MTH500

Assignment 1

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October 6, 2017

I hereby declare that I am the sole author of this work.

Question 1

a)

$\Pr_{X,Y}(X,Y)$	$y=1$	$y=2$	$\Pr_X(x)$
$x=1$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{4}{9}$
$x=2$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{5}{9}$
$\Pr_Y(y)$	$\frac{7}{18}$	$\frac{11}{18}$	1

Recall, X & Y are independent if $\Pr_{X|Y}(x|y) = \Pr_Y(y)$ or $\Pr_{X\cap Y}(x \cap y) = \Pr_Y(y)\Pr_X(x)$ $\forall$ x & y.

$\times$	$\Pr_Y(1) = \frac{7}{18}$	$\Pr_Y(2) = \frac{11}{18}$
$\Pr_X(1) = \frac{4}{9}$	$\frac{14}{81}$	$\frac{22}{81}$
$\Pr_X(2) = \frac{5}{9}$	$\frac{35}{162}$	$\frac{55}{162}$

Hence, X & Y are not independent random variables.

b)

$$\begin{aligned}\text{Recall,}&\\ Cov(X,Y) = \sigma_{XY} &= E[(X-\mu_x)(Y-\mu_Y)]\\ &= \sum_y \sum_x (x - E(X))\,(y - E(Y)) \Pr_{X\cap Y}(x \cap y)\end{aligned}$$

for discrete random variables.

Now,

$$E(X) = \sum_x x \Pr_X(x) = 1\left(\frac{4}{9}\right) + 2\left(\frac{5}{9}\right) = \frac{14}{9} \qquad E(Y) = \sum_y y \Pr_Y(y) = 1\left(\frac{7}{18}\right) + 2\left(\frac{11}{18}\right) = \frac{29}{18}$$

and,

$$\begin{aligned}Cov(X,Y) &= \left(1-\frac{14}{9}\right)\left(1-\frac{29}{18}\right)\left(\frac{1}{6}\right) + \left(1-\frac{14}{9}\right)\left(2-\frac{29}{18}\right)\left(\frac{5}{18}\right)\\ &+ \left(2-\frac{14}{9}\right)\left(1-\frac{29}{18}\right)\left(\frac{2}{9}\right) + \left(2-\frac{14}{9}\right)\left(2-\frac{29}{18}\right)\left(\frac{1}{3}\right)\\ &= \frac{55}{972} - \frac{175}{2916} - \frac{44}{729} + \frac{14}{243} = -\frac{1}{162}\end{aligned}$$

Hence, X,Y have negative correlation.

c)

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V[X]Var[Y]}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-\frac{1}{162}}{\sqrt{\frac{14}{9}\frac{29}{18}}}$$
$$\approx -0.0255$$

Now,

$$Var[X] = \sum_x [x_i - E[X]]^2 \Pr(x)$$
$$= \left(1 - \frac{14}{9}\right)^2 \left(\frac{4}{9}\right)$$
$$+ \left(2 - \frac{14}{9}\right)^2 \left(\frac{5}{9}\right)$$
$$= \frac{20}{81}$$

$$Var[Y] = \sum_y [y_i - E[X]]^2 \Pr(y)$$
$$= \left(1 - \frac{29}{18}\right)^2 \left(\frac{7}{18}\right)$$
$$+ \left(2 - \frac{29}{18}\right)^2 \left(\frac{11}{18}\right)$$
$$= \frac{77}{324}$$

then,

$$\rho(X,Y) = \frac{\frac{12907}{2916}}{\sqrt{\frac{20}{81}\frac{77}{324}}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

d)

$$Var[X+2Y] = Var[X] + 4Var[Y] + 2(2)Var[X,Y] = \sigma_X^2 + 4\sigma_Y^2 + 4\sigma_{xy}$$
$$= \left(\frac{20}{81}\right) + 4\left(\frac{77}{324}\right) + 4\left(-\frac{1}{162}\right) \qquad = \frac{95}{81}$$

Question 2

$$
\begin{aligned}
P(X,Y) &= \int_0^1 \int_0^1 f_{X\cap Y}(x\cap y)dydx = \int_0^1 \int_0^1 (4xy)dydx \\
&= \int_0^1 2xy^2|_{y=0}^{y=1} dx = \int_0^1 2xdx \\
&= x|_{x=0}^{x=1} \\
&= 1
\end{aligned}
$$

a i)

$$
\begin{aligned}
P(Y<X) &= \int_0^1 \int_0^x f_{X\cap Y}(x\cap y)dydx = \int_0^1 \int_0^x (4xy)dydx \\
&= \int_0^1 2xy^2|_{y=0}^{y=x} dx = \int_0^1 2x^3dx \\
&= \frac{1}{2}x^4|_{x=0}^{x=1} \\
&= \frac{1}{2}
\end{aligned}
$$

a ii)

$$
\begin{aligned}
f_X(x) &= \int_0^1 f_{X\cap Y}(x\cap y)dy \\
&= \int_0^1 (4xy)dy \\
&= 2xy^2 \\
0 &< y < 1
\end{aligned}
\qquad\qquad
\begin{aligned}
f_Y(y) &= \int_0^1 f_{X\cap Y}(x\cap y)dx \\
&= \int_0^1 (4xy)dx \\
&= 2x^2y \\
0 &< x < 1
\end{aligned}
$$

hence,

$$
\begin{aligned}
f_X(x) &= 2xy^2|_{y=0}^{y=1} \\
&= 2x
\end{aligned}
\qquad\qquad
\begin{aligned}
f_Y(y) &= 2x^2y|_{x=0}^{x=1} \\
&= 2y
\end{aligned}
$$

a iii)

$$
f_{X\cap Y}(x\cap y) = 4xy \qquad\qquad f_X(x)f_Y(y) = (2x)(2y) = 4xy
$$

hence, X and Y are independent.

a iv)

$$\begin{aligned}E[X] &= \int_0^1 \int_0^1 x f_{X\cap Y}(x \cap y) dx dy = \int_0^1 \int_0^1 (4x^2 y) dx dy \\ &= \int_0^1 \frac{4}{3} x^3 y |_{x=0}^{x=1} dy = \int_0^1 \left(\frac{4}{3} y\right) dy \\ &= \frac{2}{3} |_{y=0}^{y=1} \\ &= \frac{2}{3}\end{aligned}$$

Similarly,

$$E[Y] = \frac{2}{3}$$

b)

$$f_{X\cap Y}(x \cap y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{if } otherwise \end{cases}$$

$$\begin{aligned}f_X(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{X\cap Y}(x \cap y) dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\ &= \frac{2\sqrt{1-x^2}}{\pi}\end{aligned}$$

$$\begin{aligned}f_{Y|X}(y|x) = \frac{f_{X\cap Y}(x \cap y)}{f_X(x)} &= \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-x^2}}{\pi}} \\ &= \frac{1}{2\sqrt{1-x^2}}\end{aligned}$$

c)

$$f_Y(y) = \int \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, exp\left(-\frac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \left(\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho)^2}\right) dx$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right) \int_{-\infty}^{\infty} exp\left(-\frac{\left(x-(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)^2}{2(1-\rho^2)\sigma_1^2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma_2} exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right)$$

now, $$f_{X|Y}(x|y) = \frac{f_{X\cap Y}(x\cap y)}{f_Y(y)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_2} exp\left(-\frac{\left(x-(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)^2}{2(1-\rho^2)\sigma_1^2}\right)$$

hence,

$$(X|Y=y) \sim \mathcal{N}\left((\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2), (1-\rho^2)\sigma_1^2\right)$$

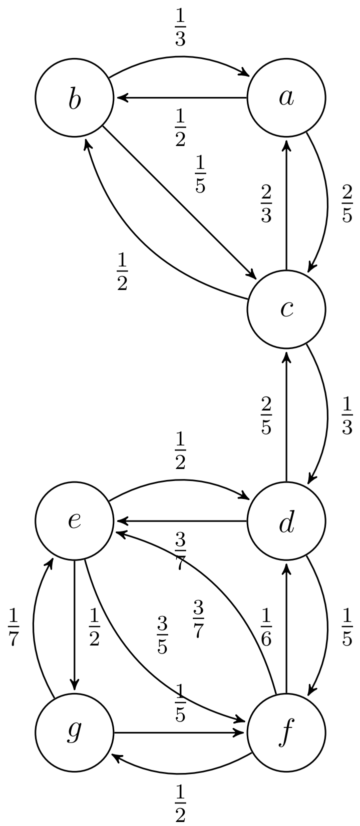
if X and Y are independent, then

$$f_{X\cap Y}(x\cap y) = f_X(x)f_Y(y)$$

I give up

Question 3

(a Weighted graph:



(b) Matlab code

```
clear all
close all
clc
P = [0 1/3 2/3 0 0 ;
    1/2 0 1/2 0 0 ;
    2/5 1/5 0 2/5 0;
    0 0 1/3 0 1/2 ;
    0 0 0 3/7 0 ];
I =  eye(5);
A = inv(I - P);
```

```
S = sum(A,2)
```

Matlab output

```
S =

   13.6000
   13.8000
   12.0000
    7.0000
    4.0000
```

Hence, the expected number of steps from a to f is 13.6

I give up