Brief Introduction to Integrability and Thermodynamic Bethe Ansatz

Ting-Kai Hsu

National Taiwan University,
Taipei, Taiwan

E-mail: b11901097@ntu.edu.tw

ABSTRACT: This final report is mainly based on the early review papers[1, 2]. We presents an introduction to quantum integrability through the example of the $1D$ interacting Bose gas, known as the Lieb–Liniger model. Starting from the two-body scattering problem, we derive the exact S-matrix and illustrate how factorized scattering leads to the Yang–Baxter equation and an infinite set of conserved charges. These structures provide the microscopic origin of integrability in one dimension. We then introduce the thermodynamic Bethe ansatz (TBA) formalism and derive the integral equation governing the pseudo-energy. Finally, we solve the TBA equation numerically and analyze the resulting momentum and energy distributions at finite temperature. The numerical results demonstrate the qualitative effects of interactions and are consistent with physical expectations for $1D$ quantum systems.

Contents

1 Introduction

Integrable quantum systems occupy a special position in theoretical physics. Unlike generic interacting many-body systems, integrable models possess an infinite number of conserved charges, which severely constrain their dynamics and render them exactly solvable. These systems provide rare examples in which nonperturbative phenomena can be analyzed analytically, offering valuable insights into strongly correlated quantum matter. In one spatial dimension, interactions and kinematics conspire to produce particularly rich integrable structures. Because particles cannot pass through one another, scattering processes are highly constrained and often reduce to elastic, factorized two-body collisions. This remarkable simplification underlies the exact solvability of many $1D$ models, including the Lieb–Liniger Bose gas, the Heisenberg spin chain, and relativistic integrable quantum field theories. The Lieb–Liniger model describes bosons interacting via a repulsive contact potential in one dimension. Despite its apparent simplicity, the model exhibits highly nontrivial physics and serves as a paradigmatic example of quantum integrability. Its exact solution, originally obtained by Bethe ansatz techniques, reveals how integrability emerges from the consistency of many-body scattering, encoded in the Yang–Baxter equation. The purpose of this report is twofold. First, we aim to illustrate the physical origin of integrability by deriving the two-body S-matrix and explaining the role of factorized scattering, the Yang–Baxter equation, and infinite conserved charges. Second, we study the thermodynamic properties of the model using the thermodynamic Bethe ansatz. By solving the TBA equations numerically, we explore how interactions modify the momentum and energy distributions at finite temperature and compare the results with the free-boson case.

2 The Lieb-Liniger Model: Interacting Bose Gas

First, let's consider a system of N interacting bosons with a replusive force

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i>j} \delta(x_i - x_j) \tag{2.1}$$

We further assume that the boundary of this system is closed, that is, the $1D$ system is set on a circle of circumference L. In this paper, we set $\hbar = 1$, $k_B = 1$, and the mass of the boson $m = 1/2$.

2.1 Integrability

The interaction between bosons at x_i and x_j is localized at $x_i = x_j$, which allows us to divide the configuration space into N sectors. Within each sector, the Hamiltonian is free Hamiltonian.

For simplicity, let's consider two-boson system, where the positions of bosons are denoted as x_1 and x_2. There are two sectors $x_1 > x_2$ and $x_1 < x_2$, and the solution of Schrodinger equation within each sector are plane-wave solutions

$$\psi(x_1, x_2) = \begin{cases} A_{12}e^{i(k_1 x_1 + k_2 x_2)} + A_{21}e^{i(k_1 x_2 + k_2 x_1)}, & \text{for } x_1 > x_2. \\ B_{12}e^{i(k_1 x_1 + k_2 x_2)} + B_{21}e^{i(k_1 x_2 + k_2 x_1)}, & \text{for } x_1 < x_2. \end{cases} \tag{2.2}$$

The wave-function should be continuous at $x_1 = x_2$, so we have $A_{12} + A_{21} = B_{12} + B_{21}$. At $x_1 = x_2$, there is a scattering of p_1 and p_2 due to the contact interaction. Consider a change of variables $X \equiv (x_1 + x_2)/2$, $r = x_1 - x_2$, and the Schrodinger equation becomes

$$\left[-\left(\frac{1}{2}\partial_X^2 + 2\partial_r^2 \right) + 2c\delta(r) \right] \psi(X, r) = E\psi(X, r). \tag{2.3}$$

Consider integration around $r = 0$, we obtain another condition

$$\int_{-\epsilon}^{\epsilon} dr \hat{H}\psi(X, r) - E\psi(X, r) = 0,$$

$$\left[(\partial_1 - \partial_2)\psi(x_1, x_2) \right]_{x_1 = x_2^+} - \left[(\partial_1 - \partial_2)\psi(x_1, x_2) \right]_{x_1 = x_2^-} = 2c\psi(x_2, x_2). \tag{2.4}$$

When crossing the boundary $x_1 = x_2$, the ordering of particle coordinates is exchanged. Since in $1D$ the worldlines cannot pass through each other, this crossing corresponds to an exchange of the momenta carried by the two worldlines. Consequently, the amplitudes in adjacent sectors are related by $A_{12} = B_{21}$, $A_{21} = B_{12}$. We define the two-body S-matrix element as the ratio of amplitudes A_{21} and A_{12}, which can be derived from the continuous condition and Eq. (2.4)

$$S(k_1 - k_2) \equiv \frac{A_{21}}{A_{12}} = \frac{k_1 - k_2 - ic}{k_1 - k_2 + ic}. \tag{2.5}$$

An important observation is that the S-matrix element only depends on the difference of momenta. In $1D$ system, because of the conservation of energy and momentum, the scattering process corresponds to a permutation of particle momenta. Moreover, because the interaction in this model is localized at $x_i = x_j$, any many-body scattering process would decompose into a sequence of two-body scattering. The factorization property has profound consequences for many-body scattering. Consider three particles undergoing a sequence of pairwise collisions. Since the final state must be independent of the order in which these binary collisions occur, the S-matrix must satisfy a consistency condition

$$S(p_1 - p_2)S(p_1 - p_3)S(p_2 - p_3) = S(p_2 - p_3)S(p_1 - p_3)S(p_1 - p_2) \qquad (2.6)$$

This consistency condition is known as the Yang–Baxter equation and ensures that the outcome of a three-body scattering is independent of the order in which two-body scatterings occur. It guarantees that many-body scattering processes are well-defined and independent of the order in which two-body scatterings occur. Physically, the Yang–Baxter equation expresses the associativity of many-body scattering: it guarantees that asymptotic many-body states are well-defined and that no dynamical ambiguity arises when particles undergo successive collisions. A direct consequence of factorized and elastic scattering is the existence of infinitely many conserved charges. Since scattering only permutes the set of particle momenta, any symmetric function of $\{p_j\}$ is conserved. In particular, one can define an infinite family of conserved quantities

$$Q_n = \sum_{j=1}^{N} p_j^n. \qquad (2.7)$$

Here n is an integer, and Q_1 and Q_2 correspond to the total momentum and energy. These conserved quantities commute with the Hamiltonian and among themselves, reflecting the highly constrained dynamics of integrable systems. As a result, the phase space of the system is strongly constrained, preventing conventional thermalization and underlying the emergence of generalized equilibrium descriptions.

From this simple example we can summarize some properties of integrable systems:

- factorized scatterings

- no particle production in scattering processes

- infinite number of conserved charges

2.2 Thermodynamic Bethe Ansatz Approach

The thermodynamic Bethe ansatz, which depends on the boundary condition we set, provides a framework to describe the macroscopic properties of the system. First,

the closed boundary condition indicates that the summation of phase-shift should be unity, and hence we have the following *Bethe* equations for N-bosons system

$$e^{ip_j L} \prod_{k \neq j}^{N} S(p_j - p_k) = 1 \tag{2.8}$$

Here L is the circumference. Taking the logarithm of this equation

$$p_j L - i \sum_{k \neq j}^{N} \log S(p_j - p_k) = 2\pi n_j \tag{2.9}$$

Here n_j is an integer. From this equation, we can define the *counting function* $C(p)$, which counts the number of allowed momentum states below p

$$LC(p) = \frac{L}{2\pi} p + \frac{1}{2\pi i} \sum_k \log S(p - p_k). \tag{2.10}$$

Here p is a continuous variable, and the counting function satisfies the quantization condition $LC(p_j) = n_j$. Its derivative gives the density of quantum states in momentum space.

The collection of bosons can be characterized by a set of quantum number $\{n_j\}$. Any allowed quantum number that is not in this set, $m \notin \{n_j\}$, is called a *hole*. Let the densities of the particles and holes in the momentum space

$$L\rho(p) = \frac{\#\text{of particles}}{\Delta p}, \quad L\bar{\rho}(p) = \frac{\#\text{ of holes}}{\Delta p}. \tag{2.11}$$

The densities of particles and holes are not linear-independent; in the large N limit (thermodynamic limit), we have

$$\rho(p) + \bar{\rho}(p) = \frac{dC(p)}{dp}. \tag{2.12}$$

In this sense, the summation in the counting function can be written as integration

$$C(p) = \frac{p}{2\pi} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} dp' \rho(p') \log S(p - p'). \tag{2.13}$$

Therefore, the condition becomes

$$\rho(p) + \bar{\rho}(p) = \frac{1}{2\pi} + \frac{1}{2\pi i} K * \rho(p) \tag{2.14}$$

Here the kernel $K(q) = \frac{d}{dq} \log S(q)$ and $*$ is the convolution: $K * \rho(p) \equiv \int_{-\infty}^{\infty} dp' K(p - p') \rho(p')$. In this model with temperature T, the free energy is

$$F(T) \equiv U - TS = \int_{-\infty}^{\infty} dp \left\{ E\rho(p) - T \left[\frac{dC(p)}{dp} \log \left(\frac{dC(p)}{dp} \right) - \rho \log \rho - \bar{\rho} \log \bar{\rho} \right] \right\} \tag{2.15}$$

Here the free-boson energy $E=p^2$. The thermodynamic equilibrium is reached by the condition that the free energy is extremized with respect to the particle and hole densities $\rho(p)$ and $\bar{\rho}(p)$. The variation of the densities is subjected to the condition that the total density of states is fixed

$$\delta\bar{\rho}(p) = -\delta\rho(p) + \frac{1}{2\pi i}K * \delta\rho(p) \tag{2.16}$$

Thus the variantion of the free energy

$$\delta F = \int_{-\infty}^{\infty} dp \left[E\delta\rho(p) - T\left(\delta\rho\log\left(\frac{\rho+\bar{\rho}}{\rho}\right) + \delta\bar{\rho}\log\left(\frac{\rho+\bar{\rho}}{\bar{\rho}}\right)\right)\right],$$
$$= \int_{-\infty}^{\infty} dp\,\delta\rho(p)\left[E - T\left(\log\left(\frac{\bar{\rho}}{\rho}\right) + \log\left(1+\frac{\rho}{\bar{\rho}}\right)\tilde{*}K\right)\right] \tag{2.17}$$

Here we have changed the integral order of the last term and define the operation $\tilde{*}$: $f\tilde{*}K \equiv \int_{-\infty}^{\infty} dp' f(p')K(p'-p)$. Next, we observe that the condition of thermodynamic equilibrium $\delta F=0$ becomes a equation of $\bar{\rho}/\rho$, which is the definition of pseudo-energy $\epsilon(p)$

$$\epsilon(p) \equiv T\log\left(\frac{\bar{\rho}}{\rho}(p)\right), \tag{2.18}$$

and then the thermodynamic equilibrium becomes the thermodynamic Bethe ansatz equation

$$\epsilon(p) = E(p) - T\log\left(1 + e^{-\epsilon/T}\tilde{*}K\right). \tag{2.19}$$

This equation can be solved by numerically iteration, which we introduce in the next subsection.

2.3 Numerical Simulation

To solve the thermodynamic Bethe ansatz equation by iterations, we start with a guess of pseudo-energy $\epsilon_0(p)=E(p)$, and use this initial condition to compute the R.H.S.

$$\epsilon_{n+1}(p) = E(p) - T\log\left(1 + e^{-\epsilon_n/T}\tilde{*}K\right). \tag{2.20}$$

The numerical solution requires several technical choices. We set a finite momentum cutoff to make the integration domain tractable, introduce a small chemical potential to regulate the infrared divergence in the free case, and use damping to stabilize the iterative convolution. Specifically:

- chemical potential $=0.05$

- momentum cutoff $=6$

- damping $=0.1$, which is added on numerical convolution

- tolerance $=10^{-5}$

- maximum of iterations $=1000$

With temperature T set to 100 and the interaction strength c set to 10, let's plot the numerical results of energy distribution and momentum distribution for interacting and free bosons. In the free-boson case, the momentum distribution diverges as $p\to0$,

(a) Momentum distribution $\rho(p)$ in free-boson system.

(b) Momentum distribution $\rho(p)$ in interacting-boson system.
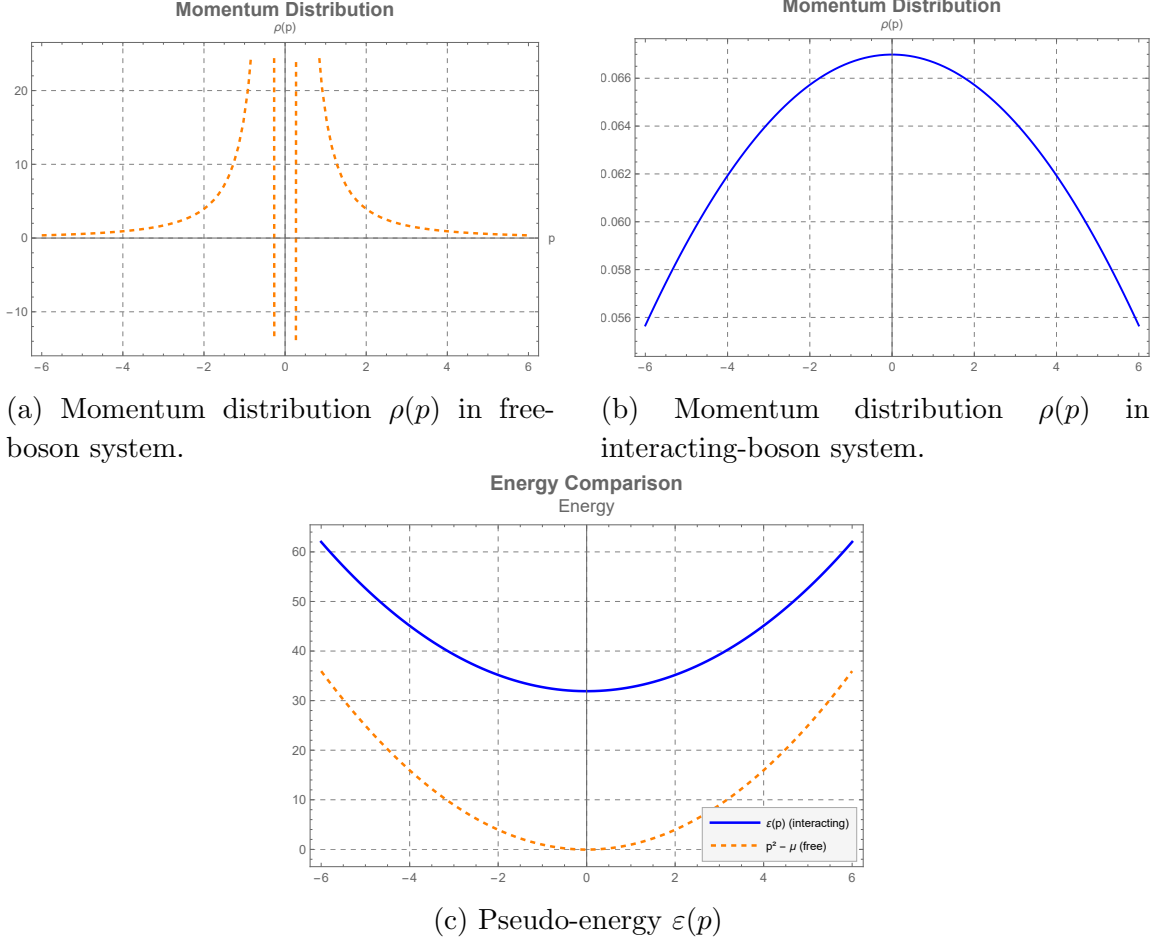
(c) Pseudo-energy $\varepsilon(p)$

Figure 1: Thermodynamic Bethe ansatz solution of the Lieb–Liniger model. (a) Momentum distribution $\rho(p)$ in interacting-boson system. (b) Momentum distribution $\rho(p)$ in free-boson system. (c) Pseudo-energy $\varepsilon(p)$.

consistent with the Bose distribution

$$n(p)=\frac{1}{\exp\left((p^2-\mu)/T\right)-1}.$$

This infrared divergence reflects the instability of the $1D$ free Bose gas. The divergence of the momentum distribution near $p = 0$ signals the breakdown of a conventional thermodynamic description. Additionally, repulsive interactions suppress low-momentum occupation and lead to a broader pseudo-energy profile, reflecting enhanced effective kinetic pressure, as shown in the figure. We emphasize that no

sharp phase transition is expected in this model. As a $1D$ system with short-range interactions, the Lieb–Liniger gas exhibits smooth crossover behavior rather than genuine thermodynamic phase transitions.

These numerical results demonstrate the power of the TBA formalism: starting from the exact S-matrix, we have obtained the finite-temperature thermodynamics without any approximations beyond discretizing the integral equations. The method generalizes to other integrable models and provides exact benchmarks for understanding thermalization and relaxation dynamics in $1D$ quantum systems.

References

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[2] Stijn J. van Tongeren. Introduction to the thermodynamic Bethe ansatz. *J. Phys. A*, 49(32):323005, 2016.