ETC3430 / ETC5343 Financial Mathematics under Uncertainty

2022 Semester One Class Test

Format:	It is an individual assessment. Collaboration / collusion / plagiarism between students is strictly prohibited.
Date and Time:	12pm, Wednesday 27th April 2022
Time Allowed:	90 minutes (including download and upload time)
Number of Questions:	Answer all FIVE questions. The total mark is 20.
Marks:	The mark for each question is stated at the start of that question.
Numerical Details:	The missing details in Questions 1, 4, and 5 (in bold) are provided in the separate spreadsheet (specifically for each student).
Working:	Show all your steps clearly. Start each question on a new page.
Formulae and Tables:	Formulae sheet and statistical tables are provided.
Submission:	Type up your answers in Word and / or write up and scan your answers. Submit the answers via Moodle within the time limit. Late submission will not be accepted.
Assessment:	The class test contributes 20% to the total mark of the unit.

Question 1 [4 marks]

Consider a Poisson process with rate **X**, starting from time 0.

(a) What distribution does the inter-event time follow?

(b) What is the expected inter-event time?

(c) Compute the expected time of the **A**th event.

(d) Compute the probability that the 3rd event occurs **B** units of time or more after the 2nd.

(e) Compute the probability that the 2nd event occurs after time **C**.

Question 2 [4 marks]

(a) Consider a Markov chain with two states 1 and 2. Suppose that $p_{1,2} = a$ and $p_{2,1} = b$. For which values of a and b do we obtain an absorbing Markov chain? Explain.

(b) Consider a continuous-time Markov process with two states 1 and 2. Suppose that $a_{1,2} = a$ and $a_{2,1} = b$. For which values of a and b do we obtain an absorbing Markov process? Explain.

(c) Consider a Markov chain with two states. If the chain alternates between these two states, does the long-term distribution of the chain exist? Explain.

Question 3 [4 marks]

Consider a Markov chain on the vertices of a triangle: the chain moves from one vertex to one of the other two with probability 1/2 in each step.

(a) Write down the transition probability matrix of this Markov chain.

(b) What is the probability that the chain starts from one vertex and then returns to this vertex in two steps?

(c) Suppose the Markov chain starts with equal chance of being at each vertex. Ultimately, what is the distribution of the Markov chain? Does it matter if the Markov chain starts, instead of following a discrete uniform distribution, from one particular vertex?

Question 4 [4 marks]

For a certain population, suppose that ${}_5q_{55} = \mathbf{X}$, ${}_{10}q_{55} = \mathbf{Y}$, ${}_{15}q_{55} = \mathbf{Z}$, and the force of mortality follows Makeham's Law. Compute the corresponding model parameters. Then calculate the value of ${}_6q_{58}$ for this population.

Question 5 [4 marks]

In a mortality investigation for Population A over a period of three years, there are a total of **X** deaths and the total initial exposed to risk is **Y** (in years). Use the Poisson model to estimate its force of mortality and the corresponding 95% confidence interval. Then suppose that Population B has a constant size of **P** over the next few years and is subject to **Q** times the force of mortality of Population A. Estimate the probability that there will be more than **R** deaths in Population B during the next two years.

Formulae & Statistical Tables

Random Walk

$X_t = X_{t-1} + \varepsilon_t$

Strict Stationarity

$F(x_{t_1+k}, x_{t_2+k}, ..., x_{t_n+k}) = F(x_{t_1}, x_{t_2}, ..., x_{t_n})$

White Noise

$Z_t \sim \text{Normal}(0, \sigma^2)$ independent and identically distributed

Weak Stationarity

$\text{E}(X_t)$ is constant for all t

$\text{Cov}(X_t, X_{t+k})$ depends only on lag k

Independent Increments

$X_{t+h} - X_t$ is independent of past X_s

Markov Property

$\Pr(X_t \in A \mid X_{s_1} = x_1, X_{s_2} = x_2, ..., X_s = x) = \Pr(X_t \in A \mid X_s = x)$ for $s_1 < s_2 < ... < s < t$

Poisson Process

$N_t \sim \text{Poisson}(\lambda t)$

$N_0 = 0$

$N_s \le N_t$ when $s < t$

$N_{t_2} - N_{t_1}, ..., N_{t_n} - N_{t_{n-1}}$ are mutually independent

$\Pr(N_{t_2+h} - N_{t_1+h} = k) = \Pr(N_{t_2} - N_{t_1} = k)$

$N_t - N_s \sim \text{Poisson}(\lambda(t-s))$

$\tau \sim \text{Exponential}(\lambda)$

$\Pr(X_{t+h} = i+1 \mid X_t = i) = \lambda h + o(h)$

$\Pr(X_{t+h} = i \mid X_t = i) = 1 - \lambda h + o(h)$

$\text{P}_{i,j}^{(h)} = 1 - \lambda h + o(h)$ if $j = i$

$\text{P}_{i,j}^{(h)} = \lambda h + o(h)$ if $j = i+1$

$\text{P}_{i,j}^{(h)} = 0$ otherwise

$\mu_{i,j} = -\lambda$ if $j = i$

$\mu_{i,j} = \lambda$ if $j = i+1$

$\mu_{i,j} = 0$ otherwise

Compound Poisson Process

$S_t = \sum_{i=1}^{N_t} X_i$

Markov Property

$$\Pr(Z_{n+1}=j \mid Z_n=i_n, Z_{n-1}=i_{n-1},\dots,Z_0=i_0)=\Pr(Z_{n+1}=j \mid Z_n=i_n)$$

Transition Matrix (discrete time, time homogeneous, discrete state space)

$$\mathrm{P}_{i,j}=\Pr\left(Z_n=j \mid Z_{n-1}=i\right)$$

$$\sum_j \mathrm{P}_{i,j}=1$$

Transition Matrix (discrete time, discrete state space)

$$\mathrm{P}_{i,j}^{m,n}=\Pr\left(X_n=j \mid X_m=i\right)$$

$$\pi_n=\pi_0\mathrm{P}^{0,n}=\pi_0\mathrm{P}^{0,1}\mathrm{P}^{1,2}\dots\mathrm{P}^{n-1,n}$$

$$\pi_n=\pi_m\mathrm{P}^{m,n}=\pi_m\mathrm{P}^{m,m+1}\mathrm{P}^{m+1,m+2}\dots\mathrm{P}^{n-1,n}$$

Chapman-Kolmogorov Equation

$$\mathrm{P}_{i,j}^{m,n}=\sum_k \mathrm{P}_{i,k}^{m,l}\mathrm{P}_{k,j}^{l,n}$$

n-Step Transition Matrix (discrete time, time homogeneous, discrete state space)

$$\mathrm{P}_{i,j}^{(n)}=\Pr(X_{n+m}=j \mid X_m=i)$$

$$\mathrm{P}^{(n)}=\mathrm{P}^n$$

$$\pi_n=\pi_0\mathrm{P}^n$$

Stationary Distribution

$$\pi=\pi\mathrm{P}$$

Discrete-Time Markov Chain

$$f_{ii}=\Pr(X_n=i,\text{for some } n\geq 1 \mid X_0=i)$$

$\Pr(V=\infty \mid X_0=i)=1$ (recurrent state)

$V \mid X_0=i \sim \text{Geometric}(1-f_{ii})$ (transient state)

Limiting Distribution

$$\pi_j^\infty=\lim_{n\to\infty}\Pr(X_n=j \mid X_0=i)$$

$$\sum_j \pi_j^\infty=1$$

$\pi^\infty=\pi^\infty\mathrm{P}$ (stationary distribution)

Markov Jump Process (continuous time, time homogeneous, discrete state space)

$$\Pr(X_{t+s} = j \mid X_s = i) = \Pr(X_t = j \mid X_0 = i)$$

$$\mathrm{P}_{i,j}^{(t+s)} = \sum_k \mathrm{P}_{i,k}^{(s)} \mathrm{P}_{k,j}^{(t)}$$

$$\mathrm{P}^{(t+s)} = \mathrm{P}^{(s)} \mathrm{P}^{(t)}$$

$$\mu_{i,j} = \frac{d}{dt} \mathrm{P}_{i,j}^{(t)} \big|_{t=0} = \lim_{t \to 0} \frac{\mathrm{P}_{i,j}^{(t)} - \delta_{i,j}}{t}$$

$$\mu_{i,i} = -\sum_{j \neq i} \mu_{i,j}$$

Healthy-Sick-Death Model

$$A = \begin{bmatrix} -\mu-\sigma & \sigma & \mu \\ \rho & -\rho-\nu & -\nu \\ 0 & 0 & 0 \end{bmatrix} \qquad \mu_{H,S} = \sigma \qquad \mu_{H,D} = \mu \qquad \mu_{S,H} = \rho \qquad \mu_{S,D} = \nu$$

$\frac{d}{dt} \mathrm{P}^{(t)} = \mathrm{P}^{(t)} A$ (forward differential equation)

$\frac{d}{dt} \mathrm{P}^{(t)} = A \mathrm{P}^{(t)}$ (backward differential equation)

$\pi A = 0$ (stationary distribution)

$$\hat{\mu} = \frac{d}{v} \qquad \hat{\nu} = \frac{u}{w} \qquad \hat{\sigma} = \frac{s}{v} \qquad \hat{\rho} = \frac{r}{w}$$

$$\hat{\mu}_{km} \pm 1.96 \sqrt{\frac{\hat{\mu}_{km}}{t_k}}$$

Poisson Distribution

$$\Pr(N = n) = \frac{e^{-\lambda} \lambda^n}{n!} \qquad \mathrm{E}(N) = \lambda \qquad \mathrm{Var}(N) = \lambda$$

Exponential Distribution

$$f(x) = \lambda e^{-\lambda x} \qquad F(x) = 1 - e^{-\lambda x} \qquad \mathrm{E}(X) = \frac{1}{\lambda} \qquad \mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Maximum Likelihood Estimate

$$\tilde{\theta} = \hat{\theta}(X_1, \dots, X_n)$$

$$\tilde{\theta} \overset{a}{\sim} N(\theta, I^{-1})$$

$$I_{i,j} = -\mathrm{E}\left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \log L(\theta; X_1, \dots, X_n) \right)$$

Central Limit Theorem

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \overset{a}{\sim} N(\mu,\sigma^2)$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \overset{a}{\sim} N(\mu,\Sigma)$$

Slutsky's Theorem

Let $\tilde{\theta}_1 \overset{a}{\sim} N(\theta_1,\sigma_1^2)$ and $\tilde{\theta}_2 \approx c$

$$\tilde{\theta}_1 - \tilde{\theta}_2 \overset{a}{\sim} N(\theta_1 - c,\sigma_1^2)$$

$$\tilde{\theta}_1 \cdot \tilde{\theta}_2 \overset{a}{\sim} N(c\theta_1, c^2\sigma_1^2)$$

$$\frac{\tilde{\theta}_1}{\tilde{\theta}_2} \overset{a}{\sim} N\left(\frac{\theta_1}{c},\frac{\sigma_1^2}{c^2}\right)$$

Confidence Interval

Let $\tilde{\theta} \overset{a}{\sim} N(\theta,\sigma_n^2)$

$$\hat{\theta} \pm 1.96\sigma_n$$

$$\hat{\theta} \pm 1.96\hat{\sigma}_n$$

Survival Models

$$F_x(t) = \Pr(T_x \le t) = {}_t q_x$$

$$S_x(t) = \Pr(T_x > t) = {}_t p_x$$

$${}_{s+t} p_x = {}_t p_x \; {}_s p_{x+t}$$

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr(T_0 \le x + dx \mid T_0 > x)$$

$$\mu_x dx \approx \Pr(T_0 \le x + dx \mid T_0 > x) = \Pr(T_x \le dx)$$

$$f_x(t) = {}_t p_x \; \mu_{x+t}$$

$$\frac{d}{dt}{}_t p_x = -{}_t p_x \; \mu_{x+t}$$

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$

$${}_t q_x = \int_0^t {}_s p_x \; \mu_{x+s} ds$$

$$m_x = \frac{q_x}{\int_0^1 {}_t p_x dt} = \frac{\int_0^1 {}_t p_x \ \mu_{x+t} dt}{\int_0^1 {}_t p_x dt}$$

$$\Pr\left(K_x = k\right) = {}_k p_x \ q_{x+k}$$

$$\overset{\circ}{e}_x = \mathrm{E}\left(T_x\right) = \int_0^\infty t \ {}_t p_x \ \mu_{x+t} dt = \int_0^\infty {}_t p_x dt$$

$$e_x = \mathrm{E}\left(K_x\right) = \sum_{k=0}^{\infty} k \ {}_k p_x \ q_{x+k} = \sum_{k=1}^{\infty} {}_k p_x$$

$$\overset{\circ}{e}_x \approx \frac{1}{2} + e_x$$

UDD

$${}_t q_x = t \ q_x$$

$${}_t q_{x+s} = \frac{t \ q_x}{1 - s \ q_x}$$

Balducci

$${}_{1-t} q_{x+t} = \left(1-t\right) q_x$$

$${}_t q_x = \frac{t \, q_x}{1 - \left(1-t\right) q_x}$$

Gompertz' Law

$$\mu_x = Bc^x$$

$${}_t p_x = \exp\left(-\frac{Bc^x\left(c^t - 1\right)}{\ln c} \right)$$

Makeham's Law

$$\mu_x = A + Bc^x$$

$${}_t p_x = \exp\left(-A \, t - \frac{Bc^x\left(c^t - 1\right)}{\ln c} \right)$$

Binomial Model

$D_i \sim \text{Bernoulli}({}_{b_i - a_i} q_{x+a_i})$

$$E_x = \sum_{\text{survivors}} (b_i - a_i) + \sum_{\text{deaths}} (1 - a_i) = \sum_{\text{survivors}} (b_i - a_i) + \sum_{\text{deaths}} (t_i - a_i) + \sum_{\text{deaths}} (1 - t_i)$$

$$E_x^C = \sum_{\text{survivors}} (b_i - a_i) + \sum_{\text{deaths}} (t_i - a_i)$$

$$E_x = E_x^C + \sum_{i=1}^{N} d_i (1 - t_i) \approx E_x^C + \frac{d}{2}$$

$$\hat{q}_x = \frac{d}{E_x} \approx \frac{d}{E_x^C + \frac{d}{2}}$$

$$\mathrm{E}(\tilde{q}_x) = q_x$$

$$\mathrm{Var}(\tilde{q}_x) \approx \frac{q_x(1 - q_x)}{E_x}$$

$\tilde{q}_x$ is approximately normally distributed asymptotically

Poisson Model

$D \sim \text{Poisson}(E^C \mu)$

$$\hat{\mu} = \frac{d}{E^C}$$

$$\mathrm{E}(\tilde{\mu}) = \mu$$

$$\mathrm{Var}(\tilde{\mu}) = \frac{\mu}{E^C}$$

$\tilde{\mu}$ is normally distributed asymptotically

Trapezium Approximation

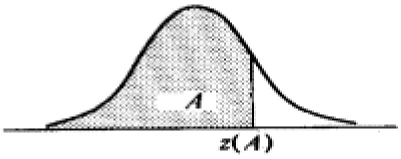
$$E_x^C = \int_0^{K+1} P_{x,t}\, dt \approx \sum_{t=0}^{K} \frac{P_{x,t} + P_{x,t+1}}{2}$$

${}^{(1)}E_x^C \approx \sum_{t=0}^{K} \frac{P_{x,t}^{(1)} + P_{x,t+1}^{(1)}}{2}$ where $P_{x,t}^{(1)} \approx \frac{P_{x,t}^{(2)} + P_{x+1,t}^{(2)}}{2}$ or $P_{x,t}^{(1)} = P_{x+1,t}^{(3)}$

${}^{(2)}E_x^C \approx \sum_{t=0}^{K} \frac{P_{x,t}^{(2)} + P_{x,t+1}^{(2)}}{2}$ where $P_{x,t}^{(2)} \approx \frac{P_{x-1,t}^{(1)} + P_{x,t}^{(1)}}{2}$ or $P_{x,t}^{(2)} \approx \frac{P_{x,t}^{(3)} + P_{x+1,t}^{(3)}}{2}$

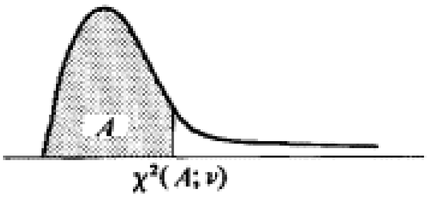
${}^{(3)}E_x^C \approx \sum_{t=0}^{K} \frac{P_{x,t}^{(3)} + P_{x,t+1}^{(3)}}{2}$ where $P_{x,t}^{(3)} = P_{x-1,t}^{(1)}$ or $P_{x,t}^{(3)} \approx \frac{P_{x-1,t}^{(2)} + P_{x,t}^{(2)}}{2}$

Entry is area A under the standard normal curve from $-\infty$ to $z(A)$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Entry is $\chi^2(A;\, \nu)$ where $P\{\chi^2(\nu) \le \chi^2(A;\, \nu)\} = A$



	A									
ν	.005	.010	.025	.050	.100	.900	.950	.975	.990	.995
1	0.0^4393	0.0^3157	0.0^3982	0.0^2393	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.60
3	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84
4	0.207	0.297	0.484	0.711	1.064	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.145	1.61	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.4	104.2
80	51.17	53.54	57.15	60.39	64.28	96.58	101.9	106.6	112.3	116.3
90	59.20	61.75	65.65	69.13	73.29	107.6	113.1	118.1	124.1	128.3
100	67.33	70.06	74.22	77.93	82.36	118.5	124.3	129.6	135.8	140.2