We illustrate exhaustive search by applying it to three important problems: the traveling salesman problem, the knapsack problem, and the assignment problem.

Traveling Salesman Problem

The ***traveling salesman problem (TSP)*** has been intriguing researchers for the last 150 years by its seemingly simple formulation, important applications, and interesting connections to other combinatorial problems. In layman's terms, the problem asks to find the shortest tour through a given set of n cities that visits each city exactly once before returning to the city where it started. The problem can be conveniently modeled by a weighted graph, with the graph's vertices representing the cities and the edge weights specifying the distances. Then the problem can be stated as the problem of finding the shortest ***Hamiltonian circuit*** of the graph. (A Hamiltonian circuit is defined as a cycle that passes through all the vertices of the graph exactly once. It is named after the Irish mathematician Sir William Rowan Hamilton (1805–1865), who became interested in such cycles as an application of his algebraic discoveries.)

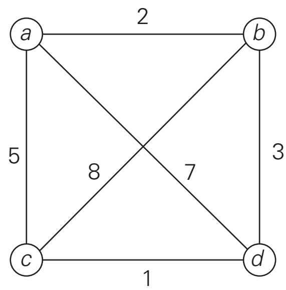
It is easy to see that a Hamiltonian circuit can also be defined as a sequence of $n+1$ adjacent vertices $v_{i_0}, v_{i_1}, \ldots, v_{i_{n-1}}, v_{i_0}$, where the first vertex of the sequence is the same as the last one and all the other $n-1$ vertices are distinct. Further, we can assume, with no loss of generality, that all circuits start and end at one particular vertex (they are cycles after all, are they not?). Thus, we can get all the tours by generating all the permutations of $n-1$ intermediate cities, compute the tour lengths, and find the shortest among them. Figure 3.7 presents a small instance of the problem and its solution by this method.

An inspection of Figure 3.7 reveals three pairs of tours that differ only by their direction. Hence, we could cut the number of vertex permutations by half. We could, for example, choose any two intermediate vertices, say, b and c, and then consider only permutations in which b precedes c. (This trick implicitly defines a tour's direction.)

This improvement cannot brighten the efficiency picture much, however. The total number of permutations needed is still $\frac{1}{2}(n-1)!$, which makes the exhaustive-search approach impractical for all but very small values of n. On the other hand, if you always see your glass as half-full, you can claim that cutting the work by half is nothing to sneeze at, even if you solve a small instance of the problem, especially by hand. Also note that had we not limited our investigation to the circuits starting at the same vertex, the number of permutations would have been even larger, by a factor of n.

Knapsack Problem

Here is another well-known problem in algorithmics. Given n items of known weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$ and a knapsack of capacity W, find the most valuable subset of the items that fit into the knapsack. If you do not like the idea of putting yourself in the shoes of a thief who wants to steal the most



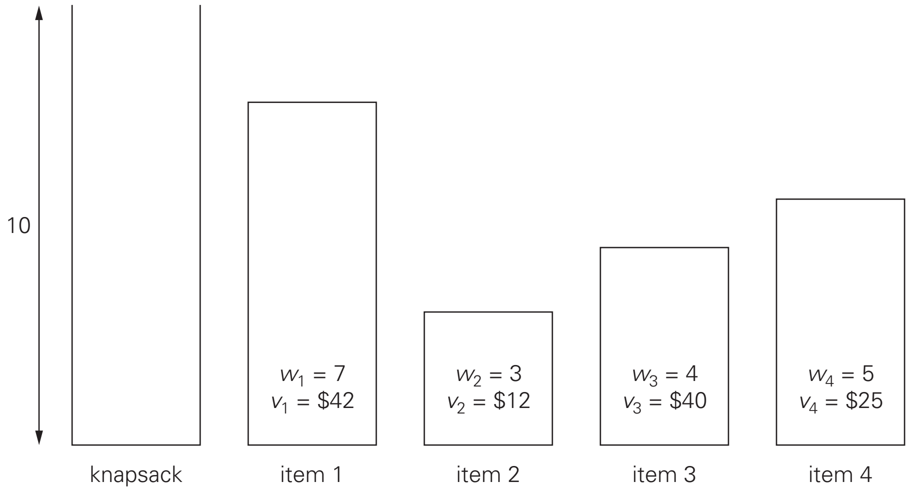
Tour	Length	
$a \dashrightarrow b \dashrightarrow c \dashrightarrow d \dashrightarrow a$	$l = 2 + 8 + 1 + 7 = 18$	
$a \dashrightarrow b \dashrightarrow d \dashrightarrow c \dashrightarrow a$	$l = 2 + 3 + 1 + 5 = 11$	optimal
$a \dashrightarrow c \dashrightarrow b \dashrightarrow d \dashrightarrow a$	$l = 5 + 8 + 3 + 7 = 23$	
$a \dashrightarrow c \dashrightarrow d \dashrightarrow b \dashrightarrow a$	$l = 5 + 1 + 3 + 2 = 11$	optimal
$a \dashrightarrow d \dashrightarrow b \dashrightarrow c \dashrightarrow a$	$l = 7 + 3 + 8 + 5 = 23$	
$a \dashrightarrow d \dashrightarrow c \dashrightarrow b \dashrightarrow a$	$l = 7 + 1 + 8 + 2 = 18$	

FIGURE 3.7 Solution to a small instance of the traveling salesman problem by exhaustive search.

valuable loot that fits into his knapsack, think about a transport plane that has to deliver the most valuable set of items to a remote location without exceeding the plane's capacity. Figure 3.8a presents a small instance of the knapsack problem.

The exhaustive-search approach to this problem leads to generating all the subsets of the set of n items given, computing the total weight of each subset in order to identify feasible subsets (i.e., the ones with the total weight not exceeding the knapsack capacity), and finding a subset of the largest value among them. As an example, the solution to the instance of Figure 3.8a is given in Figure 3.8b. Since the number of subsets of an n-element set is 2^n, the exhaustive search leads to a $\Omega(2^n)$ algorithm, no matter how efficiently individual subsets are generated.

Thus, for both the traveling salesman and knapsack problems considered above, exhaustive search leads to algorithms that are extremely inefficient on every input. In fact, these two problems are the best-known examples of so-called ***NP-hard problems***. No polynomial-time algorithm is known for any *NP*-hard problem. Moreover, most computer scientists believe that such algorithms do not exist, although this very important conjecture has never been proven. More-sophisticated approaches—backtracking and branch-and-bound (see Sections 12.1 and 12.2)—enable us to solve some but not all instances of these and



(a)

Subset	**Total weight**	**Total value**
∅	0	$ 0
{1}	7	$42
{2}	3	$12
{3}	4	$40
{4}	5	$25
{1, 2}	10	$54
{1, 3}	11	not feasible
{1, 4}	12	not feasible
{2, 3}	7	$52
{2, 4}	8	$37
{3, 4}	**9**	**$65**
{1, 2, 3}	14	not feasible
{1, 2, 4}	15	not feasible
{1, 3, 4}	16	not feasible
{2, 3, 4}	12	not feasible
{1, 2, 3, 4}	19	not feasible

(b)

FIGURE 3.8 (a) Instance of the knapsack problem. (b) Its solution by exhaustive search. The information about the optimal selection is in bold.

similar problems in less than exponential time. Alternatively, we can use one of many approximation algorithms, such as those described in Section 12.3.

Assignment Problem

In our third example of a problem that can be solved by exhaustive search, there are n people who need to be assigned to execute n jobs, one person per job. (That is, each person is assigned to exactly one job and each job is assigned to exactly one person.) The cost that would accrue if the ith person is assigned to the jth job is a known quantity $C[i, j]$ for each pair $i, j = 1, 2, \ldots, n$. The problem is to find an assignment with the minimum total cost.

A small instance of this problem follows, with the table entries representing the assignment costs $C[i, j]$:

	Job 1	**Job 2**	**Job 3**	**Job 4**
Person 1	9	2	7	8
Person 2	6	4	3	7
Person 3	5	8	1	8
Person 4	7	6	9	4

It is easy to see that an instance of the assignment problem is completely specified by its cost matrix C. In terms of this matrix, the problem is to select one element in each row of the matrix so that all selected elements are in different columns and the total sum of the selected elements is the smallest possible. Note that no obvious strategy for finding a solution works here. For example, we cannot select the smallest element in each row, because the smallest elements may happen to be in the same column. In fact, the smallest element in the entire matrix need not be a component of an optimal solution. Thus, opting for the exhaustive search may appear as an unavoidable evil.

We can describe feasible solutions to the assignment problem as n-tuples $\langle j_1, \ldots, j_n \rangle$ in which the ith component, $i = 1, \ldots, n$, indicates the column of the element selected in the ith row (i.e., the job number assigned to the ith person). For example, for the cost matrix above, $\langle 2, 3, 4, 1 \rangle$ indicates the assignment of Person 1 to Job 2, Person 2 to Job 3, Person 3 to Job 4, and Person 4 to Job 1. The requirements of the assignment problem imply that there is a one-to-one correspondence between feasible assignments and permutations of the first n integers. Therefore, the exhaustive-search approach to the assignment problem would require generating all the permutations of integers $1, 2, \ldots, n$, computing the total cost of each assignment by summing up the corresponding elements of the cost matrix, and finally selecting the one with the smallest sum. A few first iterations of applying this algorithm to the instance given above are shown in Figure 3.9; you are asked to complete it in the exercises.

$$C = \begin{bmatrix} 9 & 2 & 7 & 8 \\ 6 & 4 & 3 & 7 \\ 5 & 8 & 1 & 8 \\ 7 & 6 & 9 & 4 \end{bmatrix}$$

<1, 2, 3, 4>	cost = 9 + 4 + 1 + 4 = 18	
<1, 2, 4, 3>	cost = 9 + 4 + 8 + 9 = 30	
<1, 3, 2, 4>	cost = 9 + 3 + 8 + 4 = 24	
<1, 3, 4, 2>	cost = 9 + 3 + 8 + 6 = 26	etc.
<1, 4, 2, 3>	cost = 9 + 7 + 8 + 9 = 33	
<1, 4, 3, 2>	cost = 9 + 7 + 1 + 6 = 23	

FIGURE 3.9 First few iterations of solving a small instance of the assignment problem by exhaustive search.

Since the number of permutations to be considered for the general case of the assignment problem is $n!$, exhaustive search is impractical for all but very small instances of the problem. Fortunately, there is a much more efficient algorithm for this problem called the ***Hungarian method*** after the Hungarian mathematicians König and Egerváry, whose work underlies the method (see, e.g., [Kol95]).

This is good news: the fact that a problem domain grows exponentially or faster does not necessarily imply that there can be no efficient algorithm for solving it. In fact, we present several other examples of such problems later in the book. However, such examples are more of an exception to the rule. More often than not, there are no known polynomial-time algorithms for problems whose domain grows exponentially with instance size, provided we want to solve them exactly. And, as we mentioned above, such algorithms quite possibly do not exist.

Exercises 3.4

1. **a.** Assuming that each tour can be generated in constant time, what will be the efficiency class of the exhaustive-search algorithm outlined in the text for the traveling salesman problem?

   b. If this algorithm is programmed on a computer that makes ten billion additions per second, estimate the maximum number of cities for which the problem can be solved in

   i. 1 hour. **ii.** 24 hours. **iii.** 1 year. **iv.** 1 century.

2. Outline an exhaustive-search algorithm for the Hamiltonian circuit problem.

3. Outline an algorithm to determine whether a connected graph represented by its adjacency matrix has an Eulerian circuit. What is the efficiency class of your algorithm?

4. Complete the application of exhaustive search to the instance of the assignment problem started in the text.

5. Give an example of the assignment problem whose optimal solution does not include the smallest element of its cost matrix.