frac{\partial h_{\mu\nu}}{\partial x^{\beta}}\right) = 0.$$

by Faddeev-Popov method.

$S_{GF}[\bar{h}, H] = \int d^4x \, \frac{1}{2} f^2[\bar{h}, H](x)$, f is gauge fixing.

3. Remaining problem: $d\tau_a = dt\sqrt{1 - v_a^2}$, $v_a \equiv \frac{dx_a}{dt}$

but not $= dt_a \sqrt{1 - v_a^2}$ where $t_1 \neq t_2$.

4. Why

from $-\frac{1}{2} H_{00}$ and $H_{0i} v_i^a$, $P_{00;0i} = \frac{1}{2}\left(\eta_{00}\eta_{0i} + \eta_{0i}\eta_{00} - \eta_{00}\eta_{0i}\right) = 0$.