Solve the wave equation using the finite difference scheme

Consider the wave equation

$$\frac{\partial^2 \phi(x)}{\partial x^2} + (k^2 n(x)^2 - \beta^2)\phi(x) = 0$$

where the refractive index distribution $n(x)$ is given, we want to solve for $\phi(x)$ and its corresponding propagation constant β.

First, we discretize the equation using the finite difference scheme, assuming the x axis is discretized into N points $x_1, x_2, \ldots x_n, \ldots, x_N$ with equal spacing Δx between adjacent points. The first-order derivative of f with respect to x at $x_{n+1/2}=(x_n+x_{n+1})/2$ can be approximated by

$$\left.\frac{\partial \phi}{\partial x}\right|_{x_{n+1/2}} = \frac{\phi(x_{n+1}) - \phi(x_n)}{\Delta x} = \frac{\phi_{n+1} - \phi_n}{\Delta x}$$

The second-order derivative of ϕ with respect to x at x_n can thus be approximated by

$$\begin{aligned} \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x_n} &= \frac{\partial\phi/\partial x|_{x_{n+1/2}} - \partial\phi/\partial x|_{x_{n-1/2}}}{\Delta x} \\ &= \frac{\frac{\phi_{n+1}-\phi_n}{\Delta x} - \frac{\phi_n - \phi_{n-1}}{\Delta x}}{\Delta x} = \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{(\Delta x)^2} \end{aligned}$$

With the finite difference expression for the second-order derivative, the wave equation at x_n can be written as

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{(\Delta x)^2} + (k^2 n_n^2 - \beta^2)\phi_n = 0$$

where $n_n=n(x_n)$ is the refractive index at x_n. The above equation can be rearranged into the following form

$$\frac{1}{(\Delta x)^2}\left\{\phi_{n+1} + [k^2 n_n^2 (\Delta x)^2 - 2]\phi_n + \phi_{n-1}\right\} = \beta^2 \phi_n$$

Define $\mu_n = k^2 n_n^2 (\Delta_x)^2 - 2$, we have for each point x_n

$$\frac{1}{(\Delta x)^2}(\phi_{n+1} + \mu_n \phi_n + \phi_{n-1}) = \beta^2 \phi_n$$

So, the original differential equation now results in N coupled linear equations with N unknowns ϕ_1, ϕ_2,…,ϕ_N. The system of equations can be written in matrix form as

$$\mathbf{MV} = \lambda \mathbf{V}$$

where

$$\mathbf{M} = \frac{1}{(\Delta x)^2}\begin{bmatrix} \mu_1 & 1 & 0 & \cdots & 0 \\ 1 & \mu_2 & 1 & \ddots & \vdots \\ 0 & 1 & \mu_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & \mu_N \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_N \end{bmatrix} \quad \lambda = \beta^2$$

So, the solution $\phi(x)=\mathbf{V}$ is an eigenvector of **M**, and λ is the corresponding eigenvalue. By constructing the matrix **M** from known waveguide parameters, the solution of the wave equation can be obtained by solving for the eigenvectors and eigenvalues of **M**.