ECN702

Homework 2

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I hereby declare that I am the sole author of this work.

1.i

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad \vec{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \vec{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix} \quad \beta = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$\vec{Y} = \mathbf{X}\beta + \vec{\epsilon}$$

Let $\vec{x_i} = [1, x_i]$
So, $\epsilon_i = y_i - \beta x_i$.

Let $\hat{\mathbf{b}}$ denote the estimates of β such that it minimizes the sum of squared residuals $\hat{\epsilon}$.

$$\begin{aligned} \hat{\epsilon}^T\hat{\epsilon} &= (\vec{Y} - \mathbf{X}\hat{\mathbf{b}})^T(\vec{Y} - \mathbf{X}\hat{\mathbf{b}}) \\ &= \vec{Y}^T\vec{Y} - 2\vec{Y}^T\mathbf{X}\hat{\mathbf{b}} + \hat{\mathbf{b}}^T\mathbf{X}^T\mathbf{X}\hat{\mathbf{b}} \end{aligned}$$

The necessary condition for a minimum is

$$\begin{aligned} &\frac{\partial}{\partial \hat{\mathbf{b}}}\left(\vec{Y}^T\vec{Y} - 2\vec{Y}^T\mathbf{X}\hat{\mathbf{b}} + \hat{\mathbf{b}}^T\mathbf{X}^T\mathbf{X}\hat{\mathbf{b}}\right) \\ &= -2\mathbf{X}^T\vec{Y} + 2\mathbf{X}^T\mathbf{X}\hat{\mathbf{b}} = 0 \end{aligned}$$

If an inverse matrix exists, then $\hat{\mathbf{b}}$ satisfies the given least squares equation.

$$\begin{aligned} \mathbf{X}^T\mathbf{X}\hat{\mathbf{b}} &= \mathbf{X}^T\vec{Y} \\ (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{X}\hat{\mathbf{b}} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\vec{Y} \\ \hat{\mathbf{b}} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\vec{Y} \quad (*) \end{aligned}$$

1.ii

$$
\begin{aligned}
\hat{\epsilon} &= \vec{Y} - \mathbf{X}\hat{\mathbf{b}} \quad \text{By substitution of } (*) \\
&= \vec{Y} - \mathbf{X}\left((\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\vec{Y}\right) \\
&= \left(\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\vec{Y} \quad (**)
\end{aligned}
$$

$$
\begin{aligned}
\hat{Y} &= \vec{Y} - \hat{\epsilon} \\
&= \vec{Y} - \left(\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\vec{Y} \quad \text{By substitution of } (**) \\
&= \left(\mathbf{I} - \left(\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\right)\vec{Y} \\
&= \left(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\vec{Y}
\end{aligned}
$$

$$
\begin{aligned}
\vec{Y} &= \hat{Y} + \hat{\epsilon} \\
&= \left(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\vec{Y} + \left(\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\vec{Y}
\end{aligned}
$$

$$
\text{If, } \left(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\left(\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)^T = 0
$$

I give up...

1.ii

$$\begin{aligned} \widehat{y_i} &= 1.4 & +2.7x_i \\ & (0.8) & (1.1) \end{aligned}$$

$$\begin{aligned} t^{act} &= \frac{2.7 - 0}{1.1} \\ &\approx 2.45 \\ |t^{act}| &> 1.96 \end{aligned}$$

3.a

True. Correlation between regressors and error terms means that the OLS estimator is inconsistent.

3.a

True. Potential threats to external validity arise from differences between the population and setting studied and the population and setting of interest.