Week 3 Tutorial Solutions

2021

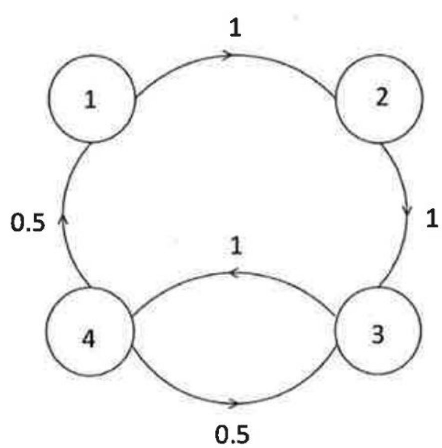
1. Mike starts with 0% discount in 2015. He makes no claims in 2015 or 2016, and so has a 40% discount in 2017. He makes exactly one claim that year so he falls back to 20% discount for 2018. So we are looking for the probability of being at the 20% discount level in 2020, given that in 2018 the discount was also 20%.

   The transition matrix is:

$$P = \begin{pmatrix} 0.1 & 0.9 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0 \\ 0.025 & 0.075 & 0 & 0.9 \\ 0.025 & 0 & 0.075 & 0.9 \end{pmatrix}$$

   We want the (20%,20%) entry (or (2,2)th entry) in p^2 . Multiplying the 2nd row of P by the 2nd column gives: $0.l \times 0.9 + 0.9 \times 0.075 = 0.1575$

2. The transition graph for this Markov chain is shown below:



   We can see from this that the chain is irreducible, so all the states have the same period.

We only need to find the period of one of the states. The chain can return to state 1 having started in state 1 after 4, 6, 8, 10, ... moves. The highest common factor of these numbers is 2, so the period of all the states in the chain is 2.

3. (a) **Markov property** The Markov property means a lack of dependence on the past of the process:

$$P[X_n = j | X_0 = i_0, X_1 = i_1, \ldots, X_{m-1} = i_{m-1}, X_m = i] = P[X_n = j | X_m = i]$$

   for all integer times $n > m$ and states $i_0, i_1, \cdots, i_{m-1}, i, j$ in S.

   (b) **Probabilities**

   i.
$$P[X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = q_{i_0} p^{(0,1)}_{i_0 i_1} p^{(1,2)}_{i_1 i_2} \cdots p^{(n-1,n)}_{i_{n-1} i_n}$$

   This is the probability of the process taking a unique given path.

   ii.
$$P[X_4 = i] = \sum_{i_0 \in S} \sum_{i_1 \in S} \sum_{i_2 \in S} \sum_{i_3 \in S} q_{i_0} p^{(0,1)}_{i_0 i_1} p^{(1,2)}_{i_1 i_2} p^{(2,3)}_{i_2 i_3} p^{(3,4)}_{i_3 i_4}$$

   Here we need to sum over all the possible starting points and then over all paths from these starting points to end up in state i at time 4.

4. (a) **Probabilities**

   i. $\alpha^5(1-\alpha)$

   ii. $(1-\beta)\beta + \beta\alpha = \beta(\alpha - \beta + 1)$

   (b) **Explanation**

   Because the student's performance depends only upon whether he passed or failed the last exam, we can think of the problem as a Markov chain on the state space F,P representing 'failed the last exam' and 'passed the last exam' respectively. The transition matrix is:

$$\begin{matrix} 1-\beta & \beta \\ 1-\alpha & \alpha \end{matrix}$$

   In (a)(i) we are considering the probability of the unique path:

$$P \to P \to P \to P \to P \to P \to F$$

   given that we start in P.
   In (b)(i) we are considering the probabilities of the paths:

$$F \to F \to P \text{ and } F \to P \to P$$

where the F at the start of these sequences represents the event that he fails the third exam.

Alternatively, we could view this as the transition probability $p_{FP}^{(2)}$, which is the FP entry (ie (1,2)th entry) in the matrix:

$$\begin{pmatrix} 1-\beta & \beta \\ 1-\alpha & \alpha \end{pmatrix}^2 = \begin{pmatrix} (1-\beta)^2 + \beta(1-\alpha) & (1-\beta)\beta + \beta\alpha \\ (1-\alpha)(1-\beta) + (1-\alpha)\alpha & (1-\alpha)\beta + \alpha^2 \end{pmatrix}$$

5. (a) **Probabilities**

i.

$$\begin{aligned} P(X_5 = 1{,}001) &= P(Z_5 + Z_4 = 1{,}001) \\ &= P(Z_5 = 1, OOO, Z_4 = 1) + P(Z_5 = 1, Z_4 = 1{,}000) \\ &= qp + pq = 2pq \end{aligned}$$

ii. Using the result in (a)(i) to evaluate the denominator:

$$\begin{aligned} P(X_5 = 1{,}001 | X_4 = 1{,}001) &= \frac{P(X_5 = 1{,}001, X_4 = 1{,}001)}{P(X_4 = 1{,}001)} \\ &= \frac{P(Z_5 + Z_4 = 1{,}001, Z_4 + Z_3 = 1{,}001)}{2pq} \\ &= \frac{P(Z_5 = 1{,}000, Z_4 = 1, Z_3 = 1{,}000)}{2pq} \\ &\quad + \frac{P(Z_5 = 1, Z_4 = 1{,}000, Z_3 = 1)}{2pq} \\ &= \frac{q^2 p + p^2 q}{2pq} \\ &= \frac{pq(q+p)}{2pq} \\ &= \frac{1}{2} \quad \text{because } p + q = 1 \end{aligned}$$

iii. Using the expression for the numerator in (b)(i) to evaluate the denominator:

$$
\begin{aligned}
&P(X_5 = 1,001 | X_4 = 1,001, X_3 = 1,001) \\
&\quad = \frac{P(X_5 = 1,001, X_4 = 1,001, X_3 = 1,001)}{P(X_4 = 1,001, X_3 = 1,001)} \\
&\quad = \frac{P(Z_5 + Z_4 = 1,001, Z_4 + Z_3 = 1,001, Z_3 + Z_2 = 1,001)}{pq(p+q)} \\
&\quad = \frac{P(Z_5 = Z_3 = 1,000, Z_4 = Z_2 = 1)}{pq(p+q)} + \frac{P(Z_5 = Z_3 = 1, Z_4 = Z_2 = 1,000)}{pq(p+q)} \\
&\quad = \frac{2p^2q^2}{pq(p+q)} \\
&\quad = \frac{2pq}{(p+q)} \\
&\quad = 2pq \quad \text{because } p+q=1
\end{aligned}
$$

(b) **Markov?** If $\{X_t\}$ had the Markov property, the probabilities in (b)(i) and (c)(i) would be the same. Since they are not, it doesn't. (Note that $2pq < \frac{1}{2}$ when $q < p$.)

6. We are solving:

$$
\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\
0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \\
\frac{1}{2} & \frac{3}{10} & \frac{1}{5} & 0 & 0
\end{pmatrix}
= \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix}
$$

This gives five equations:

$$
\begin{aligned}
\frac{1}{2}\pi_5 &= \pi_1 \\
\frac{1}{5}\pi_2 + \frac{4}{5}\pi_4 + \frac{3}{10}\pi_5 &= \pi_2 \\
\frac{4}{5}\pi_2 + \frac{1}{5}\pi_4 + \frac{1}{5}\pi_5 &= \pi_3 \\
\frac{1}{2}\pi_1 + \frac{1}{3}{}_{-}pi_3 &= \pi_4 \\
\frac{1}{2}\pi_1 + \frac{2}{3}{}_{-}pi_3 &= \pi_5
\end{aligned}
$$

Rearranging we obtain:

$$
\begin{aligned}
-\pi_1 + \frac{1}{2}\pi_5 &= 0 \\
-\frac{4}{5}\pi_2 + \frac{4}{5}\pi_4 + \frac{3}{10}\pi_5 &= 0 \\
\frac{4}{5}\pi_2 - \pi_3 + \frac{1}{5}\pi_4 + \frac{1}{5}\pi_5 &= 0 \\
\frac{1}{2}\pi_1 + \frac{1}{3}\pi_3 - \pi_4 &= 0 \\
\frac{1}{2}\pi_1 + \frac{2}{3}\pi_3 - \pi_5 &= 0
\end{aligned}
$$

We will ignore the third equation since one equation is always redundant. So we are trying to solve:

$$
\begin{aligned}
-\pi_1 + \frac{1}{2}\pi_5 &= 0 \\
-\frac{4}{5}\pi_2 + \frac{4}{5}\pi_4 + \frac{3}{10}\pi_5 &= 0 \\
\frac{1}{2}\pi_1 + \frac{1}{3}\pi_3 - \pi_4 &= 0 \\
\frac{1}{2}\pi_1 + \frac{2}{3}\pi_3 - \pi_5 &= 0
\end{aligned}
$$

We will choose π_1 as the working variable. From the first equation we have $\pi_5 = 2\pi_1$. Substituting this in the fourth equation gives $\pi_3 = \frac{9}{4}\pi_1$. Using the third we can then obtain $\pi_4 = \frac{1}{2}\pi_1 + \frac{1}{3} \times \frac{9}{4}\pi_1 = \frac{5}{4}\pi_1$.

Finally from the second equation we see that:

$\pi_2 = \frac{5}{4}\left(\frac{4}{5} \times \frac{5}{4}\pi_1 + \frac{3}{10} \times 2\pi_1\right) = 2\pi_1$

Thus our solution in terms of π_1 is $(1, 2, \frac{9}{4}, \frac{5}{4}, 2)\pi_1$. Now apply the condition of summing to 1 to get:

$$\pi_1 = \frac{1}{(1 + 2 + \frac{9}{4} + \frac{5}{4} + 2)} = \frac{2}{17}$$

and therefore the stationary distribution is $\left(\frac{2}{17}, \frac{4}{17}, \frac{9}{34}, \frac{5}{34}, \frac{4}{17}\right)$

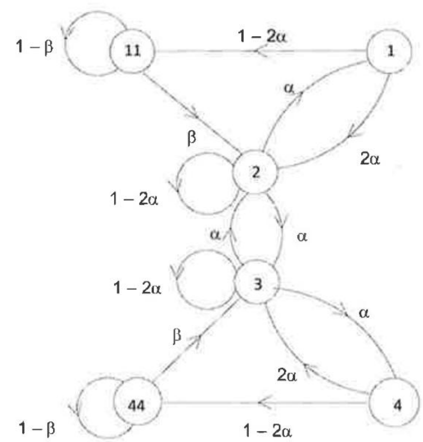
This chain has a finite state space and is irreducible, so it has a unique stationary distribution.

7. (a) **Markov chain** We need to subdivide the top and bottom quartiles in order to satisfy the Markov property. This results in the following 6 states:

   State 11: Funds in the 1st quartile this year and last year

State 1: Funds in the 1st quartile this year but not last year
State 2: Funds in the 2nd quartile this year
State 3: Funds in the 3rd quartile this year
State 4: Funds in the 4th quartile this year but not last year
State 44: Funds in the 4th quartile this year and last year

The labels for the states need not match the ones given here.
The transition diagram then looks like this:



(b) **Transition matrix**

The transition matrix is:

$$
\begin{array}{cccccc} 11 & 1 & 2 & 3 & 4 & 44 \end{array}
$$

$$
P = \begin{array}{c} 11\\ 1\\ 2\\ 3\\ 4\\ 44 \end{array} \begin{pmatrix} 1-\beta & & \beta & & & \\ 1-2\alpha & & 2\alpha & & & \\ & \alpha & 1-2\alpha & \alpha & & \\ & & & 2\alpha & & 1-2\alpha \\ & & & \beta & & 1-\beta \end{pmatrix}
$$

(c) **Irreducible and periodic?**

This chain is irreducible since it is possible to move from each state to any other, eg by following the route $\cdots \to 11 \to 2 \to 3 \to 4 \to 44 \to 3 \to 2 \to 1 \to 11 \to \cdots$.

A periodic chain is one in which a state can only be revisited at multiples of some fixed number $d > 1$. State 11 is aperiodic as it can be revisited after any number of steps. Also, since this chain is irreducible, all the states have the same periodicity. So the chain is aperiodic.

(d) **Proof**

If a stationary distribution exists with a quarter of the funds in each quartile, then the stationary probabilities π_i; must satisfy:

$$\pi_1 11 + \pi_1 = \pi_2 = \pi_3 = \pi_4 + \pi_{44} = \frac{1}{4}$$

The stationary probabilities also satisfy the matrix equation $\pi = \pi P$
The first column of this matrix equation tells us that:

$$(1-\beta)\pi_1 11 + (1-2\alpha)\pi_1 = \pi_1 1 \Rightarrow -\beta\pi_{11} + (1-2\alpha)\pi_1 = 0 \Rightarrow \pi_{11} = \frac{(1-2\alpha)}{\beta}\pi_1$$

But we want $\pi_{11} + \pi_1 = \frac{1}{4}$. So:

$$\frac{(1-2\alpha)}{\beta}\pi_1 + \pi_1 = \frac{1}{4} \quad ie \quad \left(1 + \frac{1-2\alpha}{\beta}\pi_1\right) = \frac{1}{4}$$

The second column of this matrix equation tells us that:

$$\alpha\pi_2 = \pi_1 \quad ie \quad \alpha \times \frac{1}{4} = \pi_1$$

Combining these two equations gives:

$$\left(1 + \frac{1-2\alpha}{\beta}\right)\alpha \times \frac{1}{4} = \frac{1}{4} \Rightarrow \left(1 + \frac{1-2\alpha}{\beta}\right)\alpha = 1 \Rightarrow \beta = \frac{\alpha(1-2\alpha)}{1-\alpha}$$

(e) **Estimated probability**

The probability of a fund in the second quartile moving up to the top quartile is α . So we estimate $\hat{\alpha} = 0.2$. Hence the probability of the fund remaining in the top quartile for a third consecutive year is estimated to be:

$$1 - \hat{\beta} = 1 - \frac{\hat{\alpha}(1-2\hat{\alpha})}{1-\hat{\alpha}} = 1 - \frac{0.2 \times 0.6}{0.8} = 0.85$$

8. (a) **Transition matrix**

The one-step transition matrix is:

$$P = \frac{1}{10}\begin{pmatrix} 3 & 7 & 0 & 0 \\ 3 & 0 & 7 & 0 \\ 1 & 2 & 0 & 7 \\ 1 & 0 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.1 & 0 & 0.2 & 0.7 \end{pmatrix}$$

(b) **Two-step transition probabilities**

We use the fact that $p_{ij}^{(2)} = \left(P^2\right)_{ij}$

$$P^2 = \frac{1}{100}\begin{pmatrix} 3 & 7 & 0 & 0 \\ 3 & 0 & 7 & 0 \\ 1 & 2 & 0 & 7 \\ 1 & 0 & 2 & 7 \end{pmatrix}\begin{pmatrix} 3 & 7 & 0 & 0 \\ 3 & 0 & 7 & 0 \\ 1 & 2 & 0 & 7 \\ 1 & 0 & 2 & 7 \end{pmatrix} = \frac{1}{100}\begin{pmatrix} 30 & 21 & 49 & 0 \\ 16 & 35 & 0 & 49 \\ 16 & 7 & 28 & 49 \\ 12 & 11 & 14 & 63 \end{pmatrix}$$

(c) **Probability of being at maximum discount in 5 years**

We shall represent the states 0%, 20%, 40% and 60% by 0,1,2 and 3 respectively. In order to calculate $p_{0,3}^5$ we can use $\left(P^5\right)_{0,3} = \sum_{k=0}^{3}\left(P^2\right)_{0,k}\left(P^3\right)_{k,3}$. So we can first calculate the fourth column of P^3 :

$$P^3 = \frac{1}{1{,}000}\begin{pmatrix} 30 & 21 & 49 & 0 \\ 16 & 35 & 0 & 49 \\ 16 & 7 & 28 & 49 \\ 12 & 11 & 14 & 63 \end{pmatrix}\begin{pmatrix} 3 & 7 & 0 & 0 \\ 3 & 0 & 7 & 0 \\ 1 & 2 & 0 & 7 \\ 1 & 0 & 2 & 7 \end{pmatrix} = \frac{1}{1{,}000}\begin{pmatrix} * & * & * & 343 \\ * & * & * & 343 \\ * & * & * & 539 \\ * & * & * & 539 \end{pmatrix}$$

Now we have:

$$\begin{aligned}\left(P^5\right)_{0,3} = \sum_{k=0}^{3}\left(P^2\right)_{0,k}\left(P^3\right)_{k,3} &= \frac{1}{100{,}000}(30\times 343 + 21\times 343 + 49\times 539) \\ &= \frac{43{,}904}{100{,}000} = 0.43904\end{aligned}$$

(d) **Long-term proportions on each discount level**

This is equivalent to finding the stationary distribution, ie solving the matrix equation. :

This matrix equation is equivalent to the simultaneous equations:

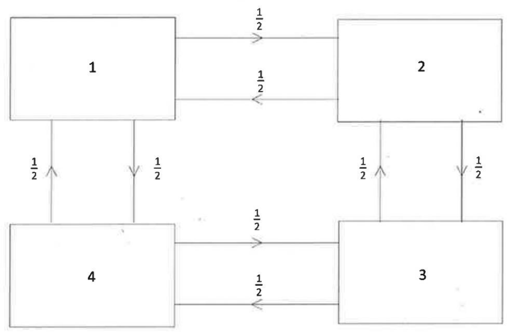
$$\begin{pmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3\end{pmatrix}\begin{pmatrix} 3 & 7 & 0 & 0 \\ 3 & 0 & 7 & 0 \\ 1 & 2 & 0 & 7 \\ 1 & 0 & 2 & 7 \end{pmatrix} = 10\begin{pmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3\end{pmatrix}$$

So we have the stationary distribution $(118, 112, 147, 343)\frac{\pi_3}{343}$. Since the probabilities must sum to 1, the stationary distribution is:

$$\frac{1}{720}(118, 112, 147, 343) = (0.1639, 0.1556, 0.2042, 0.4764)$$

9. **Chain I**

(a) **Transition diagram**



(b) **Probabilities**

The initial distribution is (1,0,0,0). Repeated postmultiplication of this vector by the transition matrix for Chain I gives: $(1,0,0,0) \rightarrow (0, \frac{1}{2}, 0, \frac{1}{2}) \rightarrow (\frac{1}{2}, 0, \frac{1}{2}, 0) \rightarrow (0, \frac{1}{2}, 0, \frac{1}{2})(\frac{1}{2}, 0, \frac{1}{2}, 0) \rightarrow \cdots$
So:

i. $P(X_2 = 1|X_0 = 1) = \frac{1}{2}$
ii. $P(X_4 = 1|X_0 = 1) = \frac{1}{2}$

Alternatively, because we are only asked about particular probabilities, we could evaluate all the possible paths corresponding to each event and add their probabilities.

$P(X_2 = 1 | X_0 = 1)$

Time	0	1	2	Probability
Path	1	2	1	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
Path	1	4	1	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(X_4 = 1 | X_0 = 1)$

Time	0	1	2	3	4	Probability
Path	1	2	1	2	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	2	3	2	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	4	1	4	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	4	3	4	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	2	1	4	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	4	1	2	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	2	3	4	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	4	3	2	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

(c) **Irreducible and/or aperiodic?**

Chain I is irreducible since every state can be reached from every other state.

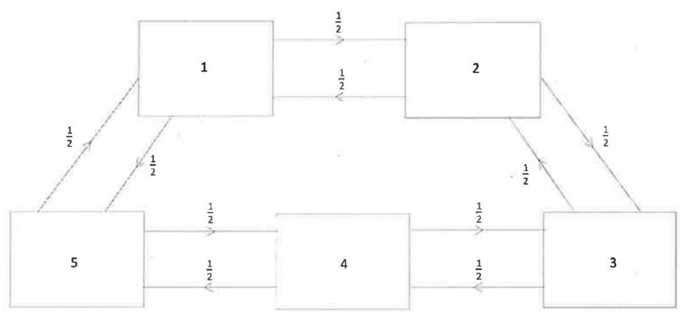
Because the chain is irreducible every state will have the same period. It is possible to return to state 1 in 2, 4, 6, 8 ... moves. State 1 has a period of 2 and so every state has a period of 2. The chain is not aperiodic.

(d) **Will the process converge to a stationary distribution?**

The process has a finite number of states and is irreducible, so it has a unique stationary distribution, but this process will not converge to its stationary distribution. In the solution to part (ii), we saw that the distribution will alternate between $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(0, \frac{1}{2}, 0, \frac{1}{2})$. [1]

Chain II

(a) **Transition diagram**



(ii) Probabilities

The initial distribution is (1,0,0, 0, 0). Repeated postmultiplication of this ve tor by the transition matrix for Chain II gives:

$$(1,0,0,0,0) \rightarrow \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}\right) \rightarrow \left(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}, 0\right) \rightarrow \left(0, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}\right) \rightarrow \left(\frac{3}{8}, \ldots, \ldots, \ldots, \ldots\right)$$

So:

i. $P(X_2 = 1|X_0 = 1) = \frac{1}{2}$

ii. $P(X_4 = 1|X_0 = 1) = \frac{3}{8}$

Alternatively, because we are only asked about particular prababilities, we could evaluate all the possible paths corresponding to each event and add their probabilities.

$P(X_2 = 1 | X_0 = 1)$

Time	0	1	2	Probability
Path	1	2	1	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
Path	1	5	1	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(X_4 = 1 | X_0 = 1)$

Time	0	1	2	3	4	Probability
Path	1	2	1	2	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	2	3	2	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	5	1	5	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	2	1	5	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	5	1	2	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
Path	1	5	4	5	1	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

(b) **Irreducible and/or aperiodic?**

Chain II is irreducible since every state can be reached from every other state.

Because the chain is irreducible every state will have the same period. It is possible to return to state 1 in 2, 4, 5, 6, 7, 8 ... moves. State 1 has a period of 1 (it is aperiodic) and so every state is aperiodic. The chain is aperiodic.

(c) **Will the process converge to a stationary distribution?**

Yes. The chain has a finite number of states, is irreducible and is aperiodic. So there will be a unique stationary distribution that the process will conform to in the long term. By symmetry, this stationary distribution is $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$.

10. (a) **Markov chain**

The process has the Markov property since the probability of moving on to the next chapter does not depend on the number of chapters currently written (so it is not dependent on the past history of the process).

In fact, we have:

$$X_k = \begin{cases} X_{k-1} + 1 & \text{with probability } 0.75 \\ X_{k-l} & \text{with probability } 0.25 \end{cases}$$

for $X_{k-1} \neq 20$ and $P(X_k = 20 | X_{k-1} = 20) = 1$

X_t has a discrete state space, namely 0,1,2,...,20, and a discrete time set since the value of the process is recorded at the end of each week. So the process is a Markov chain.

(b) **Probability**

To calculate the probability that the book is finished in exactly 25 weeks, we need the probability that the last chapter is completed in the 25th week and, in the first 24 weeks there were 5 chapters rewritten. So the probability is:

$$\begin{pmatrix} 24 \\ 5 \end{pmatrix} 0.25^5 \times 0.75^{19} \times 0.75 = 0.13163$$

(c) **Expected number of weeks until completion**

Let m_k be the expected time until the book is finished, given that there are currently k chapters completed. Then, for k = 0,1,...,19:

$$m_k = 1 + 0.75m_{k+1} + 0.25m_k$$

That is, in one week's time, there is a 75% chance of having $k+1$ completed chapters and a 25% chance of still having k completed chapters.

Rearranging this equation, we get:

$$0.75m_k = 1 + 0.75m_{k+l}$$

or:

$$m_k = \frac{1}{0.75} + m_{k+1}$$

Since $m_20 = 0$, we have:

$$m_{19} = \frac{1}{0.75}$$

$$m_{18} = \frac{1}{0.75} + \frac{1}{0.75} = \frac{2}{0.75}$$

$$m_{17} = \frac{1}{0.75} + \frac{2}{0.75} = \frac{3}{0.75}$$

and so on. In general, we have:

$$m_k = \frac{20-k}{0.75}$$

So the expected time until the book is completed is:

$$m_0 = \frac{20}{0.75} = 26.67 \text{ weeks}$$

Alternatively, let N denote the number of weeks it takes to complete the book. The possible values of N are 20, 21, 22, ... and:

$$P(N=20)=0.75^{20}$$

$$P(N=21)=\binom{20}{1}0.25\times 0.75^{20}=\binom{20}{19}0.25\times 0.75^{20}$$

$$P(N=22)=\binom{21}{2}0.25^2\times 0.75^{20}=\binom{21}{19}0.25^2\times 0.75^{20}$$

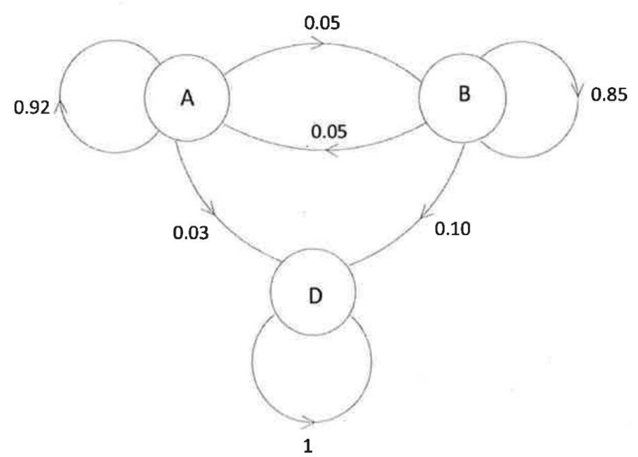
and soon. So N hasaTypelnegativebinomialdistributionwith k=20 and p=0.75. Hence:

$$E(N)=\frac{k}{p}=\frac{20}{0.75}=26.67 \text{ weeks}$$

11. This question is Subject CT4, September 2006, Question A4.

(a) **Probability of never being rated Bin the future**

We have the following transition diagram:



A company that is never rated Bin the future will:
(a) remain in State A for some period of time, and
(b) then move to State D and remain there.
So we can sum over all future times at which the single transition from State A to State D can take place. This gives us the following expression:

$$0.03+0.92\times 0.03+(0.92)^2\times 0.03+(0.92)^3\times 0.03+\cdots$$

This is an infinite geometric progression, whose sum is:

$$\frac{0.03}{1-0.92}=0.375$$

So the probability that a company is never rated Bin the future is 0.375.

(b) i. **Second-order transition probabilities**

The second-order transition probabilities are given by:

$$P^2 = \begin{pmatrix} 0.92 & 0.05 & 0.03 \\ 0.05 & 0.85 & 0.1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.92 & 0.05 & 0.03 \\ 0.05 & 0.85 & 0.1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.8489 & 0.0885 & 0.0626 \\ 0.0885 & 0.7250 & 0.1865 \\ 0 & 0 & 1 \end{pmatrix}$$

ii. **Expected number of defaults**

The probability that a company rated A at time zero is in State D at time 2 is 0.0626. So the expected number of companies in this state out of 100 is 6.26.

(c) **Expected number of defaults**

For this manager we use the original matric P. After one year, the expected number of companies in each state will be:

$$\begin{pmatrix} 100 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.92 & 0.05 & 0.03 \\ 0.05 & 0.85 & 0.1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 92 & 5 & 3 \end{pmatrix}$$

If the five state B's are moved to State A and the process repeated, we have:

$$\begin{pmatrix} 97 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0.92 & 0.05 & 0.03 \\ 0.05 & 0.85 & 0.1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 89.24 & 4.85 & 5.91 \end{pmatrix}$$

So the expected number of defaults by the end of the second year under this arrangement is 5.91.

(d) **Comment**

The downgrade trigger strategy will reduce the expected number of defaults, as we have seen. However, the return on the portfolio will also be a function of the yields on the debt. Companies rated Bare likely to have bonds with a higher yield (because of the higher risk), so excluding these may in fact reduce the yield on the portfolio.

Also, the actual number of defaults may not match the expected number. The return depends on the actual progress of the portfolio, rather than the expected outcome. There will also be a cost incurred when buying and selling bonds.