Week 5 Tutorial Solutions

2021

1. (a) **Distribution of the inter-arrival time**

   The distribution of the inter-arrival time random variable is $Exp(\lambda)$.

   (b) **Probability of exactly one event**

   The distribution of the number of occurrences in a time period of length t is $Poisson(\lambda t)$. So the probability of exactly one event is $\lambda t e^{-\lambda t}$

2. (a) **Expected waiting time until the first claim of amount 30**

   Claims of amount 30 occur according to a Poisson process with a mean of $0.1 \times 5 = 0.5$ per day. So the waiting time until the first claim of amount 30 has an $Exp(0.5)$ distribution and the expected waiting time is 2 days

   (b) **Probability that there ore at least 10 claims during the first 2 days, given that there were exactly 6 claims during the first day**

   Let $N(t)$ denote the number of claims during the interval $[0, t]$. Then:

   $$\begin{aligned} P\left(N(2) \geq 10 \middle| N(1) = 6\right) &= P\left(N(2) - N(1) \geq 4 \middle| N(1) - N(0) = 6\right) \\ &= P\left(N(2) - N(1) \geq 4\right) \end{aligned}$$

   since $N(0) = 0$ and the numbers of claims in non-overlapping time intervals are independent. Now $N(2) - N(1)$ $Poisson(5)$, so:

   $$\begin{aligned} P\left(N(2) \geq 10 \middle| N(1) = 6\right) &= P\left(Poisson(5) \geq 4\right) \\ &= 1 - P\left(Poisson(5) \leq 3\right) \\ &= 1 - e^{-5}\left(1 + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!}\right) \\ &= 0.73497 \end{aligned}$$

   (c) **Probability that there are at least 2 claims of amount 20 during the first day and at least 3 claims of amount 20 during the first 2 days**

   Let $N_{20}(t)$ denote the number of claims of amount 20 in the interval $[0, t]$. We want:

$$P(N_{20}(1) \geq 2, N_{20}(2) \geq 3)$$

If we have 3 or more claims during the first day, then the second condition is automatically satisfied. lfwe have exactly 2 claims on the first day, then we need at least 1 claim on the second day. So the required probability is:

$$P(N_{20}(1) \geq 3) + P(N_{20}(1) = 2, N_{20}(2) - N_{20}(1) \geq 1$$

Now $N_{20}(1)$ and $N_{20}(2) - N_{20}(1)$ are both Poisson with mean $0.7 \times 5 = 3.5$. Also, $N_{20}(1)$ and
$N_{20}(2) - N_{20}(1)$ are independent. So:

$$P(N_{20}(1) \geq 3) = 1 - P(N_{20}(1) \leq 2) = 1 - e^{-3.5}\left(1 + \frac{3.5^1}{1!} + \frac{3.5^2}{2!}\right) = 0.679153$$

and:

$$\begin{aligned} P(N_{20}(1) = 2, N_{20}(2) - N_{20}(1) \geq 1) &= P(N_{20}(1) = 2) P(N_{20}(2) - N_{20}(1) \geq 1) \\ &= P(N_{20}(1) = 2)\left[1 - P(N_{20}(2) - N_{20}(1) = 0)\right] \\ &= \frac{e^{-3.5}\, 3.5^2}{2!}\left[1 - e^{-3.5}\right] \\ &= 0.179374 \end{aligned}$$

The required probability is:

$$P(N_{20}(1) \geq 3) + P(N_{20}(1) = 2, N_{20}(2) - N_{20}(1) \geq 1) = 0.679153 + 0.179374 = 0.85853$$

(d) **Conditional variance**

Let $N_j(t), j = 10, 20, 30$ denote the number of claims of amount j in the interval [O,t]. Then:

$$N(1) = N_{10}(1) + N_{20}(1) + N_{30}(1)$$

and:

$$\begin{aligned} \text{var}\left[N(1) \mid N_{10}(1) = 2\right] &= \text{var}\left[N_{10}(1) + N_{20}(1) + N_{30}(1) \mid N_{10}(1) = 2\right] \\ &= \text{var}\left[2 + N_{20}(1) + N_{30}(1)\right] \\ &= \text{var}\left[N_{20}(1) + N_{30}(1)\right] \\ &= \text{var}\left[N_{20}(1)\right] + \text{var}\left[N_{30}(1)\right] \end{aligned}$$

by independence.

Now, since $N_{20}(1) \sim \textit{Poisson}(3.5)$ and $N_{30}(1) \sim \textit{Poisson}(0.5)$:

$$\text{var}\left[N(1) \mid N_{10}(1) = 2\right] = 3.5 + 0.5 = 4$$

3. The process defined is a Poisson process with parameter λ .

(a)
$$p_{ij}(t) = \frac{e^{-\lambda t}(\lambda t)^{j-i}}{(j-i)!} \quad \text{for } j \geq i$$

(b) The holding times are inter-event times. In other words, the time spent in a particular state between transitions. For the process given, the i th holding time T_{i-1} is the time spent in state $i-1$ before the transition to state i.

(c) We have:

$$P[T_0 > t | X_0 = 0] = P[X_t = 0 | X_0 = 0] = p_{00}(t) = e^{-\lambda t}$$

and it follows that T_0 has an $Exp(\lambda)$ distribution.

(d) $X_0 = 1$ since we choose the sample paths to be right-continuous. So at time T_0 it has just jumped to 1.

(e) Consider T_i:

$$P\left[T_i > t \,|\, X_0 = 0, \sum_{j=0}^{i-1} T_j = s\right] = P\left[X_{t+s} - X_s = 0 \,|\, X_0 = 0, \sum_{j=0}^{i-1} T_j = s\right]$$
$$= P[X_t - X_0 = 0] = P_{00}(t) = e^{-\lambda t}$$

The second equality is due to the increments being independent and stationary. Hence T_i also has an exponential distribution with parameter λ.

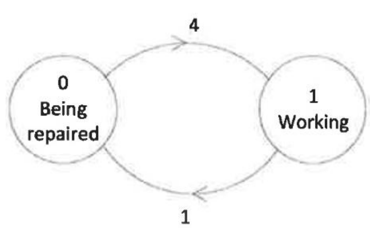
4. (a) Assumptions

We are assuming that the process is Markov and that the transition rates are constant.

The Markov property of the underlying jump chain can be tested using a chi-squared test based on triplets of successive observations.

A chi-squared test can also be used to test whether the waiting times are exponentially distributed with constant parameter.

(b) Transition graph



(c) Kolmogorov's differential equations

The backward differential equation is:

$$\frac{d}{dt} P_{0,0}(t) = -4P_{0,0}(t) + 4P_{1,0}(t)$$

and the forward differential equation is:

$$\frac{d}{dt}P_{0,0}(t) = P_{0,0}(t)\times(-4) + P_{0,1}(t)\times 1 = -4P_{0,0}(t) + P_{0,1}(t)$$

(d) Proof

We have the differential equation:

$$\frac{d}{dt}P_{0,0}(t) = 1 - 5P_{0,0}(t)$$

together with the boundary condition $P_0, 0(0) = 1$

We can solve this equation and boundary condition using an integrating factor of e^{5t}:

$$\frac{d}{dt}P_{0,0}(t)e^{5t} + 5P_{0,0}(t)e^{5t} = e^{5t}$$

$$\Rightarrow \frac{d}{dt}\left(P_{0,0}(t)e^{5t}\right) = e^{5t}$$

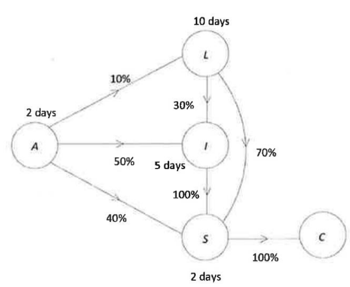
$$\Rightarrow P_{0,0}(t)e^{5t} = \frac{1}{5}e^{5t} + K$$

$$\Rightarrow P_{0,0}(t) = \frac{1}{5} + Ke^{-5t}$$

Applying the boundary condition $P_{0,0}(0) = 1 \Rightarrow K = \frac{4}{5}$. So we have the required result.

5. (a) Generator matrix

The information given in the question about the occupancy times in each state and the transition probabilities in the Markov jump chain can be summarised as:



Note that this is not a proper transition diagram, as a transition diagram must show the forces of transition.

The total force out of each state is equal to the reciprocal of the expected holding time. The percentages indicate how the total force is divided between destination states.

The generator matrix is:

$$
\begin{array}{c}
\begin{array}{ccccc} A & L & I & S & C \end{array} \\
\begin{bmatrix}
-0.5 & 0.5\times 0.1 & 0.5\times 0.5 & 0.5\times 0.4 & 0 \\
0 & -0.1 & 0.1\times 0.3 & 0.1\times 0.7 & 0 \\
0 & 0 & -0.2 & 0.2\times 1 & 0 \\
0 & 0 & 0 & -0.5 & 0.5\times 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
= \begin{bmatrix}
-0.5 & 0.05 & 0.25 & 0.2 & 0 \\
0 & -0.1 & 0.03 & 0.07 & 0 \\
0 & 0 & -0.2 & 0.2 & 0 \\
0 & 0 & 0 & -0.5 & 0.5 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(b) Proportion of claims that require further details from the insured

We can list all the paths that correspond to the event of visiting state l if the process starts in state A.

These are $A \to L \to I \to S \to C$ and $A \to I \to S \to C$.

The probabilities of these paths are $0.10\times 0.30\times 1\times 1 = 0.03 and 0.50\times 1 \times 1 = 0.50$. The total probability is 0.53.

Alternatively we can use a more general approach. This has the advantage of working in more complicated situations where the path counting approach becomes very cumbersome.

Let $p_i = P[nevervisitstatel|currentlyinstatei]$, then using the Markov jump chain transition matrix:

$$
\begin{bmatrix}
0 & 0.10 & 0.50 & 0.40 & 0 \\
0 & 0 & 0.30 & 0.70 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

we can write:

$$p_A = 0.1p_L + 0.5p_I + 0.4p_S = 0.07 + 0.40 = 0.47$$

$$p_L = 0.3p_I + 0.7p_S = 0.7$$

$$p_I = 0$$

$$p_S = 1$$

$$p_C = 1$$

The required probability is l- PA = l - 0.4 7 = 0.53.

(c) Probability that a claim is yet to be logged and classified by a claims administrator at time t

The Chapman-Kolmogorov equation is:

$$P_{AA}(t+h) = P_{AA}(t)P_{AA}(h)$$

This assumes that the process satisfies the Markov property.

Then the law of total probability allows us to write:

$$P_{AA}(h)+P_{AL}(h)+P_{AI}(h)+P_{AS}(h)+P_{AC}(h)=1$$

The definition of the transition rates gives:

$$P_{AL}(h)=0.05h+o(h)$$

$$P_{AI}(h)=0.25h+o(h)$$

$$P_{AS}(h)=0.20h+o(h)$$

Also, $P_{AC}(h)=o(h)$ because it involves more than one transition.

Substituting we obtain:

$$P_{AA}(h)=1-0.05h-0.25h-0.20h+o(h)$$

$$\Rightarrow P_{AA}(t+h)=P_{AA}(t)\left(1-0.05h-0.25h-0.20h+o(h)\right)$$

$$\Rightarrow \frac{P_{AA}(t+h)-P_{AA}(t)}{h}=-0.50P_{AA}(t)+\frac{o(h)}{h}$$

$$\Rightarrow \frac{d}{dt}P_{AA}(t)=-0.50P_{AA}(t)$$

Separating the variables:

$$\frac{\frac{d}{dt}P_{AA}(t)}{P_{AA}(t)}=\frac{d}{dt}\ln P_{AA}(t)=-0.50$$

Then integrating with respect to t:

$$\ln P_{AA}(t)=-0.50t+C$$

where C is a constant of integration. Using the initial condition $P_{AA}(0) = 1$ I we see that $C = 0$ and hence:

$$P_{AA}(t) = e^{-0.50t}$$

6. (a) Equivalent first condition for stationary

   Since $A = P'(0)$ and the vector $\underline{\pi}$ is constant, differentiating the equation $\underline{\pi}P(t) - \underline{\pi}$. with respect to t gives and setting t = 0 gives:

$$\underline{\pi}A = \underline{0}$$

   (b) Modelling as o Markov process

   This is a 3-state Markov jump process. The states are (1) level 1; (2) level 2; (3) left the company.

   We have made the Markov assumption, ie that the probability of jumping to any particular state depends only on knowing the current state that is occupied.

   We have assumed that transition rates between states are constant over time.

   (c) Generator matrix

   The average waiting time in each state, i is exponentially distributed with mean $\frac{1}{\lambda_j}$, . The mean times in states 1 and 2 are 2 and 5 years respectively. The values of the exponential parameters are:

$$\lambda_1 = \frac{1}{2} \quad \lambda_2 = \frac{1}{5}$$

The transition matrix of the jump chain, p_{ij} is:

$$\begin{array}{c} \begin{array}{ccc} \textit{level}\ 1 & \textit{level}\ 2 & \textit{left} \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

The off-diagonal elements of the generator matrix of transition rates, $_{ij}$ are given by:

$$\mu_{ij} = \lambda_i p_{ij}$$

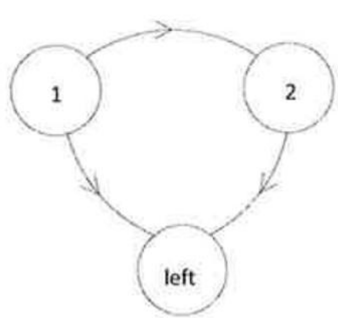
The diagonal elements are chosen to make each row of the matrix sum to O .

The generator matrix (matrix of transition rates) is:

$$\begin{array}{c} \begin{array}{ccc} \textit{level}\ 1 & \textit{level}\ 2 & \textit{left} \end{array} \\ \begin{bmatrix} -0.50 & 0.25 & 0.25 \\ 0 & -0.20 & 0.20 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

(d) Distribution of employees in five years

The model is:



We are given the initial state as [600 400 0]. We can use the five-year transition probabilities to estimate the numbers in each state in five years' time. The number in state 1 will be:

$$600P_{11}(5)$$

The number in state 2 will be:

$$600P_{12}(5) + 400P_{22}(5)$$

and the number of employees who have left the company can be obtained by deducting the numbers in states 1 and 2 from 1,000.

The occupancy probabilities for states 1 and 2 are given by:

$$P_{11}(t) = P_{\overline{11}}(t) = e^{-0.5t}$$

$$P_{22}(t) = P_{\overline{22}}(t) = e^{-0.2t}$$

Using the generator matrix, we can write the forward differential equation for $P_{12}(t)$:

$$\frac{d}{dt}P_{12}(t) = 0.25P_{11}(t) - 0.2P_{12}(t)$$

$$\Rightarrow \quad \frac{d}{dt}P_{12}(t) + 0.2P_{12}(t) = 0.25e^{-0.5t}$$

This can be solved using the integrating factor method. The integrating factor is $e^{0.2t}$.

Multiplying through by the integrating factor gives:

$$e^{0.2t}\frac{d}{dt}P_{12}(t) + 0.2e^{0.2t}P_{12}(t) = 0.25e^{-0.5t}e^{0.2t} = 0.25e^{-0.3t}$$

Integrating both sides:

$$e^{0.2t}P_{12}(t) = -\frac{5}{6}e^{-0.3t} + C$$

The boundary condition is $P_{12}(0) = 0$. So:

$$0 = -\frac{5}{6} + C \Rightarrow C = \frac{5}{6}$$

Simplifying then gives:

$$P_{12}(t) = \frac{5}{6}\left(e^{-0.2t} - e^{-0.5t}\right)$$

So the number of employees on level 1 in 5 years' time is:

$$600P_{11}(5) = 600e^{-2.5} = 49$$

and the number of employees on level 2 in 5 years' time is:

$$600P_{12}(5) + 400P_{22}(5) = 600 \times \frac{5}{6}\left(e^{-1} - e^{-2.5}\right) + 400e^{-1} = 290$$

The number of lives who have left the company is:

$$1{,}000 - 49 - 290 = 661$$

(e) In this question you have to be very careful not to mix up v (v for victor) and ν (the Greek letter 'nu').

i. A. Assumptions

We are assuming that the transition probabilities depend only upon the individual's current state. They do not depend upon the previous transitions for the individual, so we are assuming that the Markov property holds.

We are also assuming that the probability of two or more transitions in a short time interval of length h is o(h)...

... and for small values of h and $59 \leq x \leq x + t \leq 60$:

$${}_hp_x^{ad} = h\mu + o(h)$$

and: $${}_hp_x^{ar} = h\upsilon + o(h)$$

B. Proof

A life who remains active for t + h years must first remain active for t years, then remain active for a further h years (where h represents a short time interval). Expressed in terms of probabilities, this is:

$${}_{t+h}p_x^{aa} = {}_tp_x^{aa} \times {}_hp_{x+t}^{aa}$$

This follows from the Markov property, ie that the probabilities in different time periods are independent of each other. During a short time period (t, t + h), an active life must remain active, die or retire. So:

$$ {}_hp^{aa}_{x+t} + {}_hp^{ad}_{x+t} + {}_hp^{ar}_{x+t} = 1 $$

Using the formulae given in (i)(a), this becomes:

$$ {}_hp^{aa}_{x+t} + h\mu + h\upsilon + o(h) = 1 $$

So:

$$ {}_{t+h}p^{aa}_x = {}_tp^{aa}_x \times [1 - h(\mu+\upsilon) + o(h)] $$

Rearranging and letting $h \to 0$ gives:

$$ \frac{\partial}{\partial t}\,{}_tp^{aa}_x = -(\mu+\upsilon)\,{}_tp^{aa}_x $$

since $\lim_{h\to 0}\frac{o(h)}{h} = 0$.

Rearranging this, we see that:

$$ \frac{\frac{\partial}{\partial t}\,{}_tp^{aa}_x}{{}_tp^{aa}_x} = \frac{\partial}{\partial t}\log {}_tp^{aa}_x = -(\mu+\upsilon) $$

Integrating with respect to t with limits of 0 and s:

$$ \left[\log {}_tp^{aa}_x\right]_0^s = -(\mu+\upsilon)s $$

Since ${}_0p^{aa}_x = 1$, it follows that:

$$ {}_sp^{aa}_x = e^{-(\mu+\upsilon)s} $$

for $59 \le x \le x+s \le 60$.

C. Likelihood

Here are two possible approaches to this part.

During the year, individual i will either survive to the end, die or retire. Using the result in (i)(b) and writing t; for this individual's waiting time in the active state, the likelihood corresponding to each of these is:

Survival:	$e^{-(\mu+\upsilon)t_i}$
Death:	$e^{-(\mu+\upsilon)t_i} \times \mu$
Retirement:	$e^{-(\mu+\upsilon)t_i} \times \upsilon$

Since the experiences of the individuals are assumed to be independent, the overall likelihood for all the lives will be:

$$ L(\mu,\upsilon) = \prod_{survivors} e^{-(\mu+\upsilon)t_i} \times \prod_{deaths} \mu e^{-(\mu+\upsilon)t_i} \times \prod_{retirements} \upsilon e^{-(\mu+\upsilon)t_i} $$

This can be simplified to give:

$$ L(\mu,\upsilon) = \prod_{all\ lives} e^{-(\mu+\upsilon)t_i} \times \prod_{deaths} \mu \times \prod_{retirements} \upsilon $$

$$ = e^{-(\mu+\upsilon)\Sigma t_i} \times \mu^d \times \upsilon^r = e^{-(\mu+\upsilon)v} \times \mu^d \times \upsilon^r $$

Alternatively, we can write down the probability density/mass function for life i as a single function:

$$f_i(d_i, r_i, v_i) = \begin{cases} {}_{v_i}p_x & (d_i = 0, r_i = 0) \\ {}_{v_i}p_x\,\mu & (d_i = 1) \\ {}_{v_i}p_x\,\upsilon & (r_i = 1) \end{cases}$$

Here d_i and r_i represent the number of deaths and retirements experienced by this individual during the year (which will be O or 1).
We can then express these three 'combinations' in a single formula as:

$$f_i(d_i, r_i, v_i) = {}_{v_i}p_x \times \mu^{d_i} \times \upsilon^{r_i} = \exp[-(\mu+\upsilon)\,v_i] \times \mu^{d_i} \times \upsilon^{r_i}$$

So the joint likelihood for the whole group will be:

$$L = \prod_{i=1}^{N} \exp[-(\mu+\upsilon)\,v_i] \times \mu^{d_i} \times \upsilon^{r_i} = \exp[-(\mu+\upsilon)v]\mu^d\upsilon^r$$

D. Formulae
The MLE ofν is $\tilde{\nu} = \frac{R}{V}$
Asymptotically, this has moments:

Mean: $E(\tilde{\upsilon}) = \upsilon$

Estimated standard error: $ese(\tilde{\upsilon}) = \sqrt{\frac{\upsilon}{v}}$

Recall that the standard error of an estimator is the square root of its variance.

ii. A. Likelihood
The likelihood function is now found by combining the likelihood of observing d deaths during the year with the likelihood of observing r retirements out of the m lives who survived to age 60.
This second part is a binomial probability, and we get:

$$e^{-\mu v}\mu^d \times \binom{m}{r} k^r (1-k)^{m-r}$$

B. Maximum likelihood estimate of k
Since we have m lives at age 60 and r are observed to retire, the maximum likelihood estimate
of k is the binomial proportion, $\hat{k} = \frac{r}{m}$

(f) i. Distribution of the time spent in each state
In a continuous-time Markov jump process, the times spent in each state are exponentially distributed.

ii. Generator matrix
lfwe measure times in minutes, the generator matrix (with zeros omitted) is:

$$
\begin{array}{c}
\\ W \\ A \\ M \\ S \\ H
\end{array}
\begin{array}{c}
\begin{array}{ccccc} W & A & M & S & H \end{array} \\
\begin{bmatrix}
-\frac{1}{15} & \frac{1}{15} & & & \\
 & -\frac{1}{20} & \frac{3}{400} & \frac{1}{400} & \frac{1}{25} \\
 & & -\frac{1}{30} & & \frac{1}{30} \\
 & & & -\frac{1}{180} & \frac{1}{180} \\
 & & & &
\end{bmatrix}
\end{array}
$$

If you work in hours, all these entries need to be multiplied by 60.

(g) i. Kolmogorov forward differential equations

The general formula for the Kolmogorov forward differential equation in the time-homogeneous case is:

$$\frac{d}{dt}p_{ij}(t) = \sum_k p_{ik}(t)\mu_{kj}$$

Applying this, with i = W and j = M, we get:

$$\frac{d}{dt}p_{WM}(t) = p_{WA}(t)\mu_{AM} + p_{WM}(t)\mu_{MM} = \frac{3}{400}p_{WA}(t) - \frac{1}{30}p_{WM}(t)$$

Similarly:

$$\frac{d}{dt}p_{WA}(t) = \frac{1}{15}p_{WW}(t) - \frac{1}{20}p_{WA}(t)$$

ii. Verify formula

In order to check that the formula given in the question satisfies the differential equation just stated, we first need a formula for $P_{ww}(t)$. Since it is not possible to return to state W once it has been left,$p_{WW}(t)$ is the same as $p_{\overline{WW}(t)}$, which we can work out as:

$$p_{WW}(t) = p_{\overline{WW}}(t) = e^{-t/15}$$

Substituting the formula given in the question for $p_{WA}(t)$ into the Kolmogorov equation, we see that:

$$LHS = \frac{d}{dt}p_{WA}(t) = \frac{d}{dt}\left(4e^{-t/20} - 4e^{-t/15}\right) = -\frac{1}{5}e^{-t/20} + \frac{4}{15}e^{-t/15}$$

and:

$$
\begin{aligned}
RHS &= \frac{1}{15}p_{WW}(t) - \frac{1}{20}p_{WA}(t) \\
&= \frac{1}{15}e^{-t/15} - \frac{1}{20}\left(4e^{-t/20} - 4e^{-t/15}\right) \\
&= -\frac{1}{5}e^{-t/20} + \frac{4}{15}e^{-t/15}
\end{aligned}
$$

So the differential equation is satisfied.

We also need to check the boundary conditi on. Su bst itutin g t = 0 into the formula given, we get:

$$p_{WA}(0) = 4e^0 - 4e^0 = 0$$

This is the correct value since the process cannot move from state W to state A in zero time .

iii. Derive an expression for $P_{WM}(t)$

We can now use the formula for $P_{WM}(t)$ from part (b)(iii) in conjunction with the first differential equation from part (a)(iii) to find a formula for $P_{WM}(t)$. We have:

$$\frac{d}{dt}p_{WM}(t) = \frac{3}{400}p_{WA}(t) - \frac{1}{30}p_{WM}(t)$$

$$= \frac{3}{400}\left(4e^{-t/20} - 4e^{-t/15}\right) - \frac{1}{30}p_{WM}(t)$$

$$= \frac{3}{100}\left(e^{-t/20} - e^{-t/15}\right) - \frac{1}{30}p_{WM}(t)$$

We can solve this using an integrating factor. We first need to rearrange it in the form:

$$\frac{d}{dt}p_{WM}(t) + \frac{1}{30}p_{WM}(t) = \frac{3}{100}\left(e^{-t/20} - e^{-t/15}\right)$$

The integrating factor is:

$$\exp\left(\int \frac{1}{30}dt\right) = e^{t/30}$$

Multiplying through by the integrating factor, we get:

$$e^{t/30}\frac{d}{dt}p_{WM}(t) + \frac{1}{30}e^{t/30}p_{WM}(t) = \frac{3}{100}\left(e^{-t/60} - e^{-t/30}\right)$$

So:

$$\frac{d}{dt}\left[e^{t/30}p_{WM}(t)\right] = \frac{3}{100}\left(e^{-t/60} - e^{-t/30}\right)$$

Now we can integrate to get:

$$e^{t/30}p_{WM}(t) = \frac{3}{100}\left(-60e^{-t/60} + 30e^{-t/30}\right) + c = -\frac{9}{5}e^{-t/60} + \frac{9}{10}e^{-t/30} + c$$

When $t = 0$, this becomes:

$$0 = -\frac{9}{5} + \frac{9}{10} + c = -\frac{9}{10} + c \quad \Rightarrow c = \frac{9}{10}$$

So we have:

$$e^{t/30}p_{WM}(t) = -\frac{9}{5}e^{-t/60} + \frac{9}{10}e^{-t/30} + \frac{9}{10}$$

Dividing through by the integrating factor gives us the final answer:

$$p_{WM}(t) = -\frac{9}{5}e^{-t/20} + \frac{9}{10}e^{-t/15} + \frac{9}{10}e^{-t/30}$$

(h) i. If a vehicle is currently in state W, it will wait 15 minutes on average in that state before moving to state A (definitely), after which it will wait on average a 'turthenime T_A before it can be driven home. So the average time T_W before it can be driven home is $15 + T_A$.

ii. Equations for T_A, T_M and T_S

Using similar logic, a vehicle in state A will wait 20 minutes on average in that state before moving either to state H (with probability 0.8) or to state M (with probability 0.15) or to state

S (with probability 0.05). So the corresponding equation is:

$$T_A = 20 + 0.8 \times 0 + 0.15T_M + 0.05T_S$$

Since we know that $T_M = 30$ and $T_S = 180$, this gives $T_A = 33.5$ minutes.

iii.

$$T_W = 15 + T_A = 15 + 33.5 = 48.5 \ minutes$$