Formulae & Statistical Tables

Random Walk

$X_t = X_{t-1} + \varepsilon_t$

Strict Stationarity

$F(x_{t_1+k}, x_{t_2+k}, ..., x_{t_n+k}) = F(x_{t_1}, x_{t_2}, ..., x_{t_n})$

White Noise

$Z_t \sim \text{Normal}(0, \sigma^2)$ independent and identically distributed

Weak Stationarity

$\text{E}(X_t)$ is constant for all t

$\text{Cov}(X_t, X_{t+k})$ depends only on lag k

Independent Increments

$X_{t+h} - X_t$ is independent of past X_s

Markov Property

$\Pr(X_t \in A \mid X_{s_1} = x_1, X_{s_2} = x_2, ..., X_s = x) = \Pr(X_t \in A \mid X_s = x)$ for $s_1 < s_2 < ... < s < t$

Poisson Process

$N_t \sim \text{Poisson}(\lambda t)$

$N_0 = 0$

$N_s \le N_t$ when $s < t$

$N_{t_2} - N_{t_1}, ..., N_{t_n} - N_{t_{n-1}}$ are mutually independent

$\Pr(N_{t_2+h} - N_{t_1+h} = k) = \Pr(N_{t_2} - N_{t_1} = k)$

$N_t - N_s \sim \text{Poisson}(\lambda(t-s))$

$\tau \sim \text{Exponential}(\lambda)$

$\Pr(X_{t+h} = i+1 \mid X_t = i) = \lambda h + o(h)$

$\Pr(X_{t+h} = i \mid X_t = i) = 1 - \lambda h + o(h)$

$\text{P}_{i,j}^{(h)} = 1 - \lambda h + o(h)$ if $j = i$

$\text{P}_{i,j}^{(h)} = \lambda h + o(h)$ if $j = i+1$

$\text{P}_{i,j}^{(h)} = 0$ otherwise

$\mu_{i,j} = -\lambda$ if $j = i$

$\mu_{i,j} = \lambda$ if $j = i+1$

$\mu_{i,j} = 0$ otherwise

Compound Poisson Process

$S_t = \sum_{i=1}^{N_t} X_i$

Markov Property

$$\Pr(Z_{n+1} = j \mid Z_n = i_n, Z_{n-1} = i_{n-1}, \ldots, Z_0 = i_0) = \Pr(Z_{n+1} = j \mid Z_n = i_n)$$

Transition Matrix (discrete time, time homogeneous, discrete state space)

$$\mathrm{P}_{i,j} = \Pr\left(Z_n = j \mid Z_{n-1} = i\right)$$

$$\sum_j \mathrm{P}_{i,j} = 1$$

Transition Matrix (discrete time, discrete state space)

$$\mathrm{P}_{i,j}^{m,n} = \Pr\left(X_n = j \mid X_m = i\right)$$

$$\pi_n = \pi_0 \mathrm{P}^{0,n} = \pi_0 \mathrm{P}^{0,1}\mathrm{P}^{1,2}\ldots\mathrm{P}^{n-1,n}$$

$$\pi_n = \pi_m \mathrm{P}^{m,n} = \pi_m \mathrm{P}^{m,m+1}\mathrm{P}^{m+1,m+2}\ldots\mathrm{P}^{n-1,n}$$

Chapman-Kolmogorov Equation

$$\mathrm{P}_{i,j}^{m,n} = \sum_k \mathrm{P}_{i,k}^{m,l}\mathrm{P}_{k,j}^{l,n}$$

n-Step Transition Matrix (discrete time, time homogeneous, discrete state space)

$$\mathrm{P}_{i,j}^{(n)} = \Pr(X_{n+m} = j \mid X_m = i)$$

$$\mathrm{P}^{(n)} = \mathrm{P}^n$$

$$\pi_n = \pi_0 \mathrm{P}^n$$

Stationary Distribution

$$\pi = \pi \mathrm{P}$$

Discrete-Time Markov Chain

$$f_{ii} = \Pr(X_n = i, \text{for some } n \geq 1 \mid X_0 = i)$$

$$\Pr(V = \infty \mid X_0 = i) = 1 \qquad \text{(recurrent state)}$$

$$V \mid X_0 = i \sim \text{Geometric}(1 - f_{ii}) \qquad \text{(transient state)}$$

Limiting Distribution

$$\pi_j^\infty = \lim_{n\to\infty} \Pr(X_n = j \mid X_0 = i)$$

$$\sum_j \pi_j^\infty = 1$$

$$\pi^\infty = \pi^\infty \mathrm{P} \qquad \text{(stationary distribution)}$$

Markov Jump Process (continuous time, time homogeneous, discrete state space)

$$\Pr(X_{t+s}=j\mid X_s=i)=\Pr(X_t=j\mid X_0=i)$$

$$\mathrm{P}_{i,j}^{(t+s)}=\sum_k \mathrm{P}_{i,k}^{(s)}\mathrm{P}_{k,j}^{(t)}$$

$$\mathrm{P}^{(t+s)}=\mathrm{P}^{(s)}\mathrm{P}^{(t)}$$

$$\mu_{i,j}=\frac{d}{dt}\mathrm{P}_{i,j}^{(t)}\big|_{t=0}=\lim_{t\to 0}\frac{\mathrm{P}_{i,j}^{(t)}-\delta_{i,j}}{t}$$

$$\mu_{i,i}=-\sum_{j\neq i}\mu_{i,j}$$

Healthy-Sick-Death Model

$$A=\begin{bmatrix}-\mu-\sigma & \sigma & \mu\\ \rho & -\rho-\nu & -\nu\\ 0 & 0 & 0\end{bmatrix}\qquad \mu_{H,S}=\sigma\qquad \mu_{H,D}=\mu\qquad \mu_{S,H}=\rho\qquad \mu_{S,D}=\nu$$

$$\frac{d}{dt}\mathrm{P}^{(t)}=\mathrm{P}^{(t)}A$$ (forward differential equation)

$$\frac{d}{dt}\mathrm{P}^{(t)}=A\mathrm{P}^{(t)}$$ (backward differential equation)

$$\pi A=0$$ (stationary distribution)

$$\hat{\mu}=\frac{d}{v}\qquad \hat{\nu}=\frac{u}{w}\qquad \hat{\sigma}=\frac{s}{v}\qquad \hat{\rho}=\frac{r}{w}$$

$$\hat{\mu}_{km}\pm 1.96\sqrt{\frac{\hat{\mu}_{km}}{t_k}}$$

Poisson Distribution

$$\Pr(N=n)=\frac{e^{-\lambda}\lambda^n}{n!}\qquad \mathrm{E}(N)=\lambda\qquad \mathrm{Var}(N)=\lambda$$

Exponential Distribution

$$f(x)=\lambda e^{-\lambda x}\qquad F(x)=1-e^{-\lambda x}\qquad \mathrm{E}(X)=\frac{1}{\lambda}\qquad \mathrm{Var}(X)=\frac{1}{\lambda^2}$$

Maximum Likelihood Estimate

$$\tilde{\theta}=\hat{\theta}(X_1,\ldots,X_n)$$

$$\tilde{\theta}\overset{a}{\sim}N(\theta,I^{-1})$$

$$I_{i,j}=-\mathrm{E}\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log L(\theta;X_1,\ldots,X_n)\right)$$

Central Limit Theorem

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \overset{a}{\sim} N(\mu,\sigma^2)$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \overset{a}{\sim} N(\mu,\Sigma)$$

Slutsky's Theorem

Let $\tilde{\theta}_1 \overset{a}{\sim} N(\theta_1,\sigma_1^2)$ and $\tilde{\theta}_2 \approx c$

$$\tilde{\theta}_1 - \tilde{\theta}_2 \overset{a}{\sim} N(\theta_1 - c,\sigma_1^2)$$

$$\tilde{\theta}_1 \cdot \tilde{\theta}_2 \overset{a}{\sim} N(c\theta_1, c^2\sigma_1^2)$$

$$\frac{\tilde{\theta}_1}{\tilde{\theta}_2} \overset{a}{\sim} N\left(\frac{\theta_1}{c},\frac{\sigma_1^2}{c^2}\right)$$

Confidence Interval

Let $\tilde{\theta} \overset{a}{\sim} N(\theta,\sigma_n^2)$

$$\hat{\theta} \pm 1.96\sigma_n$$

$$\hat{\theta} \pm 1.96\hat{\sigma}_n$$

Survival Models

$$F_x(t) = \Pr(T_x \le t) = {}_t q_x$$

$$S_x(t) = \Pr(T_x > t) = {}_t p_x$$

$${}_{s+t} p_x = {}_t p_x \ {}_s p_{x+t}$$

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx}\Pr(T_0 \le x + dx \mid T_0 > x)$$

$$\mu_x dx \approx \Pr(T_0 \le x + dx \mid T_0 > x) = \Pr(T_x \le dx)$$

$$f_x(t) = {}_t p_x \ \mu_{x+t}$$

$$\frac{d}{dt}{}_t p_x = -{}_t p_x \ \mu_{x+t}$$

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$

$${}_t q_x = \int_0^t {}_s p_x \ \mu_{x+s} ds$$

$$m_x = \frac{q_x}{\int_0^1 {}_t p_x dt} = \frac{\int_0^1 {}_t p_x \ \mu_{x+t} dt}{\int_0^1 {}_t p_x dt}$$

$$\Pr\left(K_x = k\right) = {}_k p_x \ q_{x+k}$$

$$\overset{\circ}{e}_x = \mathrm{E}\left(T_x\right) = \int_0^\infty t \ {}_t p_x \ \mu_{x+t} dt = \int_0^\infty {}_t p_x dt$$

$$e_x = \mathrm{E}(K_x) = \sum_{k=0}^\infty k \ {}_k p_x \ q_{x+k} = \sum_{k=1}^\infty {}_k p_x$$

$$\overset{\circ}{e}_x \approx \frac{1}{2} + e_x$$

UDD

$${}_t q_x = t \ q_x$$

$${}_t q_{x+s} = \frac{t \ q_x}{1 - s \ q_x}$$

Balducci

$${}_{1-t} q_{x+t} = (1-t) q_x$$

$${}_t q_x = \frac{t \ q_x}{1 - (1-t) q_x}$$

Gompertz' Law

$$\mu_x = Bc^x$$

$${}_t p_x = \exp\left(- \frac{Bc^x \left(c^t - 1\right)}{\ln c} \right)$$

Makeham's Law

$$\mu_x = A + Bc^x$$

$${}_t p_x = \exp\left(-A \ t - \frac{Bc^x \left(c^t - 1\right)}{\ln c} \right)$$

Binomial Model

$D_i \sim \text{Bernoulli}({}_{b_i - a_i} q_{x+a_i})$

$E_x = \sum_{\text{survivors}} (b_i - a_i) + \sum_{\text{deaths}} (1 - a_i) = \sum_{\text{survivors}} (b_i - a_i) + \sum_{\text{deaths}} (t_i - a_i) + \sum_{\text{deaths}} (1 - t_i)$

$E_x^C = \sum_{\text{survivors}} (b_i - a_i) + \sum_{\text{deaths}} (t_i - a_i)$

$$E_x = E_x^C + \sum_{i=1}^{N} d_i (1 - t_i) \approx E_x^C + \frac{d}{2}$$

$$\hat{q}_x = \frac{d}{E_x} \approx \frac{d}{E_x^C + \frac{d}{2}}$$

$\text{E}(\tilde{q}_x) = q_x$

$$\text{Var}(\tilde{q}_x) \approx \frac{q_x (1 - q_x)}{E_x}$$

$\tilde{q}_x$ is approximately normally distributed asymptotically

Poisson Model

$D \sim \text{Poisson}(E^C \mu)$

$$\hat{\mu} = \frac{d}{E^C}$$

$\text{E}(\tilde{\mu}) = \mu$

$$\text{Var}(\tilde{\mu}) = \frac{\mu}{E^C}$$

$\tilde{\mu}$ is normally distributed asymptotically

Trapezium Approximation

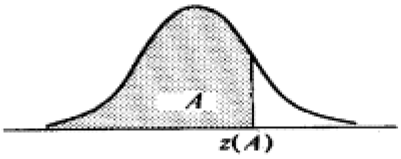
$$E_x^C = \int_0^{K+1} P_{x,t}\, dt \approx \sum_{t=0}^{K} \frac{P_{x,t} + P_{x,t+1}}{2}$$

$${}^{(1)}E_x^C \approx \sum_{t=0}^{K} \frac{P_{x,t}^{(1)} + P_{x,t+1}^{(1)}}{2} \qquad \text{where } P_{x,t}^{(1)} \approx \frac{P_{x,t}^{(2)} + P_{x+1,t}^{(2)}}{2} \text{ or } P_{x,t}^{(1)} = P_{x+1,t}^{(3)}$$

$${}^{(2)}E_x^C \approx \sum_{t=0}^{K} \frac{P_{x,t}^{(2)} + P_{x,t+1}^{(2)}}{2} \qquad \text{where } P_{x,t}^{(2)} \approx \frac{P_{x-1,t}^{(1)} + P_{x,t}^{(1)}}{2} \text{ or } P_{x,t}^{(2)} \approx \frac{P_{x,t}^{(3)} + P_{x+1,t}^{(3)}}{2}$$

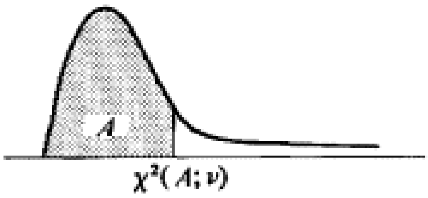
$${}^{(3)}E_x^C \approx \sum_{t=0}^{K} \frac{P_{x,t}^{(3)} + P_{x,t+1}^{(3)}}{2} \qquad \text{where } P_{x,t}^{(3)} = P_{x-1,t}^{(1)} \text{ or } P_{x,t}^{(3)} \approx \frac{P_{x-1,t}^{(2)} + P_{x,t}^{(2)}}{2}$$

Entry is area A under the standard normal curve from $-\infty$ to $z(A)$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Entry is $\chi^2(A;\,\nu)$ where $P\{\chi^2(\nu) \le \chi^2(A;\,\nu)\} = A$



ν	A: .005	.010	.025	.050	.100	.900	.950	.975	.990	.995
1	0.0^4393	0.0^3157	0.0^3982	0.0^2393	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.60
3	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84
4	0.207	0.297	0.484	0.711	1.064	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.145	1.61	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.4	104.2
80	51.17	53.54	57.15	60.39	64.28	96.58	101.9	106.6	112.3	116.3
90	59.20	61.75	65.65	69.13	73.29	107.6	113.1	118.1	124.1	128.3
100	67.33	70.06	74.22	77.93	82.36	118.5	124.3	129.6	135.8	140.2