ETC3430: Financial mathematics under uncertainty

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Outline

Time homogeneous: Holding times

Theorem

The holding time, τ^i, of a time homogeneous Markov jump process with transition rates $\mu_{i,j}$ given that its initial state is i, is an exponentially distributed random variable with $\lambda_i = -\mu_{i,i}$,

$$\mathbb{P}(\tau^i > t) = \mathbb{P}(T_1 > t | X_0 = i) = \exp^{-\lambda_i t}.$$

Further more

$$\mathbb{P}(X_{T_1} = j | X_0 = i) = \frac{\mu_{i,j}}{\lambda_i}.$$

independent of T_1.

Here λ_i is the total force out of state j.

Proof.

We have

$$
\begin{aligned}
&\mathbb{P}(T_1 > t | X_0 = i) \\
=&\prod_{j=0}^{n-1} \mathbb{P}(X_s > i, \frac{tj}{n} < s \leq \frac{tj+t}{n} | X_{\frac{tj}{n}} = i) \text{ for all n} \\
=&\mathbb{P}(X_s > i, 0 < s \leq \frac{t}{n} | X_0 = i)^n \text{ time-homogeneity} \\
=&\lim_{n\to\infty} (1 - \lambda_i \frac{t}{n} + o(\frac{t}{n}))^n = \exp^{-\lambda_i t}
\end{aligned}
$$

Hence $\tau^i \sim Exp(\lambda_i)$. □

Proof.

Probability jump is from i to j for $j \neq i$ is

$$\begin{aligned}
\lim_{h\to 0} \mathbb{P}(X_{t+h} = j | X_t = i, X_{t+h} \neq i) &= \lim_{h\to 0} \frac{\mathbb{P}(X_{t+h} = j X_t = i)}{\mathbb{P}(X_{t+h} \neq i | X_t = i)} \\
&= \lim_{h\to 0} \frac{\mathbb{P}^{(h)}_{i,j}}{\sum_{k\neq i} \mathbb{P}^{(h)}_{i,k}} \\
&= \frac{\mu_{i,j}}{\sum_{k\neq i} \mu_{i,k}}
\end{aligned}$$

□

Counting Processes

Definition (Counting Process)

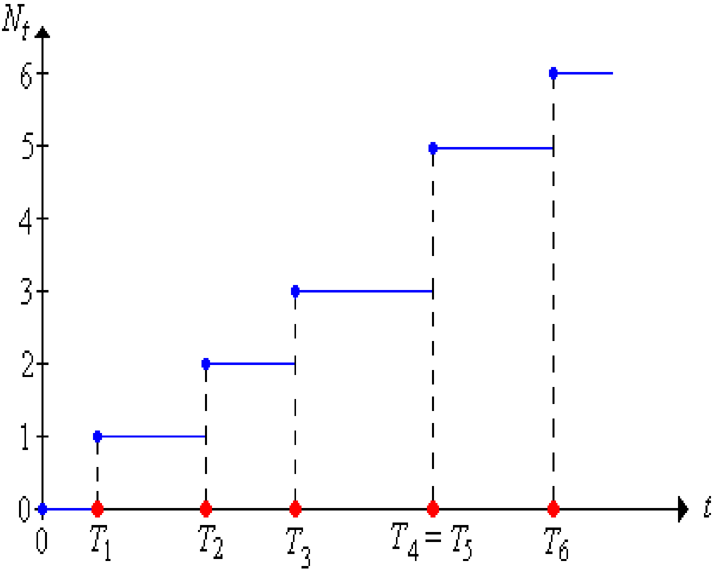
A counting process in continuous time, $\{N_t, t \geq 0\}$, has jumps of size $+1$ only, and whose paths are constant in between two jumps, i.e.

$$N_t = \sum_{k=1}^{\infty} k \mathbb{I}_{[T_k, T_{k+1})}(t) = \sum_{k=1}^{\infty} \mathbb{I}_{[T_k, \infty)}(t)$$

where $(T_k)_{k\geq 1}$ is the increasing family of jump times such that $\lim_{k\to\infty} = +\infty$.

Notice, we can also recover the jump times from the counting process

$$T_k = \inf\{t \in \mathbb{R}_+ : N_t = k\}, k \geq 1.$$


N_t
6
5
4
3
2
1
0
0
T_1
T_2
T_3
$T_4 = T_5$
T_6
t

Poisson Random Variables

Here, we briefly review some properties of the Poisson random variable that we have discussed in the previous chapters. Remember that a discrete random variable X is said to be a Poisson random variable with parameter λ, f its range is $R_X = \{0, 1, 2, 3, ...\}$, with

$$P_X(k) = \begin{cases} \frac{e^{-\mu}\mu^k}{k!} & \text{for } k \in R_X \\ 0 & \text{otherwise} \end{cases}$$

Here are some useful facts that we have seen before:

- its mean and variance are equal to λ
- the sum of independent Poissons is Poisson distributed with its parameter being the sum of the λ's

Poisson Process

Definition

A time homogeneous Poisson Process is a counting process satisfies the following conditions

1. *Independent increments, i.e.* $N_{t_4} - N_{t_3}$ *is independent of* $N_{t_2} - N_{t_1}$ *as long as* $[t_1, t_2)$ *and* $[t_3, t_4)$ *are disjoint time intervals in* $\mathbb{R}_+$.
2. *Stationary increments, i.e.* $N_{t+h} - N_{s+h}$ *has the same distribution as* $N_t - N_s$ *for all* $h > 0$ *and* $0 \leq s \leq t$.

Distribution of N_t

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Theorem

The increment of a time homogeneous Poisson process follows a Poission distribution that

$$N_t - N_s \sim Poisson(\lambda(t-s)) \text{ with the intensity } \lambda = \lim_{h \to 0} \frac{1}{h} \mathbb{P}(N_t = 1).$$

Sum of independent Poisson Process

Given Poisson random variables $N^a \sim Poisson(\lambda_a)$ and $N^a \sim Poisson(\lambda_a)$, their sum

$$N = N^a + N^b \sim Poisson(\lambda_a + \lambda_b).$$

[1] Hence, the natural extension of this result is that independent Poisson Processes is also a Poisson process with intensity equals the sum of the original intensities.

[1] Verify it via MGF.

Weighting Time

Theorem

Let $W_1 = T_1$, and $W_i = T_i - T_{i-1}$ for $i = 2, 3, ...$ denote the sequence of weighting times of a Poisson process with intensity λ. This sequence $\{W_i\}$ are i.i.d exponential random variables with parameter λ.

Proof.

To show they are exponential, we have

$$\mathbb{P}(W_1 \leq t) = \mathbb{P}(N_t > 0) = 1 - \exp^{-\lambda t}$$

$$\mathbb{P}(W_i \leq t) = 1 - \mathbb{P}(N_{T_{i-1}+t} - N_{T_{i-1}} = 0) = 1 - \mathbb{P}(N_t = 0) = 1 - \exp^{-\lambda t}$$

Weighting Time

Proof.

To show independence, we consider

$$
\begin{aligned}
&\mathbb{P}(W_i > s_i | W_1 = s_1, W_2 = s_1, \dots W_{i-1} = s_{i-1}) \\
=&\mathbb{P}(N_{\sum_{j=1}^{i} s_j} = i - 1 | W_1 = s_1, W_2 = s_1, \dots W_{i-1} = s_{i-1}) \\
=&\mathbb{P}(N_{\sum_{j=1}^{i} s_j} - N_{\sum_{j=1}^{i-1} s_j} = 0) \\
=&\exp^{-\lambda s_i} = \mathbb{P}(W_i > s_i).
\end{aligned}
$$

Question

Consider a two state Markov chain with $\mu_{1,1} = \mu_{2,2} = -\lambda$, find the transition matrix $\mathbb{P}(t)$.

This model has a very simple structure, assume $X_0 = 0$, $X_t = 0$ if and only if there is even number of transitions. Since the intensity of transition from 0 to 1 and 1 to 0 are the same, hence the time between each transition is $exp(\lambda)$. This implies a Poisson process of parameter λ.

$$
\begin{aligned}
P_{00}(t) &= P(X(t) = 0 | X(0) = 0) \\
&= P\big(\text{an even number of arrivals in } [0, t]\big) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{2n}}{(2n)!} \\
&= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{2n}}{(2n)!} \\
&= e^{-\lambda t} \left[\frac{e^{\lambda t} + e^{-\lambda t}}{2} \right] = \frac{1}{2} + \frac{1}{2} e^{-2\lambda t}.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
\sinh(x) &= \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \\
\cosh(x) &= \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.
\end{aligned}
$$

Transition Rate of Poisson Process

The Poisson Process is that

$$\mathbb{P}(X_{t+h} = i + 1 | X_t = i) = \lambda h + o(h)$$

$$\mathbb{P}(X_{t+h} = i | X_t = i) = 1 - \lambda h + o(h).$$

Consider $\mathbb{S} = \{0, 1, 2, 3, ...\}$, we have

$$\mathbb{P}_{i,j}^{(h)} = \begin{cases} 1 - \lambda h + o(h) \text{ if } j = i \\ \lambda h + o(h) \text{ if } j = i + 1 \\ 0 \text{ otherwise} \end{cases}$$

and

$$\mu_{i,j} = \begin{cases} -\lambda \text{ if } j = i \\ \lambda \text{ if } j = i + 1 \\ 0 \text{ otherwise} \end{cases}$$

Example

The number of customers arriving at a grocery store can be modeled by a Poisson process with intensity $\lambda = 10$customers per hour. Find the probability that there are 2 customers between 10:00 and 10:20. Find the probability that there are 3 customers between 10:00 and 10:20 and 7 customers between 10:20 and 11.

First $X \sim Pois(\frac{10}{3})$,

$$P(X=2) = \frac{e^{-\frac{10}{3}}\left(\frac{10}{3}\right)^2}{2!} \approx 0.2.$$

Then, we have two non-overlapping intervals,

$$\begin{aligned}
&P\Big(3 \text{ arrivals in } I_1 \quad \text{and} \quad 7 \text{ arrivals in } I_2\Big) \\
=&P\Big(3 \text{ arrivals in } I_1\Big) \cdot P\Big(7 \text{ arrivals in } I_2\Big) \\
=&\frac{e^{-\frac{10}{3}}\left(\frac{10}{3}\right)^3}{3!} \cdot \frac{e^{-\frac{20}{3}}\left(\frac{20}{3}\right)^7}{7!} \\
&\approx 0.0325
\end{aligned}$$

Example

Let N_t be a Poisson process with intensity $\lambda = 2$, and let $X_1, X_2, ..$ be the corresponding interarrival times.

1. *Find the probability that the first arrival occurs after $t > 0.5$*
2. *Given that we have had no arrivals before $t = 1$, find $\mathbb{P}(X_1 > 3)$*
3. *Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$.*
4. *I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. In other words, T is the first arrival after $t = 10$. Find $\mathbb{E}[T]$ and $var(T)$*

1. $X_1 \sim Exp(2)$,

$$P(X_1 > 0.5) = e^{-(2\times 0.5)} \approx 0.37$$

2.

$$\begin{aligned} P(X_1 > 3|X_1 > 1) &= P(X_1 > 2) \text{ (memoryless property)} \\ &= e^{-2\times 2} \approx 0.0183 \end{aligned}$$

3. $X_4 \sim Exp(2)$,

$$\begin{aligned} P(X_4 > 2|X_1 + X_2 + X_3 = 2) &= P(X_4 > 2) \text{ (independence of the } X_i\text{'s)} \\ &= e^{-2\times 2} \approx 0.0183 \end{aligned}$$

4. When I start watching the process at time $t = 10$, I will see a Poisson process. Thus, the time of the first arrival from $t = 10$ is $Exp(2)$. In other words, we can write $T = 10 + X$ with $X \sim Exp(2)$. Thus, $\mathbb{E}[T] = 10.5$ and $Var(T) = 0.25$.