Week 5 Tutorial Questions

2021

1. For a Poisson process with rate λ:
   (a) state the distribution of the inter-arrival time random variable, T
   (b) give an expression for the probability that exactly one event will occur during a finite time interval of length t .

2. Claims on a portfolio of policies occur according to a Poisson process with a mean rate of 5 claims per day. Claim amounts are 10, 20 or 30. 20% of claims are of amount 10, 70% are of amount 20 and 10% are of amount 30.
   (a) Calculate the expected waiting time until the first claim of amount 30.
   (b) Calculate the probability that there are at least 10 claims during the first 2 days, given that there were exactly 6 claims during the first day.
   (c) Calculate the probability that there are at least 2 claims of amount 20 during the first day and at least 3 claims of amount 20 during the first 2 days.
   (d) Calculate the conditional variance of the number of claims during the first day, given that there are 2 claims of amount 10 during the first day.

3. $\{X_t\}$ is a Markov jump process with state space $S = \{0, 1, 2, \ldots\}$ and $X_0 = 0$. The transition rates are given by:

$$\mu_{ij} = \begin{cases} \lambda & \text{if } j = i + 1 \\ -\lambda & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

   (a) Write down the transition probabilities $P_{ij}(t)$.
   (b) Define the term holding time.
   (c) Find the distribution of the first holding time T_0 .
   (d) State the value of X_{T_0}

(e) Given that the increments are stationary and independent, state the distributions of $T_0, T_1, T_2, \ldots$ Justify your answer.

4. A particular machine is in constant use. Regardless of how long it has been since the last repair, it tends to break down once a day (ie once every 24 hours of operation) and on average it takes the repairman 6 hours to fix.

   You are modelling the machine's status as a time-homogeneous Markov jump process $\{X(t) : t \geq 0\}$ with two states: 'being repaired' denoted by 0, and 'working' denoted by 1.

   Let $P_{i,j}(t)$ denote the probability that the process is in state j at time t given that it was in state i at time ci and suppose that t is measured in days.

   (a) State the two main assumptions that you make in applying the model and discuss briefly how you could test that each of them holds.

   (b) Draw the transition graph for the process, showing the numerical values of the transition rates.

   (c) State Kolmogorov's backward and forward differential equations for the probability $P_{0,0}(t)$

   (d) Solve the forward differential equation in (c) to show that:

$$P_{0,0}(t) = \frac{1}{5} + \frac{4}{5}e^{-5t}$$

5. Claims on an insurer's travel insurance policies arriving in the claims department (state A) wait for an average of two days before being logged and classified by a claims administrator as requiring:

   - investigation by a loss adjuster (state L),
   - more details from the insured (state I),
   - no further information is required and the claim should be settled immediately (state S).

   Only one new claim in ten is classified as requiring investigation by a loss adjuster, and five in ten require further information from the insured.

   If needed, investigation by a loss adjuster takes an average of 10 days, after which 30% of cases require further information from the insured and 70% are sent for immediate settlement.

   Collecting further information from the insured takes an average of 5 days to complete, and immediate settlement takes an average of 2 days before the settlement process is complete (state C).

   It is suggested that a time-homogeneous Markov process with states A, L, I, Sand C could be used to model the progress of claims through the settlement system with the ultimate aim of reducing the average time to settle a claim.

(a) Calculate the generator matrix, $\{_{ij}; i,j = A,l,l,S,C\}$, of such a model.

(b) Calculate the proportion of claims that eventually require more details from the insured.

(c) Derive a forward differential equation for the probability that a claim is yet to be logged and classified by a claims administrator at time t . Solve this equation to obtain an expression for the probability.

6. An n -state, time-homogeneous Markov jump process with transition probability matrix $P(t)$ over a period of length t, is said to have a stationary distribution, $\underline{\pi} = (\pi_1, \ldots, \pi_n)$, if:

- $\underline{\pi} P(t) = \underline{\pi}$
- $0 \le \pi_i \le 1$ for each $i = 1, 2, \ldots, n$
- $\sum_{i=1}^{n} \pi_i = 1$

(a) Explain why the first condition is equivalent to the condition $\underline{\pi} A = \underline{0}$ where A is the generator matrix and $\underline{0}$ is an n -dimensional vector whose entries are all 0.

In a particular company the salary scale has only two different levels. On average, an employee spends 2 years at level 1 before moving on to the higher level, or leaving the company. An employee at the maximum level spends an average of 5 years before leaving. Nobody is demoted, promotion can occur at any time, and mortality can be ignored.

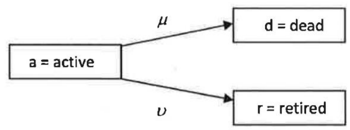
Upon leaving level 1, the probability that an employee moves to level 2 is 50%.

(b) Explain how you could model this as a Markov process, commenting on any assumptions that you make.

(c) Derive the generator matrix of the Markov jump process.

(d) The company currently has 1,000 employees. The proportions at levels 1 and 2 are 60% and 40% respectively. Use a forward differential equation to determine the distribution of these employees in five years' time. You should assuming that nobody joins the company in the future.

7. (a) The following multiple state model has been suggested as a representation of deaths and retirements between the ages of 59 and 60. There are no other decrements and the forces of decrement μ and ν are constant. Let ${}_tp_x^{ij}$ denote the probability that a life is in state j at age $x + t$ given that it was in state i at age x.



i. State the assumptions underlying the above model.
ii. Show thatt P 0 = -e (+v)t for 59 x x+t 60.
iii. Suppose that you make the following observations in respect of n identical and statistically independent lives:
   - v = time spent in the active state
   - d = number of deaths
   - r = number of retirements

   Assuming that lives are only observed to the earlier of death or retirement, show that the likelihood for μ and ν given these observations is:

$$L(\mu, \nu) = e^{-(\mu+\nu)v\mu^d\nu^r}$$

iv. Give formulae (without proof) for:
   - the maximum likelihood estimator of the parameter v
   - the asymptotic expected value of the estimator
   - an estimated standard error of the estimator. (16)

(b) Suppose that you learn that retirements can only take place on a birthday, so that r is the number of retirements at exact age 60. In addition to v, d and r you also observe:

m = number of lives attaining exact age 60, where $m \leq n$. Suppose that any life attaining exact age 60 will retire with probability k, where $0 < k < l$.

i. State the likelihood for μ and k, given v, d, r and m.
ii. Give a formula (without proof) for the maximum likelihood estimate of the parameter k.

8. Vehicles in a certain country are required to be assessed every year for road-worthiness. At one vehicle assessment centre, drivers wait for an average of 15 minutes before the road-worthiness assessment of their vehicle commences. The assessment takes on average 20 minutes to complete. Following the assessment, 80% of vehicles are passed as road-worthy allowing the driver to drive home. A further 15% of vehicles are categorised as a 'minor fail'; these vehicles require on average 30 minutes of repair work before the driver is allowed to drive home. The remaining 5% of vehicles are categorised as a 'significant fail'; these vehicles require on average three hours of repair work before the driver can go home.

A continuous-time Markov model is to be used to model the operation of the vehicle assessment centre, with states W (waiting for assessment), A (assessment taking place), M (minor repair taking place), S (significant repair taking place) and H (travelling home).

(a) Identify the distribution of the time spent in each state.

(b) Write down the generator matrix for this process.

(c) i. Use Kolmogorov's forward equations to write down differential equations satisfied by $p_{WM}(t)$ and by $p_{WA}(t)$.

ii. Verify that $p_{WM}(t) = 4e^{-t/20} - 4e^{-t/15}$ for $t \geq 0$, where t is measured in minutes.

iii. Derive an expression for $p_{WM}(t)$ for $t \geq 0$

(d) Let T_i be the expected length of time (in minutes) until the vehicle can be driven home given that the assessment process is currently in state i .

i. Explain why $T_W = 15 + T_A$.

ii. Derive corresponding equations for T_A, T_M and T_S

iii. Calculate T_W .