MTH719

Assignment 1

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I hereby declare that I am the sole author of this work.

1.a

$$\text{Given,} \quad uy''(t) + vy'(t) + wy(t) = f(t) \quad 1 < t < n+1$$
$$\text{and step size } h = \frac{a-b}{n}$$

$$\mathbf{A} = \begin{bmatrix} q & r & & 0 \\ p & \ddots & \ddots & \\ & \ddots & \ddots & r \\ 0 & & p & q \end{bmatrix}$$

$$\begin{aligned} \text{Where } p &= -u - v \times \frac{h}{2} \\ q &= 2 \times u - w \times h^2 \\ r &= -u + v \times \frac{h}{2} \end{aligned}$$

$$\vec{X} = \begin{bmatrix} y(2) \\ \vdots \\ y(n) \end{bmatrix} \quad \vec{B} = \begin{bmatrix} -f(2) \times h^2 - p \times \alpha \\ -f(3) \times h^2 \\ \vdots \\ -f(n-1) \times h^2 \\ -f(n) \times h^2 - p \times \beta \end{bmatrix}$$

$$\begin{aligned} \text{Where } \alpha &= y(1) \\ \beta &= y(n+1) \end{aligned}$$

and, $f(j)$ are known values

1.c

If computed values are $\hat{X} = \mathbf{A}^{-1}\vec{B}$

and actual values are $\vec{X}$

then accuracy for computed values is $\epsilon = ||\vec{X} - \hat{X}||$

Where the smaller ϵ is preferred.

2.a

We can convert the room diagram into a transition matrix, where the (i, j)-entry of the matrix corresponds to the probability that one person moves from from the jth room to the ith room.

$$\mathbf{A} = \begin{bmatrix} 0.3333 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0.3333 & 0.2500 & 0.3333 & 0 & 0.2000 & 0 & 0 & 0 & 0 \\ 0 & 0.2500 & 0.3333 & 0 & 0 & 0.2500 & 0 & 0 & 0 \\ 0.3333 & 0 & 0 & 0.2500 & 0.2000 & 0 & 0.3333 & 0 & 0 \\ 0 & 0.2500 & 0 & 0.2500 & 0.2000 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & 0.3333 & 0 & 0.2000 & 0.2500 & 0 & 0 & 0.3333 \\ 0 & 0 & 0 & 0.2500 & 0 & 0 & 0.3333 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & 0.2000 & 0 & 0.3333 & 0.2500 & 0.3333 \\ 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0.3333 \end{bmatrix}$$

Note that all of the columns add up to one.

Let the state of our system be represented by a probability vector

$$\vec{x_0} = \begin{bmatrix} x_1 \\ \vdots \\ x_9 \end{bmatrix}$$

where each entry represents the probability of being in that room at t_0.

If each time interval is represented by 15 minutes, then at t_1

$$\vec{x_1} = A^1 \times \vec{x_0}$$

$$=A^1 \times \begin{bmatrix} 30/100 \\ 40/100 \\ 0 \\ 0 \\ 10/100 \\ 0 \\ 0 \\ 20/100 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2000 \\ 0.2200 \\ 0.1000 \\ 0.1200 \\ 0.1700 \\ 0.0200 \\ 0.0500 \\ 0.0700 \\ 0.0500 \end{bmatrix}$$

And the number of people in each room is

$$= 100 \times \begin{bmatrix} 0.2000 \\ 0.2200 \\ 0.1000 \\ 0.1200 \\ 0.1700 \\ 0.0200 \\ 0.0500 \\ 0.0700 \\ 0.0500 \end{bmatrix} = \begin{bmatrix} 20 \\ 22 \\ 10 \\ 12 \\ 17 \\ 2 \\ 5 \\ 7 \\ 5 \end{bmatrix}$$

2.b

We know that $x_n = Ax_{n-1} = \ldots = A^n x_0$. Given x_j, we need to find x_{j-1}.

$$x_j = A \times x_{j-1} \qquad\qquad x_j = \begin{bmatrix} 0.1295 \\ 0.0518 \\ 0.1554 \\ 0.0777 \\ 0.2850 \\ 0.1036 \\ 0.1295 \\ 0.0518 \\ 0.0155 \end{bmatrix}$$

So, the probability distribution of the rooms at t_{j-1} is $x_{j-1} = A^{-1} \times x_j$ and number of people in the each room at t_{j-1} is $1100 \times x_{j-1}$

$$1100 \times x_{j-1} = 1100 \times A^{-1} x_j = \begin{bmatrix} 125 \\ 50 \\ 150 \\ 75 \\ 275 \\ 100 \\ 125 \\ 50 \\ 150 \end{bmatrix}$$

2.c

We know that $x_n = A^n x_0$.

We set $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then at t_4 is the first time that room 9 is filled.

$x_4 = A^4 \times x_0 = \begin{bmatrix} 0.1644 \\ 0.1654 \\ 0.0913 \\ 0.1654 \\ 0.1578 \\ 0.0669 \\ 0.0913 \\ 0.0669 \\ 0.0306 \end{bmatrix}$. Where each row is the distribution of the population at t_4.

2.d

We see that

$$\lim_{n\to\infty} A^n =$$

$$\begin{bmatrix} 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 \\ 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 \\ 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 \\ 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 \\ 0.1515 & 0.1515 & 0.1515 & 0.1515 & 0.1515 & 0.1515 & 0.1515 & 0.1515 & 0.1515 \\ 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 \\ 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 \\ 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 & 0.1212 \\ 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 & 0.0909 \end{bmatrix}$$

Hence the Markov chain is convergent, ie regardless of the initial vector, all nonzero vectors will converge to $\lim_{n\to\infty} x_n = \lim_{n\to\infty} A^n \times x_0$

$\lim_{x\to\infty} x_n = \begin{bmatrix} 0.0909 \\ 0.1212 \\ 0.0909 \\ 0.1212 \\ 0.1515 \\ 0.1212 \\ 0.0909 \\ 0.1212 \\ 0.0909 \end{bmatrix}$. Thus it is also the distribution of resources for each room.