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Adaptive Routing and Recharging Policies for Electric Vehicles

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Abstract. Planning a trip with an electric vehicle requires consideration of both battery dynamics and the availability of charging infrastructure. Recharging costs for an electric vehicle, which increase as the battery's charge level increases, are fundamentally different than refueling costs for conventional vehicles, which do not depend on the amount of fuel already in the tank. Furthermore, the viability of any route requiring recharging is sensitive to the availability of charging stations along the way. In this paper, we study the problem of finding an optimal adaptive routing and recharging policy for an electric vehicle in a network. Each node in the network represents a charging station and has an associated probability of being available at any point in time or occupied by another vehicle. We develop efficient algorithms for finding an optimal a priori routing and recharging policy and then present solution approaches to an adaptive problem that build on a priori policy. We present two heuristic methods for finding adaptive policies—one with adaptive recharging decisions only and another with both adaptive routing and recharging decisions. We then further enhance our solution approaches to a special case of the grid network. We conduct numerical experiments to demonstrate the empirical performance of our solutions and provide insights to our findings.

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1. Introduction

Battery electric vehicles (EVs) have become a practical and affordable option in recent years for environmentally conscious drivers looking to reduce their carbon footprint (Klabjan and Sweda 2011). EVs are powered solely by electricity and recharge by plugging into an outlet or charging station. This feature offers the potential for significant fuel-cost savings as well as a host of other benefits, such as fewer greenhouse gas emissions, reduced dependence on foreign oil, and improved power systems management (if vehicle-to-grid services are enabled; Sioshansi and Denholm 2009, Sovacool and Hirsh 2009). Despite the numerous advantages that EVs have over conventional gasoline-powered vehicles, range anxiety—the worry that an EV's range is insufficient for a driver's commuting needs—remains a chief concern among many potential purchasers (Klabjan and Sweda 2011). Without the ability to easily recharge away from home, EV owners are restricted primarily to short-distance trips. The expansion of public charging infrastructure to increase opportunities for EV drivers to recharge their vehicles outside their homes has been one popular method for alleviating range anxiety and encouraging greater EV adoption (Bakker 2011). However, since each charging station can usually only recharge one or two vehicles at a time, and charge times can be on the order of hours, a driver who arrives at a fully occupied station may incur significant inconvenience (e.g., a long wait time) if no other, nearby charging station is available. Thus, EV drivers can greatly benefit from accounting for charging station availability and anticipating wait times at the stations while planning their routes, which is the focus of this paper.

We study an optimal EV routing problem in a network where trips may require multiple recharging stops along the way. Since the availability of charging stations is a critical factor in the planning of long-distance routes, an EV driver must not only select which path to take to arrive at the destination as quickly as possible but also decide where to recharge and what to do in case a desired charging station is unavailable (i.e., wait or seek an alternative station). The uncertainty of charging station availability and wait times within the network, as well as the driver's ability to adaptively make routing and recharging decisions, are unique and critical features of our problem. Furthermore, since the availability of each station may differ, the selection of stopping locations must be part of the routing decision.

In addition to station availability and waiting time, overcharging costs, incurred when an EV's battery is charged near its maximum capacity, are important to consider when creating EV recharging policies for a number of reasons. First, recharging an EV battery while it is already at a high state of charge takes place at a slower rate than when it is more depleted, and storing high levels of charge for prolonged periods of time can shorten the lifespan of the battery. A couple of models describing this relation can be found in Millner (2010) and Serrao et al. (2011). In addition, overcharging causes battery degradation because of greater stresses from being charged near full capacity and excess heat generated during recharging. In the refueling problem for conventional vehicles, the main disadvantage of traveling with a full tank of fuel is the limited ability to take advantage of lower fuel prices further along the route. Thus, optimal solutions tend to favor filling large quantities and making fewer stops (assuming that stopping costs are considered), but the opposite is often true for optimal EV recharging policies. Therefore, the trade-off among total recharging time, number of charging stops, station availabilities, and wait times must be captured in our EV routing and recharging policy. We consider the problem on a given network with charging stations positioned at every node, each with a corresponding probability of being available and expected waiting time until becoming available (known a priori to the driver). Our goal is to determine an adaptive routing and recharging policy that minimizes the sum of all traveling, waiting, and recharging costs. We assume that whenever the vehicle stops to recharge, it incurs a fixed stopping cost, a charging cost based on the total amount it recharges, and an additional cost when the battery becomes overcharged.

In this paper, we develop and analyze a variety of models depending on the type of decisions to be made and the amount and timing of information available to the EV driver while traveling. In our first model, we consider an optimal adaptive recharging problem for a fixed path, where information about a station's availability is revealed to the driver on arriving at that location, and the driver decides at that time whether to stop. The second model aims to find an optimal adaptive routing and recharging policy for a general network, where path selection is added to the decision space. This model captures more proactive behavior that some drivers might exhibit to reduce their waiting time at the stations. As the computational complexity of the general network setting makes the problem intractable, we develop heuristic solution approaches. First, we find an a priori policy corresponding to an optimal route and charging locations that minimizes the total expected cost. Then, we propose two solution methods for adaptive decision making based on the a priori policies: (1) the driver is committed to a path corresponding to the optimal a priori policy but can change the charging locations along the way; and (2) the driver can adapt both the recharging locations and path in response to the realized availability of charging stations. For the special class of problems with a grid network, we perform an in-depth analysis and further improve the proposed solution methods by taking advantage of the grid structure. A numerical study demonstrates the performance and solution quality of our algorithms using simulated charging infrastructure networks and also provides some insights to the problem at hand.

This paper is the first in the literature to consider adaptive routing and recharging (or refueling) for range-constrained vehicles. It is also the first to implement two features together that are unique to EVs: overcharging costs and uncertain charging station availability. It contrasts our earlier work on optimal EV recharging policies for a fixed path in a deterministic setting (Sweda, Dolinskaya, and Klabjan 2017) by introducing stochastic station availability, which motivates the adaptive decision making studied here. In addition, the heterogeneous charging stations with their distinct probabilities of being available and expected waiting times make the path selection nontrivial as the EV balances the trade-off between longer paths and opportunities to visit more promising charging stations. Thus, the main contributions of this paper are: (i) properties of optimal adaptive and a priori recharging policies that consider EV overcharging characteristics and uncertain charging station availability; (ii) efficient solution procedures for obtaining a priori and adaptive routing and recharging policies in a network; (iii) models capturing and analyzing various levels of adaptive decision making and information timing; and (iv) an illustrative example and numerical study that establish the value of adaptive decision making in a variety of settings.

The remainder of the paper is organized as follows. Section 2 provides an overview of the existing literature related to EV recharging as it pertains to routing decisions. Section 3 describes the general problem setting and provides a motivating example. In Section 4, we present an efficient solution method to find an optimal recharging locations policy for a fixed path. Section 5 analyzes an adaptive routing and recharging policy for a general network. Here, the presented problem complexity motivates heuristic solution approaches that build on a priori routing and recharging policies (also discussed in Section 5). To further improve the proposed solution methods, we study a special class of grid network problems in Section 6. Section 7 presents some discussion on model extensions. Subsequently, a numerical study is exhibited in Section 8 to demonstrate the performance of the solution algorithms presented in the earlier sections. Finally, Section 9 presents the conclusion of this paper.

2. Literature Review

The vehicle refueling problem was first introduced in 1981 (Ichimori, Ishii, and Nishida 1981) with the objective of finding a shortest path from an origin to a destination, where a vehicle might need to visit some refueling nodes along the way to avoid running out of fuel. No refueling costs are considered in this model other than the detour required to visit a refueling node. This problem was later extended to a setting with a fixed stopping cost for each refueling visit in Mirchandani, Adler, and Madsen (2014). A variation of the vehicle refueling problem in which a driver must decide at which nodes to refuel as well as how much to refuel to minimize the total cost of fuel, has been well studied. It is shown in Khuller, Malekian, and Mestre (2007) and Lin, Gertsch, and Russell (2007) that the optimal refueling policy along a fixed path can be solved easily with dynamic programming when fuel prices at each node are static and deterministic. For such a problem, the optimal decision at each node is always one of the following: do not refuel, refuel completely, or refuel just enough to reach the next node where refueling occurs. An algorithm for simultaneously finding the optimal path and refueling policy in a network is detailed in Lin (2008a), and some combinatorial properties of the optimal policies are explored in Lin (2008b). Specifically, it is proven that the problem of finding all-pairs optimal refueling policies reduces to an all-pairs shortest path problem that can be solved in polynomial time. However, all of the aforementioned analyses consider only fuel costs and not stopping or other costs. We include these additional costs in our analysis since they can comprise a significant portion of the total travel cost for an EV and therefore can influence optimal policies.

Several existing models have expanded on the vehicle refueling problem by introducing costs for stopping to refuel and traveling to refueling stations. A generic model for vehicle refueling is presented in Suzuki (2008) that attempts to capture such aspects, penalizing longer routes and routes with more refueling stops. Similar to other papers that study the vehicle refueling problem, it assumes that fuel prices at each station are static and deterministic. An analogous model is proposed in Suzuki and Dai (2013), which seeks to minimize both the fuel cost and travel distance simultaneously. The model in Arslan, Yildiz, and Karaşan (2015) studies a similar problem for plug-in hybrid EVs in which the vehicle must stop at both refueling and recharging stations. It seeks to minimize the sum of fuel and energy costs, stopping costs, depreciation costs (as a function of distance traveled), and battery degradation costs, which are explicitly captured.

Approaches for finding optimal refueling policies when fuel prices are stochastic are presented in Klampfl et al. (2008) and Suzuki (2009). In Klampfl et al. (2008), a forecasting model for predicting future fuel prices is used to generate parameters for a deterministic mixed-integer program, and in Suzuki (2009), a dynamic programming framework is presented that is designed to grant drivers greater autonomy to select the stations where they refuel. These models are difficult to solve analytically, and the authors develop heuristics for obtaining reasonable solutions. In addition, just like the other models of the vehicle refueling problem, these do not include any costs that are analogous to battery overcharging costs for EVs, which can be an important component (Millner 2010, Serrao et al. 2011). Sweda and Klabjan (2012) address this issue by introducing generalized charging cost functions; however, some restrictive assumptions are required to perform insightful analysis. A more specific, yet tractable overcharging cost function is analyzed in Sweda, Dolinskaya, and Klabjan (2017), and optimal recharging policies along a fixed path are determined. We apply the overcharging cost function from Sweda, Dolinskaya, and Klabjan (2017) to a realistic problem setting on a general network, incorporating the effects of heterogeneous and uncertain station availabilities on EV decision making in adaptive path finding and adaptive recharging policies.

To solve the problem of finding a path for an EV within a network with recharging considerations, one thread of research has taken an entirely different approach: having vehicles recharge via regenerative braking rather than by recharging at stations along their paths. As an EV decelerates, it can recapture some of its lost kinetic energy as electrical energy, which can then be used to recharge the battery. It is therefore possible in some cases for an EV's state of charge to *increase* while traveling rather than decrease, such as when the vehicle is coasting and braking downhill. In Artmeier et al. (2010), the authors model the problem of finding the most energy-efficient path for an EV in a network as a shortest path problem with constraints on the charge level of the vehicle, such that the charge level can never be negative and cannot exceed the maximum charge level of the battery. Edge weights are permitted to be negative to represent energy recapturing from regenerative braking and yet no negative cycles exist. A simple algorithm for solving the problem is provided, and more efficient algorithms are presented in Eisner, Funke, and Storandt (2011) and Sachenbacher et al. (2011). It is shown in Eisner, Funke, and Storandt (2011) that the battery capacity constraints can be modeled as cost functions on the edges, and a transformation of the edge cost functions permits the application of Dijkstra's algorithm. The approach described in Sachenbacher et al. (2011) avoids the use of preprocessing techniques so that edge costs can be calculated dynamically, and it achieves an order of magnitude reduction in the time complexity of the algorithm from Artmeier et al. (2010). In practice, however, the amount of energy recovered by regenerative braking is insignificant when compared with the amount that must be recharged at charging stations, and these papers do not model recharging decisions at nodes. Consequently, they also do not capture overcharging costs considered in the presented work. All of the aforementioned models assume that there is no downtime for refueling or recharging stations, and that a vehicle never needs to wait to use a station. Although this may be valid for gasoline stations, which can typically accommodate multiple vehicles at the same time and often have negligible wait times, it is not necessarily the case for electric charging stations. Most charging stations have only one or two plugs, and due to the long time that it takes to recharge a vehicle, the availability of a charging station can become an issue when multiple vehicles are present.

As the application of EVs emerges for commercial sectors such as taxi services and urban pickups and deliveries, the academic research follows. To facilitate the introduction of EVs to taxi service providers, problems such as where to locate charging stations and how to handle other aspects of taxi operations have been recently studied in the literature. Simulations of EV taxi operations based on current demands and potential charging station locations in Korea (Jung, Jayakrishnan, and Choi 2012), Munich, Germany (Sellmair and Hamacher 2014), and Vienna, Austria (Asamera et al. 2016, Hess et al. 2012), to name a few, have assessed the feasibility of their adoptions. However, these studies center on an aggregate or planning level of the taxi operations and assess the economic potential of such services. As a result, they make numerous simplifying assumptions about the EV recharging function. The simulation model in Asamera et al. (2016) does not explicitly capture recharging characteristics of an EV or the travel time or paths taken to those stations. Hess et al. (2012), Sellmair and Hamacher (2014); and Zhu et al. (2014) assume a linear charging time model as a function of charge level, while Jung, Jayakrishnan, and Choi (2012) consider a fixed time to charge or swap a depleted battery every time an EV visits a charging station. While most of the existing simulations do not incorporate charging station capacity constraints, some do. More specifically, the simulation model presented in Hess et al. (2012) requires an EV to travel to the closest charging station whenever the state of charge drops below a certain threshold level, queueing at the station if necessary and waiting until it receives service. Niu and Zhang (2015) use the actual electric taxi data from Shenzhen, China, to develop a method to guide the taxi to appropriate charging stations to balance the loads among the stations. While their work integrates a metric to capture more preferential stations for each taxi as part of their objective function, no explicit routing to those stations is considered by the model. Zhu et al. (2014) go even further and develop a scheduling algorithm that assigns each EV to a charging station with the goal of minimizing the total vehicle traveling, waiting, and recharging time, again neglecting the unique characteristics of EV battery recharging dynamics.

Over the past few years, considerable research has also been conducted to study EV routing problems (EVRP), often with time window constraints on when each customer must be serviced. (For example, see Pelletier, Jabali, and Laporte 2015 for a survey specific to goods distribution with EVs.) This body of work extends the well-studied vehicle routing problem by introducing a limit on how far a vehicle can travel before it runs out of charge (or, more generally, fuel) and has to visit one of the refueling stations. However, all of the work on EVRP assumes that charging stations are always available, and only deterministic problem settings are considered. In addition, as in the case of the electrical taxi problem, the authors make simplifying assumptions on the recharging function and do not capture the unique characteristics of an EV's battery. More specifically, Conrad and Figliozzi (2011) and Erdogan and Miller-Hooks (2012) assume a constant charging cost (or time), while Felipe et al. (2014), Hiermann et al. (2016), Desaulniers et al. (2016), Keskin and Catay (2016); and Schneider, Stenger, and Goeke (2014) assume a linear cost as a function of the quantity to be refueled or recharged. Furthermore, Conrad and Figliozzi (2011) assumes that the vehicle always recharges to a threshold value of 80% of battery capacity, and Erdogan and Miller-Hooks (2012), Hiermann et al. (2016); and Schneider, Stenger, and Goeke (2014) assume that the vehicle always recharges to the maximum capacity level.

Aside from the work specifically focused on EVs, substantial literature exists on stochastic variations of the shortest path problem (Frank 1969), traveling salesman problem (Cordeau et al. 2007, Gendreau, Laporte, and Seguin 1996, Jaillet 1985, Toriello, Haskell, and Poremba 2014), vehicle routing problem (Bertsimas 1992, Cordeau et al. 2007, Ferrucci 2013, Gendreau, Laporte, and Seguin 1996, Jaillet, Qi, and Sim 2016), and orienteering problem (Campbell, Gendreau, and Thomas 2011, Gupta et al. 2012, Ilhan, Iravani, and Daskin 2008). While the objectives, constraints, and sources of uncertainty differ among these problems, the general approach is similar. Distributions for arc travel times, service times, time windows, demand, or other problem parameters are given.

Most of the solution methods handle uncertainty in one of three ways. The first approach finds an a priori solution with a certain expected performance measure over all possible realizations of uncertainty or a mean value over sampled realizations. Alternatively, when recourse actions are allowed, the problems are modeled in two stages: an a priori–like solution is constructed first; then, all stochastic information is revealed at the beginning of the second stage, and the agent has an option to alter its initial solution at a penalty cost. In the third variation of this class of problems, the stochastic parameters are assumed to be realized locally when the agent traverses an arc or arrives at a node. The solution approaches developed for these problems find a policy that optimizes the expected or worst case objective function values, or guarantees a certain level of service, such as achieving a benchmark level with maximum probability. The approach used in our work falls into the adaptive decision making category where information about a charging station is realized on arrival. However, none of the existing work considers the problem setting with adaptive routing and recharging policies for EVs that is studied in this paper.

The body of work that is perhaps most relevant to the study presented in this paper involves anticipatory routing, where the goal is to anticipate information before it becomes available and minimize the total expected travel cost (e.g., see surveys by Larsen, Madsen, and Solomon 2008, Pillac et al. 2013, Psaraftis, Wen, and Kontovas 2015). In Thomas and White (2004), a vehicle travels from an origin to a destination. Along the way, customers may request service according to known distributions. Customer requests may be realized at any point along the vehicle's path, and the vehicle must respond to all requests. An optimal route, therefore, is not the shortest path between the origin and destination, but instead includes detours to approach potential customers in the event they request service. A similar model in which congestion along arcs is the anticipated information is studied in Thomas and White (2007), and in both papers, structural properties of optimal anticipatory policies are derived. However, the models only consider routing decisions and not refueling decisions.

3. Problem Setting and Motivation

In this section, we detail the setting of our adaptive routing and recharging problem for EVs. We also provide a motivating example to demonstrate the potential benefits of adaptive routing and recharging.

3.1. Problem Setting

Consider an EV with zero initial charge that must travel within a network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ from origin node o to destination node n, where o and n are in the node set $\mathcal{N}$, and $\mathcal{E}$ is the set of edges. Every node $i \in \mathcal{N}$ has a charging station, but the availability of each station is uncertain (e.g., due to being occupied by another vehicle) and is denoted by a random variable A_i. When the vehicle arrives at node i and finds the station available (i.e., $A_i = 1$), it may begin recharging immediately. If the charging station is unavailable (i.e., $A_i = 0$), the driver must wait a time interval W_i until the station becomes available before recharging, where W_i is also a random variable. We assume that for each $i \in \mathcal{N}$, the driver knows the probability of station i being available, $P(A_i = 1)$, and the expected waiting time when it is unavailable, $E[W_i \mid A_i = 0]$, a priori. Since there is no waiting time when the station is available (i.e., $P(W_i = 0 \mid A_i = 1) = 1$), we let $\mathscr{W}_i = E[W_i \mid A_i = 0]$ to simplify the notation. The values of the random variables A_i and W_i are realized when the EV arrives at node i and observes the state of the charging station, and they are assumed to be independent of previous realizations observed by the vehicle if it visited the node previously.

As the driver selects a path to travel from o to n, she must decide which nodes to visit, whether to stop and recharge at each visited node (station), and how much to recharge at each stop. The vehicle's charge level can never exceed the maximum capacity of the battery, $q_{\max}$, and it can never drop below zero. We also do not allow the vehicle to discharge energy back to the grid. We let q_i denote the charge level of the vehicle when it arrives at node i, and for any pair of nodes i and j such that $(i, j) \in \mathcal{E}$, we let $t_{ij} > 0$ and $h_{ij} > 0$ denote the time and amount of charge, respectively, required to traverse (i, j). We make the following monotonicity assumption for quantities t_{ij} and h_{ij}.

Assumption 1. *For any pair of edges $(i, j), (k, \ell) \in \mathcal{E}$ we have $t_{ij} \leq t_{k\ell}$ if and only if $h_{ij} \leq h_{k\ell}$.*

Then, the set of feasible charging amounts at node i prior to traveling along an edge (i, j), which we denote by $R_{ij}(q_i)$, is

$$R_{ij}(q_i) = [(h_{ij} - q_i)^+,\ q_{\max} - q_i]. \tag{1}$$

We assume that $h_{ij} \leq q_{\max}$ for all $(i, j) \in \mathcal{E}$ to ensure feasibility.

Each time the vehicle stops to recharge, it incurs a fixed *stopping cost* s. The vehicle also incurs a *recharging cost*, and we let $c(r, q)$ denote the cost of recharging amount of energy r when the vehicle's initial charge level is q. The cost function $c(r, q)$ includes an *overcharging cost* corresponding to recharging beyond the point at which the charging voltage reaches its maximum value and the charging current begins to decrease. Thus, the overcharging cost accounts for the additional time per unit of energy recharged (caused by the decreasing current). As a result, the cost function

$c(r,q)$ is assumed to be monotone increasing and convex. Such properties are reflective of actual battery dynamics that make it undesirable to regularly overcharge the battery. (A more detailed model of EV battery recharging is described in Sweda, Dolinskaya, and Klabjan 2017.) We assume that all charging stations have identical cost function $c(\cdot,\cdot)$, which is a reasonable assumption since most public charging stations in existence today have similar hardware configurations (most recharge at 220 volts, with the exception of a few "fast charging" stations that recharge at 440 volts; U.S. Department of Energy 2014a). Furthermore, regional variations in electricity rates are minimal.

We formulate our problem using a dynamic programming (DP) approach as follows.

Objective: Find a minimum-cost routing and recharging policy from an origin node o to a destination node n. The total cost is measured in units of time and is equal to the sum of all travel time, waiting time for available charging stations, and charging costs (i.e., stopping, recharging, and overcharging costs).

State space: The state space, which we denote by $\mathcal{S}$, is defined as

$$\mathcal{S} = \{(i, q_i, a_i)\colon i \in \mathcal{N},\, q_i \in [0, q_{\max}],\, a_i \in \{0,1\}\}.$$

Each state consists of three components: the current vehicle position (node i), the vehicle's charge level q_i on arrival at the node, and the realized availability of the charging station at the node, denoted by a_i.

Action space: We denote the action space by

$$\mathcal{A}_{(i,q_i)} = \{(j, r_i)\colon (i,j) \in \mathcal{E},\, r_i \in R_{ij}(q_i)\},$$

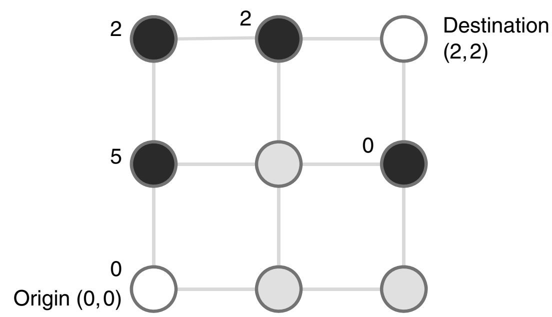
where each action consists of the next node to visit, j, along with the amount to recharge at the current node, r_i. The next node must be adjacent to the current node, and the recharging amount must be feasible (where $R_{ij}(q_i)$ is as defined in (1)) to allow the vehicle to reach the next node.

Value function: We let $V(i, q_i, a_i)$ denote the value function, which represents the minimum expected cost of traveling to the destination n from node i given that the current state is (i, q_i, a_i). We also let $I_{r_i>0}$ denote the indicator function that equals one if $r_i > 0$ (i.e., if the vehicle stops to recharge at node i) and zero otherwise. Then, the value function can be defined recursively as

$$\begin{aligned} V(i,q_i,a_i) &= \min_{(j,r_i)\in\mathcal{A}_{(i,q_i)}} \{t_{ij} + (s + E[W_i \mid A_i = a_i] \\ &\quad + c(r_i,q_i))I_{r_i>0} + E_{A_j}[V(j, q_i + r_i - h_{ij}, A_j)]\} \\ &= \min_{(j,r_i)\in\mathcal{A}_{(i,q_i)}} \{t_{ij} + (s + (1-a_i)\mathcal{W}_i)I_{r_i>0} + c(r_i,q_i) \\ &\quad + [P(A_j = 1)]V(j, q_i + r_i - h_{ij}, 1) \\ &\quad + [1 - P(A_j = 1)]V(j, q_i + r_i - h_{ij}, 0)\}, \end{aligned} \qquad (2)$$

since $E[W_i \mid A_i = 1] = 0$ and $c(0, q_i) = 0$ for any value of q_i. We set $V(n,\cdot,\cdot) = 0$ and seek to evaluate

Figure 1. Example Network with Origin at $(0,0)$ and Destination at $(2,2)$



Notes. Labels represent $\mathcal{W}_i$, and $P(A_i = 1) = 0.5$ for all black nodes and the origin (gray nodes do not have charging stations; i.e., $P(A_i = 1) = 0$ and $\mathcal{W}_i = \infty$).

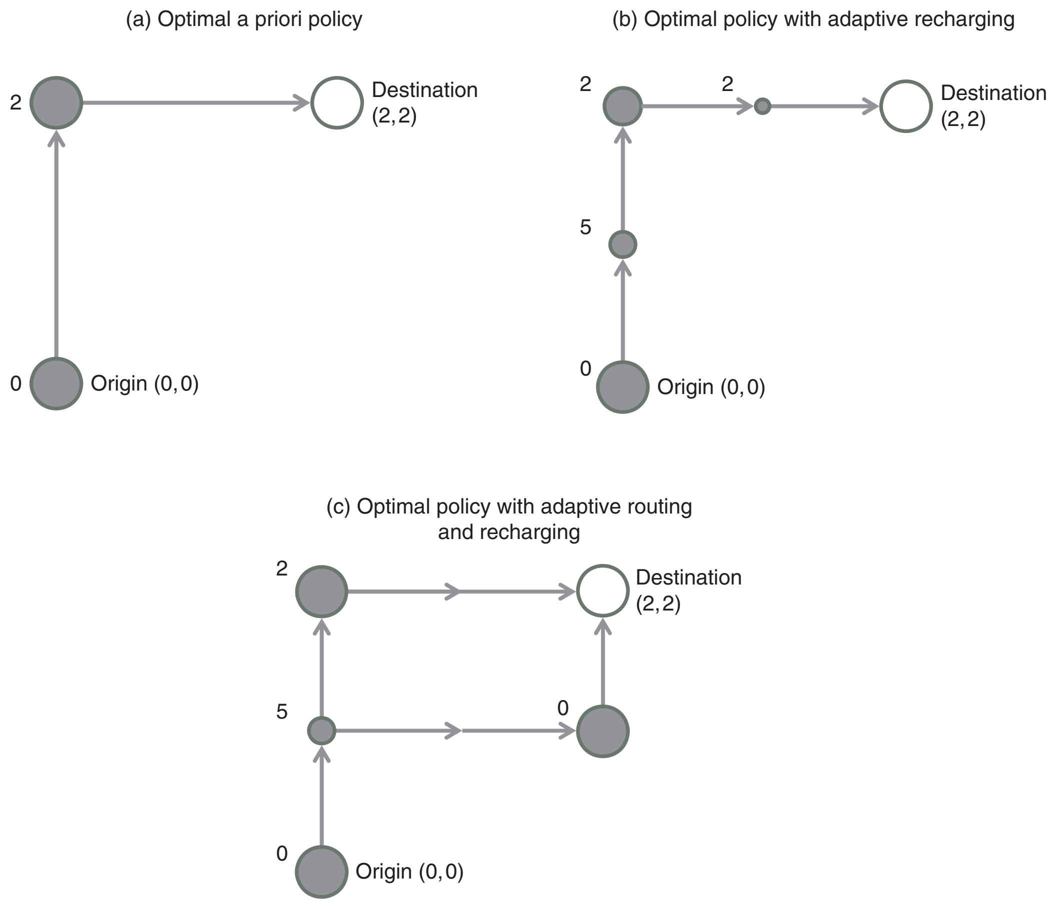
$E[V(o, 0, A_o)]$, the minimum expected total cost of traveling from o to n when the vehicle's initial charge level is zero.

3.2. Motivation for Adaptive Decision Making

To see the benefits of adaptive routing and recharging decisions, consider the example shown in Figure 1. The road network consists of nine nodes arranged in a square grid, with the origin $(0,0)$ and destination $(2,2)$ at opposite corners, and $q_{\max} = 2h$, where h is the amount of charge necessary to traverse any edge. The label next to each node on the figure corresponds to the expected wait time when the EV finds the station unavailable on arrival (i.e., $\mathcal{W}_i$), and the availability probabilities for each node are 0.5 (i.e., $P(A_i = 1) = 0.5$). Furthermore, the nodes without any labels (those shaded in light grey) do not have any recharging facilities; that is, $P(A_i = 1) = 0$ and $\mathcal{W}_i = \infty$. For illustration purposes, we consider only waiting costs and neglect any stopping, recharging, overcharging, and traveling costs.

In this setting, an optimal a priori policy, shown in Figure 2(a), is to recharge $2h$ at the origin and $2h$ at node $(0,2)$ with a total expected waiting cost of $0.5\cdot(0) + 0.5\cdot(2) = 1$. Now, consider an adaptive policy where we allow the driver to change the recharging decisions dynamically, but we fix the path to an a priori optimal path, as shown in Figure 2(b). Thus, we allow the vehicle to recharge at any node along the path in response to the observed station availability on arrival at the node. The optimal policy in such a case is to recharge $2h$ at the origin, and if the charging station at node $(0,1)$ is unavailable when the vehicle arrives, then the vehicle should recharge zero at $(0,1)$, $2h$ at $(0,2)$, and zero at $(1,2)$. Otherwise, if the driver finds the station at node $(0,1)$ to be available, then the vehicle should recharge h at node $(0,1)$. The EV then does not have to recharge at node $(0,2)$ and will not do so unless that station is also available on arrival. That is, the driver will recharge h at node $(0,2)$ if the station there

Figure 2. Optimal Policies with (a) No Adaptivity, (b) Adaptive Recharging Only, and (c) Adaptive Routing and Recharging



Notes. The size of each node corresponds to the probability that the vehicle recharges at the node under the specified policy given that the vehicle visits the node, with larger sizes representing higher probabilities.

is available (and then zero at $(1,2)$) or will recharge h at $(1,2)$ and zero at $(0,2)$ if the station at node $(0,2)$ is unavailable. The cost of such a policy is the same as that of the optimal a priori policy when the station at node $(0,1)$ is unavailable; yet, when that station is available, the only scenario in which the expected waiting cost is nonzero is if the stations at both $(0,2)$ and $(1,2)$ are unavailable, in which case the expected waiting cost is two. The overall expected waiting cost of the optimal policy with adaptive recharging decisions is therefore $0.5\cdot(1)+0.5\cdot(0.75\cdot(0)+0.25\cdot(2))=0.75$, which is 0.25 less than the cost of the optimal a priori policy.

In the model where both adaptive routing and recharging decisions are allowed, note that the optimal policy when the station at node $(0,1)$ is available is to recharge h there and then proceed to node $(2,1)$, recharging zero at $(1,1)$ and h at $(2,1)$ at zero total cost to reach the destination (see Figure 2(c)). Otherwise, when the station at node $(0,1)$ is unavailable, the optimal policy is the same as the optimal a priori policy. The cost of the optimal policy with adaptive routing and recharging is thus $0.5\cdot(0)+0.5\cdot(1)=0.5$, which is 0.25 less than the cost of the optimal policy with adaptive recharging only and 0.5 less than the cost of the optimal a priori policy. Therefore, both adaptive routing and recharging decisions can lead to a significant reduction in waiting cost.

4. Adaptive Recharging Policy for a Fixed Path

In this section, we solve for an optimal recharging policy along a fixed path $\mathcal{N}=(0,\ldots,n)$ consisting of a sequence of $n+1$ nodes. To simplify the notation, we let h_i and t_i denote the energy and time, respectively, required to travel from node i to node $i+1$. Because there are no routing decisions to be made and the vehicle always travels forward along the path, the choice of which node to visit next is fixed and thus the action space reduces to

$$\mathcal{A}_{(i,q_i)}=\{(j,r_i)\colon j=i+1,\ r_i\in[(h_i-q_i)^+,q_{\max}-q_i]\}.$$

If we instead embed the fixed path in the value function representation from (2) and use

$$\mathcal{A}^R_{(i,q_i)}=\{r_i\colon r_i\in[(h_i-q_i)^+,q_{\max}-q_i]\}$$

to denote the part of the action space consisting only of the feasible recharging amounts at each node, then we can rewrite the value function as

$$\begin{aligned}V(i,q_i,a_i) = \min_{r_i \in \mathcal{A}^R_{(i,q_i)}} \{ & t_i + (s + (1-a_i)\mathcal{W}_i) I_{r_i>0} + c(r_i,q_i) \\ & + [P(A_{i+1}=1)] V(i+1, q_i + r_i - h_i, 1) \\ & + [1 - P(A_{i+1}=1)] V(i+1, q_i + r_i - h_i, 0)\}.\end{aligned}$$

To denote a feasible recharging policy for the path, we use $\pi^R_{\mathcal{N}}: \mathcal{S} \to \mathcal{A}^R$, with $\pi^R_{\mathcal{N}}(i,q_i,a_i) \in \mathcal{A}^R_{(i,q_i)}$ representing the recharging amount specified by the policy $\pi^R_{\mathcal{N}}$ for the state (i,q_i,a_i). The expected cost of reaching the end of the path under such a policy for a given state (i,q_i,a_i), which we denote $C_{\pi^R_{\mathcal{N}}}(i,q_i,a_i)$, satisfies

$$\begin{aligned}& C_{\pi^R_{\mathcal{N}}}(i,q_i,a_i) \\ &\quad = t_i + (s + (1-a_i)\mathcal{W}_i) I_{\pi^R_{\mathcal{N}}(i,q_i,a_i)>0} + c(\pi^R_{\mathcal{N}}(i,q_i,a_i), q_i) \\ &\qquad + [P(A_{i+1}=1)] C_{\pi^R_{\mathcal{N}}}(i+1, q_i + \pi^R_{\mathcal{N}}(i,q_i,a_i) - h_i, 1) \\ &\qquad + [1 - P(A_{i+1}=1)] C_{\pi^R_{\mathcal{N}}}(i+1, q_i + \pi^R_{\mathcal{N}}(i,q_i,a_i) - h_i, 0),\end{aligned}$$

where $I_{\pi^R_{\mathcal{N}}(i,q_i,a_i)>0}$ equals one if $\pi^R_{\mathcal{N}}(i,q_i,a_i) > 0$ and zero otherwise. We let $\Pi^R_{\mathcal{N}}$ denote the set of all feasible recharging policies for the path $\mathcal{N}$, and thus the value function can be expressed in terms of the cost function $C_{\pi^R_{\mathcal{N}}}(\cdot)$ as

$$V(i,q_i,a_i) = \min_{\pi^R_{\mathcal{N}} \in \Pi^R_{\mathcal{N}}} C_{\pi^R_{\mathcal{N}}}(i,q_i,a_i).$$

Finding an optimal recharging policy is not practical when the vehicle's charge level can take on any value in the interval $[0, q_{\max}]$ since this corresponds to a continuous action space and state space. However, without loss of optimality, we can restrict the possible values for the recharging amounts at each node to a discrete set. We then consider only finitely many values for the vehicle's charge level, thereby making the problem of finding an optimal recharging policy tractable. The following two lemmas show that there exists an optimal recharging policy such that all of the recharging amounts (and also incoming charge levels at each node) can be expressed as sums of the h_i values.

Lemma 1. *There exists an optimal recharging policy $\pi^{R^*}_{\mathcal{N}}$: $\mathcal{S} \to \mathcal{A}^R$ such that*

$$\pi^{R^*}_{\mathcal{N}}(i,q_i,a_i) \in \left\{ \left(\sum_{\ell=i}^{k-1} h_\ell - q_i \right)^+ : k \in (i+1, \ldots, n-1), \sum_{\ell=i}^{k-1} h_\ell \le q_{\max} \right\}$$

for all $(i,q_i,a_i) \in \mathcal{S}$.

Proof. See Online Appendix A.2.1. □

Since we assume that the vehicle's charge level is zero at the beginning of the path, it follows from Lemma 1 that there exists an optimal recharging policy such that every incoming charge level and recharging amount can be expressed as a summation of the h_ℓ parameters. We prove this claim in the following lemma.

Lemma 2. *Let*

$$\mathcal{H}(i) = \left\{ \sum_{\ell=i}^{k-1} h_\ell : k \in (i, \ldots, n-1), \sum_{\ell=i}^{k-1} h_\ell \le q_{\max} \right\}$$

denote the set of feasible charge levels at node i that can be expressed as a summation of the h_ℓ parameters. Then, there exists an optimal recharging policy $\pi^{R^*}_{\mathcal{N}}: \mathcal{S} \to \mathcal{A}^R$ such that $q_i = \sum_{\ell=i}^{k-1} h_\ell \in \mathcal{H}(i)$ for some $k \in (i, \ldots, n-1)$ and $\pi^{R^*}_{\mathcal{N}}(i,q_i,a_i) \in \mathcal{H}(k)$ for any realized state $(i,q_i,a_i) \in \mathcal{S}$, given that the initial state is $(0,0,\cdot)$.

Proof. See Online Appendix A.2.2. □

By Lemma 2, without loss of optimality, we can create a value function representation that considers only a discrete set of values for both the charge level and recharging amount at each node. If we let $V(i,k,a_i)$ denote the value function representing the minimum expected cost of reaching node n when we arrive at node i with charge level $\sum_{\ell=i}^{k-1} h_\ell$ (i.e., just enough to reach node k) and the station availability is a_i, then we can define the value function recursively as

$$\begin{aligned}V(i,k,a_i) = \min_{j \in (k,\ldots,n):\, \sum_{\ell=i}^{j-1} h_\ell \le q_{\max}} \Bigg\{ & t_i + (s + (1-a_i)\mathcal{W}_i) I_{j>k} \\ & + c\left(\sum_{\ell=k}^{j-1} h_\ell, \sum_{\ell=i}^{k-1} h_\ell \right) + [P(A_{i+1}=1)] V(i+1, j, 1) \\ & + [1 - P(A_{i+1}=1)] V(i+1, j, 0) \Bigg\},\end{aligned} \tag{3}$$

where $I_{j>k}$ equals one if $j > k$ (i.e., if the vehicle stops to recharge) and zero otherwise, and $\sum_{\ell=k}^{j-1} h_\ell$ is the recharging amount. Then, solving the recursive Equation (3) until we find $V(0,0,\cdot)$ will result in an optimal recharging policy along the given path.

Proposition 3. *The computational time required to find an optimal policy that solves recursive Equation (3) is $O(n^3)$.*

Proof. For each pair of nodes $i, k \in \mathcal{N}$ and charging station availability $a_i \in \{0,1\}$, Equation (3) chooses a node j from a subset of $\mathcal{N}$. Since $|\mathcal{N}| = n$, the recursive equation can be solved using Dijkstra's algorithm (1959) in $O(n^3)$. □

5. Adaptive Routing and Recharging for a General Network

Although solving the dynamic programming formulation in (3) to obtain an optimal recharging policy along a fixed path is straightforward, applying a similar solution approach to find an optimal policy within a network is more difficult due to the need to incorporate routing decisions as well. The decision space increases to include the next node to visit on the EV route, and the set of states to be considered increases to include potentially all nodes of the road network. Furthermore, since the order of the visited nodes is not predetermined as it is in the fixed path setting, value function updates could require reevaluation of previously explored paths and policies. At the same time, the routing decisions between recharging stops are integrated into the adaptive decision making since they impact the nodes that the EV visits as potential recharging locations if they happen to be available. These facts lead to significant computational inefficiencies unless the number of calculations can be reduced. In Section 5.1, we present an in-depth analysis of the computational complexity of adaptive routing and recharging in a general network, which motivates our solution approach. Rather than attempt to solve for an optimal policy directly, we develop heuristic approaches that build on our analysis and findings for a priori policies. Section 5.2 presents these optimal a priori policies, and Section 5.3 describes the heuristics for adaptive policies.

5.1. Problem Complexity Analysis

To find an adaptive routing and recharging policy for a general network, we return to functional Equation (2). The first obstacle to recursively solving (2) that we must address is the infinite set of actions $\mathcal{A}_{(i,q_i)}$ (and as a result, the infinite set of states $\mathcal{S}$) due to the continuous set of values that the recharging amount r_i and charge level q_i can take. To overcome this challenge, we extend the results of Lemma 2 to the general network routing and recharging problem. Before we do so, however, we introduce some additional notation and definitions.

For any two nodes $i, j \in \mathcal{N}$, let $\mathcal{P}_q(i,j)$ denote the set of paths from i to j with a total required charge level of less than or equal to q (where $q \le q_{\max}$). As will be discussed later in further detail, we allow paths in $\mathcal{P}_q(i,j)$ to contain cycles. Then, let $\mathcal{Q}_q(i,j)$ denote the set of charge levels required by paths in $\mathcal{P}_q(i,j)$ (by construction, the quantities in this set will be bounded by zero and q). We let $\mathcal{P}_q(i,j) = \varnothing$ and $\mathcal{Q}_q(i,j) = \varnothing$ if no such paths exist, and we define

$$\mathcal{Q}_q(i) := \bigcup_{j \in \mathcal{N}} \mathcal{Q}_q(i,j).$$

Finally, as before, we denote a feasible routing and recharging policy by $\pi_{\mathcal{N}}\colon \mathcal{S} \to \mathcal{A}$, with $\pi_{\mathcal{N}}(i, q_i, a_i) \in \mathcal{A}_{(i,q_i)}$ representing the next node to visit and the recharging amount specified by the policy $\pi_{\mathcal{N}}$ for the state (i, q_i, a_i). We are now ready to state the lemma.

Lemma 4. *There exists an optimal policy $\pi^*_{\mathcal{N}}\colon \mathcal{S} \to \mathcal{A}$ for the recursive Equation (2) such that*

$$\pi^*_{\mathcal{N}}(i, q_i, a_i) \in \{(j, r_i)\colon (i,j) \in \mathcal{E}, r_i + q_i - h_{ij} \in \mathcal{Q}_{(q_{\max} - h_{ij})}(j)\}$$

for all $(i, q_i, a_i) \in \mathcal{S}$.

In words, Lemma 4 states that there exists a policy such that we would only recharge an amount corresponding to the exact charge required to reach some node in $\mathcal{N}$ from the current node. The proof for Lemma 4 is similar to the proof for Lemma 1 and is therefore omitted.

Since we assume the initial charge level to be zero, we can also restrict the set of charge levels considered in our state space $\mathcal{S}$ to a discrete set, without loss of optimality. We do so in the following corollary.

Corollary 5. *There exists an optimal policy $\pi^*_{\mathcal{N}}\colon \mathcal{S} \to \mathcal{A}$ to the recursive Equation (2) such that $q_i \in \mathcal{Q}_{q_{\max}}(i)$ for any realized state $(i, q_i, a_i) \in \mathcal{S}$, given that the initial state is $(0, 0, \cdot)$.*

While Lemma 4 and Corollary 5 deliver finite action and state spaces, the sizes of those sets remain large due to the fact that an optimal policy might include cycles. The following propositions establish the cycling property of the adaptive routing and recharging policies for a general network and the functional Equation (2).

Proposition 6. *An optimal policy satisfying the properties established in Lemma 4 and Corollary 5 may visit a node in $\mathcal{N}$ more than once. That is, a vehicle might cycle on the network.*

Proof. Consider an undirected network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = \{0,1,2,3\}$, $\mathcal{E} = \{(0,1),(1,2),(1,3)\}$, node 0 is the origin, and node 3 is the destination (see Figure 3). Let $q_{\max} \le h_{01} + h_{13}$, $h_{01} + h_{12} \le q_{\max}$, and $h_{12} + h_{13} \le q_{\max}$. That is, having a full charge level is not sufficient to get from the origin to the destination without recharging along the way, and the vehicle must recharge either at node 1 or 2 before proceeding to node 3. Furthermore, we let $\mathcal{W}_1 > 0$ and $\mathcal{W}_2 = 0$. Then, consider the following two policies, π^1 and π^2.

- Policy π^1: Recharge $r_0 = h_{01}$ at node 0 and travel to node 1. Recharge $r_1 = h_{13}$ at node 1 (with expected waiting time $\mathcal{W}_1$ if $A_1 = 0$) and travel directly to node 3.
- Policy π^2: Recharge $r_0 = h_{01} + h_{12}$ at node 0 and travel to node 1. If $A_1 = 1$ (i.e., the recharging station at node 1 is available), then recharge $r_1 = h_{13} - h_{12}$ at node 1 and travel directly to node 3. Otherwise, if $A_1 = 0$, then do not recharge at node 1, but instead travel to node 2, recharge $r_2 = h_{12} + h_{13}$ there, and complete the trip by traveling to node 1 and then to node 3.

Note that π^1 is the minimum cost policy among all of the feasible policies that do not include a cycle on network $\mathcal{G}$, and policy π^2 contains a cycle.

Then, the costs of policies π^1 and π^2, respectively, are

$$\begin{aligned}
C_{\pi^1} &= s + c(h_{01}, 0) + t_{01} + [1 - P(A_1 = 1)]\mathscr{W}_1 \\
&\quad + s + c(h_{13}, 0) + t_{13}, \\
C_{\pi^2} &= s + c(h_{01} + h_{12}, 0) + t_{01} \\
&\quad + P(A_1 = 1)[s + c(h_{13} - h_{12}, h_{12})] \\
&\quad + [1 - P(A_1 = 1)][2t_{12} + s + c(h_{12} + h_{13}, 0)] + t_{13} \\
&= 2s + [c(h_{01}, 0) + c(h_{12}, h_{01})] + t_{01} + P(A_1 = 1) \\
&\quad \cdot [c(h_{13}, 0) - c(h_{12}, 0)] + [1 - P(A_1 = 1)] \\
&\quad \cdot [2t_{12} + c(h_{13}, 0) + c(h_{12}, h_{13})] + t_{13}.
\end{aligned}$$

Then,

$$\begin{aligned}
C_{\pi^1} - C_{\pi^2} &= [1 - P(A_1 = 1)][\mathscr{W}_1 - 2t_{12} - c(h_{12}, h_{13})] \\
&\quad - c(h_{12}, h_{01}) + P(A_1 = 1)c(h_{12}, 0) \\
&= [1 - P(A_1 = 1)][\mathscr{W}_1 - 2t_{12} - c(h_{12}, h_{13}) \\
&\quad - c(h_{12}, 0)] - c(h_{12}, h_{01}) + c(h_{12}, 0) \\
&\geq [1 - P(A_1 = 1)]\mathscr{W}_1 - 2t_{12} - c(h_{12}, h_{13}) \\
&\quad - c(h_{12}, h_{01}) \quad (\text{since } 0 \leq P(A_1 = 1) < 1) \\
&> 0 \quad \text{if } [1 - P(A_1 = 1)]\mathscr{W}_1 \\
&\qquad\qquad > 2t_{12} + c(h_{12}, h_{13}) + c(h_{12}, h_{01}).
\end{aligned}$$

That is, it is advantageous to cycle between nodes 1 and 2 if the expected waiting time at node 1 is greater than the extra travel and charge time required for the detour. Since policy π^1 is the minimum-cost cycle-free feasible policy, this proves the required statement. □

The cycle illustrated in the proof of Proposition 6 (Figure 3) could represent a scenario in which the station at node 2 is private (i.e., reserved for a driver's exclusive use), and the driver would detour to use it only if the public station at node 1 is already occupied by another vehicle. If the station at node 2 were instead public with nonzero expected waiting time and the driver finds it also occupied on arrival, then assuming the driver had charged more than $h_{01} + h_{12}$ at the origin, it may be optimal for the driver to depart node 2 without recharging and check node 1 again. This cycle could be repeated multiple times, as long as the driver's vehicle has sufficient charge remaining to travel between the two nodes. In a practical setting, a driver may exhibit this behavior in anticipation of one station becoming available sooner than the other and

Figure 3. Illustrative Example of Network $\mathscr{G} = (\mathscr{N}, \mathscr{E})$ for the Proof of Proposition 6

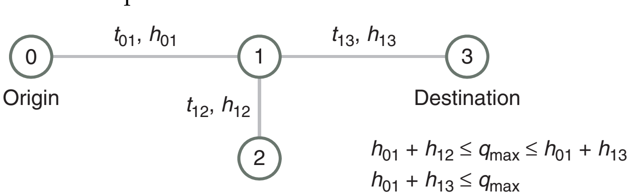
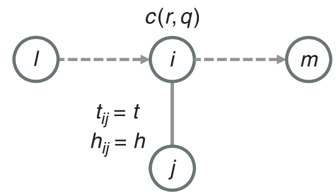


Figure 4. Illustrative Example for the Proof of Proposition 7



to mitigate the risk of encountering an extremely long wait. We formalize this claim in Proposition 7 and the following proof.

Proposition 7. *An optimal policy might traverse a given cycle in network $\mathscr{G}$ more than once.*

Proof. Consider a policy where an EV arrives at node i from node l with charge level q, recharges r, and continues to travel to node m (Figure 4). We show that for some parameter settings (characterized by inequalities (4) and (5)), it is advantageous to detour to node j if the station at node i is unavailable rather than wait at i for the station there to become available. The vehicle recharges at j if the station there is available; otherwise, it returns to node i. This statement holds true when

$$\begin{aligned}
&\mathscr{W}_i + c(r, q) \\
&\quad > 2t + P(A_j = 1)c(r + 2h, q - h) + (1 - P(A_j = 1)) \\
&\qquad \cdot [c(r + 2h, q - 2h) + (1 - P(A_i = 1))\mathscr{W}_i], \\
&\mathscr{W}_i + c(r, q) \\
&\quad > 2t + P(A_j = 1)[c(h, q - h) + c(r, q) + c(h, q + r)] \\
&\qquad + (1 - P(A_j = 1))[c(2h, q - 2h) + c(r, q) \\
&\qquad\qquad + (1 - P(A_i = 1))\mathscr{W}_i], \\
&\mathscr{W}_i > 2t + P(A_j = 1)[c(h, q - h) + c(h, q + r)] \\
&\qquad + (1 - P(A_j = 1)) \\
&\qquad \cdot [c(2h, q - 2h) + (1 - P(A_i = 1))\mathscr{W}_i], \\
&\mathscr{W}_i > 2t + c(h, q - h) + P(A_j = 1)c(h, q + r) \\
&\qquad + (1 - P(A_j = 1))c(h, q - 2h) \\
&\qquad + (1 - P(A_j = 1))(1 - P(A_i = 1))\mathscr{W}_i
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\mathscr{W}_j + c(r + 2h, q - h) \\
&\quad > c(r + 2h, q - 2h) + (1 - P(A_i = 1))\mathscr{W}_i, \\
&\mathscr{W}_j + c(r + h, q - h) + c(h, q + r) \\
&\quad > c(h, q - 2h) + c(r + h, q - h) + (1 - P(A_i = 1))\mathscr{W}_i, \\
&\mathscr{W}_j + c(h, q + r) > c(h, q - 2h) + (1 - P(A_i = 1))\mathscr{W}_i.
\end{aligned} \tag{5}$$

Note that because the cost function $c(\cdot)$ is monotone increasing and convex, we know that $c(h, q + r) \geq c(h, q - 2h)$.

When conditions (4) and (5) are satisfied, a vehicle can improve the current policy by cycling to node j and back. Applying this argument recursively, we can then replace $\mathscr{W}_i$ on the right-hand side of inequality (4) once again with the second cycle to j and back in the case that the charging station at node i is unavailable after the first cycle. This recursion may continue until the station at either node i or j becomes available, or until the charge required to complete a cycle exceeds the vehicle's remaining available charge. Using this approach it can be shown that an optimal policy might traverse a given cycle more than once. □

It is worth observing that when the cost function $c(\cdot)$ is linear, the inequalities (4) and (5) simplify to

$$\mathscr{W}_i > 2t + 2c(h,\cdot) + (1 - P(A_j = 1))(1 - P(A_i = 1))\mathscr{W}_i \quad (6)$$

$$\text{and} \quad \mathscr{W}_j > (1 - P(A_i = 1))\mathscr{W}_i. \quad (7)$$

These simplified conditions provide insights to an intuitive interpretation of the settings favorable to repeating cycles. It is beneficial to deviate from node i to node j if the waiting time at node i exceeds the time required to travel the cycle $i \to j \to i$ and the extra time required to recharge the depleted energy by this cycle, in addition to $\mathscr{W}_i$ weighted by the probability of returning to i and still having to wait there (inequality (6)). Furthermore, the vehicle should not wait at node j and return to node i if it finds the station unavailable when $\mathscr{W}_j$ exceeds the waiting time at node i weighted by the probability of station i being unavailable (inequality (7)).

We are now ready to conclude the computational complexity analysis of solving recursive Equation (2). To build a discrete action set $\mathscr{A}_{(i,q_i)}$, we first construct a set of paths $\mathscr{P}_{q_{\max}}(i,j)$ for all $j \in \mathscr{N}$. As a result of Proposition 7, we know that paths in $\mathscr{P}_{q_{\max}}(\cdot,\cdot)$ might contain numerous and repeating cycles. Therefore, the same nodes from set $\mathscr{N}$ may appear on a single path multiple times and the only possible theoretical bound on the total number of nodes visited by a path is $q_{\max}/\min_{(i,j)\in\mathscr{E}}(h_{ij})$, which in the extreme case might approach infinity as $\min_{(i,j)\in\mathscr{E}}(h_{ij}) \to 0$. Intuitively, as the arc lengths become shorter, the number of possible paths that may be traversed with a charge level of $q_{\max}$ increases and, in the limit, approaches infinity. Furthermore, even if we build a discrete action set, solving the dynamic program that allows cycles and has a large action space is prohibitively expensive and no polynomial computational bound can be derived. It is important to note that there are some practical limits to the number of repeating cycles. First, the probability of considering n additional trips on the cycle $i \to j \to i$ is equal to $[(1 - P(A_j = 1))(1 - P(A_i = 1))]^n$, which approaches zero as $n \to \infty$. Second, the ability to make additional trips $i \to j \to i$ is constrained by the remaining charge level of the vehicle.

This analysis motivates our heuristic solution approaches to an adaptive routing and recharging problem for a general network, discussed in Section 5.3. We propose two heuristic methods that aim to restrict the set of paths considered by the adaptive policy to improve the problem's tractability. We use an a priori policy to guide the decision of what the restricted set of paths should be. The a priori policies are easy to implement, identify promising charging locations for an adaptive policy to visit, and allow us to benchmark our adaptive approach to assess the value of adaptivity. The first heuristic method uses an a priori policy to select a path to travel. Then, a vehicle commits to the path but can dynamically adapt recharging locations along this path. In the second solution approach, the EV is allowed to adapt its path and recharging locations in response to realized station availabilities while using an a priori policy to restrict the set of paths considered. Next, we discuss the a priori policies (Section 5.2) before detailing the heuristics that build on them (Section 5.3).

5.2. Optimal A Priori Policies

In this section, we identify optimal a priori policies where both the vehicle's route and recharging locations are selected before departing the origin and do not change en route based on the realized availability of charging stations. Not only is an optimal a priori policy easier to obtain, but it is also a feasible policy when adaptive decision making is permitted. As a result, the value of such a policy serves as an upper bound on the value of an optimal adaptive policy. We apply this fact to our later analysis (Section 5.3) of the adaptive decision making and use the optimal a priori policy as a starting point for the solution procedure to find an optimal adaptive policy. In this section, we present algorithms for finding an optimal a priori policy along a path (Section 5.2.1) and within a general network (Section 5.2.2).

5.2.1. Optimal A Priori Policy for a Fixed Path. Here, we examine how to determine an optimal a priori recharging policy along a fixed path $\mathscr{N} = (0, \ldots, n)$ consisting of a sequence of $n+1$ nodes. As in Section 4, we let h_i and t_i denote the energy and time, respectively, required to travel from node i to node $i+1$. We let $\pi^0_{\mathscr{N}} = (r_0, \ldots, r_{n-1})$ denote an a priori policy, which consists of the sequence of amounts to recharge at each node along the path $\mathscr{N}$ (except node n, which is the end of the path). Note that the amounts to recharge at each node are not dependent on the realized availability of charging stations. In such a policy, the expected waiting time at a node if the vehicle recharges there is

$$E[W_i] = [P(A_i = 1)]E[W_i \mid A_i = 1] + [1 - P(A_i = 1)] \cdot E[W_i \mid A_i = 0] = [1 - P(A_i = 1)]\mathscr{W}_i$$

for all $i \in (1, \ldots, n)$, where $E[W_i \mid A_i = 1] = 0$ since the vehicle does not need to wait when the charging station is available. We show in the following lemma that there exists an optimal a priori policy in which the vehicle only stops to recharge when its charge level is zero. (It is worth noting that while these results are analogous to the zero-inventory-ordering policies in the inventory control literature, they do not directly follow from that literature since such policies are not even optimal in the presence of replenishment constraints; Florian, Lenstra, and Rinnooy Kan 1980.)

Lemma 8. *There exists an optimal a priori recharging policy such that for all $i \in (0, \dots, n-1)$, $r_i > 0$ if and only if $q_i = 0$.*

Proof. Let $\pi^0_{\mathcal{N}} = (r_0, \dots, r_{n-1})$ be an a priori recharging policy. Suppose the vehicle recharges at node j in the policy, and k is the next node along the path where the vehicle recharges (i.e., $k = \min\{\ell: \ell > j, r_\ell > 0\}$). Let $\Delta = r_j - \sum_{\ell=j}^{k-1} h_\ell$ denote the amount recharged at node j that is not used by the time the vehicle reaches node k, where $\Delta \geq 0$. Then, a policy $\hat{\pi}^0_{\mathcal{N}} = (\hat{r}_1, \dots, \hat{r}_n)$ that satisfies

$$\hat{r}_i = \begin{cases} r_j - \Delta, & i = j \\ r_k + \Delta, & i = k \\ r_i, & \text{otherwise} \end{cases}$$

is also feasible. Furthermore, by the convexity of $c(\cdot)$, the cost of the policy $\hat{\pi}^0_{\mathcal{N}}$ is no greater than that of $\pi^0_{\mathcal{N}}$. Therefore, there exists an optimal policy in which $r_i > 0$ implies $q_i = 0$ for all $i \in (0, \dots, n-1)$. Conversely, $q_i = 0$ necessarily implies that $r_i > 0$ to ensure feasibility. It follows that there exists an optimal a priori recharging policy such that for all $i \in (0, \dots, n-1)$, $r_i > 0$ if and only if $q_i = 0$. □

If we let $V(i)$ denote the value function representing the minimum expected cost of reaching node n (the end of the path) when the vehicle arrives at node i with zero charge level, then we can define the value function recursively as

$$V(i) = \min_{j \in (i+1, \dots, n):\, \sum_{\ell=i}^{j-1} h_\ell \leq q_{\max}} \left\{ s + [1 - P(A_i = 1)]\mathcal{W}_i + c\left(\sum_{\ell=i}^{j-1} h_\ell, 0\right) + \sum_{\ell=i}^{j-1} t_\ell + V(j) \right\}, \tag{8}$$

where $V(n) = 0$. Solving recursive Equation (8) for $V(0)$ yields an optimal recharging policy for a given path. This computation can be done using Dijkstra's algorithm (Dijkstra 1959) in time $O(n^2)$. Note that when the quantity $[1 - P(A_i = 1)]\mathcal{W}_i$ is the same for all $i \in (0, \dots, n-1)$, the problem is identical to the one studied in Sweda, Dolinskaya, and Klabjan (2017) and can be solved more efficiently by removing from the action space values of j for which $\sum_{\ell=i}^{j-1} h_\ell \leq \alpha q_{\max}$. Nevertheless, one of the contributions of this present work is to capture the heterogeneous charging stations availability when making routing and recharging decisions.

5.2.2. Optimal A Priori Policy for a General Network.

We return to the general network problem defined in Section 3.1 on a network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. An optimal a priori policy in a general network must specify not only where to recharge on the way from an origin node o to a destination node n but also the route that the vehicle takes. Since an a priori policy commits to a sequence of recharging locations regardless of whether the vehicle finds the charging stations available or not, the functional Equation (2) is modified to remove station availability from the state space, resulting in

$$V(i, q_i) = \min_{(j, r_i) \in \mathcal{A}(i, q_i)} \{ t_{ij} + (s + [1 - P(A_i = 1)]\mathcal{W}_i + c(r_i, q_i)) I_{r_i > 0} + V(j, q_i + r_i - h_{ij}) \}. \tag{9}$$

Despite adding routing decisions to the problem scope, the properties from Section 5.2.1 can be easily extended to the network setting. Since an a priori policy returns a single path (we still must find that path, but a path nevertheless), the property of Lemma 8 can be extended to the network.

Lemma 9. *There exists an optimal a priori policy for functional Equation (9) such that for all $i \in \mathcal{N}$, $r_i > 0$ if and only if $q_i = 0$.*

Proof. The proof follows from Lemma 8 since for any path returned from solving (9), we can apply Lemma 8, considering that path as a fixed path. □

From Lemma 9, it follows that we can further simplify the recursive Equation (9) by redefining the DP state as the next node i from $\mathcal{N}$ where the EV recharges. Then, the action corresponds to selecting a node $j \in \mathcal{N}$ where the vehicle recharges next. From Property 1 we know that, without loss of optimality, a vehicle should follow the shortest (or minimum required charge) path from i to j and recharge at node i the exact amount needed to traverse the path. We let $t^*_{p(i,j)}$ and $h^*_{p(i,j)}$ denote the travel time and charge amount, respectively, for such a path. The resulting functional equation one must solve to find an optimal a priori policy on a general network is

$$V(i) = \min_{j \in \mathcal{N}:\, h^*_{p(i,j)} \leq q_{\max}} \{ t^*_{p(i,j)} + s + [1 - P(A_i = 1)]\mathcal{W}_i + c(h^*_{p(i,j)}, 0) + V(j) \}. \tag{10}$$

Because of the fact that the a priori policy commits to recharging locations and does not adapt to realized station availabilities, the a priori DP functional Equation (10) will only visit each DP state (as well as each node in set $\mathcal{N}$) at most once. As a result, the computational complexity of recursively solving (10) is $O(n^4)$ where $n = |\mathcal{N}|$ since for each node in set $\mathcal{N}$, our action space is a subset of $\mathcal{N}$, and we must solve a shortest path problem from the current node to the next recharging node to find $t^*_{p(ij)}$ and $h^*_{p(ij)}$, which can be done using Dijkstra's algorithm in time $O(n^2)$. Note that while the solution to (10) returns the ordered list of recharging locations, we can also recover the actual paths traversed between those locations at no additional computational cost since we already compute those paths to find the values $t^*_{p(ij)}$ and $h^*_{p(ij)}$.

5.3. Heuristic Solution Adaptive Routing and Recharging for a General Network

As seen in Section 5.2, finding an optimal a priori policy for a general network is computationally tractable and can be done in polynomial time, unlike the case of an adaptive policy. Therefore, we use a priori policies to guide our heuristic solution approaches.

5.3.1. Heuristic Approach to Adaptive Recharging. Our first heuristic finds an optimal policy when the vehicle is only allowed to adapt the charging locations dynamically but not the path while en route. This restricted decision space has a practical application, as some drivers might not be interested in deviating from a path just to search for a more favorable charging station. The heuristic is broken down into a two-stage solution approach: at the first stage, we select which path the EV should follow from origin to destination, and then the second stage selects an adaptive recharging policy for the chosen path. As shown in Section 4, an optimal recharging policy along a fixed path can be obtained using the formulation (3). Thus, the two-step solution approach of our heuristic follows.

Step 1. Solve recursive Equation (10) to find an optimal a priori path from origin o to destination n.

Step 2. For the path found in Step 1, solve recursive Equation (3) to find an optimal recharging policy.

The computational complexity of Heuristic 1 is $O(n^4)$ since Step 1 is performed once in $O(n^4)$, and Step 2 is executed once in $O(n^3)$. The solution found in Step 1 is an upper bound on the adaptive recharging policy since the EV is always able to recharge at the locations identified by the a priori policy. At the same time, if the vehicle encounters an available station before reaching an a priori charging stop, it may be beneficial to recharge there to avoid a potential wait in the future. Furthermore, the adaptive policy will also capture the trade-off between recharging sooner than planned with zero waiting cost and potentially longer charging times and additional stops later because of opportunity charging of a non-empty battery.

One might improve the found solution by considering a larger number of paths in Step 1. Since we still want to ensure that the evaluated paths do not include large detours, finding k-shortest paths (e.g., Yen 1971) using an a priori approach and then solving recursive Equation (3) for each of them can result in an overall better solution.

5.3.2. Heuristic Approach to Adaptive Recharging and Routing. As we saw from an earlier example (Section 3.2), it might be beneficial not only to adapt recharging locations along a path but also to deviate from the path when the vehicle finds an intended recharging location unavailable. While it is intractable to find an optimal adaptive policy to recursive Equation (2), we can use a variety of approaches to restrict the subset of paths considered by the policy. Here, we propose a few solution approaches that can be applied to a general network problem. However, without additional structure of the network $\mathcal{G}$, it is not clear which approach is more advantageous, especially when considering the trade-off among implementation complexity, runtime, and performance. In Section 6, we present further in-depth analysis and solution approaches to a network with a grid structure, and in Section 8 we conduct a numerical analysis in a realistic setting.

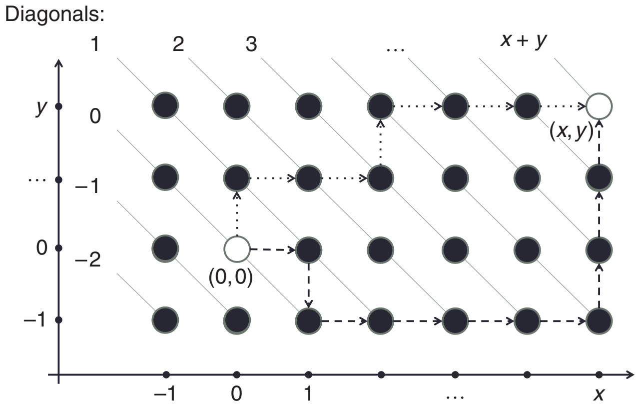
Recursive a priori path. Note that for the adaptive recharging policy found using the solution approach described in Section 5.3.1, when an EV stops to recharge at an available station that differs from an optimal a priori location, the optimal a priori policy from that node forward might change. Thus, we can improve the found solution by executing Steps 1 and 2 of the adaptive recharging policy again to allow the vehicle to also adapt its path along the way.

Value function approximation. Since an optimal a priori policy found by solving (10) provides an upper bound on the value function for an optimal adaptive policy in recursive Equation (2), we can use it to approximate the value function and apply approximate dynamic programming techniques (Powell 2007) to heuristically solve for an adaptive policy. Note that since the a priori solution provides an upper bound, and not a lower bound, we cannot implement an A^* heuristic.

6. Adaptive Policies for a Grid Network

In this section, our analysis of the optimal EV routing is carried out on a square grid network where the amount of charge required to traverse each arc is a constant and denoted by h (i.e., $h_{ij} = h, \forall (i,j) \in \mathcal{E}$). We consider this special case due to several favorable properties of the square grid network that lend themselves well to a dynamic programming framework and broad set of applications. Specifically, consider an x-, y-coordinate system imposed on our grid network where, without loss of generality, we assume that the origin o has coordinates $(0,0)$ and the destination n has coordinates (x,y) $(x \geq 0,\ y \geq 0)$. Then, the minimum distance between two nodes can easily be calculated as the sum of the absolute differences between their coordinates. Furthermore, network nodes can be partitioned by the total sum of their x and y coordinates such that all nodes in one partition have the same sum values. As a result of such a procedure, all nodes along each

Figure 5. Nodes Partitioned Diagonally; All Nodes Along the Same Diagonal have Coordinates That Sum to the Same Value



Notes. The dotted line is an example of a *direct path* from the origin to the destination, and the dashed line is an example of an *indirect path*.

$(-\pi/4)$-angled diagonal line are grouped together (see Figure 5). Indexing the partitions by the values of the sum of the two coordinates of their member nodes creates a natural ordering. In addition, it can be observed that whenever the vehicle travels to an adjacent node, the index of the diagonal passing through its location either increases by one (when the vehicle travels up or to the right) or decreases by one (when it travels down or to the left). By our convention, the vehicle starts its travel on diagonal 0 and ends on diagonal $x + y$. It follows that for any *direct path* (i.e., shortest or minimum-charge path) between origin and destination in which the vehicle only travels either up or to the right, the distance is simply the difference between the partition indices of the two nodes, or $x + y$. This notion of a direct path (as well as *indirect path*—a path with total distance greater than the minimum) will be used in our subsequent analysis.

Besides adding structure to our adaptive routing and recharging problem, a grid network also naturally lends itself to application. Most urban networks (e.g., New York City, Chicago) have grid-like road networks where each node corresponds to an intersection. In more suburban and rural settings where road networks are more sparse, a grid still can be used to approximate the road network by setting edge costs, station availability probabilities, and waiting times to arbitrary large values for the absent road links and intersections. A similar approximation scheme can be used to represent the national road network and recharging infrastructure for long-range trips.

Because of computational demands described in Section 5, we develop heuristic solution approaches to estimate the value of adaptive decision making and use an optimal a priori policy to benchmark the improvement of EV performance resulting from introducing adaptivity. The grid network structure allows us to develop a more efficient algorithm to find an optimal a priori policy, which we present in Section 6.1. Then, in Section 6.2, we present two heuristics for adaptive policies: the first allows us to compute the benefit of adaptive recharging decisions only, and then we determine the additional savings that result from adaptive routing decisions and recharging strategies.

6.1. Optimal A Priori Policy for a Grid Network

6.1.1. Algorithm for Optimal A Priori Policy for a Grid Network. When searching for an optimal a priori policy in a network, it is important to establish a tight upper bound quickly on its value. This would allow us to significantly reduce the number of network nodes to be considered as part of an optimal path and prevent us from making unnecessary value function updates at nodes that are located far from both the origin and destination. Our solution approach outlined in Algorithm 1 takes advantage of the grid network structure and quickly computes an upper bound on the value of an optimal a priori policy. It then tightens this bound by searching for policies corresponding to longer paths with potential savings in waiting times at the charging stations. The algorithm terminates once it is established that the current upper bound is also a lower bound and the corresponding policy is optimal.

Algorithm 1 (Finding an optimal a priori policy in a grid network)

Input: network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$; destination node (x, y) $(x, y \geq 0)$; max travel distance $k = \lfloor q_{\max}/h \rfloor$
Output: sequence of recharging stops $P^*_{\mathcal{G},(x,y)} = [(i_1^*, j_1^*), \ldots, (i_p^*, j_p^*)]$ with corresponding cost $V(0,0)$
Initialize: $V(x, y) = 0$; $V(i, j) = \infty$ for every $(i, j) \in \mathcal{N} \backslash (x, y)$; $check(i, j) = 0$ for every $(i, j) \in \mathcal{N}$; $succ(i, j) = (x, y)$ for every $(i, j) \in \mathcal{N}$

```
FindOptDirectPath(𝒢, x, y, k)
MinDiag = 0
MaxDiag = x + y
ScanType = reverse
while ScanType ≠ null do
    if ScanType = reverse then
        ScanReverse(𝒢, x, y, k, V, check, pred)
    else if ScanType = forward then
        ScanForward(𝒢, x, y, k, V, check, pred)
    end if
end while
(i, j) = (0, 0)
while (i, j) ≠ (x, y) do
    P*_{𝒢,(x,y)} = [P*_{𝒢,(x,y)} succ(i, j)]
    (i, j) = succ(i, j)
end while
```

An upper bound on the total cost of an optimal a priori policy is first calculated by the `FindOptDirectPath` procedure (detailed in Online Appendix A.1), which finds an optimal policy over all direct paths between the origin and destination. Such calculations can be performed efficiently since the vehicle is restricted to traveling either up or to the right toward the destination node, never detouring outside of the bounding box formed by the origin and destination or moving in a direction away from the destination. In addition, the identified policy corresponds to a shortest path since the length of any direct path is $x + y$. Thus, the `FindOptDirectPath` procedure finds for each of xy nodes the next node at which to recharge, which is bounded by the distance from the current node by $k = \lfloor q_{\max}/h \rfloor$. The nodes considered for the next recharging location therefore are restricted to $O(k^2)$ or $O(xy)$, whichever is smaller. The procedure then runs in $O(\min(xyk^2, (xy)^2))$, and since $xy \le n = |\mathcal{N}|$, it is $O(n^2)$. In the worst-case scenario where the origin and destination are located in the opposite corners of the grid network, the `FindOptDirectPath` procedure is performed in $O(n^2)$, which is consistent with the earlier-found computational complexity of recursive Equation (10) for a general network, without the need to run Dijkstra's algorithm to find the optimal path between recharging nodes (as that procedure is trivial in a grid network). However, the `FindOptDirectPath` procedure is most beneficial for instances where $xy \ll |N|$ and $k \ll x + y$, or in other words, where the nodes traversed by direct paths constitute a small subset of the entire network node set and the battery size significantly limits the set of nodes that can be considered for the next recharging location.

Algorithm 1 then searches the network for policies with a lower cost by considering paths with a longer distance but overall lower cost. It carries this out by alternating between the `ScanReverse` and `ScanForward` procedures until there are no further value function improvements at any of the network nodes. The bound found by `FindOptDirectPath` significantly limits the number of nodes that are explored during each scan step and improves the computational efficiency. Starting with the diagonal partition containing the destination node (indexed $x + y$), the `ScanReverse` procedure updates the value function for other nodes along that diagonal, as well as for the nodes on the diagonals with a lower index yet within a distance of $k = \lfloor q_{\max}/h \rfloor$ from an updated node. The procedure then advances to the diagonal with the next highest index, updating nodes along that diagonal as well as along diagonals with lower indexes, and it continues until a diagonal along which no nodes have updated value functions is reached. From this diagonal, the `ScanForward` procedure operates analogously to `ScanReverse` but in the forward direction, searching for nodes to update along the same diagonal or along diagonals with a higher index. The final value at the origin on termination of the algorithm represents the total cost of an optimal a priori policy. (See Online Appendix A.1 for detailed pseudocodes for procedures `FindOptDirectPath`, `ScanReverse`, and `ScanForward`.) We let B correspond to the number of times Algorithm 1 performs the pair of `ScanReverse` and `ScanForward` procedures, and the total run time for the algorithm is $O(Bn^2)$. Note that we cannot find a proper bound on B (see Section 6.1.2 for more discussion), and in the worst case, solving recursive Equation (10) adapted to a grid network, which can be done in $O(n^2)$, is a better solution approach. However, larger B values correspond to greater deviation from a direct path, and in a realistic setting, drivers are not interested in deviating too far from a direct path on the way to their destination (corresponding to small B) and $xy \ll |N|$. Therefore, in most practical applications, Algorithm 1 delivers significant computational benefit.

The following theorem establishes optimal termination of the algorithm in finite time.

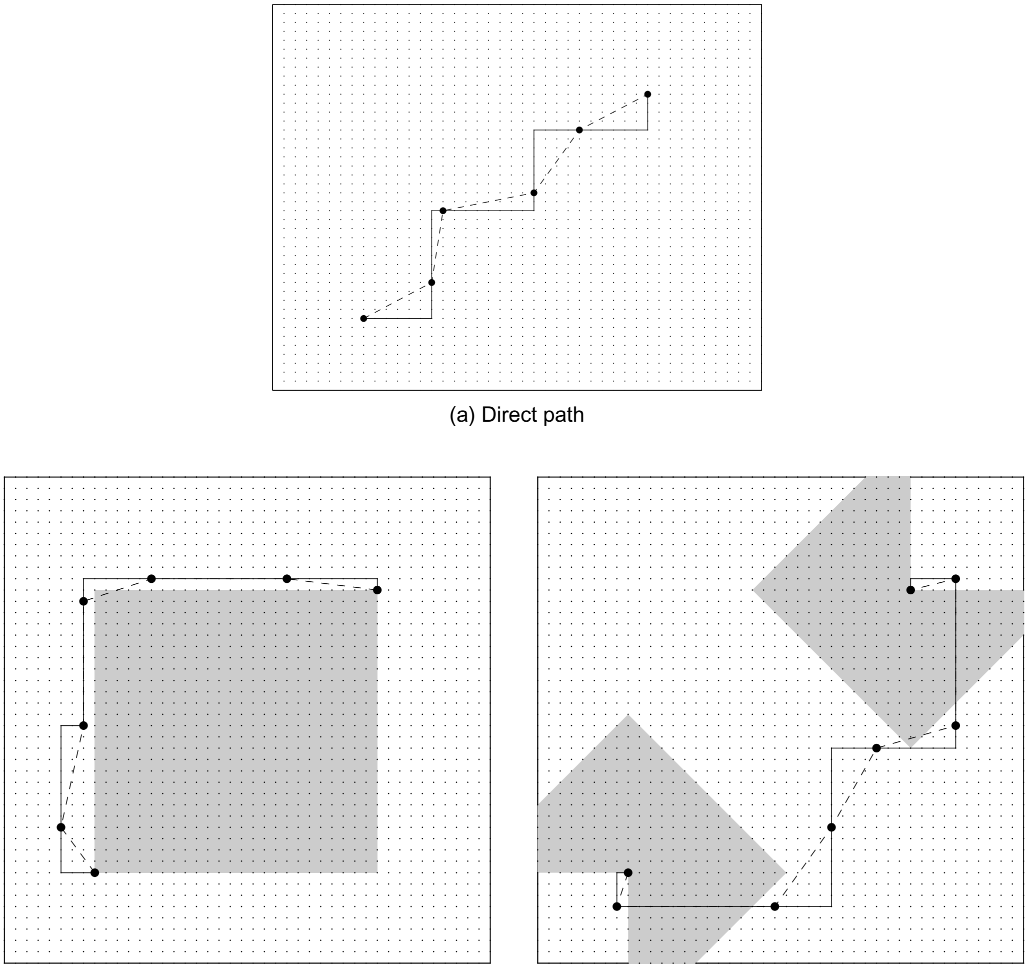
Theorem 10. *Algorithm 1 returns an optimal a priori policy in finite time.*

Proof. The LB function in `ScanReverse` and `ScanForward` procedures of Algorithm 1 (see Online Appendix A.1) computes the lower bound on the minimum travel time from a given node in the network to the destination node. LB is positive and monotone increasing with respect to the distance between a node and the destination and it restricts the number of nodes to be explored by the algorithm to a finite set. Furthermore, the number of iterations of `ScanReverse` and `ScanForward` is finite because of the suboptimality of cycles. Therefore, Algorithm 1 returns an optimal a priori policy in finite time. □

Remark 1. It is important to note that Algorithm 1 returns an optimal sequence of recharging locations and corresponding recharge amounts, but not an explicit path. Because of the grid network structure, there is always more than one shortest path between any two distinct nodes, as long as they are not horizontally or vertically aligned, and the routing policy corresponding to the identified optimal charging policy is not unique. Therefore, as long as the EV driver follows any direct path between each pair of consecutive recharge locations, the resulting routing and recharging policy will be optimal for our a priori setting.

6.1.2. Illustrative Examples of Optimal A Priori Policies. It is interesting to note that the total number of scan procedures completed (both `ScanReverse` and `ScanForward`) in Algorithm 1 for which the objective value continues to improve directly corresponds to the shape of an optimal path. For example, if

Figure 6. Examples of A Priori Policies in a Network



(a) Direct path

(b) Indirect path (nondecreasing diagonals)

(c) Indirect path (decreasing/increasing diagonals)

Notes. Highlighted nodes represent recharging stops (with a sample path shown as a solid line), and shaded areas represent regions where charging stations are never available.

an optimal policy is obtained immediately after the `FindOptDirectPath` procedure, then the corresponding path is a direct path between the origin and destination. An improvement to the objective value after one call to the `ScanReverse` procedure implies that the optimal path is not direct but the sequence of diagonals corresponding to each stop is nondecreasing. Additional improvements after the initial `ScanForward` procedure indicate that the sequence of diagonals has at least one sign change, with further improvements signaling even more sign changes.

Some numerical examples of optimal a priori policies are illustrated in Figure 6. Here, the white area corresponds to nodes with randomly generated node availability probabilities and expected wait times from uniform distributions $U(0,1)$ and $U(0,240)$, respectively, while shaded areas denote regions where charging stations are never available. In the first example (Figure 6(a)), the optimal a priori policy corresponds to a direct path between the origin and destination. This policy is found within the `FindOptDirectPath` procedure, and no improvements to it are made after calls to the `ScanReverse` and `ScanForward` procedures. The second example (Figure 6(b)) illustrates a policy corresponding to an indirect path. Whereas a direct path would have all recharging stops within the bounding box between the origin and destination, the shown path includes stops outside of the bounding box. However, note that the indices of the diagonals corresponding to the sequence of recharging stops are nondecreasing. This implies that the optimal a priori policy is obtained after the first `ScanReverse` procedure. The third example illustrates an optimal policy that is obtained after multiple calls to the `ScanReverse` and `ScanForward` procedures (Figure 6(c)). In this case, the sequence of diagonals corresponding to the recharging stops has two sign changes (i.e., decreases, then increases and decreases again), implying that the policy was obtained after two `ScanReverse` procedures (and one `ScanForward` procedure).

6.2. Heuristics for Adaptive Policies for a Grid Network

We now describe two heuristic methods for obtaining adaptive policies in a grid network that are based on a priori policies and the solution approach presented in Section 6.1. The first heuristic generates multiple sample paths to quickly estimate the value of adaptive recharging decisions and selects the best path, while the second heuristic captures additional adaptivity with regard to path selection to provide further improvement.

6.2.1. Heuristic 1: Adaptive Recharging Only. Analogous to the heuristic solution method presented in Section 5.3.1, we use a two-stage solution approach: first selecting a path for the EV to follow, and then selecting an adaptive recharging policy for the fixed path. We solve recursive formulation (3) to "price" each potential path considered by the first-stage problem and bound the set of paths to be considered using the optimal a priori policy found by Algorithm 1. Unlike most of the general network problem instances studies in Section 5, an optimal a priori policy in a grid network does not always uniquely specify a path. Instead, Algorithm 1 only identifies the optimal recharging locations to be used by the EV, and there are numerous paths that can be followed between each pair of consecutive recharging locations (see Remark 1). Since the number of actual paths corresponding to an optimal a priori solution can be large, we randomly select only a subset of these paths to be considered in the second stage. That is, in our first heuristic, outlined in Algorithm 2, we construct *numPaths* different randomly sampled paths that contain the nodes corresponding to the recharging stops in the optimal a priori policy. With the aim to minimize travel and recharging costs, we require that our sampled paths be direct paths between each consecutive pair of a priori stop locations with no detours or cycles. We then calculate the value of an optimal adaptive policy along each path and choose the path with the lowest value.

Algorithm 2 (Improving optimal a priori policies in a grid network with adaptive recharging decisions)

Input: network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$; $P(A_{(i,j)} = 1)$ and $E[W_{(i,j)} \mid A_{(i,j)} = 0]$ for all $(i,j) \in \mathcal{N}$; origin node $(0,0)$ and destination node (x,y) $(x, y \geq 0)$; max travel distance $k = \lfloor q_{\max}/h \rfloor$; optimal a priori recharging stops $P^*_{\mathcal{G},(x,y)} = [(i_1^*, j_1^*), \ldots, (i_p^*, j_p^*)]$ with corresponding cost V_0^*; number of paths to construct *numPaths*
Output: value of adaptive recharging policy v^*

Initialize: $V(x, y, \cdot, \cdot) = 0$; $V(i, j, q, a) = \infty$ for all $(i, j) \in \mathcal{N} \backslash (x, y)$, $q \in \{0, h, \ldots, (k-1)h\}$, and $a \in \{0, 1\}$
set $v^* = V_0^*$
for $path = 1, \ldots, numPaths$ **do**
  construct path $P = [(0,0), \ldots, (x,y)]$ between $(0,0)$ and (x,y) that contains all nodes in $P^*_{\mathcal{G},(x,y)}$ and is
  a direct path between each pair of consecutive nodes in $P^*_{\mathcal{G},(x,y)}$
  for $a = 0, 1$ **do**
    evaluate $V(0,0,0,a)$ for path P using Equation (3)
  end for
  set $v^* = \min\{v^*, E_{A_{(0,0)}}[V(0,0,0,A_{(0,0)})]\}$
end for

The computational complexity of Heuristic 1 can be analyzed by adding the run time of Algorithm 1 ($O(Bn^2)$) and Algorithm 2, which is $O(numPaths \cdot n^2)$. Then, Heuristic 1 finds an adaptive recharging policy for a grid network in $O(\max(B, numPaths)n^2)$. Once again, note that in most instances the number of nodes traversed by the considered path between the origin and destination is significantly smaller than $|N|$, and the runtime parameter n corresponds to the number of nodes along the path, resulting in a significantly more efficient algorithm than $O(\max(B, numPaths)n^2)$ makes it appear.

Because we choose *numPaths* to be less than the value that would be required for total enumeration over all possible paths, we are not guaranteed to find the path with the lowest optimal value. In fact, in extreme instances where the optimal path consists of nodes with extremely high station availability probabilities (and thus the total expected waiting time is negligible), the worst case performance of the heuristic relative to optimal can be infinite. The runtime of the algorithm increases linearly with the number of paths explored. At the same time, the expected marginal improvement that each additional path delivers diminishes fairly rapidly—a fact supported by our numerical results. The goal of this heuristic is to generate a quick and reasonable policy with adaptive recharging decisions. Our next heuristic incorporates adaptive path selection to provide an improved adaptive policy with a lower cost.

6.2.2. Heuristic 2: Adaptive Routing and Recharging. In our second heuristic, outlined in Algorithm 3, we include both adaptive routing and recharging decisions. As with Heuristic 1, we require that the vehicle visits (but not necessarily recharge at) every stop location identified by the optimal a priori policy and travels along a direct path between consecutive a priori stop locations. Rather than fix the entire path prior to solving for the optimal policy, however, we allow the vehicle to adjust its trajectory as it travels from the origin $(0,0)$ to the destination (x,y). Working backwards from the destination (x,y), Algorithm 3 finds an optimal policy between consecutive a priori stop locations over all possible states (i.e., not just those for which the vehicle's charge level at a stop location is 0). It considers all direct paths between each pair of nodes and permits adaptive decision making for both routing and recharging. Because this method takes advantage of the grid structure as it searches for an optimal policy, it is very efficient and terminates quicker than Algorithm 2 in many cases, even when *numPaths* is small.

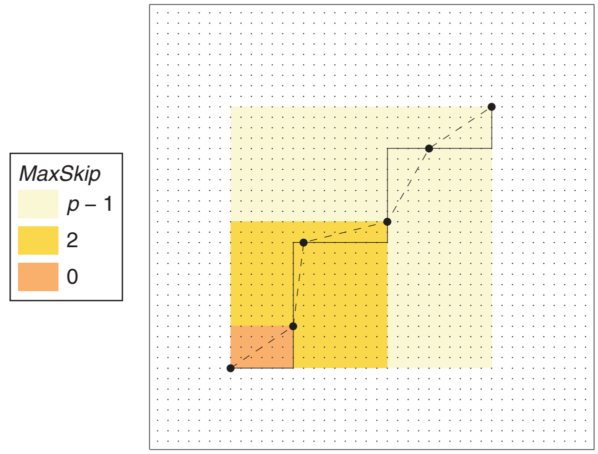
Algorithm 3 (Improving optimal a priori policies in a grid network with adaptive routing and recharging decisions)

Input: network $\mathcal{G}=(\mathcal{N},\mathcal{E})$; $P(A_{(i,j)}=1)$ and $E[W_{(i,j)} \mid A_{(i,j)}=0]$ for all $(i,j)\in\mathcal{N}$; origin node $(0,0)$ and destination node (x,y) $(x,y\geq 0)$; max travel distance $k=\lfloor q_{\max}/h\rfloor$; optimal a priori recharging stops $P^*_{\mathcal{G},(x,y)}=[(i_1^*,j_1^*),\ldots,(i_p^*,j_p^*)]$ with corresponding cost V_0^*; maximum consecutive a priori stops to skip *maxSkip*
Output: value of adaptive routing and recharging policy v^*
Initialize: $V(x,y,\cdot,\cdot)=0$; $V(i,j,q,a)=\infty$ for all $(i,j)\in\mathcal{N}\backslash(x,y)$, $q\in\{0,h,\ldots,(k-1)h\}$, and $a\in\{0,1\}$

```
define (i*_{p+1}, j*_{p+1}) = (x, y)
for skip = 0, ..., maxSkip do
    for ℓ = p − skip, p − skip − 1, ..., 1 do
        define G_ℓ as the subnetwork consisting
          of all nodes and edges that are part
          of a direct path between
        (i*_ℓ, j*_ℓ) and (i*_{ℓ+skip}, j*_{ℓ+skip})
        for q = 0, h, ..., (k − 1)h and a = 0, 1 do
            evaluate V(i*_ℓ, j*_ℓ, q, a) for subnetwork
              G_ℓ using Equation (2)
        end for
    end for
end for
set v* = E_{A(0,0)}[V(i*_1, j*_1, 0, A(0,0))]
```

To relax the restriction that every stop location in the optimal a priori policy be visited, the parameter *maxSkip* can be increased from zero up to a maximum of $p-1$ (where p is the number of stops in the optimal a priori policy). The value of *maxSkip* corresponds to the maximum number of consecutive a priori stop locations that may be skipped to find an adaptive routing and recharging policy with an even lower cost (see Figure 7). For example, when $maxSkip=0$, every stop location in the optimal a priori policy must be visited (although the vehicle need not stop to recharge at each location). In the case when $maxSkip=2$, a policy may skip one or two consecutive a priori stop locations at a time, provided that the vehicle follows a direct path between the a priori stop locations it does visit. Then, when $maxSkip=p-1$, any policy in which the vehicle travels along a direct path between the origin $(0,0)$ and destination (x,y) is permissible. Although the motivating example discussed in Section 3.2 suggests that increasing *maxSkip* can significantly lower the total cost, our numerical results indicate that in most cases the improvement is negligible and not worth the significant increase in computational time. Therefore, in Section 7, we fix $maxSkip=0$ so that only policies that include all a priori stop locations are compared.

Figure 7. (Color online) Example of the Subsets of Network Considered by Algorithm 3 for *maxSkip* Values $0,2$, and $p-1$



Note. Highlighted nodes represent recharging stops for optimal a priori policy.

Like Heuristic 1, the worst-case performance of this heuristic relative to optimal is also infinite. As a simple illustrative example, one can envision an instance in which an indirect path between consecutive stop locations in the optimal a priori policy consists of stations with station availability probabilities close to 1. Heuristic 2 does not consider indirect paths and may instead select a routing and recharging policy that benefits much less from adaptivity.

7. Model Extensions

7.1. Path Switching Cost

The two types of adaptive decision making studied in this paper (i.e., (1) adaptive recharging along a fixed path and (2) adaptive routing and recharging) capture distinct levels of adaptivity that a driver is willing to integrate while traveling towards her destination. However, one might argue that, in practice, a driver would be unlikely to change her path due to a charging station being occupied, but would instead continue on the present route and plan to stop at one of the future charging stations.

While the two presented solution methods capture these distinct driver preferences, another way to model driver aversion to path deviation is to integrate a path switching cost into the optimization model. To do so, one would find an optimal a priori path from the origin to the destination as well as the adaptive recharging policy along this path (e.g., implement Algorithm 2 for a grid network). Then, when the EV encounters an unavailable station, a detour path is considered from that node to the destination with an additional penalty captured by the cost parameter for switching from the original path. By comparing the costs of the original and newly considered paths, the next optimal decision is selected. The value of the switching cost parameter in such an approach would depend on the individual driver's preferences. Note that in the case of a sufficiently high value setting for the switching cost parameter, the EV would never deviate from the original selected path, making it an optimal solution to the adaptive routing and recharging problem setting.

7.2. Conditional Availability Probabilities and Waiting Times

Throughout the analysis of this paper, the probability of a station being available and the expected waiting time when a station is unavailable are assumed to be independent of all external factors except for the station itself. These assumptions were made to simplify the analysis while capturing the distinct characteristics of the stations. In practice, these parameters might depend on a number of other factors, which can be easily integrated into the dynamic programming framework. For example, the waiting times and probabilities might be conditional on the time of day, the number of cars observed at the station, or previously observed availabilities at other visited stations. Then, the dynamic programming state would need to be augmented to integrate this additional information. As a result, the state space, decision policy, and algorithm computation time would grow exponentially because of the curse of dimensionality (Powell 2007), resulting in even more computationally challenging models. Identifying a subset of more valuable factors to include in such extended models and developing novel solution methods to overcome such challenges is part of our future work.

7.3. Elimination of Cycles

Aside from posing difficulties for optimal solution methods, the possibility of cycles in an optimal routing and recharging policy (Proposition 6), especially repeated cycles (Proposition 7), may run contrary to how a driver would act in practice. Even if cycling could reduce a driver's expected waiting time to recharge, the driver may prefer to wait patiently at one station rather than drive around to check the availability of other stations.

There are several ways to prevent cycles from occurring in an optimal policy. The simplest is to require that the road network be acyclic, although such a model would not be representative of realistic networks. Alternatively, one could augment the DP state to include the history of visited nodes and then remove those nodes from the action space. As mentioned in Section 7.2, however, this would increase the computational complexity of the model. Another option would be to preprocess the network to determine each node's minimum distance from the destination and only allow the driver to travel to nodes that are closer than the driver's current location to the destination (similar to the notion of a direct path as described in Section 6).

A different approach to reducing the likelihood of cycles requires reconsideration of the assumption that the random station availability A_i and waiting time W_i for $i \in \mathcal{N}$ are independent of previous realizations observed by the driver at node i. Under this assumption, if a driver arrives at a node where the charging station is unavailable, she may cycle to a nearby node and back repeatedly to refresh the realizations of the station availabilities until one becomes available. This is unrealistic in practice, as the availability of a station is unlikely to change over a short period; therefore, the benefits of cycling are much lower. To prevent such cycling from occurring, one could construct the road network such that adjacent charging stations are sufficiently far apart from each other. Doing so would also provide greater validation of the assumption that A_i and W_i are independent of previous realizations since the time between possible visits at a node would be increased, and the influence of earlier observations would be less. To discourage cycling while still permitting charging stations to be close to each other, one could instead add a history of observed station availabilities to the DP state (both the values and the times when they were observed) and revise the parameters $E[W_i \mid A_i = 0]$ and $P(A_i = 1)$ for $i \in \mathcal{N}$ to be conditional on this history. Although such a modeling approach would not eliminate all cycles (and would also be difficult to implement per our discussion in Section 7.2), it would render those that defy practical considerations suboptimal.

8. Numerical Results

We implemented Algorithms 1–3 for a variety of simulated settings to compare their performance. We use as our network a 500-by-500–node grid with 5 miles between adjacent nodes. Expected waiting times and station availabilities at each node are generated from separate uniform distributions as given in Table 1. In Scenario 1, charging stations are available, on average, half of the time and have expected wait times (if occupied) of around two hours; whereas in Scenario 2, charging stations are more likely to be available but also have longer expected wait times. Thus, the average unconditional expected wait time for a station is

Table 1. Distributions of Expected Wait Times (Measured in Minutes) and Availability Probabilities at Each Charging Station

	Distributions	
Scenario	$E[W_{i,j} \mid A_{i,j}=0]$	$P(A_{i,j}=1)$
1	$U(0,240)$	$U(0,1)$
2	$U(240,480)$	$U(0.8,1)$

similar under each scenario, but the variances are different. For each scenario, we consider a set of 50 random origin–destination pairs separated by distances of between 100 and 300 miles, representing trips that an EV driver could reasonably complete in a day, and also another set of 50 random origin–destination pairs with distances ranging from 500 to 1,000 miles that require multiple recharging stops.

The charging cost function $c(r,q)$ presented in Sweda, Dolinskaya, and Klabjan (2017) is used for our numerical analysis. We test each algorithm over every scenario and set of trips, and for all possible combinations of the parameter settings listed in Table 2. We fix the value for α (overcharging threshold) as 0.8 since that is the most commonly used value in practice (Nissan USA 2014), and the range of values for $q_{\max}$ corresponds to the maximum driving range of the 2014 Nissan Leaf under different traffic and environmental conditions (U.S. Department of Energy 2014b). For the overcharging cost function, we use

$$f(z) = 2 \cdot (\exp\{z/5\} - 1), \tag{11}$$

where z is the amount overcharged at a given stop. The expression in (11) captures the real-world dynamics of battery charging and is calibrated based on one driver's experience with overcharging a Leaf (Laur 2013).

8.1. Value of Adaptive Decision Making

In this section, we measure the value of adaptive decision making by comparing the costs of the policies obtained by the two heuristic methods with that of an optimal a priori policy. Note that the primary benefit of an adaptive policy is a reduction in the total expected waiting time for a given trip. Therefore, rather than compare the aggregate cost savings obtained from Algorithms 2 and 3 to the total cost of a policy found by Algorithm 1, we instead compare them only to the total expected waiting cost of an optimal a priori policy. This provides a better reference for understanding the potential savings that adaptivity allows.

The value of adaptive decision making in the two heuristics is illustrated in Figure 8 for trips of different lengths and for both grid scenarios. Because Heuristic 2 allows for greater adaptivity than Heuristic 1, it always provides a higher benefit. The benefit is not always so pronounced, however, since the system is ergodic and the vehicle still must visit all of the a priori stop locations (because $maxSkip = 0$), and thus the ability to explore alternative paths is limited. In addition, the minimal marginal improvement of the adaptive routing and recharging heuristic (Algorithm 3) in comparison to Algorithm 2 might be explained by the fact that a well selected path by Algorithm 2 already ensures that sufficient alternative recharging locations are contained on such a path. That is, our Heuristic 1 finds a path with advantageous adaptive recharging locations, such that little benefit can be obtained by deviating from the path (in Heuristic 2).

Observation 1. *Adaptive recharging can provide nearly the same improvement in EV performance as adaptive routing and recharging together.*

Observation 2. *The advantage of adaptive routing and recharging decisions is greater for longer trips and scenarios with higher station availability probabilities.*

From the numerical results, we further observe that the value of adaptivity is greater for longer trips and for Scenario 2. Long trips have more opportunities for the vehicle to find available charging stations without having to wait, and there is also greater flexibility with regard to path selection due to the higher number of nodes between the origin and destination. Comparing the two grid scenarios, the value of adaptivity is higher in Scenario 2 because the expected wait times at unavailable charging stations are longer than in Scenario 1 despite the unconditional expected wait times ($E[W_i]$) being similar. The vehicle is more likely to encounter an available charging station before running out of charge and being forced to stop at a station regardless of its availability, whereas in Scenario 1 there is less incentive to recharge earlier than necessary.

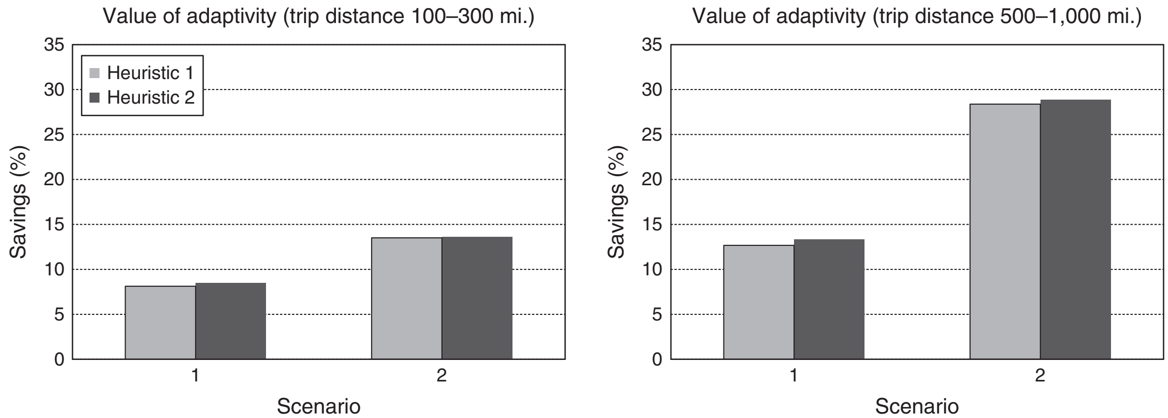
8.2. Number of Stops Comparison

As a result of the vehicle being able to make adaptive recharging decisions and take advantage of opportunities to recharge at available charging stations, the stopping locations tend to occur sooner along the route than in the optimal a priori policy. When this effect is compounded over the length of the entire path, it can

Table 2. Model Parameter Settings

Parameter	Value(s)	Units
α	0.8	N/A
γ	1	Time
h	5	Energy
$q_{\max}$	$\{70, 80, 90, 100\}$	Energy
s	$\{1, 10, 100\}$	Time
t	$\{5, 10, 15\}$	Time
numPaths	5	N/A
maxSkip	0	N/A

Figure 8. Value of Adaptive Decision Making as a Percentage of the Total Expected Waiting Time in an Optimal A Priori Policy



lead to instances in which the number of stops in an adaptive policy is greater than the number of stops in an a priori policy. Figure 9 shows the additional number of stops in the heuristic policies relative to that of the optimal a priori policy. Understandably, the longer trips tend to have a greater number of stops when adaptive decisions are permitted since the combined effect of recharging earlier than necessary is greater and is more likely to lead to an extra stop.

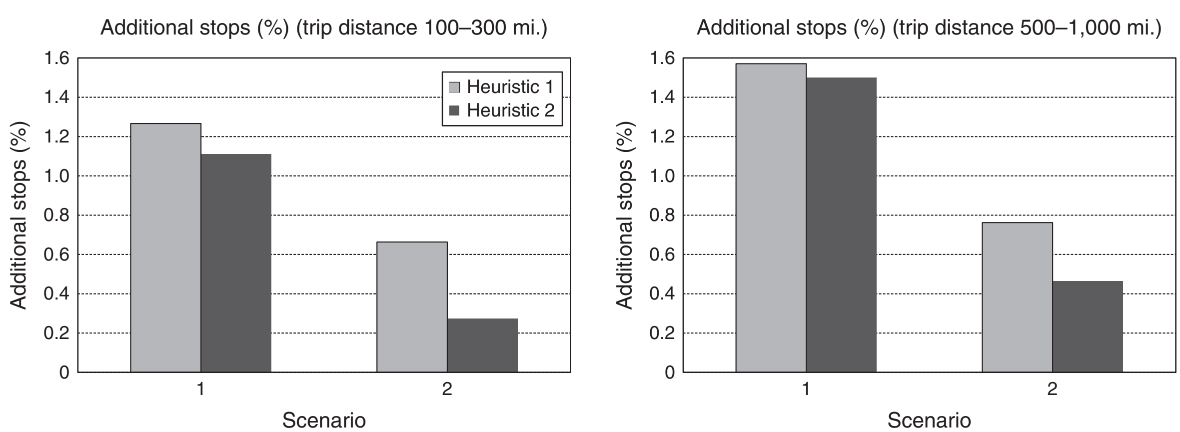
Observation 3. *The advantage of adaptive routing and recharging decisions is achieved with only a minor increase in the number of additional recharging stops. Furthermore, this number is large for scenarios with lower station availability probabilities.*

Averaged over all instances, the net additional increase in the number of stops is less than 2%. However, it can be seen that the effect is greater for Scenario 1 than for Scenario 2, which implies that the adaptive policies are more likely to stop to recharge if a charging station is available. This is due to the fact that the average probability of a charging station being available is lower in Scenario 1 than in Scenario 2, and thus the vehicle is more likely to charge sooner than necessary to avoid the chance of having to stop at an unavailable station. Heuristic 1 also tends to have more stops than Heuristic 2 because when the path is fixed, there are fewer stations from which to choose to delay recharging. With adaptive routing, the vehicle has several options and can pick the path that offers the greatest opportunity to delay recharging and reduce the likelihood of having to make an additional stop later along its route. Such opportunities are more prevalent in Scenario 2 than in Scenario 1, as the mean station availability is higher, and thus the difference in the number of stops between the two heuristics is greater in Scenario 2.

9. Conclusion

In this paper, we explore the problem of finding an optimal routing and recharging policy in a network with uncertain charging station availability. We begin with an adaptive recharging problem on a fixed path and then generalize the setting to include routing decisions on a general network. In the network setting, we identify properties of optimal policies and evaluate the computational complexity of the problem. We use these properties to develop efficient algorithms for

Figure 9. Average Difference in Number of Stops Between Heuristic and Optimal A Priori Policies



finding an optimal a priori policy and then present solution approaches to an adaptive problem that builds on a priori policy. We further enhance our solution approaches to a special case of the grid network. We present and implement two heuristics for obtaining an adaptive policy for a grid network, and we conduct detailed numerical experiments to evaluate the performance of our solution methods and estimate the value of adaptivity for a variety of problem settings.

The uncertainty of charging station availability within the network and the ability of the vehicle to adaptively choose which route to follow and where to recharge are unique and important features of our model. Our inclusion of a convex recharging cost function further improves the suitability of this work for capturing the costs incurred by EV drivers. As long as charging stations remain scarce and range anxiety persists, it is critical for EV drivers to be equipped with the necessary decision-support tools to minimize their time spent waiting for charging stations to become available and also make optimal routing and recharging decisions.

As part of our future work, we plan to further explore the value of adaptivity. We will refine our solution methods to identify additional opportunities within a network setting for an EV driver to benefit from adaptive decision making. We also intend to determine the value of an EV driver knowing not only the availability of the charging station at her current location but also the availability of other, nearby charging stations. Having such online information could potentially be of great benefit by enabling the driver to anticipate the likelihood of distant charging stations being available and plan her route accordingly. However, the exponential increase in computational time required to incorporate distant stations into a driver's decisions would need to be addressed.

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