ETC3430: Financial mathematics under uncertainty

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Outline

Recap

So far, we have discussed discrete-time Markov chains in which the chain jumps from the current state to the next state after one unit of time. That is, the time that the chain spends in each state is a positive integer. It is equal to 1 if the state does not have a self-transition

$$\mathbb{P}_{i,i} = 0,$$

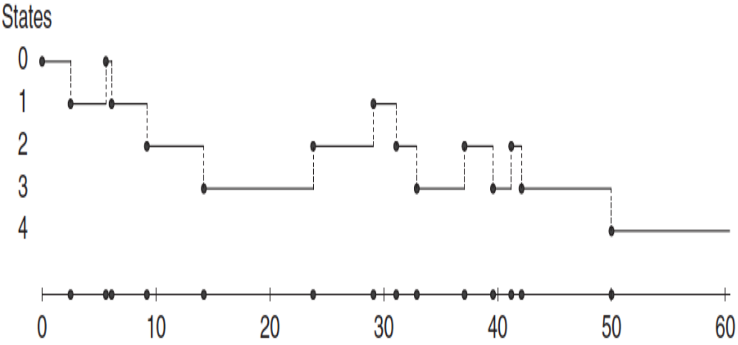
or it is a

$$Geometric(1 - \mathbb{P}_{i,i})$$

random variable if $\mathbb{P}_{i,i} > 0$. Here, we would like to discuss continuous-time Markov chains where the time spent in each state is a continuous random variable.

CTMC

A Continuous Time Markov Chain makes transitions from state to state at any instant of time rather than at fixed intervals, independent of the past,: once entering a state remains in that state, independent of the past, for an exponentially distributed amount of time before changing state again.



A gas station has a single pump and no space for vehicles to wait (if a vehicle arrives and the pump is not available, it leaves). Vehicles arrive to the gas station following a Poisson process with a rate of Î≫=3/20 vehicles per minute, of which 75% are cars and 25% are motorcycles. The refuelling time can be modelled with an exponential random variable with mean 8 minutes for cars and 3 minutes for motorcycles, that is, the services rates are $\mu_c = 1/8$ cars and $\mu_m = 1/3$ motorcycles per minute respectively.

Can we model my son's health condition via a CTMC? If yes, how?

Definition (Markov jump process)

Let $X = (X_t)_{t \geq 0}$ be a family of random variables taking values in a finite or countable state space S, which we can take to be a subset of the integers. X is a continuous-time Markov chain (CTMC) if it satisfies the markov property

$$P(X_{t_n} = x_n | X_{t_1} = x_1, ...X_{t_{n-1}} = x_{n-1})$$

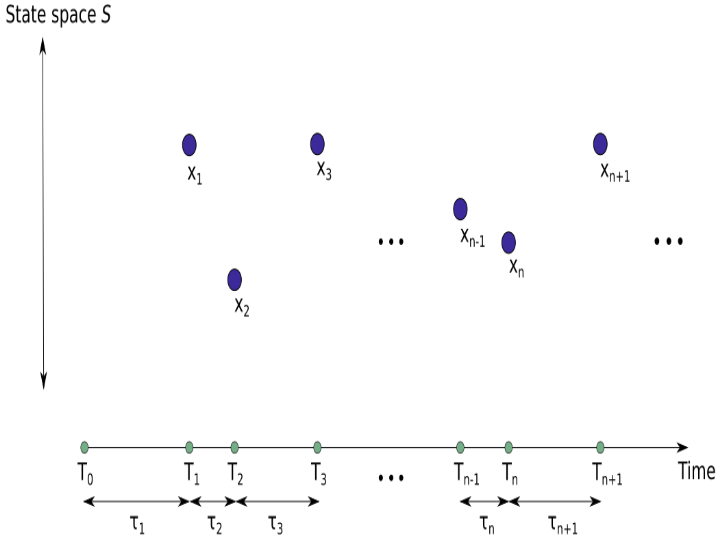
The process is time-homogeneous if the conditional probability does not depend on the current time, so that:

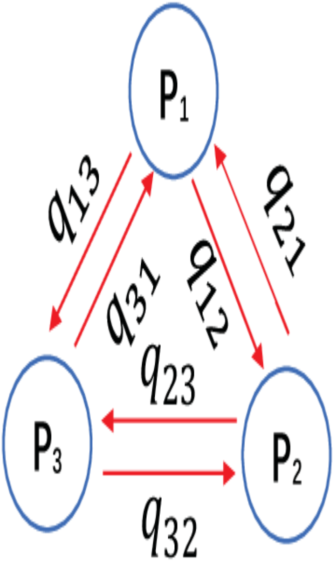
$$P(X_{t+s} = j | X_s = i) = P(X_t = j | X_0 = i), s > 0.$$

We will consider only time-homogeneous processes in this lecture.

More specifically, we will consider a random process $\{X_t, t \in [0, \infty)\}$. If $X_0 = i$, then X_t stay in state i for a random amount of time, say τ_1, where τ_1 is a continuous random variable. At the time τ_1, the process jumps to a new state j and will spend a random amount of time τ_2 in that state, and so on. As it will be clear shortly, the random variables τ_1, τ_2...have an exponential distribution. In this cases, the $T_i = \sum_{j=1}^{i} \tau_i$ denote the time of the jump. [1]

[1] Sometimes, W_i is used to denote the waiting times.


State space S
X_1
X_2
X_3
X_{n-1}
X_n
X_{n+1}
T_0 T_1 T_2 T_3 T_{n-1} T_n T_{n+1}
Time
τ_1 τ_2 τ_3 τ_n τ_{n+1}



$$Q_{ESS} = \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix}$$

$$Q_{NESS} = \begin{pmatrix} -1 & 1/3 & 2/3 \\ 2/3 & -1 & 1/3 \\ 1/3 & 2/3 & -1 \end{pmatrix}$$

Exponential Holding time

Our first question of continuous-time time-homogeneous Markov chains, and one whose answer will eventually lead to a general construction/simulation method, is:

How long will this process remain in a given state, say $X_0 = i \in \mathbb{S}$

$$
\begin{aligned}
&\mathbb{P}(T_1 > s+t | T_1 > s) \\
=&\mathbb{P}(X_v = i, \text{ for } v \in [0, s+t] | X_v = i, \text{ for } v \in [0, s]) \\
=&\mathbb{P}(X_v = i, \text{ for } v \in [s, s+t] | X_v = i, \text{ for } v \in [0, s]) \\
=&\mathbb{P}(X_v = i, \text{ for } v \in [s, s+t] | X_s = i) \text{ Markov} \\
=&\mathbb{P}(X_v = i, \text{ for } v \in [0, t] | X_0 = i) \text{ time- homogeneity} \\
=&\mathbb{P}(T_1 > t | T_1 > 0)
\end{aligned}
$$

The memoryless property implies Exponential Distribution.

The tranistion probability matrix: A matrix function of time

Let's define the transition probability $\mathbb{P}_{i,j}^{(s,t)}$

$$\begin{aligned}\mathbb{P}_{ij}^{(s,t)}(t) &= P(X_t = j | X_s = i) \quad \text{for all } 0 < s < t < \infty \\ &= P(X(t-s) = j | X(0) = i), \text{ if time inhomogeneous}\end{aligned}$$

This can also be written in its matrix form

$$\mathbb{P}^{(t)} = \begin{bmatrix} \mathbb{P}_{11}(t) & \mathbb{P}_{12}(t) & \dots & \mathbb{P}_{1r}(t) \\ \mathbb{P}_{21}(t) & \mathbb{P}_{22}(t) & \dots & \mathbb{P}_{2r}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{P}_{r1}(t) & \mathbb{P}_{r2}(t) & \dots & \mathbb{P}_{rr}(t) \end{bmatrix}.$$

CK:Time homogeneous Process

The Chapman-Kolmogorov Equation for the time homogeneous case, [2] is given by

$$\mathbb{P}_{i,j}^{(t+s)} = \sum_{k \in \mathbb{S}} \mathbb{P}_{i,k}^{(s)} \mathbb{P}_{k,j}^{(t)}$$

In the matrix format, is

$$\mathbb{P}^{(t+s)} = \mathbb{P}^{(s)} \mathbb{P}^{(t)}$$

[2] $\mathbb{P}_{i,j}^{(t)} = \mathbb{P}(X_t = j | X_s = i) = \mathbb{P}(X_{t-s} = j | X_0 = i)$ only depends the lag

The Chapman Kolmogorov Equations in continuous time

$$\mathbb{P}^{(t+s)} = \mathbb{P}^{(t)}\mathbb{P}^{(s)},$$

This is the direct analog of the discrete-time result. Just a note on terminology: in the discrete-time case, we called the matrix $\mathbb{P}^{(n)}$ the *n*-step transition probability matrix. Because there is no notion of a time step in continuous time, we call $\mathbb{P}^{(t)}$ the matrix transition probability function. Note that it is a matrix-valued function of the continuous variable t.

Transition rates

We assume that for the homogeneous and inhomogeneous case

$$\mathbb{P}^{(t)}_{i,j}|_{t=0} = \mathbb{P}^{s,s+t}_{i,j}|_{t=0} = \delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Definition (Transition rate)

Given the transition matrix $\mathbb{P}^{(t)}$ and $\mathbb{P}^{(s,t)}$ for a homogeneous and an inhomogeneous Markov chain respectively, the generator matrix A and $A(s)$ such that their $i, j$the element is the transition rate from state i to j

$$\mu_{i,j} = \frac{d}{dt}\mathbb{P}^{(t)}_{i,j}|_{t=0} = \lim_{t \to 0} \frac{\mathbb{P}^{(t)}_{i,j} - \delta_{i,j}}{t}$$

$$\mu_{i,j}(s) = \frac{\partial}{\partial t}\mathbb{P}^{(s,t)}_{i,j}|_{t=s} = \lim_{h \to 0} \frac{\mathbb{P}^{(s,s+h)}_{i,j} - \delta_{i,j}}{h}$$

The generator matrix

The sum of each row of A is zero, i.e.

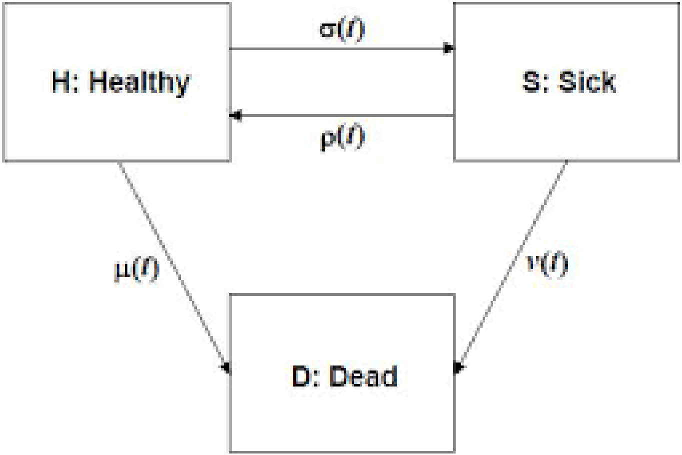
$$\mu_{i,i} = -\sum_{j \neq i} \mu_{i,j}.$$

This is simply because

$$\sum_{j \in \mathbb{S}} \mathbb{P}_{i,j}^{(t)} = 1.$$

The same result also holds for the time inhomogeneous case.

Life Insurance: Healthy-Sick-Death



Healthy-Sick-Death Model

Consider the state of a person, $\mathbb{S} = \{\textit{Healthy}, \textit{Sick}, \textit{Dead}\}$ with a constant transition such that

$$\mu_{H,S} = \sigma, \quad \mu_{H,D} = \mu, \quad \mu_{S,H} = \rho, \quad \mu_{S,D} = \nu.$$

The resulting transition is of

$$A = \begin{bmatrix} -\mu - \sigma & \sigma & \mu \\ \rho & -\rho - \nu & \nu \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} = 0 \\ = 0 \\ = 0 \end{matrix}$$

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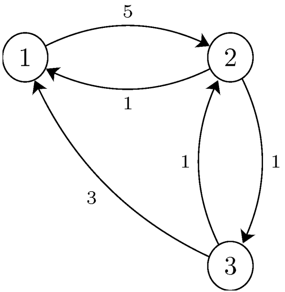
Transition Diagram

We can similar try transition diagram for continuous time Markov process, i.e.

$$A = \begin{bmatrix} -5 & 5 & 0 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix}, \tag{1}$$



The Gas pump example

The three states are car, empty and motorbike respectively

$$A = \begin{pmatrix} -\mu_{\mathrm{c}} & \mu_{\mathrm{c}} & 0 \\ p\lambda & -\lambda & (1-p)\lambda \\ 0 & \mu_{\mathrm{m}} & -\mu_{\mathrm{m}} \end{pmatrix}$$

- in the first row, given currently there is a car in the pump, the car leaves the pump with intensity μ_c
- in the last row, given currently there is a motor in the pump, the car leaves the pump with intensity μ_m
- in the middle row, given currently empty, there is an arrival rate of λ. When there is indeed an arrival, there is p chance of being a car.

The Forward Differential Equation

Theorem

The Kolmogorov forward equation for a time homogeneous Markov Jump process is

$$\frac{d}{dt}\mathbb{P}^{(t)} = \mathbb{P}^{(t)}A,$$

and that for the inhomogeneous case is given by

$$\frac{\partial}{\partial t}\mathbb{P}^{(s,t)} = \mathbb{P}^{(s,t)}A(t).$$

Ordinary Differential Equations

Definition A differential equation is an equation involving derivatives of an unknown function and possibly the function itself as well as the independent variable.

Example $y' = \sin(x), \quad (y')^4 - y^2 + 2xy - x^2 = 0, \quad y'' + y^3 + x = 0$

1st order equations — 2nd order equation

Definition The order of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation

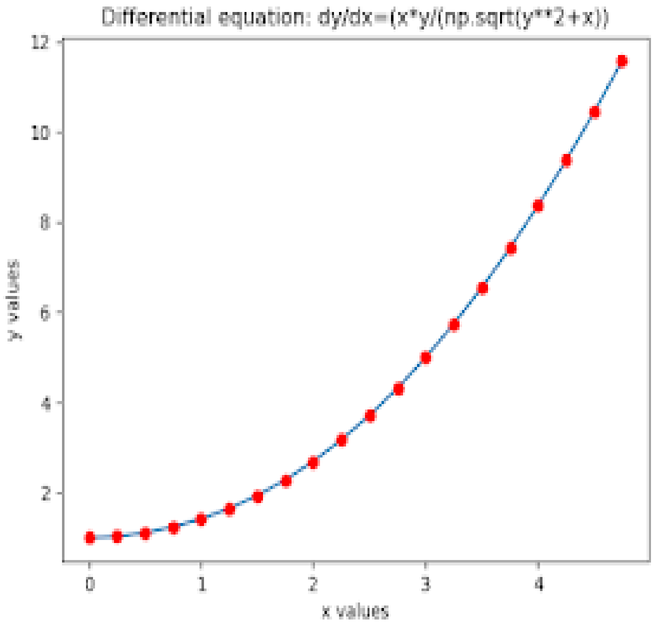
In the simplest cases, equations may be solved by direct integration.

Examples $y' = \sin(x) \Rightarrow y = -\cos(x) + C$

$$y'' = 6x + e^x \Rightarrow y' = 3x^2 + e^x + C_1 \Rightarrow y = x^3 + e^x + C_1 x + C_2$$

Observe that the set of solutions to the above 1st order equation has 1 parameter, while the solutions to the above 2nd order equation depend on two parameters.

Mika Seppälä: Differential Equations


Differential equation: dy/dx=(x*y/(np.sqrt(y**2+x))
y values
x values
0
1
2
3
4
2
4
6
8
10
12

The Forward Differential Equation

The FDE is a powerful tool for solving the transition matrix, as it constructs simultaneous differentiations. For two dimensional case,

$$A = \begin{bmatrix} -a & a \\ b & -b \end{bmatrix}$$

Hence

$$\frac{d}{dt}\mathbb{P}_{1,2}^{(t)} = a\mathbb{P}_{1,1}^{(t)} - b\mathbb{P}_{1,2}^{(t)} = a - (a+b)\mathbb{P}_{1,2}^{(t)}$$

The solution of the above ODE is

$$\mathbb{P}_{1,2}^{(t)} = \frac{a}{a+b} + C\exp^{-(a+b)t} \text{ with } \mathbb{P}_{1,2}^{(0)} = 0$$

hence

$$\mathbb{P}_{1,2}^{(t)} = \frac{a}{a+b}(1 - \exp^{-(a+b)t})$$

Kolmogorov Backward Differential Equation

Theorem

The Kolmogorov Backward Differential Equation for time homogeneous Markov Chain is

$$\frac{d}{dt}\mathbb{P}^{(t)} = A\mathbb{P}^{(t)},$$

and that of the inhomogeneous case is

$$\frac{\partial}{\partial s}\mathbb{P}^{(s,t)} = -A(t)\mathbb{P}^{(s,t)}.$$

The forward and backwards DE are equivalent as long as the sum of transition rates are bounded.

The Solution via Matrix Exponential

Theorem

In a simple time homogeneous case, we have the FKE and BKE as

$$\frac{\partial}{\partial t}\mathbb{P}^t = \mathbb{P}^t A \text{ and } \frac{\partial}{\partial t}\mathbb{P}^t = A\mathbb{P}^t.$$

Using matrix exponential, we have the solution

$$\mathbb{P}^t = \mathbb{P}^0 \exp^{tA} \text{ where } \exp Q = \sum_{i=0}^{\infty} \frac{Q^i}{i!}.$$

Though the backward and forward equations are two different sets of differential equations, with the above boundary condition they have the same solution, given by

$$\mathbb{P}^t = \exp^{tA} = \sum_{i=0}^{\infty} \frac{t^i A^i}{i!} = \mathbb{I} + tA + \frac{t^2}{2}A^2 + ...$$

We can take derivatives

$$\frac{d}{dt}\mathbb{P}^t = \sum_{i=0}^{\infty} \frac{t^{i-1} A^i}{(i-1)!} = A + tA^2 + \frac{t^2}{2}A^3 ... = A(\mathbb{I} + tA + \frac{t^2}{2}..)A^2 + ...)$$

Hence, this is $\frac{d}{dt}\mathbb{P}^t = A\mathbb{P}^t = \mathbb{P}^t A$.

Computing Matrix Exponentials

Suppose that A is an $n \times n$ matrix with n distinct eigenvectors. Then, letting D be a diagonal matrix consisting of the eigenvalues of A, we can decompose A into

$$A = QDQ^{-1}$$

where Q consists of the eigenvectors of A (ordered similarly to the order of the eigenvalues in D). In this case, we get the very nice identity

$$\exp^{At} = \sum_{i=0}^{\infty} \frac{t^i (QDQ^{-1})^i}{i!} = Q \sum_{i=0}^{\infty} \frac{D^i}{i!} Q^{-1} = Q \exp^{Dt} Q^{-1}.$$

where $\exp^{Dt}$, because D is diagonal, is a diagonal matrix with diagonal elements $\exp^{\lambda_i t}$ where λ_i the ith eigenvalue.

Stationary Distribution

Definition

For a continuous markov process X_t with $\mathbb{P}(t)$, a probability distribution π om $\mathbb{S}$ is a vector with $\pi_i \in [0,1]$ and

$$\sum_{i \in \mathbb{S}} \pi_i = 1$$

is said to be stationary distribution of X_t is

$$\pi = \pi \mathbb{P}(t) \text{ for all } t > 0.$$

Example

Consider a continuous markov process with two states with transition matrix as

$$P(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \end{bmatrix}.$$

Its stationary distribution $\pi = [\pi_0, \pi_1]$ is that

$$\pi P(t) = [\pi_0, \pi_1] \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2}e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2}e^{-2\lambda t} \end{bmatrix} = [\pi_0, \pi_1].$$

and $\pi_0 + \pi_1 = 1$. Solving the equation we have

$$\pi_0 = \pi_1 = 0.5.$$

Definition

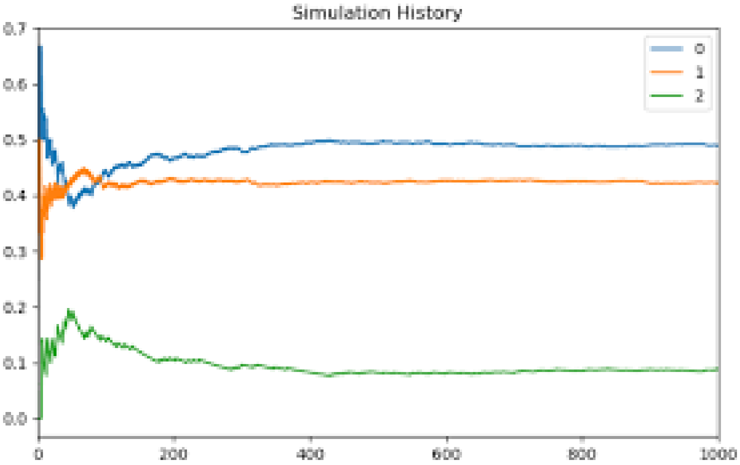
The distribution π is said to be the limiting distribution of X_t if

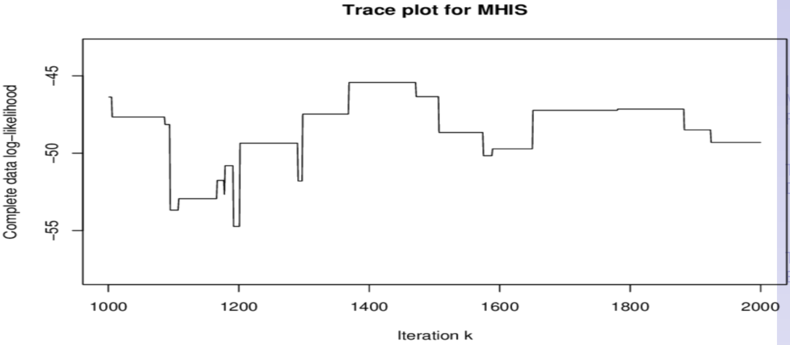
$$\pi_j = \lim_{t\to\infty} P(X(t) = j | X(0) = i)$$

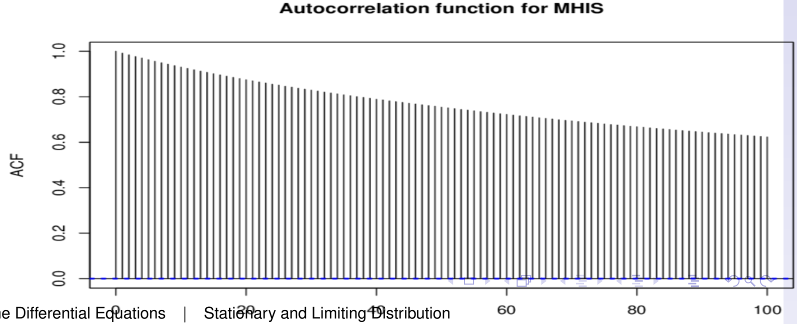
for all $i, j \in \mathbb{S}$, and

$$\sum_{i\in\mathbb{S}} \pi_i = 1.$$

For the simple example, we have the limiting distribution is the same as the stationary distribution.


Simulation History
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
200
400
600
800
1000
0
1
2


Trace plot for MHIS
Complete data log−likelihood
-45
-50
-55
1000
1200
1400
1600
1800
2000
Iteration k


Autocorrelation function for MHIS
ACF
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100

In theory, we can find the stationary (and limiting) distribution by solving $\pi\mathbb{P}(t) = \pi$ and $\lim_{t\to\infty} \mathbb{P}(t)$. However, in practice $\mathbb{P}$ is usually very complicated.

Theorem

The probability distribution π on $\mathbb{S}$ is a stationary distribution for X_t if and only if it satisfies

$$\pi A = 0.$$

Proof.
For stationary distribution, $\pi = \pi \mathbb{P}(t)$, we take derivative on both side

$$\begin{aligned} 0 &= \frac{d}{dt}[\pi P(t)] \\ &= \pi P'(t) \\ &= \pi A P(t) \quad \text{(backward equations)} \end{aligned}$$

Let $t = 0$, we have $\mathbb{P}(0) = \mathbb{I}$, hence

$$0 = \pi A.$$

$\square$

The previous simple two state example,

$$A = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}.$$

Solving

$$\pi A = [\pi_0, \pi_1] \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix} = 0.$$

which result $\pi_0 = \pi_1$, together with $\pi_0 + \pi_1 = 1$, we have $\pi_i = 0.5$.

Solve Stationary Distribution via Matrix Algebra

We need to solve

$$\pi A = 0, \text{ subject to } \pi_1 + \pi_2 + ..\pi_d = 1$$

where d is the dimension of $\mathbb{S}$. Rewrite them together in matrix form $\pi Z = b$ such that

$$Z = \begin{bmatrix} \mu_{1,1} & \cdots & \cdots & \mu_{1,d} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{1,1} & \cdots & \cdots & \mu_{1,d} \\ 1 & \cdots & \cdots & 1 \end{bmatrix}$$

and $b = [0, ..., 0, 1]$. A bit of matrix algebra gives

$$\pi = bZ^t(ZZ^t)^{-1}.$$