Future Lifetime

– (x) is a life aged exactly x

– T_x is the future lifetime of (x)

– T_x is a random variable

– T_x has a probability distribution

– T_x is in years generally

Lifetime Distribution Function

– distribution function of T_x :

$$F_x(t) = \Pr(T_x \leq t)$$

– actuarial notation :

$${}_t q_x = F_x(t)$$

– stated as q_x when $t = 1$

– q_x is called mortality rate

Survival Function

– survival function of T_x :

$$S_x(t) = \Pr(T_x > t) = 1 - F_x(t)$$

– actuarial notation :

$$_t p_x = S_x(t)$$

– stated as p_x when $t = 1$

$$_t p_x + {}_t q_x = 1$$

Survival Probabilities

$$S_x(t) = \Pr(T_x > t) = \Pr(T_0 > x + t \mid T_0 > x)$$

$$= \frac{\Pr((T_0 > x + t) \cap (T_0 > x))}{\Pr(T_0 > x)} = \frac{\Pr(T_0 > x + t)}{\Pr(T_0 > x)} = \frac{S_0(x + t)}{S_0(x)}$$

$$_t p_x = \frac{_{x+t} p_0}{_x p_0}$$

$$S_0(x + t) = S_0(x) S_x(t)$$

$$_{x+t} p_0 = {}_x p_0 \;\; {}_t p_x$$

$$_{s+t} p_x = \frac{_{x+s+t} p_0}{_x p_0} = \frac{_{x+s} p_0}{_x p_0} \frac{_{x+s+t} p_0}{_{x+s} p_0} = {}_s p_x \;\; {}_t p_{x+s}$$

$$_{s+t} p_x = {}_t p_x \;\; {}_s p_{x+t}$$

Force of Mortality

– force of mortality at age x :

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr\left(T_0 \le x + dx \mid T_0 > x\right)$$

– for small dx :

$$\mu_x dx \approx \Pr\left(T_0 \le x + dx \mid T_0 > x\right) = \Pr\left(T_x \le dx\right)$$

Relationship between μ_x and ${}_t p_x$

density function of T_x :

$$f_x(t) = \frac{d}{dt} F_x(t) = \frac{d}{dt} {}_t q_x = -\frac{d}{dt} {}_t p_x$$

$$f_x(t) = \frac{d}{dt} F_x(t) = \lim_{dx \to 0^+} \frac{1}{dx}\left(F_x(t+dx) - F_x(t)\right)$$

$$= \lim_{dx \to 0^+} \frac{1}{dx}\left(\Pr(T_x \le t+dx) - \Pr(T_x \le t)\right)$$

$$= \lim_{dx \to 0^+} \frac{1}{dx}\left(\Pr(T_0 \le x+t+dx \mid T_0 > x) - \Pr(T_0 \le x+t \mid T_0 > x)\right)$$

$$= \lim_{dx \to 0^+} \frac{1}{dx} \frac{\Pr(x < T_0 \le x+t+dx) - \Pr(x < T_0 \le x+t)}{\Pr(T_0 > x)}$$

$$= \lim_{dx \to 0^+} \frac{1}{dx} \frac{\Pr(T_0 \le x+t+dx) - \Pr(T_0 \le x) - \Pr(T_0 \le x+t) + \Pr(T_0 \le x)}{\Pr(T_0 > x)}$$

$$= \lim_{dx \to 0^+} \frac{1}{dx} \frac{\Pr(T_0 \le x+t+dx) - \Pr(T_0 \le x+t)}{\Pr(T_0 > x)}$$

$$= \frac{\Pr(T_0 > x+t)}{\Pr(T_0 > x)} \lim_{dx \to 0^+} \frac{1}{dx} \frac{\Pr(x+t < T_0 \le x+t+dx)}{\Pr(T_0 > x+t)}$$

$$= \frac{S_0(x+t)}{S_0(x)} \lim_{dx \to 0^+} \frac{1}{dx} \Pr(T_0 \le x+t+dx \mid T_0 > x+t)$$

$$= S_x(t)\mu_{x+t} = {}_t p_x \ \mu_{x+t}$$

Relationship between μ_x and ${}_t p_x$

$$\frac{d}{dt}\,{}_t p_x = -{}_t p_x \; \mu_{x+t}$$

$$\frac{1}{{}_s p_x}\frac{d}{ds}\,{}_s p_x = -\mu_{x+s}$$

$$\frac{d}{ds}\ln\left({}_s p_x\right) = -\mu_{x+s}$$

$$\int_0^t \frac{d}{ds}\ln\left({}_s p_x\right) ds = -\int_0^t \mu_{x+s} ds$$

$$\ln\left({}_t p_x\right) = -\int_0^t \mu_{x+s} ds$$

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$

Relationship between μ_x and ${}_tq_x$

distribution function of T_x :

$${}_tq_x = F_x(t) = \int_0^t f_x(s)\,ds = \int_0^t {}_sp_x\ \mu_{x+s}ds$$

Central Rate of Mortality

central rate of mortality at age x :

$$m_x = \frac{q_x}{\int_0^1 {}_t p_x dt} = \frac{\int_0^1 {}_t p_x \ \mu_{x+t} dt}{\int_0^1 {}_t p_x dt}$$

when $\mu_{x+t} = \mu_x$ for $0 < t < 1$, $m_x = \mu_x$

Curtate Future Lifetime

curtate future lifetime of (x) :

$$K_x = \left[T_x\right]$$

probability mass function of K_x :

$$\Pr\left(K_x = k\right) = \Pr\left(k \leq T_x < k+1\right) = \Pr\left(k < T_x \leq k+1\right)$$

$$= \Pr\left(T_x \leq k+1\right) - \Pr\left(T_x \leq k\right) = {}_{k+1}q_x - {}_k q_x = {}_k p_x - {}_{k+1} p_x$$

$$= {}_k p_x - {}_k p_x \ p_{x+k} = {}_k p_x \left(1 - p_{x+k}\right) = {}_k p_x \ q_{x+k}$$

Expected Value of Future Lifetime

complete expectation of life :

$$\overset{\circ}{e}_x = \mathrm{E}(T_x) = \int_0^{\infty} t \; f_x(t)\, dt = \int_0^{\infty} t \; {}_t p_x \; \mu_{x+t} dt$$

$$\overset{\circ}{e}_x = \mathrm{E}(T_x) = \int_0^{\infty} \left(1 - F_x(t)\right) dt = \int_0^{\infty} {}_t\, p_x dt$$

Expected Value of Future Lifetime

curtate expectation of life :

$$e_x = \mathrm{E}\left(K_x\right) = \sum_{k=0}^{\infty} k \ {}_k p_x \ q_{x+k}$$

$$= p_x \ q_{x+1}$$

$$+ {}_2 p_x \ q_{x+2} + {}_2 p_x \ q_{x+2}$$

$$+ {}_3 p_x \ q_{x+3} + {}_3 p_x \ q_{x+3} + {}_3 p_x \ q_{x+3}$$

$$+ \ldots$$

$$= p_x + {}_2 p_x + {}_3 p_x + \ldots$$

$$= \sum_{k=1}^{\infty} {}_k \, p_x$$

Expected Value of Future Lifetime

$$\mathring{e}_x = \int_0^\infty {}_t p_x dt = \int_0^1 {}_t p_x dt + \int_1^2 {}_t p_x dt + \int_2^3 {}_t p_x dt \ldots$$

$$\approx \frac{1}{2}(1 + p_x) + \frac{1}{2}(p_x + {}_2p_x) + \frac{1}{2}({}_2 p_x + {}_3p_x) + \ldots$$

$$= \frac{1}{2} + p_x + {}_2p_x + {}_3p_x + \ldots = \frac{1}{2} + e_x$$

Uniform Distribution of Deaths (UDD)

for $0 < t < 1$ and integral x assume :

$$ {}_t p_x = 1 - t\, q_x $$

$$ {}_t q_x = t\, q_x $$

for $0 < s$, $t < 1$ and $0 < s + t < 1$:

$$ {}_t q_{x+s} = 1 - \frac{{}_{s+t} p_x}{{}_s p_x} = \frac{{}_s p_x - {}_{s+t} p_x}{{}_s p_x} = \frac{t\, q_x}{1 - s\, q_x} $$

$$ {}_t p_x\, \mu_{x+t} = -\frac{d}{dt}\, {}_t p_x = -\frac{d}{dt}\left(1 - t\, q_x\right) = q_x $$

$$ \therefore \mu_{x+t} = \frac{q_x}{{}_t p_x} = \frac{q_x}{1 - t\, q_x} > q_x $$

Constant Force of Mortality

for $0 < t < 1$ and integral x assume :

$$\mu_{x+t} = \mu_x$$

$$_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right) = \exp\left(-\mu_x t\right)$$

for $0 < s$, $t < 1$ and $0 < s + t < 1$:

$$_t p_{x+s} = \exp\left(-\int_0^t \mu_{x+s+r} dr\right) = \exp\left(-\mu_x t\right)$$

Balducci Assumption

for $0 < t < 1$ and integral x assume :

$$ {}_{1-t}q_{x+t} = (1-t)\,q_x $$

$$ p_x = {}_t p_x \; {}_{1-t}p_{x+t} $$

$$ 1-q_x = (1-{}_t q_x)(1-{}_{1-t}q_{x+t}) = (1-{}_t q_x)(1-(1-t)\,q_x) $$

$$ {}_t p_x = \frac{1-q_x}{1-(1-t)\,q_x} $$

$$ {}_t q_x = \frac{t\,q_x}{1-(1-t)\,q_x} $$

$$ {}_t p_x \, \mu_{x+t} = -\frac{d}{dt}\,{}_t p_x = \frac{(1-q_x)\,q_x}{(1-(1-t)\,q_x)^2} $$

$$ \therefore \mu_{x+t} = \frac{q_x}{1-(1-t)\,q_x} > q_x $$

Gompertz' Law

$$\mu_x = Bc^x$$

$$ {}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right) = \exp\left(-\int_0^t Bc^{x+s} ds\right)$$

$$= \exp\left(-Bc^x \left[\frac{c^s}{\ln c}\right]_0^t\right) = \exp\left(-\frac{Bc^x\left(c^t - 1\right)}{\ln c}\right)$$

with $\ln g = -B/\ln c$:

$$ {}_t p_x = \exp\left(\ln g \;\; c^x\left(c^t - 1\right)\right) = g^{c^x\left(c^t - 1\right)}$$

Makeham's Law

$$\mu_x = A + Bc^x$$

$$_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right) = \exp\left(-\int_0^t \left(A + Bc^{x+s}\right) ds\right)$$

$$= \exp\left(-At - \frac{Bc^x\left(c^t - 1\right)}{\ln c}\right)$$

with $\ln g = -B/\ln c$ and $s = \exp(-A)$:

$$_t p_x = s^t g^{c^x\left(c^t - 1\right)}$$