# ETC3430: Financial mathematics under uncertainty

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Consultation:

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- Consultation Hour: 10:00-12:00pm Wedsdays via Zoom Meeting ID: 997 898 6063 Passcode: 299920. I will be in my office if the circumstance permits.

Assessments

- 10% Assignment One.
- 10% Assignment Two.
- 20% Class Test.
- 60% Final exam.

# Outline

1. Stochastic Processes in a Nutshell
   - Review on Random Variables
   - Stochastic Process as a collection of random variables
   - Stochastic Processes Classification
   - Mixed type Stochastic Process
2. Properties of stochastic processes
   - Conditional Distribution and Joint Distributions
   - Stationarity
   - Increments
   - Markov Property
3. Widely used Processes in Insurance
   - Poisson Process
   - Compound Poisson

# Random Variable

Suppose that to each point of a sample space, i.e. $\omega \in \Omega$, and we assign a number. We then have a function defined on the sample space, $\Omega$. This function is called a random variable, usually denoted by a capital letter such as $X$ or $Y$.

**Example**

*A coin is tossed twice so that the sample space is $\Omega = \{HH, HT, TH, TT\}$. Let $X$ represent the number of heads that can come up.*

# Discrete Vs Continuous

A random variable that takes on a finite or countably infinite number of values is called a discrete random variable. In contrast, one which takes on a non-countable and an infinite number of values is called a non-discrete random variable.

- Discrete: Binomial, Poisson, Negative Binomial...
- Continuous: Normal, Student-t, Gamma...

# Probability distribution function

**Definition**

*A probability mass function (pmf) is a function that gives the probability that a discrete random variable is exactly equal to some value. A probability density function (PDF) of a continuous random variable, is a function whose value at any given sample (or point) in the sample space can be interpreted as providing a relative likelihood that the random variable's value would equal that sample.*

# Stochastic Process

## Definition

*A stochastic process is a model for a time-dependent random experiment, that is a collection of ordered random variables, $X_t$, defined on a common probability space, taking values in a common set, one each time $t$ in some set* **J**,

$$\{X_t, t \in J\}$$

*The set of values that the random variables $X_t$ are capable of taking is called the* ***state space*** *of the process,* **S**, *and the set* **J** *is the* ***time domain*** *of the process.*

# $X_t(\omega)$: a random function

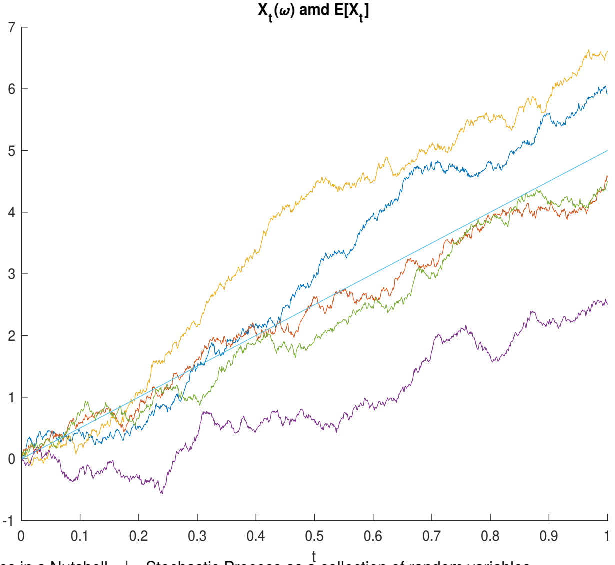
**Definition**

*A joint realization of the random variables $X_t(\omega)$ for all $t \in \mathbf{J}$ given the random outcome of the realization is called a sample path of the process, that is a deterministic function from $\mathbf{J}$ to $\mathbf{S}$.*

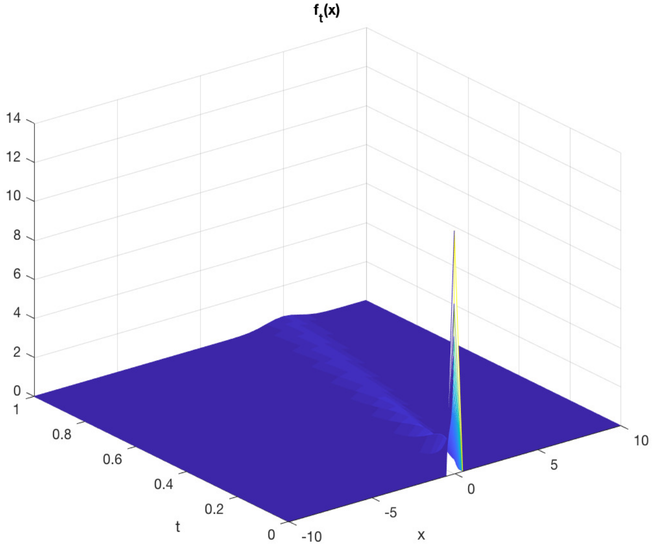
**Definition**

*Given $t \in \mathbf{J}$, $X_t$ is a random variable, i.e a function from $\Omega$ to $\mathbf{S}$, it's likelihood is described by its corresponding probability density function.*

# Stochastic Process: a Graphical view



# Stochastic Process: a Graphical view



A stochastic process is classified by the nature of the time domain(Continuous or Discrete), and of the state space **S**.

1. Continuous time and Discrete state space
2. Continuous time and Continuous state space
3. Discrete time and Continuous state space
4. Discrete time and Discrete state space

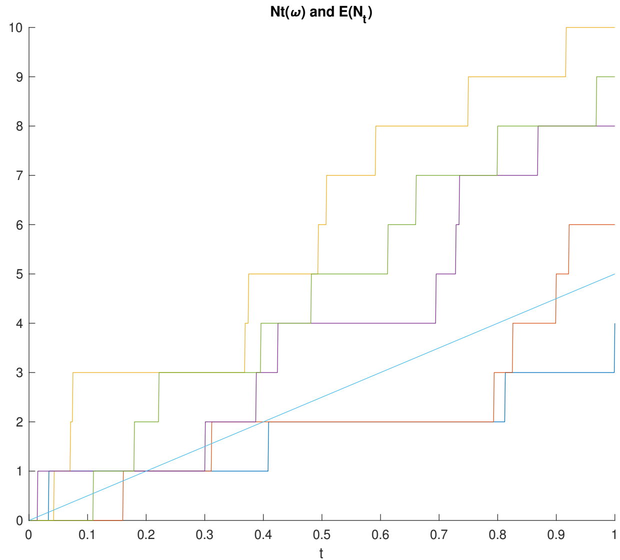
# The aggregate number of claims: $N_t$

## Example

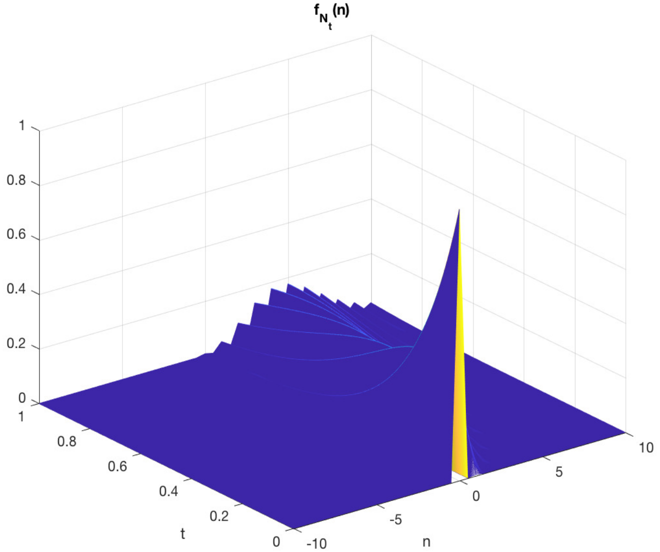
*In many examples of Life and General insurance portfolio modelling, we study the aggregate number of claims, $N_t$, that arises from the portfolio starting from time zero.*

- *Continuous time* $\mathbf{J} = [0, T)$ *and Discrete state space* $\mathbf{S} = \mathbb{N}_0$.
- *A counting process, each sample path of* $X_t$ *is a non-decreasing function of t.*

# Stochastic Process: a Graphical view



# Stochastic Process: a Graphical view



# Mixed type Stochastic Process

A mixed type stochastic process is a combination of two stochastic processes, one discrete-time set and one continuous-time set.

## Example

*The number of decrements in a pension fund with* $S = \{1, 2, 3, ...\}$ *and* $J = [0, \infty)$ *is the sum of*

- *the number of death, modelled via a continuous-time discrete state-space process*
- *the number of retirement at exact ages or members aged between 60 and 65, modelled via a dicrete-time discrete state-space process.*

A stochastic process is defined by its

1. time domain
2. state space
3. the joint distribution of $X_{t_1}, X_{t_2}...X_{t_k}$ for all $t_1, t_2, .., t_k$ and all integer k.

Typically the joint distribution is specified indirectly, i.e. via the conditionals.

# Random Walk

## Example

*Random Walk Consider a sequence of independent and identically distributed random variables $\epsilon_j$ for $j = 1, 2, ...,$ and a random walk is of the form*

$$X_t = \sum_{j=1}^{t} \epsilon_j$$

*with initial condition $X_0 = 0$.*

Hence, we can also write this as

$$X_t = X_{t-1} + \epsilon_t.$$

# Random Walk

For the special case of $\epsilon_t \sim N(0,1)$ for all $t$, we have

$$X_t|X_{t-1} \sim N(X_{t-1}, 1)$$

the joint distribution

$$\begin{aligned} f(x_t, x_{t-1} \ldots, x_1) &= f(x_t|x_{t-1}) f(x_{t-1}|x_{t-2}) \ldots f(x_2|x_1) f(x_1) \\ &= \prod_{j=1}^{t} \frac{1}{\sqrt{2\pi}} \exp^{-0.5(x_j - x_{j-1})^2} \end{aligned}$$

This is a multivariate normal distribution.
Challenge: What is the mean and covariance of this multivariate normal?

# Stationarity

A stationary process is a stochastic process whose unconditional joint probability distribution does not change when shifted in time. Hence its statistical properties such as mean and variance also do not change over time.

# Strict Stationarity

## Definition

*Let $F(x_{t_1+k}, x_{t_2+k}, ..., x_{t_n+k})$ represent the cumulative distribution function of the unconditional (i.e., with no reference to any particular starting value) joint distribution of $X_t$ at times $t_1+k, t_2+k, ..., t_n+k$, $X_t$ is said to be strictly stationary, strongly stationary or strict-sense stationary if*

$$F(x_{t_1+k}, x_{t_2+k}, ..., x_{t_n+k}) = F(x_{t_1}, x_{t_2}, ..., x_{t_n})$$

*for all $t_1+k, t_2+k, ..., t_n+k$ and $t_1, t_2, ..., t_n \in \mathbf{J}$ and $n \in \mathbb{N}$.*

# White Noise

## Definition

*White noise is a stochastic process that consists of a set of independent and identically distributed variables. The random variable can be either discrete or continuous, and the time set can be either discrete and continuous.*

White noise is the simplest example of a stationary process. Two classic examples are

$$Z_t = \begin{cases} \sigma & w.p. 0.5 \\ -\sigma & w.p. 0.5 \end{cases} \quad \text{and } Z_t \sim N(0, \sigma^2)$$

# Weak Stationarity

A weaker form of stationarity is known as weak-sense stationarity(WSS). WSS random processes only require that 1st moment (i.e. the mean) and autocovariance do not vary with respect to time and that the 2nd moment is finite for all times. Any strictly stationary process which has a defined mean and covariance is also WSS.

## Definition

*A process is weakly stationary if*

- $\mathbb{E}[X_t]$ *is constant for all* $t \in \mathbb{J}$
- $cov(X_t, X_{t+k})$ *depends only on the lag* $k$.

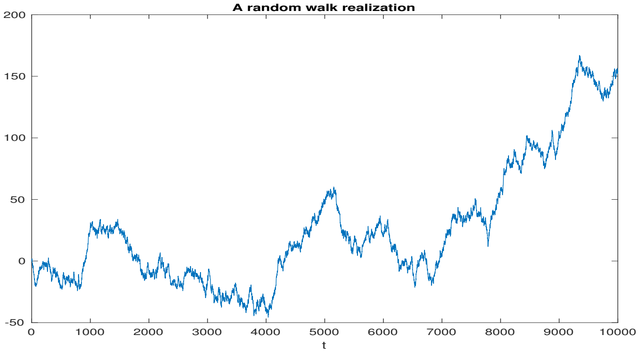
Note that the second conditional also implies constant variance, i.e. $k = 0$.

# Random Walk is not stationary

Recall that a random walk is defined by $X_t = X_{t-1} + \epsilon_t$ such that $\epsilon_j$'s are i.i.d. Hence

$$\mathbb{E}[X_t] = \mathbb{E}[\epsilon]t \text{ and } var(X_t) = var(\epsilon)t$$

the process is not even weakly stationary unless $\epsilon_t = 0$ for all $t$.



# Increments

As mentioned early, the joint distribution of the processes is sometimes difficult to specify. Instead, increments of stochastic processes often have much simpler properties.

**Definition**

*An increment of a process is its value changes over time, i.e.*

$$X_{t+h} - X_t \text{ where } h > 0.$$

For example, the increment of a random walk $X_t - X_{t-1}$ is a white noise, $\epsilon_{t-1}$.

# Stock Return

Let $S_t$ denote the stock market index process; most stock indices exhibit a non-stationary behaviour. Let's look at

- S&P 500
- ASX 200
- Hang Seng index

However, if we instead look at the log price process, $X_t = \log(S_t)$, its increment

$$X_{t+h} - X_t = \log(\frac{S_{t+h}}{S_t})$$

is typically stationary.

# Independent Increments

**Definition**

*A process is said to have independent increments if for all $t \in \mathbb{J}$ and every $h > 0$, the increment $X_{t+h} - X_t$ is independent of the past history $\{X_s\}_{0 \le s \le t}$.*

- White noise: $Z_t - Z_{t-1}$ is clearly dependent on $Z_{t-1}$
- Random Walk: $X_{t+h} - X_t = \sum_{j=t+1}^{t+h} \epsilon_j$, the independence of $\epsilon_j$'s implies independent increments.

# Markov Property

Markov property refers to the memoryless property of a stochastic process. The conditional probability distribution of future states of the process (dependent on both past and present states) depends only on the current state, not on the sequence of events that preceded it.

## Definition (Markov Property)

$$\mathbb{P}(X_t \in \mathbf{A} | X_{s_1} = x_1, X_{s_2} = x_2, ..., X_s = x) = \mathbb{P}(X_t \in \mathbf{A} | X_s = x)$$

*for all times* $s_1 < s_2 < ... < s < t$, *all states* $x_1, x_2, ...., x \in \mathbf{S}$ *and all subset* $\mathbf{A}$ *of* $\mathbf{S}$.

A stochastic process with this property is called a Markov process.

# Filtration vaguely...

For any stochastic process,

- it is mapping from the sample space $\Omega$ and the time domain $\mathbf{J}$ to the state space $\mathbf{S}$.
- $\mathbb{F}$ is a collection of events of $\Omega$, to which a probability can be signed
- at time $t$, a smaller $\mathbb{F}_t \in \mathbb{F}$ is a smaller collection of events that are known at time $t$. As time increases,

$$\mathbb{F}_s \in \mathbb{F}_t \text{ for } s \leq t.$$

Via filtration, the markov property can be formulated as

$$\mathbb{P}(X_t \in \mathbf{A} | \mathbb{F}_s) = \mathbb{P}(X_t \in \mathbf{A} | X_s).$$

# Markov and Forecasting

In predictive modelling and probabilistic forecasting, the Markov property is desirable since it greatly simplifies the mathematical structure. Such a model is known as a Markov model.

## Example

*Suppose that you'd like to predict the most probable next word in a sentence. You can gather huge amounts of statistics from the text. The most straightforward way to make such a prediction is to use the previous words in the sentence. For example, given the sentence, "The cat chased the ??", suppose you'd like to predict the next word. You could consider all of the words that have appeared up until that point in the sentence, and chose the most likely next word. Or you could look at the last one, ignoring everything that came before it. Ignoring events before a certain point in the past gives such a model the Markov property.*

# Independent increment→ Markov

**Theorem**

*A process with independent increments has the Markov property.*

**Proof.**

$$
\begin{aligned}
&\mathbb{P}(X_t \in \mathbf{A} | X_{s_1} = x_1, X_{s_2} = x_2, \ldots, X_s = x) \\
=&\mathbb{P}(X_t - X_s + x \in \mathbf{A} | X_{s_1} = x_1, X_{s_2} = x_2, \ldots, X_s = x) \\
=&\mathbb{P}(X_t - X_s + x \in \mathbf{A} | X_s = x) \ \text{ I.C} \\
=&\mathbb{P}(X_t \in \mathbf{A} | X_s = x)
\end{aligned}
$$

□

Note the above theorem is not vice versa.

# Point Process

## Definition

*A sequence of events which occur at random instants, $T_1, T_2, ....T_n, ...,$ the sequence $(T_i)_{i\geq 1}$ is called a* ***point process****.*

Classical examples are

- the arrival of claims,
- the occurrence of system breakdowns .

It is often the case that a counting process has the property of

$$0 < T_1 < T_2 < ... < T_n < ... \text{ and } \lim_{n\to\infty} T_n = \infty \quad a.s.$$

This means the registration of events begins at time 0, but 0 is not an event arrival, two events can't occur simultaneously, and the observation takes place over a long time.

# Counting Process

**Definition**

*For a point process $\{T_n\}$, the associated* ***counting process****, $\{N_t\}_{t\geq 0}$ represent the number of events that have occurred in the time interval $[0, t]$,*

$$N_t = \sup\{n : n = 0, 1, 2, ...; T_n \leq t\}, t \geq 0.$$

*The point process and associated counting process contains the same information.*

# Poisson Process

## Definition

*$N_t \sim Poisson(\lambda t)$ distribution, such that*

1. *$N_0 = 0$ and $N_s \leq N_t$ when $s < t$*
2. *Independent increment*

   *for any $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$ in $\mathbb{R}^+$, the increments*

   $$N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$$

   *are mutually independent*
3. *Stationary increment*

   *for any $0 < t_1 < t_2$ and $h > 0$ in $\mathbb{R}$*

   $$\mathbb{P}(N_{t_2+h} - N_{t_1+h} = k) = \mathbb{P}(N_{t_2} - N_{t_1} = k), \quad k \in \mathbb{N}$$

From the definition, we conclude $\{N_t; t \geq 0\}$ is a continuous time, discrete state space Markov process.

# Properties of Poisson Process

- continuous time and discrete state space
- independent increment

$$N_t - N_s \sim Poisson(\lambda(t-s)) \text{ for } 0 \leq s < t.$$

- Markov
- Non-stationary, i.e. $\mathbb{E}[N_t] = \lambda t$.

# Compound Poisson: Aggregate Claims Process

## Definition

*A compound poisson process is defined as*

$$S_t = \sum_{i=1}^{N_t} X_i$$

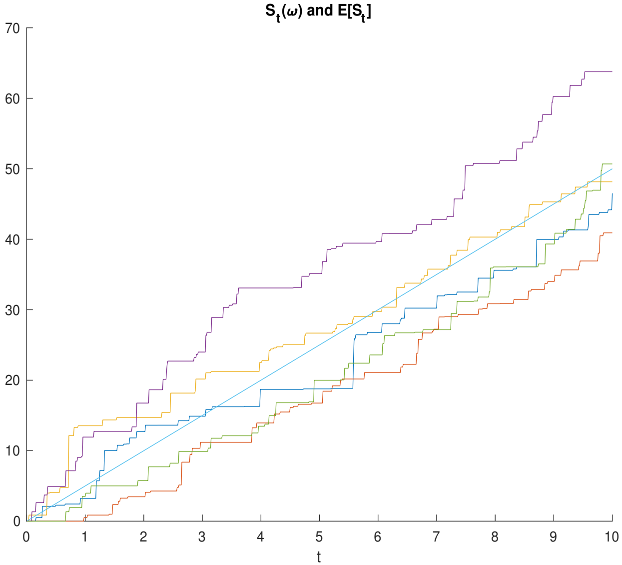
*where $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative R.V. distributed according to $F$, and $N_t$ is a Poisson Process that is independent of all $X_i$'s.*

# Properties of Compound Poisson Process

- continuous time and discretecontinuous state space(depends on the distribution of X),
- independent increment

$$S_t - S_s = \sum_{j=N_s+1}^{N_t} X_j \sim CPoisson(\lambda(t-s), F) \text{ for } 0 \leq s < t.$$

- Markov
- Non-stationary, i.e. $\mathbb{E}[S_t] = \lambda t \mathbb{E}[X]$.


$S_t(\omega)$ and $E[S_t]$
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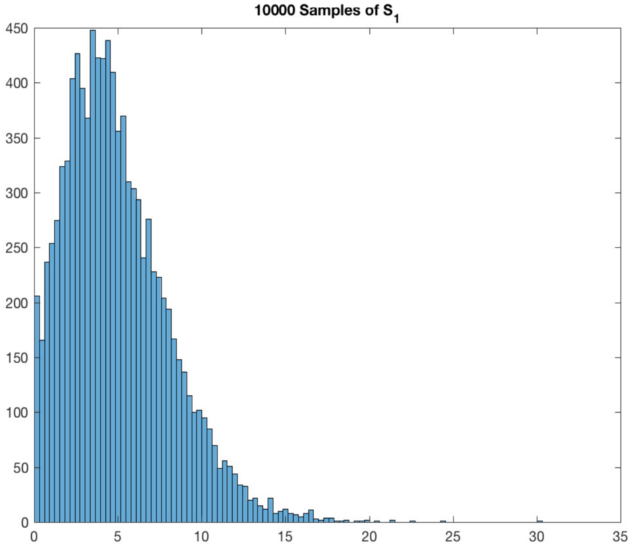
10000 Samples of S1
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