Black Hole Thermodynamics

① Temperature as imagenary time period.

Spacetime structure at asymptotic flat

at near horizon.

②. Entropy GHY boundary approach

Conical Singularity approach.

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①. Schwarzchild metric in Euclidean signature

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$\downarrow$ Wick rotation $t = -it_E$

$$ds^2 = \left(1-\frac{2GM}{r}\right)dt_E^2 + \left(1-\frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

(t_E, r, Ω) coordinate is asymptotic flat at infty, $t_E \in \mathbb{R}$.

②. Near horizon coordinate. $r = 2GM + \epsilon$, $\epsilon \ll 2GM$.

$\Rightarrow \rho^2 = 8GM\epsilon$

$$ds^2 = \left(\frac{\rho}{4GM}\right)^2 dt_E^2 + d\rho^2 + R_s^2 d\Omega^2$$

Interpretation ① horizon as origin of polar coordinate

② t_E is an angular variable.

$$\rho^2 d\left(\frac{t_E}{4GM}\right)^2 + d\rho^2, \quad t_E \cong t_E + 2\pi\cdot 4GM.$$

③ imaginary time period corresponds to Hawking temperature

$\Updownarrow$

Quantum field theory in finite temperature.

Remark: $ds^2 = dt_E^2 + dr^2 + r^2 d\Omega^2 \Rightarrow S^1 \times \mathbb{R}^3$ at infinity.

$ds^2 = \rho^2 d\left(\frac{t_E}{4GM}\right)^2 + d\rho^2 + R_s^2 d\Omega^2 \Rightarrow \mathbb{R}^2 \times S^2$ at horizon.

Entropy GHY boundary term.

partition function in QFT $Z \equiv \int [dg]\, e^{-I_E[g]}$

by imposing imaginary time period condition. $\Rightarrow Z[\beta]$ is partition function of a thermal field and can be used to compute thermal quantities.

$Z[\beta] = e^{S-\beta E}$, $\because$ 1st law. $dE = T\,dS = \frac{1}{\beta}dS$

$\Rightarrow S = \left(1 - \beta\frac{d}{d\beta}\right)\log(Z)$

To compute entropy at leading order, we can consider on-shell action.

$Z[\beta] \doteq e^{-I_E[g_{cl}]}$, $S = -\left(1-\beta\frac{d}{d\beta}\right) I_E[g_{cl},\beta]$

① Schwarzchild BH $\beta = \beta_H$

$\because$ Schwarzchild BH is vacuum sol. of Eistein's eq.

$\therefore I_E = -\frac{1}{16\pi G}\int d^4x\,\sqrt{g}\,R \approx 0.$

But $\because R \sim g^{-1}\partial^2 g$, and this actually leaves ambiguities when deriving e.o.m.

$I_1 = \frac{1}{2}\int dt\,\dot{x}^2$ and $I_2 = -\frac{1}{2}\int dt\, x\,\ddot{x}$

$\delta I_2 = -\frac{1}{2}\int dt(\delta x\,\ddot{x} + x\cdot\delta\ddot{x}) = -\int dt\,\delta x\,\ddot{x} - \frac{1}{2}(x\delta\dot{x} - \dot{x}\delta x)_{BT}$

$\hookrightarrow \frac{d}{dt}(x\,\delta\dot{x}) - \dot{x}\,\delta\dot{x} = \frac{d}{dt}(x\delta\dot{x}) - \frac{d}{dt}(\dot{x}\delta x) + \ddot{x}\,\delta x$

$\Rightarrow$ cancel the Boundary Term properly

$I_2' = I_2 + \frac{1}{2}(x\dot{x})_{BT}.$

② Derivation of GHY term.

Gauss-Codazzi theorem.

$$\sqrt{g}\,R = \sqrt{g}\,\Big[\underbrace{{}^{(3)}R + K_{\mu\nu}K^{\mu\nu} - K^2}_{\text{bulk}} \underbrace{- 2\nabla_\mu (K n^\mu + a^\mu)}_{\text{boundary}}\Big] \quad , a^\mu = n^\nu \nabla_\nu n^\mu$$

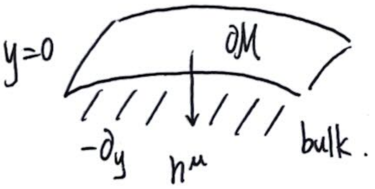
$$I_{EH}' = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}\,R - \frac{-1}{16\pi G}(-2)\int_M d^4x\sqrt{g}\,\nabla_\mu (K n^\mu + a^\mu)$$

Stoke's Thm. $\int_M d^4x\sqrt{g}\,\nabla_\mu V^\mu = \int_{\partial M} d^3x\sqrt{\gamma}\; n^\mu V_\mu .$

$\because$ $n^\mu a_\mu = 0.$, n^μ is timelike vector of spacelike surface. , $n^\mu n_\mu = 1$ (ES)

$\therefore$ $I_{EH}' = I_{EH} - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{\gamma}\,K$,

③ Hyper surface. $r = \bar{r} = \text{const}.$



$$ds^2 = dy^2 + \gamma_{ij}(x,y)\,dx^i dx^j$$

$$K_{ij} \equiv \tfrac{1}{2}\mathcal{L}_n(\gamma_{ij}) = -\tfrac{1}{2}\partial_y(\gamma_{ij})\Big|_{y=0}.$$

$$K = \gamma^{ij}K_{ij} = -\partial_y \log(\sqrt{\gamma})\Big|_{y=0}.$$

$$V(y) = \int_{\partial M} d^3x\sqrt{\gamma(x,y)}$$

$$\frac{d}{dy}V(y) = \int_{\partial M} d^3x\sqrt{\gamma(x,y)}\;\frac{\partial_y\sqrt{\gamma}}{\sqrt{\gamma}}.$$

$$\therefore\; I_{GHY}(BH) = \frac{1}{8\pi G}\frac{d}{dy}V(y)\Big|_{y=0}.$$

④ We need a cutoff of $I_{GHY}(BH)$ as $\bar{r}\to\infty$.

$\because$ $ds^2 = dt_E^2 + dr^2 + r^2 d\Omega^2 \Rightarrow S^1\times\mathbb{R}^3$ at infinity.

$\therefore$ consider I_{GHY} of $S^1\times\mathbb{R}^3$.

$$I'_{BH} \underset{\substack{\uparrow \\ \text{on-shell}}}{\approx} \lim_{\bar{r}\to\infty} I_{GHY}(BH) - I_{GHY}(S^1 \times \mathbb{R}^3)$$

More precisely, $S^1 \times \mathbb{R}^3$ needs to have the same circuference as BH at $r = \bar{r}$

$$\Rightarrow V(\mathbb{R}^3 \times S^1) = \beta_H \left(1 - \frac{2GM}{\bar{r}}\right)^{1/2} \times 4\pi r^2$$

$$V(BH) = \beta_H \left(1 - \frac{2GM}{r}\right)^{1/2} \times 4\pi r^2$$

$$\frac{d}{dy} = \begin{cases} -d/dr & \text{for } \mathbb{R}^3 \times S^1 \\ -\left(1 - \frac{2GM}{r}\right)^{1/2} d/dr & \text{for BH.} \end{cases}$$

$$I_{GHY}(BH) = \frac{-1}{8\pi G}\left(1 - \frac{2GM}{\bar{r}}\right)^{1/2} \frac{d}{dr}\Big(V(BH)\Big)$$

$$= \frac{-1}{8\pi G}\left(1 - \frac{2GM}{\bar{r}}\right)^{1/2} \beta_H \frac{8\pi\bar{r} - 12GM\pi}{\sqrt{1 - \frac{2GM}{\bar{r}}}} = \frac{\beta_H}{8\pi G}(12GM\pi - 8\pi\bar{r})$$

$$I'_{BH} \approx \frac{4\pi GM}{8\pi G}\beta_H = \frac{\beta_H M}{2} = \frac{\beta_H^2}{16\pi G} \quad \Rightarrow S = \frac{\beta_H^2}{16\pi G} = \frac{A}{4G} \ \#$$

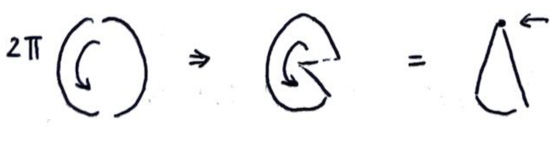
Entropy Conical Singularity.

Remark : Horizon plays no particular role.

For cases w/ no BH , we can choose arbitrary imaginary period β w/o singularity.

$$\Rightarrow I = \int d^d x\, \mathcal{L} = \int d\tau \int d^{d-1}x\, \mathcal{L} = \beta \cdot \int d^{d-1}x\, \mathcal{L} \Rightarrow S = 0.$$

On the other hand, choosing arbitrary imaginary period β cause a conical singularity at origin of polar coordinate.



conical singularity.

$\because$ $S = -(1-\beta\frac{d}{d\beta}) I_{BH}$

$\uparrow$ varying β while M is fixed.

$\therefore$ consider arbitrary β closed to β_H s.t. classical sol. still a good approx.

$R \not\approx 0$ and $\because$ $I_{GHY} \propto \beta$ $\therefore -(1-\beta\frac{\partial}{\partial\beta}) I_{GHY} = 0$.

① Regularized Metric.

for BH metric $ds^2 = (\frac{1}{\beta_H})^2 \rho^2 dt_E^2 + d\rho^2$, $t_E \cong t_E + 2\pi\beta_H$

if arbitrary period $t_E \cong t_E + 2\pi\beta$.

define $\phi \equiv \beta^{-1} t_E$

$ds^2 = (\beta/\beta_H)^2 \rho^2 d\phi^2 + d\rho^2$

Regularized by embedding formalism. $x = \alpha\rho\cos(\phi)$, $y = \alpha\rho\sin(\phi)$, $z = \sqrt{1-\alpha^2}\sqrt{\rho^2+a^2}$, $x^2+y^2+z^2 = \rho^2 + a^2(1-\alpha^2)$

$$ds_{3D}^2 = dx^2 + dy^2 + dz^2$$

$$= \alpha^2\rho^2 d\phi^2 + \alpha^2 d\rho^2 + \frac{(1-\alpha^2)\rho^2}{\rho^2+a^2} d\rho^2 = \alpha^2\rho^2 d\phi^2 + \frac{\alpha^2\rho^2 + \alpha^2 a^2 + \rho^2 - \alpha^2\rho^2}{\rho^2+a^2} d\rho^2$$

$$= \frac{\rho^2 + a^2\alpha^2}{\rho^2+a^2} d\rho^2 + \alpha^2\rho^2 d\phi^2 \quad , \quad a \to 0.$$

$$\Rightarrow R = -\frac{2a^2(\alpha-1)(\alpha+1)}{(\alpha^2 a^2 + \rho^2)^2} \quad , \quad \int R \cdot \alpha\rho \cdot \left(\frac{a^2\alpha^2+\rho^2}{\rho^2+a^2}\right)^{1/2} d\rho\, d\alpha = -4\pi(\alpha-1)$$

$$= -\frac{2(\alpha-1)}{\alpha}\delta(\rho) \quad , \quad \int \delta(\rho)\, d\rho \cdot \rho = 1.$$

induced metric at horizon

$$\Rightarrow I_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\, R = -\frac{1}{16\pi G}\int d\rho \cdot \alpha\rho \cdot d\phi \cdot \left(\frac{2(1-\alpha)}{\alpha}\right)\delta(\rho) \times (\sqrt{\gamma}\, d^2x)$$

$$= -\frac{2\pi}{16\pi G} 2(1-\alpha) A = \frac{1}{4G}\left(\beta/\beta_H - 1\right) A$$

$\Rightarrow S = -(1-\beta\frac{\partial}{\partial\beta}) I_{EH} = A/4G$ # Entropy as a response of system w/ conical singularity.