MTH500

# Assignment 2

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I hereby declare that I am the sole author of this work.

# Question 1

**a)** Let $X = N_1(t) \sim Poi(\lambda_1)$, $Y = N_2(t) \sim \lambda_2$ and $Z = N(t) = N_1(t) + N_2(t)$
$MGF_X(t) = \sum_x e^{tx} \Pr(X = x)$ in the discrete case

$$\begin{aligned}
\text{So, } MGF_X(t) &= \sum_x e^{tx} \Pr(X = x) \\
&= \sum_x^{\infty} e^{tx} \frac{\lambda^x}{x!} e^{-\lambda} \\
&= e^{-\lambda} \sum_x^{\infty} \frac{(e^t \lambda)^n)}{n!} \\
&= e^{-\lambda + \lambda e^t} \\
&= e^{\lambda(e^t - 1)} \\
MGF_Z(t) = MGF_{X+Y}(t) &= MGF_X + MGF_Y \\
&= e^{\lambda_X(e^t-1)} + e^{\lambda_Y(e^t-1)} \\
&= e^{(\lambda_X + \lambda_Y)(e^t - 1)}
\end{aligned}$$

Hence, $Z = N(t) = N_1(t) + N_2(t) \sim Poi(\lambda_X + \lambda_Y)$, by the moment generating function.

**b)** Let A := The event that a blue marble is picked

$$\begin{aligned}
\Pr(A) &= \sum_1^3 \Pr(A|B_n) \Pr(B_n) \\
&= \left(\frac{20}{50}\right)\left(\frac{1}{3}\right) + \left(\frac{25}{50}\right)\left(\frac{1}{3}\right) + \left(\frac{35}{50}\right)\left(\frac{1}{3}\right) \\
&= \frac{20}{150} + \frac{25}{150} + \frac{35}{150} = \frac{80}{150} \approx 0.53
\end{aligned}$$

**c)** Let $X = B_{(t)} \sim \mathcal{N}(0, \sqrt{t})$, $Y = B_{(t+1)} \sim \mathcal{N}(0, \sqrt{t+1})$ and $Z = B_{(t+1)} - B_{(t+1)}$

$$MGF_x(s) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{sx} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{t}}\right)^2}$$

**d)**

$$\begin{aligned}
Var[X + 2Y] &= Var[X] + 4Var[Y] + 2(2)Var[X, Y] = \sigma_X^2 + 4\sigma_Y^2 + 4\sigma_{xy} \\
&= \left(\frac{20}{81}\right) + 4\left(\frac{77}{324}\right) + 4\left(-\frac{1}{162}\right) \qquad = \frac{95}{81}
\end{aligned}$$

# Question 2

$$
\begin{aligned}
P(X,Y) &= \int_0^1 \int_0^1 f_{X\cap Y}(x\cap y)dydx = \int_0^1 \int_0^1 (4xy)dydx \\
&= \int_0^1 2xy^2|_{y=0}^{y=1} dx = \int_0^1 2xdx \\
&= x|_{x=0}^{x=1} \\
&= 1
\end{aligned}
$$

### a i)

$$
\begin{aligned}
P(Y<X) &= \int_0^1 \int_0^x f_{X\cap Y}(x\cap y)dydx = \int_0^1 \int_0^x (4xy)dydx \\
&= \int_0^1 2xy^2|_{y=0}^{y=x} dx = \int_0^1 2x^3dx \\
&= \frac{1}{2}x^4|_{x=0}^{x=1} \\
&= \frac{1}{2}
\end{aligned}
$$

### a ii)

$$
\begin{aligned}
f_X(x) &= \int_0^1 f_{X\cap Y}(x\cap y)dy \\
&= \int_0^1 (4xy)dy \\
&= 2xy^2 \\
0 &< y < 1
\end{aligned}
\qquad\qquad
\begin{aligned}
f_Y(y) &= \int_0^1 f_{X\cap Y}(x\cap y)dx \\
&= \int_0^1 (4xy)dx \\
&= 2x^2y \\
0 &< x < 1
\end{aligned}
$$

hence,

$$
\begin{aligned}
f_X(x) &= 2xy^2|_{y=0}^{y=1} \\
&= 2x
\end{aligned}
\qquad\qquad
\begin{aligned}
f_Y(y) &= 2x^2y|_{x=0}^{x=1} \\
&= 2y
\end{aligned}
$$

### a iii)

$$f_{X\cap Y}(x\cap y) = 4xy \qquad\qquad f_X(x)f_Y(y) = (2x)(2y) = 4xy$$

hence, X and Y are independent.

**a iv)**

$$
\begin{aligned}
E[X] &= \int_0^1 \int_0^1 x f_{X\cap Y}(x \cap y) dx dy = \int_0^1 \int_0^1 (4x^2 y) dx dy \\
&= \int_0^1 \frac{4}{3} x^3 y|_{x=0}^{x=1} dy = \int_0^1 \left(\frac{4}{3} y\right) dy \\
&= \frac{2}{3}|_{y=0}^{y=1} \\
&= \frac{2}{3}
\end{aligned}
$$

Similarly,

$$
E[Y] = \frac{2}{3}
$$

**b)**

$$
f_{X\cap Y}(x \cap y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{if } otherwise \end{cases}
\qquad
\begin{aligned}
f_X(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{X\cap Y}(x \cap y) dy \\
&= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\
&= \frac{2\sqrt{1-x^2}}{\pi}
\end{aligned}
$$

$$
\begin{aligned}
f_{Y|X}(y|x) = \frac{f_{X\cap Y}(x \cap y)}{f_X(x)} &= \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-x^2}}{\pi}} \\
&= \frac{1}{2\sqrt{1-x^2}}
\end{aligned}
$$

**c)**

$$
\begin{aligned}
f_Y(y) &= \int \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\ exp\left(-\frac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \left(\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho)^2}\right) dx \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\ exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right)\int_{-\infty}^{\infty} exp\left(-\frac{\left(x-(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)^2}{2(1-\rho^2)\sigma_1^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_2} exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right) \\
\text{now,}\quad f_{X|Y}(x|y) &= \frac{f_{X\cap Y}(x\cap y)}{f_Y(y)} \\
&= \frac{1}{\sqrt{2\pi}\sigma_2} exp\left(-\frac{\left(x-(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)^2}{2(1-\rho^2)\sigma_1^2}\right)
\end{aligned}
$$

hence,

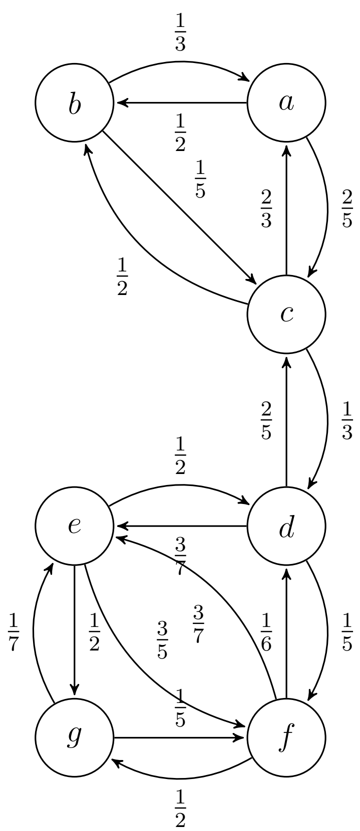
$$(X|Y=y) \sim \mathcal{N}\left((\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2), (1-\rho^2)\sigma_1^2\right)$$

if X and Y are independent, then

$$f_{X\cap Y}(x\cap y) = f_X(x)f_Y(y)$$

# Question 3

**(a** Weighted graph:



**(b)** Matlab code

```
clear all
close all
clc
P = [0 1/3 2/3 0 0 ;
    1/2 0 1/2 0 0 ;
    2/5 1/5 0 2/5 0;
    0 0 1/3 0 1/2 ;
    0 0 0 3/7 0 ];
I =  eye(5);
A = inv(I - P);
```

```
S = sum(A,2)
```

Matlab output

```
S =

   13.6000
   13.8000
   12.0000
    7.0000
    4.0000
```

Hence, the expected number of steps from a to f is 13.6

I give up