Week 1 Tutorial Solutions

2021

1. (a) **Stationary**

   A stochastic process X_n is stationary if the joint distributions of $X_{t_1}, X_{t_2}, \ldots, X_{t_m}$ and $X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_m+k}$ are identical for all $t_1, t_2, \ldots, t_m, t_1+k, \ldots, t_m+k \in J$ and all integers m.

   (b) **Weakly stationary**

   The process is weakly stationary if the expectations $E[X_t]$ are constant with respect to t and the covariances $cov(X_t, X_{t+k})$ depend only on the lag k.

   (c) **Increment**

   If t and $t+u$ are in J then the increment for duration u will be $X_{t+u} - X_t$.

   (d) **Markov property**

   The Markov property states that:

   $$P\left(X_t \in A | X_{t_1} = x_1, X_{t_2} = x_2, \ldots, X_{t_m} = x_m\right) = P\left(X_t \in A | X_{t_m} = x_m\right)$$

   for all times $t_1 < t_2 < \cdots < t_m < t \in J$, all states $x_1, x_2, \ldots, x_m \in S$ and all subsets A of S

2. (a) **Weak stationarity**

   The Z_j are independent and identlcally distributed, and the α_j are constants. So:

   $$E(X_n) = (1 + \alpha_1 + \alpha_2 + \alpha_3)E(Z) = (1 + \alpha_1 + \alpha_2 + \alpha_3) \times 0 = 0$$

   and:

   $$var(X_n) = var(Z) + \alpha_1^2 var(Z) + \alpha_2^2 var(Z) + \alpha_3^2 var(Z) = (l + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)\sigma^2$$

   which is constant.

   The covariance at lag 1 is:

$cov(X_n, X_{n+1})$
$= cov(Z_n + \alpha_1 Z_{n-1} + \alpha_2 Z_{n-2} + \alpha_3 Z_{n-3} + \alpha_1 Z_n + \alpha_2 Z_{n-1} + \alpha_3 Z_{n-2})$
$= \alpha_1 var(Z) + \alpha_1\alpha_2 var(Z) + \alpha_2\alpha_3 var(Z)$
$= (\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3)\sigma_2$

The covariance at lag 2 is:

$cov(X_n, X_{n+2})$
$= cov(Z_n + \alpha_1 Z_{n-1} + \alpha_2 Z_{n-2} + \alpha_3 Z_{n-3}, Z_{n+2} + \alpha_1 Z_{n+1} + \alpha_2 Z_n + \alpha_3 Z_{n-1})$
$= \alpha_2 var(Z) + \alpha_1\alpha_3 var(Z)$
$= (\alpha_2 + \alpha_1\alpha_3)\sigma^2$

The covariance at lag 3 is:

$cov(X_n, X_{n+3})$
$= cov(Z_n + \alpha_1 Z_{n-1} + \alpha_2 Z_{n-2} + \alpha_3 Z_{n-3}, Z_{n+3} + \alpha_1 Z_{n+2} + \alpha_2 Z_{n+1} + \alpha_3 Z_n)$
$= \alpha_3 var(Z)$
$= \alpha_3\sigma^2$

The covariances at lags 4, 5, 6 ... are 0.
So the covariance depends only on the lag and not on the value of n.
Thus the process X_n is weakly stationary.

(b) **Markov?**

For a Markov process, the value of X_n only depends on the most recently known value. However, X_n depends on the previous X values so it does not possess the Markov property.

(c) **Independent increments?**

If the increments of a process are independent, then that process must have the Markov property. Since we've said that this process is not a Markov process, it cannot have independent increments.

3. A random walk has independent increments, so it has the Markov property.

4. (a) i. **Poisson process**

A Poisson process N_t, $t \geq 0$, with rate λ is a continuous-time, integer-valued process such that:

- $N_0 = 0$
- N_t has independent increments
- $N_t - N_s \sim Poisson(\lambda(t-s))$ for $0 \leq s < t$.

ii. **Compound Poisson process**

Let $\{X_n\}_{n=1}^{\infty}$ be independent identically distributed random variables. A compound Poisson process with rate *lambda* is defined for $t \geq 0$ to be:

$$S_t = X_1 + X_2 + \cdots + X_{N_t}$$

where N_t is a Poisson process and $S_t = 0$ when $N_t = 0$.

(b) **When a compound Poisson process is also a Poisson process**

S_t is also a Poisson process if the random variables X_j can only take the value O or 1.

The situation where $X_j = l$ for all j is a special case of this.

(c) i. **Markov property**

It is sufficient to show that the compound Poisson process has independent increments, since then the Markov property must hold. However, having independent increments is part of the definition of the compound Poisson process.

ii. **Reasonableness**

This is consistent with insurance claims, since we would only expect the cumulative insurance claims by time t to depend on the most recently known value. For example, if we know the cumulative claims after day one are 1,000, and by day ten are 15,000, we wouldn't expect the older value of 1,000 to add any useful information to the more recent value of 15,000.

iii. **Weak stationarity**

The process cannot be stationary since, for example, $E(S_t)$ changes with t.

iv. **Is cumulative claim amount weakly stationary?**

We wouldn't expect $E(S_t)$ to be constant since the cumulative claims generally increases with time. This would be a constant only in the trivial case where the individual claim amounts are 0, which is rather uninteresting.

In order to show that a process is not stationary, it is sufficient to show .that any one of the conditions fails to hold.

5. A simple symmetric random walk is defined by the equation:

$$X_n = \sum_{j=1}^{n} Y_j$$

where the random variables Y_j are independent and identically distributed with common probability distribution:

$$P(Y_j = +1) = \frac{1}{2} \;\; and \;\; P(Y_j = -1) = \frac{1}{2}$$

The word 'symmetric' is important as it denotes a particular process that is equally likely to 'step' upwards or downwards in its walk.

In addition, the process starts at 0, *ie* $X_0 = 0$.

The simple symmetric random walk has a discrete state space consisting of the values, $\{\ldots, -21-1, 0, +1, +2, \ldots\}$ and a discrete time set consisting of the values $\{0, 1, 2, \ldots\}$.

6. (a) **Independent increments**

   By definition:

$$\ln X_n = \ln x_0 + \sum_{j=1}^{n} \ln U_j = \ln x_0 + \sum_{j=1}^{n} Z_j$$

   where $Z_j = \ln U_j$ is a white noise process, *ie* are a set of independent and identically distributed random variables. Then:

$$\ln X_n - \ln X_{n-1} = \ln U_n = Z_n$$

   Because $Z_n, n = 0, 1, \ldots$ are independent, $\ln X_n$ has independent increments.

   (b) **Markov process**

   $\ln X_n$ has independent increments

   $\Longrightarrow \ln X_n$ is a Markov process

   $\Longrightarrow X_n = \exp(\ln X_n)$ is a Markov process.

   because exponentiation merely rescales the state space of the process.

7. The key to many results for continuous-time stochastic processes is to realise that the random variables representing behaviour in non-overlapping time periods are independent. Here the non-overlapping time periods are (O, t) and $(t, t+s)$. So...

$$cov(X(t), X(t+s)) = cov(X(t), X(t) + (X(t+s) - X(t))) \quad = cov(X(t), X(t)) + cov(X(t), X(t+s)$$

   since $X(t) \sim Poisson(\lambda t)$

8. (a) **Proof**

   We have:

$$P[X_n = a | X_{n-m} = x, X_{n-m-1} = x_{n-m-1}, X_{n-m-2} = x_{n-m-2}, X_{n-m-3} = x_{n-m-3}, \ldots]$$
$$= P[X_n - X_{n-m} + x = a | X_{n-m} = x, X_{n-m-1} = x_{n-m-1}, X_{n-m-2} = x_{n-m-2}, \ldots]$$

for all times $m > 0$ and all states $a, x, x_{n-m-1}, x_{n-m-2}, \ldots$ in the state space, S.

$$P[X_n - X_{n-m} + x = a | X_{n-m} = x, X_{n-m-1} = x_{n-m-1}, X_{n-m-2} = x_{n-m-2}, \ldots]$$
$$= P[X_n - X_{n-m} + x = a | X_{n-m} = x]$$
$$= P[X_n = a | X_{n-m} = x]$$

if non-overlapping increments are independent. So X_n has the Markov property.

(b) **White noise process**

For a discrete time, discrete state white noise process $\{Z_n : n = 1, 2, 3, \ldots\}$, where Z_n are independent and identically distributed random variables with mean μ and variance σ^2 , we have:

$$cov(Z_n - Z_{n-1}, Z_{n-1} - Z_{n-2}) = cov(-Z_{n-1}, Z_{n-1}) = -\lambda^2$$

So non-overlapping increments are not independent. However:

$$P[Z_n = z_n | Z_{n-m} = z_{n-m}, Z_{n-m-1} = z_{n-m-1}, Z_{n-m-2} = z_{n-m-2}, \ldots] = P[Z_n = z_n]$$

and:

$$P[Z_n = z_n | Z_{n-m} - z_{n-m}] = P[Z_n = z_n]$$

because the random variables are independent. So the process satisfies the Markov property.

9. (a) **Model**

Let $N(t)$ denote the number of claims received by the insurer up to time t. $N(t)$ can be modelled as a Poisson process.

Let X_j denote the amount of the j th claim. Then the cumulative claim amount up to time t is given by:

$$S(t) = X_1 + X_2 + \cdots X_{N(t)}$$

If we assume that the random variables X_j are independent and identically distributed, and they are independent of $N(t)$, then $S(t)$ is a compound Poisson process.

(b) **Probability of ruin**

The probability of ruin for the insurer is the probability that, for some time t, its claims outgo up to time t is greater than its initial capital plus premium income up to time t. In symbols, this is:

$$P(S(t) > u + ct \text{ for some } t > 0)$$

10. This is Subject CT4, September 2005, Question A2.

(a) i. **State space**
The state space of the stochastic process $X_t : t \in J$ is the set of values that the random variables X_t can take. The state space can be discrete or continuous. []

ii. **Time set**
The time set for this stochastic process is J, which contains all points at which the value of the process can be observed. The time set can be discrete or continuous.

iii. **Sample path**
A sample path is a joint realisation of the random variables X_t for all $t \in J$.

(b) i. **Examples of stochastic processes**

Discrete state space, discrete time set
Examples include Markov chains, simple random walks and discrete-time white noise processes that have discrete state spaces;

Discrete state space, continuous time set
Examples include Markov jump processes (of which the Poisson process is a special case) and counting processes.

Continuous state space, discrete time set
Examples include general random walks and time series.

Continuous state space, continuous time set
Examples include Brownian motion, diffusion processes and compound Poisson processes where the state space is continuous.

Brownian motion and diffusion processes are covered in Subject CM2.

ii. **Examples of problems an actuary may wish to study**
Discrete state space, discrete time set
An example of this is a no claims discount system. The random variable X_t represents the discount level given to a policyholder in year $t, t = 1, 2, \ldots$

Discrete state space, continuous time set
An example of this is the health, sickness, death model, which can be used to value sickness benefits. The random variable X_t takes one of the values healthy, sick or dead for each $t \geq 0$.

Continuous state space, discrete time set

An example of this is a company's share price at the end of each trading day. Another example is the annual UK inflation rate.

Continuous state space, continuous time set
An example of this is the cumulative claim amount incurred on a portfolio of policies up to time t. Another example is a company's share price at time t, where t d. enotes time since trading began.