ECN627 Assignment 1

Saumitra Mazumder
500720916

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Problem 2.6

a)

$$
\begin{aligned}
E[Y] &= \sum_i y_i \Pr_i(Y = y_i) \\
&= 0 \times 0.068 + 1 \times 0.932 \\
&= 0.932
\end{aligned}
$$

b)

$$
\begin{aligned}
E[\overline{Y}] &= \sum_i y_i (1 - \Pr_i(Y = y_i)) \\
&= 0 \times (1 - 0.068) + 1 \times (1 - 0.932) \\
&= 0.068 \\
&= 1 - E[Y]
\end{aligned}
$$

c)

$$
\begin{aligned}
E[Y|X=1] &= \sum_i y_i \Pr(Y=y_i|X=1) \\
&= 0 \times Pr(Y=0|X=1) + 1 \times \Pr(Y=1|X=1) \\
&= 0 \times \frac{\Pr(Y=0 \cap X=1)}{\Pr(X=1)} + 1 \times \frac{\Pr(Y=1 \cap X=1)}{\Pr(X=1)} \\
&= 0 \times \frac{0.015}{0.361} + 1 \times \frac{0.346}{0.361} \\
&\approx 0.958
\end{aligned}
$$

$$
\begin{aligned}
E[Y|X=0] &= \sum_i y_i \Pr(Y=y_i|X=0) \\
&= 0 \times \Pr(Y=0|X=0) + 1 \times \Pr(Y=1|X=0) \\
&= 0 \times \frac{\Pr(Y=0 \cap X=0)}{\Pr(X=0)} + 1 \times \frac{\Pr(Y=1 \cap X=0)}{\Pr(X=0)} \\
&= 0 \times \frac{0.053}{0.639} + 1 \times \frac{0.586}{0.639} \\
&\approx 0.917
\end{aligned}
$$

d)

$$
\begin{aligned}
\Pr(Y=0 \cap X=1) = 0.015 \\
\Pr(Y=0 \cap X=0) = 0.053
\end{aligned}
$$

e)

$$
\begin{aligned}
\Pr(Y=0|X=1) &= \frac{\Pr(Y=0 \cap X=1)}{\Pr(X=1)} \\
&= \frac{0.015}{0.361} \\
&\approx 0.041
\end{aligned}
$$

$$
\begin{aligned}
\Pr(Y=0|X=0) &= \frac{\Pr(Y=0 \cap X=0)}{\Pr(X=0)} \\
&= \frac{0.053}{0.639} \\
&\approx 0.083
\end{aligned}
$$

f)

$$\begin{array}{lll} \Pr_Y(Y=0) = 0.068 & \Pr(Y=0|X=0) \approx 0.083 & \Pr(Y=0|X=1) \approx 0.042 \\ \Pr_Y(Y=1) = 0.932 & \Pr(Y=1|X=0) \approx 0.917 & \Pr(Y=1|X=1) \approx 0.984 \end{array}$$

$$\begin{array}{lll} \Pr_X(X=0) = 0.639 & \Pr(X=0|Y=0) \approx 0.779 & \Pr(X=0|Y=1) \approx 0.629 \\ \Pr_X(X=1) = 0.361 & \Pr(X=1|Y=0) \approx 0.221 & \Pr(X=1|Y=1) \approx 0.371 \end{array}$$

Since $\Pr_{X|Y}(X|Y) \neq \Pr_X(x), \Pr_X(x) > 0$ and $\Pr_{Y|X}(Y|X) \neq \Pr_Y(y), \Pr_Y(y) > 0$ $\forall$ x and y, employment status and educational achievement NOT independent.

Problem 2.8

$$\begin{aligned} \mu_Y &= 1 \\ \sigma_y^2 &= 4 \\ Z &= \frac{1}{2}(Y-1) \\ &= \frac{1}{2}(Y) - \frac{1}{2} \end{aligned}$$

Hence,

$$\begin{aligned} E[Z] &= \frac{1}{2}\mu_Y - \frac{1}{2} \\ &= \frac{1}{2}(1) - \frac{1}{2} \\ &= 0 \end{aligned}$$

and,

$$\begin{aligned} Var[Z] &= \left(\frac{1}{2}\right)^2 Var[Y] \\ &= \frac{1}{4}(4) \\ &= 1 \end{aligned}$$

Problem 2.10

(a)

$$
\begin{aligned}
Y &\sim \mathcal{N}(1,4) \\
\Pr(Y \leq y) &= \Pr\left(Z \leq \frac{y-\mu_Y}{\sigma_Y}\right) \\
\Pr(Y \leq 3) &= \Pr\left(Z \leq \frac{3-1}{2}\right) \\
&= \Pr\left(Z \leq 1\right) \\
&= \Phi(1) \\
&= 0.84134
\end{aligned}
$$

(b)

$$
\begin{aligned}
Y &\sim \mathcal{N}(3,9) \\
\Pr(Y > y) &= 1 - \Pr\left(Z \leq \frac{y-\mu_Y}{\sigma_Y}\right) \\
\Pr(Y > 0) &= 1 - \Pr\left(Z \leq \frac{0-3}{3}\right) \\
&= 1 - \Pr\left(Z \leq -1\right) \\
&= 1 - \Pr\left(Z > 1\right) \\
&= 1 - \left(1 - \Pr\left(Z \leq 1\right)\right) \\
&= \Pr\left(Z \leq 1\right) \\
&= \Phi(1) \\
&= 0.84134
\end{aligned}
$$

(c)

$$
\begin{aligned}
Y &\sim \mathcal{N}(50, 25) \\
\Pr(y_1 \leq Y \leq y_2) &= \Pr\left(\frac{y_1 - \mu_Y}{\sigma_Y} \leq Z \leq \frac{y_2 - \mu_Y}{\sigma_Y}\right) \\
\Pr(40 \leq Y \leq 52) &= \Pr\left(\frac{40 - 50}{5} \leq Z \leq \frac{52 - 50}{5}\right) \\
&= \Pr\left(-2 \leq Z \leq 0.4\right) \\
&= \Pr\left(Z \leq 0.4\right) - \Pr\left(Z \leq -2\right) \\
&= \Pr\left(Z \leq 0.4\right) - \Pr\left(Z \geq 2\right) \\
&= \Pr\left(Z \leq 0.4\right) - \left(1 - \Pr\left(Z \leq 2\right)\right) \\
&= \Pr\left(Z \leq 0.4\right) + \left(Z \leq 2\right) - 1 \\
&= \Phi(0.4) + \Phi(2) - 1 \\
&= 0.65542 + 0.97725 - 1 \\
&= 0.63267
\end{aligned}
$$

(d)

$$
\begin{aligned}
Y &\sim \mathcal{N}(5, 2) \\
\Pr(6 \leq Y \leq 8) &= \Pr\left(\frac{6 - 5}{\sqrt{2}} \leq Z \leq \frac{8 - 5}{\sqrt{2}}\right) \\
&= \Pr\left(\frac{1}{\sqrt{2}} \leq Z \leq \frac{3}{\sqrt{2}}\right) \\
&= \Pr\left(Z \leq \frac{3}{\sqrt{2}}\right) - \Pr\left(Z \leq \frac{1}{\sqrt{2}}\right) \\
&= \Phi\left(\frac{3}{\sqrt{2}}\right) + \Phi\left(\frac{1}{\sqrt{2}}\right) \\
&\approx 0.98300 - 0.76115 \\
&\approx 0.22185
\end{aligned}
$$

Problem 2.11

(a)

$$
\begin{aligned}
Y &\sim \chi_4^2 \\
\Pr(Y \leq 7.78) &= \Pr(Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \leq 7.78) \\
&= 0.90
\end{aligned}
$$

(b)

$$
\begin{aligned}
Y &\sim \chi_4^{10} \\
\Pr(Y > 18.31) &= 1 - \Pr(Y \leq 18.31) \\
&= 1 - \Pr\left(\sum_{i=1}^{10} Z_i^2 \leq 18.31\right) \\
&= 1 - 0.95 \\
&= 0.05
\end{aligned}
$$

(c)

$$
\begin{aligned}
Y &\sim F_{10,\infty} \\
\Pr(Y > 1.83) &= 1 - \Pr(Y \leq 1.83) \\
&= 1 - 0.95 \\
&= 0.05
\end{aligned}
$$

d)

The $F_{m,\infty}$ is the distribution of the χ_m^2 random variable divided by m.

(e)

$$\begin{aligned}
Y &\sim \chi_1^2 \\
\Pr(Y \leq 1.0) &= \Pr(Z_1^2 \leq 1.0) \\
\Pr(Y \leq 1.0) &= \Pr\left(-1.0 \leq \sqrt{Z_1^2} \leq 1.0\right) \\
&= \Pr(Z \leq 1.0) - \Pr(Z \leq -1.0) \\
&= \Pr(Z \leq 1.0) - \Pr(Z > 1.0) \\
&= \Pr(Z \leq 1.0) - (1 - \Pr(Z \leq 1.0)) \\
&= 2(0.84134) - 1 \\
&= 0.68268
\end{aligned}$$

Problem 2.14

$$\begin{aligned}
\overline{Y} &\sim \mathcal{N}\left(\mu_Y, \frac{\sigma_Y^2}{n}\right) \\
\overline{Y} &\sim \mathcal{N}\left(100, \frac{43}{n}\right)
\end{aligned}$$

(a)

$$\overline{Y} \sim \mathcal{N}\left(100, \frac{43}{100}\right), n \geq 30$$

$$\begin{aligned}
\Pr(\overline{Y} \leq 101) &= \Pr\left(Z \leq \frac{101 - 100}{\sqrt{\frac{43}{100}}}\right) \\
&= \Pr\left(Z \leq 1.52\right) \\
&= 0.93574
\end{aligned}$$

(b)

$$\overline{Y} \sim \mathcal{N}\left(100, \frac{43}{165}\right), n \geq 30$$

$$\begin{aligned}
\Pr(\overline{Y} > 98) &= 1 - \Pr\left(Z \leq \frac{98 - 100}{\sqrt{\frac{43}{165}}}\right) \\
&= 1 - \Pr\left(Z \leq \frac{98 - 100}{\sqrt{\frac{43}{165}}}\right) \\
&= 1 - \Pr\left(Z \leq -3.92\right) \\
&= 1 - \Pr\left(Z > 3.92\right) \\
&= 1 - \left(1 - \Pr\left(Z \leq 3.92\right)\right) \\
&= \Pr\left(Z \leq 3.92\right) \\
&= 0.99996
\end{aligned}$$

(c)

$$\overline{Y} \sim \mathcal{N}\left(100, \frac{43}{64}\right), n \geq 30$$

$$\begin{aligned}
\Pr(101 \leq \overline{Y} \leq 103) &= \Pr\left(Z \leq \frac{103 - 100}{\sqrt{\frac{43}{64}}}\right) - \Pr\left(Z \leq \frac{101 - 100}{\sqrt{\frac{43}{64}}}\right) \\
&= \Pr\left(Z \leq 3.66\right) - \Pr\left(Z \leq 1.22\right) \\
&= 0.99987 - 0.88877 \\
&= 0.1111
\end{aligned}$$

Problem 2.23

$E[Y|X](x) = E[Y|X = x]$
$Corr(X, Y) = 0$
$X \sim \mathcal{N}(1, 0)$
$Z \sim \mathcal{N}(1, 0)$
$Y = X^2 + Z$

(a)

$$
\begin{aligned}
E[Y] &= E[X^2 + Z] \\
E\left[E[Y|X]\right] &= E[X^2] + E[Z] \\
E\left[E[Y|X]\right] &= E[X^2] + 1
\end{aligned}
$$

Hence,

$$E[Y|X] = X^2$$

(b)

$$
\begin{aligned}
\mu_Y &= E[Y] \\
&= E\left[E[Y|X]\right] \\
&= E[X^2]
\end{aligned}
$$

recall,

$$
\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
(\sigma_x)^2 &= E[X^2] - (1)^2 \\
0 &= E[X^2] - (1)^2 \\
E[X^2] = 1
\end{aligned}
$$

Hence,

$$\mu_Y = 1$$

(c)

$$
\begin{aligned}
Cov(X,Y) &= E\left[(X - E[X])(Y - E[Y])\right] \\
&= E\left[(XY - XE[Y] - YE[X] + E[X]E[Y])\right] \\
&= E\left[XY - XE[Y] - Y + E[Y]\right] \\
&= E\left[XY - X - Y + 1\right] \\
&= E[XY] - E[X] - E[Y] + 1 \\
&= E[XY] - 1 - 1 + 1 \\
0 &= E[XY] - 1 \\
1 &= E[XY]
\end{aligned}
$$

(d)

$$
\begin{aligned}
\rho_{XY} = Corr(X,Y) &= \frac{Cov(X,Y)}{\sigma_x \sigma_Y} \\
0 &= \frac{Cov(X,Y)}{\sigma_x \sigma_Y} \\
0 &= Cov(X,Y)
\end{aligned}
$$