# Week 2 Tutorial Questions

2021

1. A simple no claims discount system for motor insurance has four levels of discount.- 0%, 20%, 40% and 60%. A new policyholder starts on 0% discount. At the end of each policy year, policyholders will change levels according to the following rules:

   - At the end of a claim-free year, a policyholder moves up one level, or remains on the maximum discount.
   - At the end of a year in which exactly one claim was made, a policyholder drops back one level, or remains at 0
   - At the end of a year in which more than one claim was made, a policyholder drops back to zero discount.

   For a particular driver in any year, the probability of a claim-free year is 0.9, the probability of exactly one claim is 0.075, and the probability of more than one claim is 0.025.

   Mike took out a policy for the first time on 1 January 2015, and by 1 January 2018 he had made only one claim, on 3 May 2017. Calculate the probability that he is on 20% discount in 2020.

2. A Markov chain is determined by the transition matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

   Determine the period of each of the states in this chain.

3. A Markov chain $\{X_n\}_{n=0}^{\infty}$ has a discrete state space $S$. The initial probability distribution is given by $P[x_0 = i] = q_i$. The one-step transition probabilities are denoted by

   $P\left[X_{m+1} - i_{m+1} | X_m = i_m\right] = p_{i_m i_{m+1}}^{(m,m+1)}$.

   (a) State the Markov property for such a process.

   (b) Write down expressions for the following in terms of p's and q's.

i. $P[X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n]$

ii. $P[X_4 = i]$

4. A new actuarial student is planning to sit one exam each session. He expects that his performance in any exam will only be dependent on whether he passed or failed the last exam he sat. If he passes a given exam, the probability of passing the next will be $\alpha$, regardless of the nature of the exam. If he fails an exam, the probability of passing the next will be $\beta$ .

   (a) Obtain an expression for the probability that:

      i. the first exam he fails is the seventh, given that he passes the first

      ii. he passes the fifth exam, given that he fails the first three.

   (b) Explain the results above in terms of a Markov chain, specifying the state space and transition matrix. (For the purposes of this model, assume that we are only interested in predicting passing or failing, not in the number of exams passed so far.)

5. The stochastic process $X_t$ is defined by the relationship $X_t = Z_t + Z_{t-i}$, where $Z_t$ is a sequence of independent random variables with probability function:

$$X_t = \begin{cases} 1 & \text{with probability } p \\ 1,000 & \text{with probability } q \end{cases}$$

   where $p + q = 1$ and $q < p$.

   (a) Obtain expressions in terms of $p$ and $q$ for each of the following probabilities:

      i. $P(X_5 = 1,001)$

      ii. $P(X_5 = 1,001 | X_4 = 1,001)$

      iii. $P(X_5 = 1,001 | X_4 = 1,001, X_3 = 1,001)$.

   (b) State, with reasons, whether $X_t$ has the Markov property.

6. Determine all the stationary distributions for a Markov chain with transition matrix:

$$P = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \\ \frac{1}{2} & \frac{3}{10} & \frac{1}{5} & 0 & 0 \end{pmatrix}$$

7. At the end of each year an independent organisation ranks the performance of the unit trusts invested in a particular sector, and classifies them into four quartiles (with quartile 1 relating to the best performance). Past experience has shown that, at the end of each year, a fund will either remain in the same quartile or will move to a neighbouring quartile.

   In fact, there is a probability $1 - 2\alpha$ that a fund will remain in the same quartile and, where upward or downward movements are both possible, these are equally likely. However, it has been found that a fund that has remained in the top or bottom quartile for two consecutive years has a probability of $1-\beta(\beta < \alpha)$ of remaining in the same quartile the following year.

   (a) Construct a Markov chain with six states to model this situation, defining the states in your model and drawing a transition diagram.

   (b) Write down the transition matrix for your model.

   (c) Explain whether this Markov chain is irreducible and/or periodic.

   (d) Show that, if a stationary distribution exists with a quarter of the funds in each quartile, then $\beta = \frac{\alpha(1-2\alpha)}{1-\alpha}$

   (e) Last year 20% of funds in the second quartile moved up to the top quartile. Assuming the fund rankings have reached a stationary state, estimate the probability that a fund that has been in the top quartile for the last two years will remain in the top quartile for a third consecutive year.

8. A simple NCO system has four levels of discount - 0%, 20%, 40% and 60%. A new policyholder starts on 0% discount. At the end of each policy year, policyholders will change levels according to the following rules:

   - At the end of a claim-free year, a policyholder moves up one level, or remains on the maximum discount.
   - At the end of a year in which exactly one claim was made, a policyholder drops back one level, or remains at 0%.
   - At the end of a year in which more than one claim was made, a policyholder drops back to zero discount.

   For a particular policyholder in any year, the probability of a claim-free year is $\frac{7}{10}$ , the probability of exactly one claim is $\frac{1}{5}$ and the probability of more than one claim is $\frac{1}{10}$.

   (a) Write down the transition matrix for this time-homogeneous Markov chain.

   (b) Calculate the 2-step transition probabilities from state $i$ to state $j$, $p_{ij}^{(2)}$

(c) If the policyholder starts with no discount, calculate the probability that this policyholder is at the maximum discount level 5 years later.

(d) If a large number of people having the same claim probabilities take out policies at the same time, calculate the proportion would you expect to be in each discount category in the long run.

9. Consider the following two Markov chains:

- Chain I is defined on the state space $\{1, 2, 3, 4\}$ and has transition matrix:

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

- Chain II is defined on the state space $\{1, 2, 3, 4, 5\}$ and has transition matrix:

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Let $X_t$ denote the state occupied at time $t$ . For each chain:

(a) Draw a transition diagram, including on your diagram the probability of each possible transition.

(b) Calculate:

i. $P(X_2 = 1|X_0 = 1)$

ii. $P(X_4 = 1|X_0 = 1)$

(c) Explain whether the chain is irreducible and/or aperiodic.

(d) Explain whether or not the process will converge to a stationary distribution given that it is in State 1 at time 0. If it does converge, determine the stationary distribution.

10. An author is about to start writing a book that will contain 20 chapters. The author plans to write a new chapter each week. However, when he reviews his work at the end of each week, there is a probability of0.25 (which is independent of the current state of the book) that he will not be happy with one of the chapters he has written. In this case, he will spend the following week rewriting that particular chapter instead of embarking on a new one. He may decide to rewrite any one chapter, including a new one he has just finished or one that he has previously rewritten.

Let $X_k$ denote the number of chapters that the author is happy with at the end of week $k$, and define $X_0 = 0$.

(a) Explain why $X_k$ can be modelled as a Markov chain.

(b) Calculate the probability that the author will complete the book in exactly 25 weeks.

(c) Calculate the expected number of weeks it will take the author to complete the book.

11. The credit-worthiness of debt issued by companies is assessed at the end of each year by a credit rating agency. The ratings are A (the most credit-worthy), Band D (debt defaulted). Historic evidence supports the view that the credit rating of a debt can be modelled as a Markov chain with one-year transition matrix:

$$P = \begin{pmatrix} 0.92 & 0.05 & 0.03 \\ 0.05 & 0.85 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Determine the probability that a company currently rated A will never be rated B in the future.

(b) i. Calculate the second-order transition probabilities of the Markov chain.

ii. Hence calculate the expected number of defaults within the next two years from a group of 100 companies, all initially rated A

The manager of a portfolio investing in company debt follows a 'downgrade trigger' strategy. Under this strategy, any debt in a company whose rating has fallen to Bat the end of a year is sold and replaced with debt in an A-rated company.

(c) Calculate the expected number of defaults for this investment manager over the next two years, given that the portfolio initially consists of 100 A-rated bonds. [2]

(d) Comment on the suggestion that the downgrade trigger strategy will improve the return on the portfolio.