ETC3430: Financial mathematics under uncertainty

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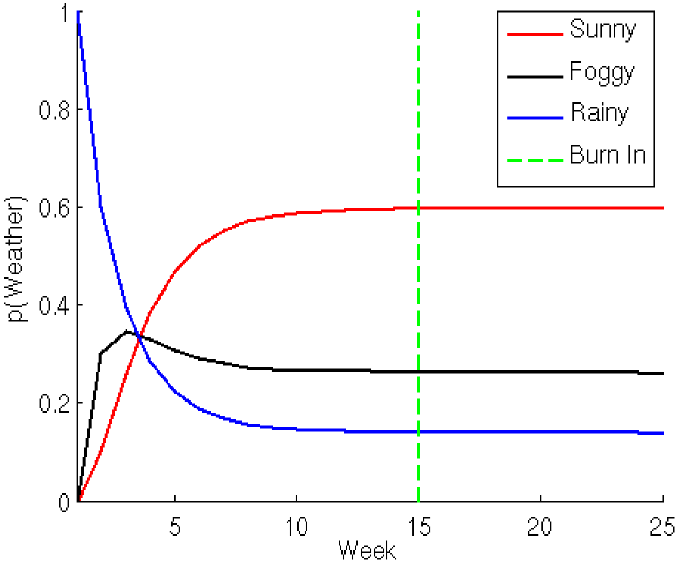
Outline

π_t and π



Sunny
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p(Weather)
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15
20
25

Communication

Definition (Accessible $i \to j$)

A state j is said to be accessible from a state i (written $i \to j$) if there exist $\infty > n_{i,j} \geq 1$ such that

$$\mathbb{P}(X_{n_{i,j}} = j | X_0 = i) > 0.$$

A state i is said to communicate with state j if both $i \to j$ and $j \to i$. A communicating class is a maximal set of states C such that every pair of states in C communicates with each other.

Acessibility

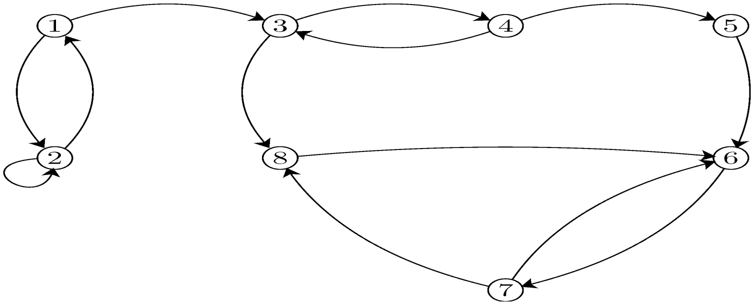
For a discrete-time Markov chain with state-space $\mathbb{S}$ and transition probabilities $\mathbb{P}$, we say that there is a possible path from state i to state j if there is a sequence of states

$$i = i_0 \rightarrow i_1 \rightarrow \ldots \rightarrow i_n = j$$

such that for all transitions along the path we have

$$\mathbb{P}_{i_{l-1}, i_l} > 0$$

We will also use the phrase that state j is accessible from state i.



$$\text{Class } 1 = \{\text{state } \mathsf{1}, \text{state } \mathsf{2}\},$$
$$\text{Class } 2 = \{\text{state } \mathsf{3}, \text{state } \mathsf{4}\},$$
$$\text{Class } 3 = \{\text{state } \mathsf{5}\},$$
$$\text{Class } 4 = \{\text{state } \mathsf{6}, \text{state } \mathsf{7}, \text{state } \mathsf{8}\}.$$

In the seven states example, we notice that there are two kinds of classes. In particular, if, at any time, the Markov chain enters Class 4, it will always stay in that class. On the other hand, for other classes, this is not true. For example, if $X_0 = 1$, then the Markov chain might stay in Class 1 for a while, but at some point, it will leave that class, and it will never return to that class again. The states in Class 4 are called recurrent states, while the other states in this chain are called transient.

Definition

For any state i, we define

$$f_{ii} = P(X_n = i, \textit{ for some } n \geq 1 | X_0 = i).$$

State i is recurrent if $f_{ii} = 1$, and it is transient if otherwise.

It is relatively easy to show that if two states are in the same class, both are recurrent, or both are transient. Thus, we can extend the above definitions to classes. A class is said to be recurrent if the states in that class are recurrent. If, on the other hand, the states are transient, the class is called transient.

Theorem

Consider a discrete-time Markov chain. Let V be the total number of visits to state i

- *If i is a recurrent state, then*

$$P(V = \infty | X_0 = i) = 1$$

- *If i is a transient state, then*

$$V | X_0 = i \ \sim \ Geometric(1 - f_{ii}).$$

Theorem

A finite state Markov chain has at least one recurrent class.

Proof.

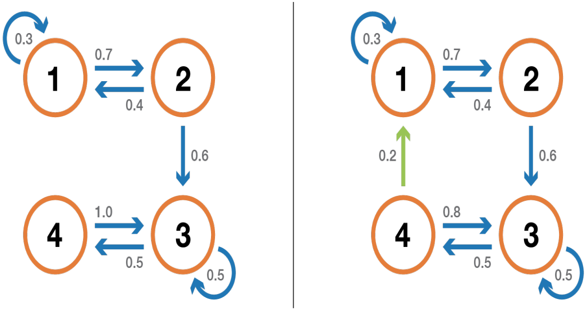
Consider a finite Markov chain with $S = \{1, 2, \cdots, r\}$. Suppose that all states are transient. Then, starting from time 0, the chain might visit state 1 several times, but at some point the chain will leave state 1 and will never return to it. That is, there exists an integer $M_1 > 0$ such that $X_n \neq 1$, for all $n \geq M_1$. Similarly, there exists an integer $M_2 > 0$, such that $X_n \neq 2$, for all $n \geq M_2$ and so on. Now, if you choose

$$n \geq \max\{M_1, M_2, \cdots, M_r\},$$

then X_n cannot be equal to any of the states $1, 2, 3..., r$. This is a contradiction, so we conclude that there must be at least one recurrent state, which means that there must be at least one recurrent class.

Definition (Irreducibility)

A Markov chain is said to be irreducible if it is possible to get to any state from any state, i.e. only one communicating class.



The left chain is not irreducible: from 3 or 4, we can't reach 1 or 2. The chain on the right (one edge has been added) is irreducible: the chain can reach each state from any other state.

Simple Random Walk is irreducible

Recall a simple random walk

$$X_n = X_{n-1} + Z_n \text{ with } Z_n = \begin{cases} -1 & w.p. \ 0.5 \\ 1 & w.p. \ 0.5 \end{cases}$$

Here $\mathbb{S} = \mathbb{Z}$, we can always reach any state from any other state, doing so step-by-step, using the fact that

$$\mathbb{P}_{i,i+1} = 0.5 \text{ and } \mathbb{P}_{i,i-1} = 0.5.$$

Since $\mathbb{P}^n_{i,j} \geq 0$ with $n = |i - j|$.

Analysing the finite transition Matrix

Consider the markov chain with $\mathbb{S} = 0, 1, 2, 3$ follow transition matrix

$$\mathbb{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.1 & 0.1 & 0.2 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice how states 0,1 keep to themselves in that whereas they communicate with each other, no other state is reachable from them (together they form an absorbing set), $C_1 = 0, 1$. Whereas every state is reachable from state two, getting to state two is impossible from any other state, $C_2 = 2$. Finally, state 3 is absorbing, $C_3 = 3$. Hence, this is not an irreducible Markov chain.

Periodicity

Definition

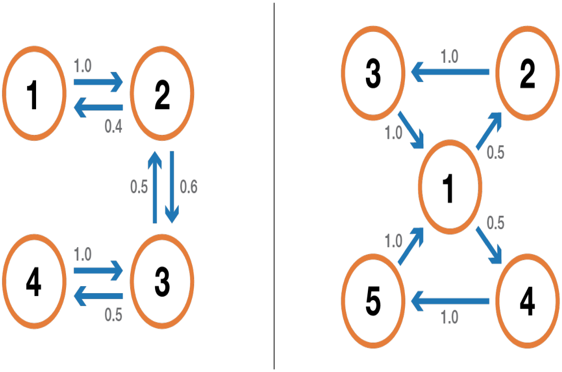
A state i has period k if any return to state i must occur in multiples of k time steps, the period of a state is defined as

$$k = gcd\{n > 0 : \mathbb{P}(X_n = i | X_0 = i) > 0\}$$

denote $d(i) = k$. If k = 1, then the state is said to be aperiodic. Otherwise, the state is said to be periodic with period k. A Markov chain is aperiodic if every state is aperiodic.

The simple random walk has a period of TWO for every state, and the last example is aperiodic.

Period



The chain on the left is 2-periodic: when leaving any state, it always takes a multiple of 2 steps to come back to it. The chain on the right is 3-periodic.

Theorem

If $i \leftrightarrow j$, $d(i) = d(j)$. An irreducible Markov chain only needs one aperiodic state to imply all states are aperiodic.

Remember that two numbers m and l are said to be co-prime if their greatest common divisor (gcd) is 1. Now, suppose that we can find two co-prime numbers l and m such that

$$\mathbb{P}^{m}_{i,i} > 0 \text{ and } \mathbb{P}^{l}_{i,i} > 0.$$

That is, we can go from state i to itself in l steps and also in m steps. Then, we can conclude state i is aperiodic. If we have an irreducible Markov chain, this means that the chain is aperiodic.

Exercise

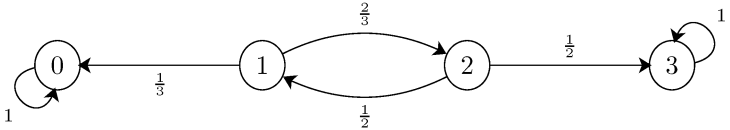
Consider the seven state markov chain

- Is Class 1 aperiodic?
- Is Class 2 aperiodic?
- Is Class 4 aperiodic?

Answer

- Yes, $\mathbb{P}_{2,2} > 0$
- No, the period is 2.
- Yes. We can go from state 6 to 6 in two steps (6-7-6) and three steps (6-7-8-6). The greatest common divisor of 2 and 3 is one. And state 6 is communicating with both 7 and 8. Hence they have the same period.

Absorption



The state transition matrix of this Markov chain is given by the following matrix.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

There are three classes: Class One consists of one state, state 0, a recurrent state. Class Two consists of two states 1 and 2, both of which are transient. Finally, class three consists of one state, state 3, which is a recurrent state.

Absorption probabilities

$$
\begin{aligned}
a_0 &= P(\text{absorption in } 0 | X_0 = 0) = 1, \\
a_1 &= P(\text{absorption in } 0 | X_0 = 1), \\
a_2 &= P(\text{absorption in } 0 | X_0 = 2), \\
a_3 &= P(\text{absorption in } 0 | X_0 = 3) = 0.
\end{aligned}
$$

To find $a_1 \& a_2$ we apply the law of total probability with recursion. The main idea is the following: if $X_n = i$, then the next state will be $X_{n+1} = k$ with p_{ik}, thus

$$
a_i = \sum_k a_k p_{ik}, \qquad \text{for } i = 0, 1, 2, 3 \tag{1}
$$

More specifically,

$$
\begin{aligned}
a_0 &= a_0 = 1, \\
a_1 &= \frac{1}{3}a_0 + \frac{2}{3}a_2, \\
a_2 &= \frac{1}{2}a_1 + \frac{1}{2}a_3, \\
a_3 &= a_3 = 0.
\end{aligned}
$$

Hence, $a_1 = 0.5 \& a_2 = 0.25$.

Hitting times

Let's define t_i as the number of steps needed until the chain hits state 0 or state 3, given $X_0 = i$. The idea is also to use the law of total probability with recursion.For example, if $X_0 = 1$, then after one step, we have $X_1 = 0$ or $X_1 = 2$. Thus, we can write

$$\begin{aligned} t_1 &= 1 + \frac{1}{3}t_0 + \frac{2}{3}t_2 \\ &= 1 + \frac{2}{3}t_2. \end{aligned}$$

Similarly, we can write

$$\begin{aligned} t_2 &= 1 + \frac{1}{2}t_1 + \frac{1}{2}t_3 \\ &= 1 + \frac{1}{2}t_1. \end{aligned}$$

Solving the above equations, we obtain

$$t_1 = \frac{5}{2}, \quad t_2 = \frac{9}{4}.$$

Mean return times

In particular, assuming the chain is in state l, we consider the expected time (number of steps) needed until the chain returns to state l. For example, consider a Markov chain for which $X_0 = 2$, and

$$X_0 = 2,\ X_1 = 1,\ X_2 = 4,\ X_3 = 3,\ X_4 = 2,\ X_5 = 3,\ X_6 = 2,\ X_7 = 3,\ \cdots,$$

then the first return to state 2 occurs at time 4. Here, we are interested in the expected value of the first return time

$$R_l = \min\{n \geq 1 : X_n = l\}.$$

then

$$r_l = E[R_l | X_0 = l].$$

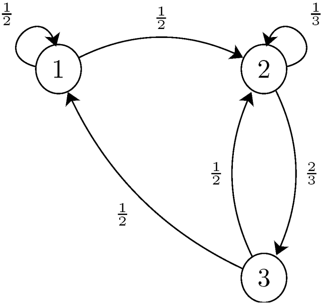
Again, let's define $t_{k,l}$ as the expected time until the chain hits state l for the first time, given that $X_0 = k$. Using the law of total probability, we can write

$$r_l = 1 + \sum_k p_{lk} t_{k,l}.$$

Specifically

$$\begin{aligned} t_{l,l} &= 0, \\ t_{k,l} &= 1 + \sum_j t_{j,l} p_{kj}, \quad \text{for } k \neq l. \end{aligned}$$

Find r_1



First we need $t_{2,1}$ and $t_{1,1} = 0$. To find $t_{2,1}$, we have

$t_{2,1} = 1 + 1/3t_{2,1} + 2/3t_{3,1}$ and $t_{3,1} = 1 + 0.5t_{1,1} + 0.5t_{2,1}$

Solving them,

$$t_{2,1} = 5 \text{ and } t_{3,1} = 3.5$$

We then

$$r_1 = 1 + 0.5t_{2,1} = 3.5.$$

The Limiting Distribution

Here, we would like to discuss the long-term behaviour of Markov chains. In particular, we would like to know the fraction of times that the Markov chain spends in each state as n becomes large. More specifically, we would like to study the distributions

$$\pi^{(n)} = \begin{bmatrix} P(X_n = 0) & P(X_n = 1) & \cdots & \end{bmatrix} \tag{2}$$

as $n \to \infty$

Example

Consider two state $\mathbb{S} = \{0, 1\}$

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}, \tag{3}$$

where $a, b \in [0, 1]$,and

$$\pi^{(0)} = \begin{bmatrix} P(X_0 = 0) & P(X_0 = 1) \end{bmatrix} = \begin{bmatrix} \alpha & 1-\alpha \end{bmatrix},$$

Example

Using induction, it can be shown that

$$P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

Then

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix},$$

and the limiting distribution

$$\lim_{n\to\infty} \pi^{(n)} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$$

π^{∞} does not depend on X_0

Notice that in the simple example, the limiting distribution does not depends on π_0 at all! In other words

$$\lim_{n\to\infty} P(X_n = 0 | X_0 = i) = \frac{b}{a+b},$$
$$\lim_{n\to\infty} P(X_n = 1 | X_0 = i) = \frac{a}{a+b}.$$

The limiting Distribution

Definition

The probability distribution $\pi^{\infty} = [\pi_0^{\infty}, \pi_1^{\infty}, \pi_2^{\infty}, \cdots]$ *is called the limiting distribution of the Markov chain if*

$$\pi_j^{\infty} = \lim_{n\to\infty} P(X_n = j | X_0 = i) \text{ for all } i, j \in \mathbb{S}$$

and we have

$$\sum_{j \in S} \pi_j^{\infty} = 1.$$

Finite state Markov Chain

In general, a finite Markov chain can consist of several transient states as well as recurrent states. As n becomes large, the chain will enter a recurrent class, and it will stay there forever. Therefore, when studying long-run behaviours, we focus only on the recurrent classes.
If a finite Markov chain has more than one recurrent class, then the chain will get absorbed in one of the recurrent classes. Thus, the first question is: in which recurrent class does the chain get absorbed?
Thus, we can limit our attention to the case where our Markov chain consists of one recurrent class. In other words, we have an irreducible Markov chain.

Finite-State Markov Chain

Theorem

A Markov chain with finite state space has at least one stationary probability distribution.

Theorem

An irreducible Markov chain with finite state space has a unique stationary probability distribution.

Another term for the stationary distribution is the steady state distribution, i.e. if X_n reached the steady state, the distribution of X_{n+1} is the same as X_n,.

The Link between stationary and limiting distribution

Recall that

$$\begin{aligned}\pi^{\infty} &= \lim_{n\to\infty} \pi^{(n)} \\ &= \lim_{n\to\infty} \left[\pi^{(0)} P^n\right].\end{aligned}$$

Similarly, we have

$$\begin{aligned}\pi^{\infty} &= \lim_{n\to\infty} \pi^{(n+1)} \\ &= \lim_{n\to\infty} \left[\pi^{(0)} P^{n+1}\right] \\ &= \lim_{n\to\infty} \left[\pi^{(0)} P^n P\right] \\ &= \left[\lim_{n\to\infty} \pi^{(0)} P^n\right] P \\ &= \pi^{\infty} P.\end{aligned}$$

Hence, when π^{∞} exist, it is also the stationary distribution of the Markov Chain.

Theorem

An irreducible, aperiodic Markov chain with a finite state space has a unique stationary probability distribution and will reach its unique distribution in the long run. That, all rows of the limiting transition

$$\mathbb{P}^{\infty} = \lim_{n \to \infty} \mathbb{P}^{n}$$

equals to π.

Verify that for the simple two state example, the stationary distribution is indeed the same as the stationary distribution.

Mean Return times

For the two state Markov chain,

$$\begin{aligned} r_0 &= E[R|X_1 = 0]P(X_1 = 0) + E[R|X_1 = 1]P(X_1 = 1) \\ &= E[R|X_1 = 0] \cdot (1 - a) + E[R|X_1 = 1] \cdot a. \end{aligned}$$

For $X_1 = 0$,then $R = 1$, and if

$$R|X_1 = 1 \sim 1 + Geo(b),$$

hence

$$r_0 = 1(1 - a) + (1 + \frac{1}{b})a = \frac{a + b}{b}.$$

Symmetry gives us,

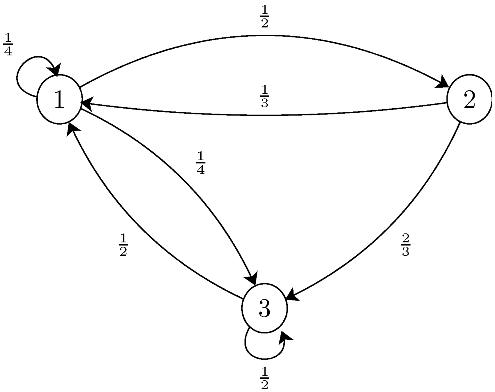
$$r_1 = \frac{a + b}{b}.$$

Theorem

Assume that the chain is irreducible and aperiodic. We have

$$r_j = \frac{1}{\pi_j}.$$

A Concluding Example



The chain is irreducible since we can go from any state to any other states in a finite number of steps. Moreover, $\mathbb{P}_{1,1} > 0$ implies that this is also aperiodic. Hence, there is a unique stationary distribution that the chain will reach in long run.

To find the stationary distribution, we need to solve

$$\begin{aligned}
\pi_1 &= \frac{1}{4}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3, \\
\pi_2 &= \frac{1}{2}\pi_1, \\
\pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 + \frac{1}{2}\pi_3, \\
\pi_1 &+ \pi_2 + \pi_3 = 1.
\end{aligned}$$

We find

$$\pi_1 = \frac{3}{8},\ \pi_2 = \frac{3}{16},\ \pi_3 = \frac{7}{16}.$$

and

$$r_1 = \frac{8}{3}, r_2 = \frac{16}{3} \text{ and } r_3 = \frac{16}{7}.$$

Takeaways so far I

The main takeaways are the following:

- random processes are collections of random variables often indexed over time, indices often represent discrete- or continuous-time
- for a random process, the Markov property says that, given the present, the probability of the future is independent of the past, this property is also called "memoryless property",
- discrete-time Markov chain are random processes with discrete-time indices, and that verifies the Markov property
- the Markov property of Markov chains makes the study of these processes much more tractable and allows to derive some interesting explicit results, i.e.,mean recurrence time, stationary distribution.

Takeaways so far II

To conclude, let's emphasise once more how powerful Markov chains are for problems modelling when dealing with random dynamics. Due to their good properties, they are used in various fields such as

- queueing theory, optimising the performance of telecommunications networks, where messages must often compete for limited resources and are queued when all ressources are already allocated,
- statistics,the well known "Markov Chain Monte Carlo" random variables generation technique is based on Markov chains,
- biology, modelling of biological populations evolution,
- computer science, hidden Markov models are important tools in information theory and speech recognition and others.