MTH712

Assignment 1

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I hereby declare that I am the sole author of this work.

Question 1

$$\begin{aligned}
\text{Let } \mu(t) \text{ be such that} \quad \mu(t)a &= \mu'(t) \\
a &= \frac{\mu'(t)}{\mu(t)} \\
&= \frac{d}{dt}\left(\ln(\mu(t))\right) \\
\text{then,} \quad \ln(\mu(t) &= \int a dt \\
&= at + C_1 \\
\text{and,} \quad \mu(t) &= C_1 e^{at} \qquad (*)
\end{aligned}$$

$$\begin{aligned}
\text{Now,} \quad \mu(t)\frac{du}{dt} + \mu(t)au &= \mu(t)e^{at} \\
\mu(t)\frac{du}{dt} + \mu'(t)u &= \mu(t)e^{at}
\end{aligned}$$

$$\text{By the product rule,} \quad \frac{d}{dt}(\mu(t)u(t)) = \mu(t)e^{at}$$

$$\begin{aligned}
\int \frac{d}{dt}(\mu(t)u(t))dt &= \int \mu(t)e^{at}dt \\
\mu(t)u(t)) + C_2 &= \int \mu(t)e^{at}dt \\
u(t) &= \frac{\int \mu(t)e^{at}dt + C_2}{\mu(t)} \\
&= \frac{\int \mu(t)e^{at}dt + C_2}{\mu(t)} \\
\text{then,} \quad u(t) &= \frac{C_1 \int e^{at}e^{at}dt + C_3}{C_1 e^{at}} \quad \text{by } (*)
\end{aligned}$$

$$\text{By substitution,} \quad u(t) = \frac{(e^{at})^2 + C_4}{2ae^{at}}$$

$$\text{Hence,} \quad u(t) = \frac{e^{at}}{K_1} + \frac{K_2}{e^{at}}$$

Question 2

$$\frac{d^2u}{dx^2} - u\gamma^2 + \gamma^2 T = 0 \quad \text{where } \gamma, T \text{ are constants}$$
$$\frac{d^2u}{dx^2} - u\gamma^2 = -\gamma^2 T \quad (*)$$

$$\text{Using the auxillary equation,} \quad U = e^{mx}$$
$$\text{Now,} \quad m^2 - \gamma^2 = 0$$
$$(m+\gamma)(m-\gamma) = 0$$
$$\text{Hence} \quad m = \pm\gamma$$

$$\text{So,} \quad U_c = C_1 e^{\gamma x} + C_2 e^{-\gamma x}$$

Using undetermined coefficients:
Let $U_p = K$, where K is a constant

Now, sub into (*)
$$\text{So,} \quad -\gamma^2 D = -\gamma^2 T$$
$$D = T$$
$$\text{and,} \quad U_p = T$$

Hence, the general solutions is,
$$U(x) = U_c + U_p = C_1 e^{\gamma x} + C_2 e^{-\gamma x} + T$$

Question 3

$$\frac{d}{dx}\left[(h+kx)\frac{dv}{dx}\right] = 0$$

$$\begin{aligned}
\int \frac{d}{dx}\left[(h+kx)\frac{dv}{dx}\right] &= \int 0 dx \\
(h+kx)\frac{dv}{dx} &= C_1 \quad \text{where } C_1 \text{ is some constant}
\end{aligned}$$

$$\begin{aligned}
\text{Now,} \quad dv &= \frac{C_1}{h+kx}dx \\
\int dv &= \int \frac{C_1}{h+kx}dx
\end{aligned}$$

$$\begin{aligned}
\text{Hence,} \quad v &= C_1 \int \frac{dx}{h+kx} \\
&= C_1 \frac{\ln|h+kx|}{k} + C_2
\end{aligned}$$

Question 4

$$t^2\frac{d^2u}{dt^2}+2t\frac{du}{dt}-q(q+1)u=0 \text{ (*)} \quad \text{Where, } t>0 \text{ , q is a positive constant}$$

$$\begin{aligned}
&\text{Now,} \quad u=t^m \\
&\text{So,} \quad \frac{du}{dt}=mt^{m-1} \\
&\text{and,} \quad \frac{d^2u}{dt^2}=(m)(m-1)t^{m-2}
\end{aligned}$$

Now substituting into (*)

$$\begin{aligned}
t^2\frac{d^2u}{dt^2}+2t\frac{du}{dt}-q(q+1)u &= (m)(m-1)t^m+2mt^m-q(q+1)t^m \\
&= t^m\left[m(m-1)+2m-q(q+1)\right] \\
&= t^m\left[m^2+m-(q^2+q)\right]=0
\end{aligned}$$

$$\text{So,} \quad t^m=0 \text{ and } m^2+m-(q^2+q)=0$$

$$\begin{aligned}
\text{Now,} \quad m^2+m &= q^2+q \\
m^2+m+\frac{1}{4} &= q^2+q+\frac{1}{4} \\
\left(m+\frac{1}{2}\right)^2 &= \left(q+\frac{1}{2}\right)^2 \\
\left[m+\frac{1}{2}+q+\frac{1}{2}\right]\left[m+\frac{1}{2}-q-\frac{1}{2}\right] &= 0
\end{aligned}$$

$$\text{So, } m=q \text{ and } m=-(q+1)$$

Hence, $u=C_1t^q+C_2t^{-(q+1)}$

Question 5

$$\frac{d^2u}{dx^2} - au = 0 \quad 0 < x < 1$$

$$\frac{du}{dx}(0) = 0$$

$$u(1) = 1$$

Using the auxillary equation, $\quad u = e^{mx}$

Now, $\quad m^2 - a = 0$

Hence $\quad m = \pm\sqrt{a}$

So, $\quad U_c = C_1 e^{\sqrt{a}x} + C_2 e^{-\sqrt{a}x} \quad (*)$

Recall,

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}\sinh x = \cosh x$$

Now, substitution into (*)

$$U_c = (C_1 + C_2)\cosh(\sqrt{a}x) + (C_1 - C_2)\sinh(\sqrt{a}x)$$

Using the initial conditions,

$$\frac{du}{dx}(0) = (C_1 + C_2)\sqrt{a}\sinh(\sqrt{a}0) + (C_1 - C_2)\sqrt{a}\cosh(\sqrt{a}0)$$

$$= (C_1 - C_2)\sqrt{a} = 0$$

So, $\quad C_1 = C_2$

Also, $\quad u(1) = (C_1 + C_2)\cosh(\sqrt{a}(1)) + (C_1 - C_2)\sinh(\sqrt{a}(1))$

So, $\quad 1 = 2C_1\cosh(\sqrt{a})$

From this, $\quad C_1 = C_2 = \dfrac{1}{2\cosh(\sqrt{a})}$

Hence, $\quad u(x) = \dfrac{\cosh(\sqrt{a}x)}{\cosh(\sqrt{a})}$