Graduation

– crude estimates of mortality rate or force of mortality progress roughly with age
– based on general experience and intuitive sense, mortality rate or force of mortality should largely be a smooth function of age
– premiums progress smoothly with age to be justifiable to customers
– assume underlying mortality rate or force of mortality does progress smoothly with age
– use graduation techniques to smooth crude estimates to obtain graduated estimates
– need to check whether graduated estimates are close to data or not

Null Hypothesis

- $\hat{q}$ and $\hat{\mu}$ are crude estimates
- $\overset{\circ}{q}$ and $\overset{\circ}{\mu}$ are graduated estimates
- check whether $\overset{\circ}{q}$ and $\overset{\circ}{\mu}$ are close to data
- null hypothesis is that $\overset{\circ}{q}$ and $\overset{\circ}{\mu}$ are true underlying mortality
- if rejected, graduated estimates are not true, adherence to data is poor, and there may be overgraduation
- D_x is total number of deaths
- d_x is observed value
- aged x last birthday
- lives of different ages are independent
- D_x 's are independent

Null Hypothesis

– binomial model :

$$D_x \sim \text{Binomial}\left(E_x \ , \ \overset{\circ}{q}_x \right)$$

$$D_x \sim \text{Normal}\left(E_x \, \overset{\circ}{q}_x \ , \ E_x \, \overset{\circ}{q}_x \left(1 - \overset{\circ}{q}_x \right) \right)$$

if E_x is very large

– Poisson model :

$$D_x \sim \text{Poisson}\left(E_x^C \, \overset{\circ}{\mu}_{x+1/2} \right)$$

$$D_x \sim \text{Normal}\left(E_x^C \, \overset{\circ}{\mu}_{x+1/2} \ , \ E_x^C \, \overset{\circ}{\mu}_{x+1/2} \right)$$

if $E_x^C \, \overset{\circ}{\mu}_{x+1/2}$ is large

Null Hypothesis

standardised deviation (binomial model) :

$$Z_x = \frac{D_x - E_x \overset{\circ}{q}_x}{\sqrt{E_x \overset{\circ}{q}_x \left(1 - \overset{\circ}{q}_x\right)}} \qquad z_x = \frac{d_x - E_x \overset{\circ}{q}_x}{\sqrt{E_x \overset{\circ}{q}_x \left(1 - \overset{\circ}{q}_x\right)}}$$

standardised deviation (Poisson model) :

$$Z_x = \frac{D_x - E_x^C \overset{\circ}{\mu}_{x+1/2}}{\sqrt{E_x^C \overset{\circ}{\mu}_{x+1/2}}} \qquad z_x = \frac{d_x - E_x^C \overset{\circ}{\mu}_{x+1/2}}{\sqrt{E_x^C \overset{\circ}{\mu}_{x+1/2}}}$$

– Z_x ’s are approximately iid N(0,1) random variables under null hypothesis
– test statistics are based on this random variable
– when a certain test statistic exceeds critical value(s), Z_x is not iid standard normal, so null hypothesis is rejected

Chi Square Test

– $\sum_{\text{all ages}} z_x^2$

– if it is larger than $\chi_n^2(0.95)$, it does not look like chi square, Z_x is not standard normal, so null hypothesis is rejected and adherence to data is poor (one-tailed)

– too small a value may indicate undergraduation, but need smoothness test to confirm this

– if $E_x \, \mathring{q}_x$ or $E_x^C \, \mathring{\mu}_{x+1/2}$ is less than 5 for certain ages, we often combine these ages

– *n* is equal to number of age groups minus a number that depends on graduation technique used

Chi Square Test – Pros & Cons

– an overall test for all ages

– does not detect a few large deviations offset by many small deviations
– does not detect deviations being positive or negative
– does not detect any clumping of signs
– need further more specific tests

Standardised Deviations Test

- $\displaystyle\sum_{\text{all intervals}} \frac{(A-E)^2}{E}$
- (–∞,–3), (–3,–2), (–2,–1), (–1,0), (0,1), (1,2), (2,3), (3,∞) for N(0,1) and *m* samples
- expected numbers are 0, 0.02*m*, 0.14*m*, 0.34*m*, 0.34*m*, 0.14*m*, 0.02*m*, 0
- count how many fall in each interval and compare with expected
- *A* is actual number and *E* is expected number
- if it is larger than $\chi_n^2(0.95)$, Z_x is not standard normal, so null hypothesis is rejected and adherence to data is poor (one-tailed)
- if expected number is less than 5 for certain intervals, we often combine these intervals
- *n* is equal to number of intervals minus one

Standardised Deviations Test – Others

- under null hypothesis approximately half should fall in (–2/3,2/3)
- too many in the tails may indicate overgraduation
- too many within central region may indicate undergraduation
- under null hypothesis there should be roughly equal numbers of positive and negative deviations
- too many positive ones indicate graduated estimates are too low compared to data, and vice versa
- can check whether there are a few large ones offset by many small ones
- this is a good test that identifies many potential problems

Cumulative Deviations Test

– $$\frac{\sum_x \left(d_x - E_x \overset{\circ}{q}_x \right)}{\sqrt{\sum_x E_x \overset{\circ}{q}_x \left(1 - \overset{\circ}{q}_x \right)}}$$ (binomial model)

$$\frac{\sum_x \left(d_x - E_x^C \overset{\circ}{\mu}_{x+1/2} \right)}{\sqrt{\sum_x E_x^C \overset{\circ}{\mu}_{x+1/2}}}$$ (Poisson model)

– if it is larger than 1.96 in absolute value, it does not look like N(0,1), so null hypothesis is rejected and adherence to data is poor (two-tailed)
– if it is too large, graduated estimates are too low compared to data, and vice versa
– can be applied to all ages or certain age ranges
– age range chosen based on say financial grounds but not on data

Cumulative Deviations Test – Pros & Cons

– detects long runs of deviations of same sign

– does not detect cumulative positive deviations over an age range offset by cumulative negative ones over another age range

– some graduation techniques force it to become zero and the test is then invalid

Signs Test

- number of positive z_x's
- under null hypothesis there should be roughly equal numbers of positive and negative deviations
- for m samples, number of positive deviations is $\text{Binomial}(m, 1/2)$
- if m is large, use $(2n_1 - m)/\sqrt{m}$
- n_1 is actual number of positive deviations
- if it is larger than 1.96 in absolute value, Z_x is not standard normal, so null hypothesis is rejected and adherence to data is poor (two-tailed)

Signs Test – Pros & Cons

- detects excessive amounts of positive or negative deviations
- too many positive deviations indicate graduated estimates are too low compared to data, and vice versa

- does not reveal extent of discrepancy

Grouping of Signs Test

- number of distinct groups of positive z_x's
- under null hypothesis z_x's are samples of iid N(0,1) and they should change sign at a reasonable frequency
- n_1 is actual number of positive deviations
- n_2 is actual number of negative deviations
- if $n_1 + n_2$ is large, use :

$$\left(g - n_1\left(n_2 + 1\right)/\left(n_1 + n_2\right)\right)\Big/\sqrt{\left(n_1 n_2\right)^2/\left(n_1 + n_2\right)^3}$$

- g is actual number of positive groups
- if it is smaller than –1.64, z_x is not standard normal, so null hypothesis is rejected and adherence to data is poor (one-tailed)

Grouping of Signs Test – Pros & Cons

– detects infrequent sign changes i.e. any clumping of deviations, and if so there may be overgraduation

– conclusion may be different if number of negative groups is used

Comparison with Standard Tables

- compare data with standard table figures
- statistical tests above can be applied here
- null hypothesis is that standard table figures are true underlying mortality
- if null hypothesis is rejected, data differ significantly from standard table experience
- comments under each test apply here except for those about graduation

Smoothness vs Adherence to Data

- usually conflicting requirements
- overgraduation : smooth rates but poor adherence to data
- undergraduation : good adherence to data but rough rates
- a fine balance is required
- when forming national standard life table, more emphasis on adherence to data, because of necessary accuracy
- when calculating premiums, more emphasis on smoothness, so no abrupt changes in premiums over age

Smoothness

– many graduation techniques give smooth results

– whether third differences are small

– whether third differences progress regularly with age

– 3rd order smoothness if :

$$\left| \Delta^3 \overset{\circ}{q}_x \right| \cdot 7^3 < \overset{\circ}{q}_x$$

– this applies similarly to force of mortality

Graduation by Reference to Standard Table

– find a link between crude estimates and standard table figures

– data and standard table have similar characteristics

– plot $\hat{q}_x$ against q_x^S :

a linear relationship may indicate

$$\mathring{q}_x = a + bq_x^S$$

– plot $-\ln(1-\hat{q}_x)$ against $-\ln(1-q_x^S)$:

a linear relationship may indicate

$$\mathring{\mu}_x = a + b\mu_x^S$$

– $\mathring{q}_x = (a+bx)q_x^S$, $\mathring{\mu}_x = \mu_x^S + k$, $\mathring{\mu}_x = \mu_{x+k}^S$

Graduation by Reference to Standard Table

- use weighted least squares to estimate parameters
- e.g. for $\overset{\circ}{q}_x = a + bq_x^S$, minimise :

$$\sum_x w_x \left(\hat{q}_x - \overset{\circ}{q}_x \right)^2 = \sum_x w_x \left(\hat{q}_x - a - bq_x^S \right)^2$$

- $w_x = E_x$ or $w_x = \dfrac{E_x}{\hat{q}_x \left(1 - \hat{q}_x\right)}$
- set first derivative to zero for each parameter or use numerical methods
- maximum likelihood as alternative

Graduation by Reference to Standard Table

- test adherence to data
- for chi square test, we deduct 1 degree of freedom for each parameter
- further deduction for choice of standard table
- may simply use a 25% deduction overall
- standard table figures should be reasonably smooth and so we may not need to test smoothness if number of parameters is small
- useful when data is scarce
- unsuitable when relationship between data and standard table is unclear
- unsuitable when there is no simple link
- unsuitable for a large amount of data where it is unlikely for data and standard table to have close characteristics

Graduation by Mathematical Formula

– fit a parametric curve to crude estimates

– $\overset{\circ}{\mu}_x = Bc^x$

– $\overset{\circ}{\mu}_x = A + Bc^x$

– $\dfrac{\overset{\circ}{q}_x}{1-\overset{\circ}{q}_x} = A + Hx + Bc^x$

– $\ln \dfrac{\overset{\circ}{q}_x}{1-\overset{\circ}{q}_x} = f(x)$

Graduation by Mathematical Formula

– maximise the (approximate) likelihood function :

$$L = \prod_x \binom{E_x}{d_x} \mathring{q}_x^{\,d_x} \left(1 - \mathring{q}_x\right)^{E_x - d_x}$$

$$L = \prod_x \frac{\exp\left(-E_x^C \, \mathring{\mu}_{x+1/2}\right)\left(E_x^C \, \mathring{\mu}_{x+1/2}\right)^{d_x}}{d_x!}$$

– set first derivative to zero for each parameter or use numerical methods

– weighted least squares as alternative

Graduation by Mathematical Formula

- test adherence to data
- for chi square test, we deduct 1 degree of freedom for each parameter
- smoothness is not an issue when number of parameters is small
- useful for producing standard table with a large amount of data
- an automated process
- only subjectivity is choice of formula
- parameter test can be performed
- fit the same formula to different or successive experiences, where changes in parameter values help identify differences or trends
- difficult to employ a single formula to all ages, e.g. infant mortality, accident hump, exponential mortality after middle age
- inappropriate when data is scarce, particularly for very old ages
- combination of formulae or combination of methods

Graduation by Cubic Spline

- fit a curve to crude estimates
- join several cubic functions
- knots refer to certain selected ages
- a cubic spline is formed when third order polynomials join at the knots
- first two derivatives are continuous at each knot
- too many knots result in strong adherence to data
- no graduation is performed if number of knots is equal to number of crude estimates
- too few knots are likely to result in a smooth curve but miss key features

Graduation by Cubic Spline

– n knots are selected at ages

$x_1 < x_2 < \ldots < x_n$

– cubic spline :

$$\mathring{q}_x = a_0 + a_1 x + \sum_{j=1}^{n-2} b_j \Phi_j(x)$$

$$\Phi_j(x) = \phi_j(x) - \frac{x_n - x_j}{x_n - x_{n-1}} \phi_{n-1}(x) + \frac{x_{n-1} - x_j}{x_n - x_{n-1}} \phi_n(x)$$

$$\phi_j(x) = \begin{cases} (x - x_j)^3 & x \geq x_j \\ 0 & \text{otherwise} \end{cases}$$

Graduation by Cubic Spline

for $x < x_1$ $\qquad \mathring{q}_x = a_0 + a_1 x$

for $x_1 \le x < x_2$ $\qquad \mathring{q}_x = a_0 + a_1 x + b_1 (x - x_1)^3$

for $x_2 \le x < x_3$ $\qquad \mathring{q}_x = a_0 + a_1 x + b_1 (x - x_1)^3 + b_2 (x - x_2)^3$

for $x_{n-1} \le x < x_n$

$$\mathring{q}_x = a_0 + a_1 x + \sum_{j=1}^{n-2} b_j (x - x_j)^3 - \frac{1}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j (x_n - x_j)\ (x - x_{n-1})^3$$

$$\mathring{q}_x = a_0 + a_1 x + \sum_{j=1}^{n-1} b_j (x - x_j)^3$$

$$b_{n-1} = -\frac{1}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j (x_n - x_j)$$

for $x \ge x_n$

$$\mathring{q}_x = a_0 + a_1 x + \sum_{j=1}^{n-1} b_j (x - x_j)^3 + \frac{1}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j (x_{n-1} - x_j)\ (x - x_n)^3$$

$$\mathring{q}_x = a_0 + a_1 x + \sum_{j=1}^{n} b_j (x - x_j)^3$$

$$b_n = \frac{1}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j (x_{n-1} - x_j)$$

Graduation by Cubic Spline

for $x \geq x_n$

$$\mathring{q}_x = a_0 + a_1 x + \sum_{j=1}^{n-2} b_j \left(x - x_j\right)^3 - \frac{1}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j \left(x_n - x_j\right) \left(x - x_{n-1}\right)^3$$

$$+ \frac{1}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j \left(x_{n-1} - x_j\right) \left(x - x_n\right)^3$$

– coefficient of x^3 is :

$$\sum_{j=1}^{n-2} b_j - \frac{1}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j \left(x_n - x_j\right) + \frac{1}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j \left(x_{n-1} - x_j\right) = 0$$

– coefficient of x^2 is :

$$-3 \sum_{j=1}^{n-2} b_j x_j + \frac{3x_{n-1}}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j \left(x_n - x_j\right) - \frac{3x_n}{x_n - x_{n-1}} \sum_{j=1}^{n-2} b_j \left(x_{n-1} - x_j\right) = 0$$

– cubic spline is linear for $x < x_1$ and for $x \geq x_n$ but consists of third order polynomials in between

Graduation by Cubic Spline

- place the knots where we expect changes in the shape of the curve
- use weighted least squares to estimate the *n* parameters a_0, a_1, b_1, b_2, … , b_{n-2}
- test adherence to data
- for chi square test, we deduct 1 degree of freedom for each knot
- further deduction if knot positions are selected by reference to data
- test smoothness, especially when there are many knots
- this method is flexible as number and positions of knots can be varied
- requires skills to choose knots
- similar for force of mortality

Other Considerations

- compare graduated estimates with prior experiences
- male mortality is higher than female mortality?
- mortality of lives holding life insurance policies is lower than that of the population as a whole?
- mortality of lives who have recently bought life insurance policies is lower than that of lives who bought insurance a long time ago?
- graduated estimates may need further adjustments
- for life insurance contracts, cannot underestimate mortality
- for annuity contracts, cannot overestimate mortality
- may need to model potential mortality improvement over time