Probability

3.1 Discrete Random Variables Basics

Alex Tsun

Agenda

- Intro to Discrete Random Variables
- Probability Mass Functions
- Expectation

Flipping two coins

Suppose you flip a fair coin twice. What is the sample space Ω?

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Usually, we don't care about the exact outcome (HT vs TH), but just the fact we got exactly one head. So we define a **random variable** as a *numeric* function of the outcome. What does this mean?

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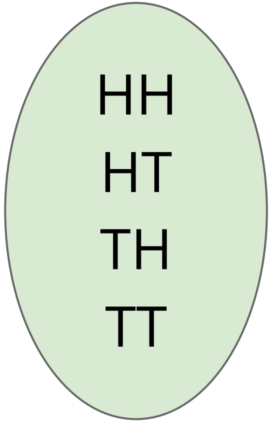

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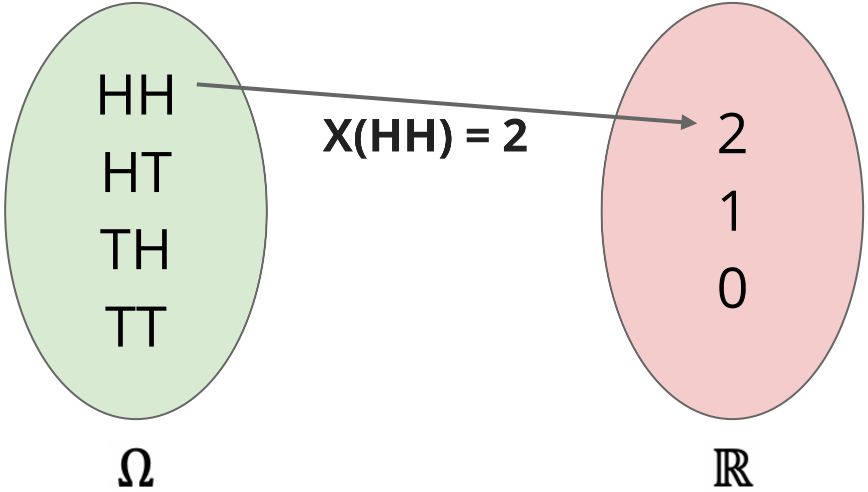

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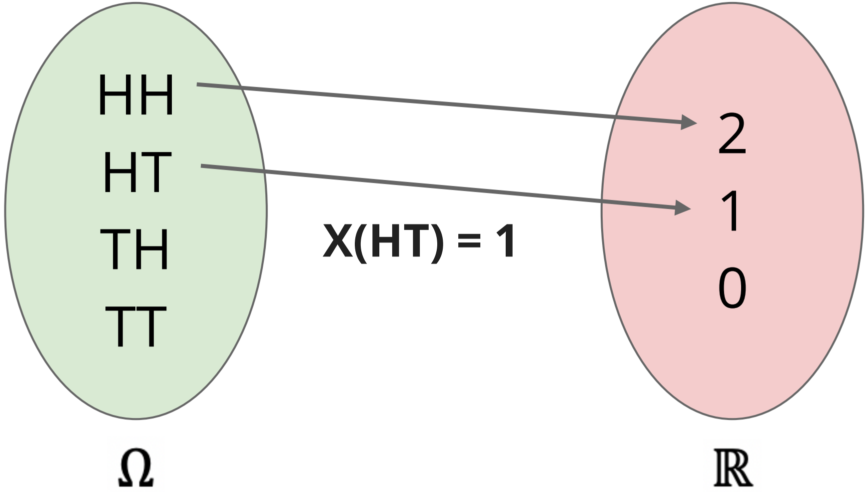

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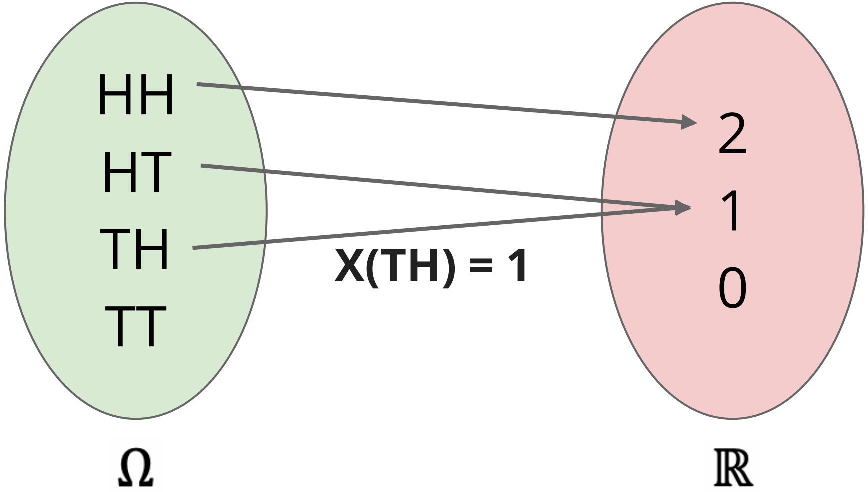

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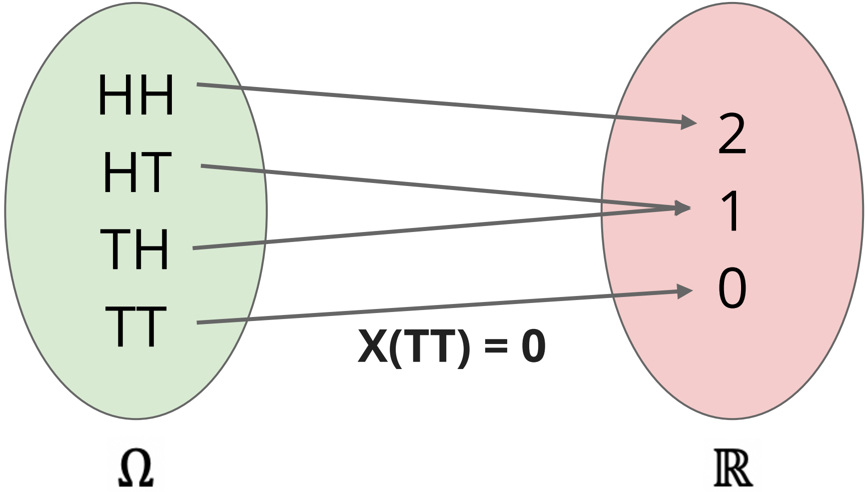

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If Ω_X is finite or countably infinite (typically integers or a subset), X is a **discrete** random variable (drv). Else if Ω_X is uncountably large (the size of real numbers), X is **continuous** random variable.

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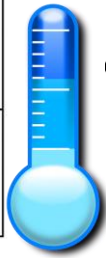

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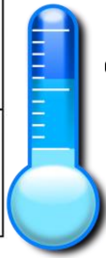

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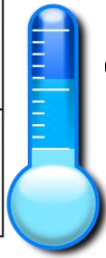

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$$\sum_{z \in \Omega_X} p_X(z) = 1$$

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Expectation

The **<u>expectation/expected value/average</u>** of a discrete random variable X is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

Or equivalently,

$$E[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$$

The interpretation is that we take an average of the values in Ω_X, but weighted by their probabilities.

Probability

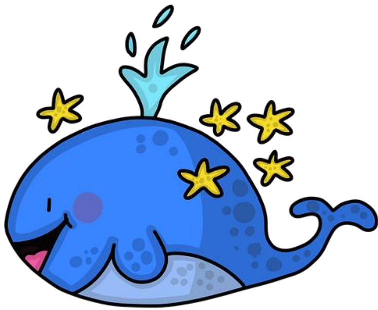
3.2 More on Expectation

Alex Tsun

Agenda

- Linearity of Expectation (LoE)
- Law of the Unconscious Statistician (Lotus)

Linearity of Expectation (Idea)



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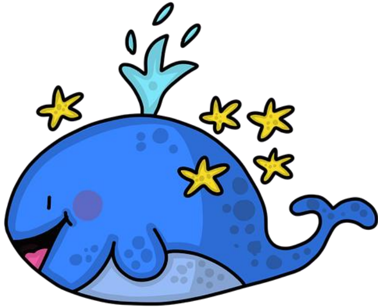
Linearity of Expectation (Idea)



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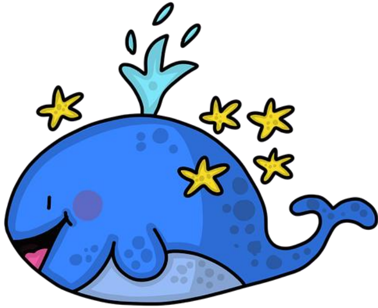

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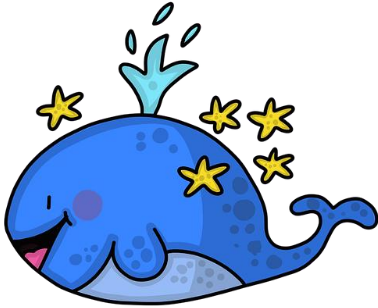

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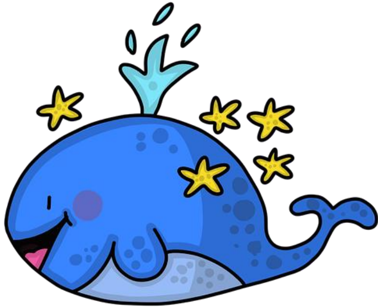

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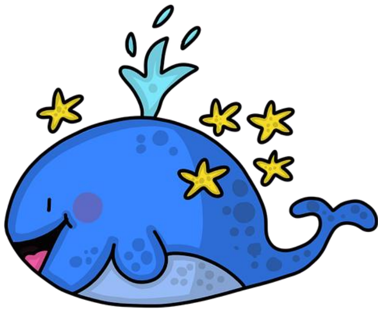
$E[Z] = E[X + Y] =$ $3 + 7 = 10$

Linearity of Expectation (Idea)



Let's say you and your friend sell fish for a living.

- Every day you catch X fish, with $E[X] = 3$.
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How many fish do the two of you bring in ($Z = X + Y$) on an average day?

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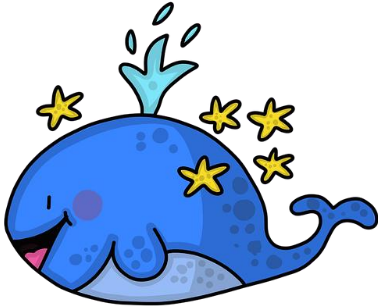

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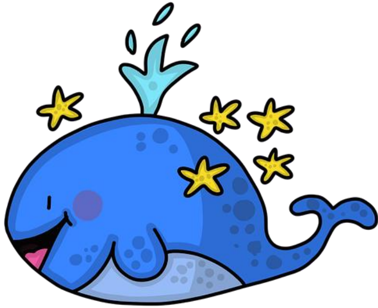

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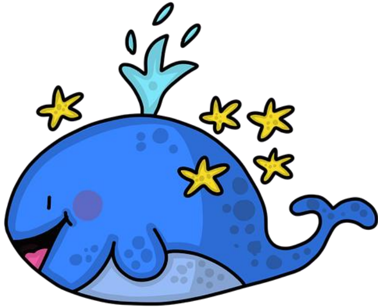

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Linearity of Expectation (LoE)

Linearity of Expectation: Let Ω be the sample space of an experiment, $X, Y: \Omega \to \mathbb{R}$ be (possibly "dependent") random variables both defined on Ω, and $a, b, c \in \mathbb{R}$ be scalars. Then,

$$E[X + Y] = E[X] + E[Y]$$

and

$$E[aX + b] = aE[X] + b$$

Combining them gives,

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Linearity of Expectation (Proof)

$$E[X] + E[Y]$$

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Frogger!

A frog starts on a 1-dimensional number line at 0. At each time step, it moves

- left with probability p_L
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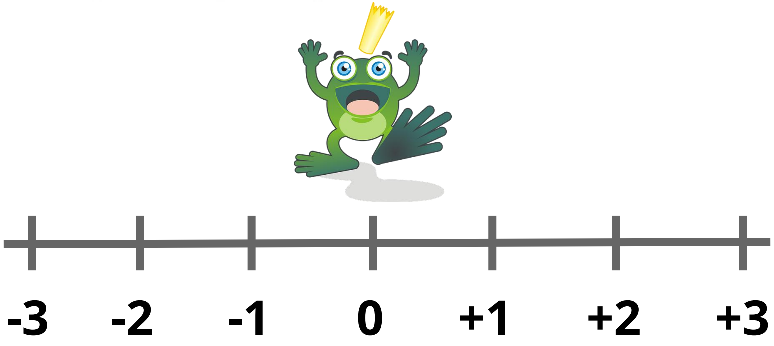

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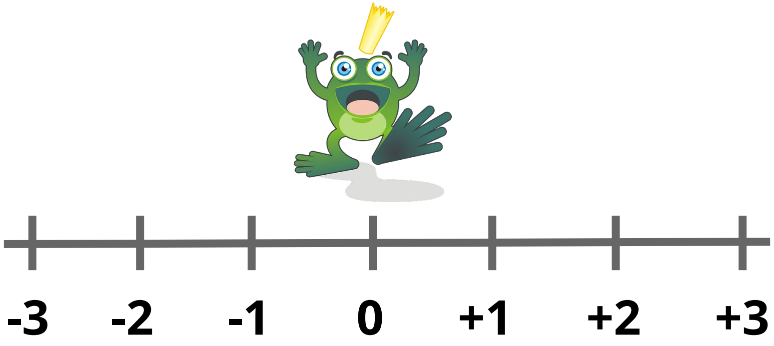

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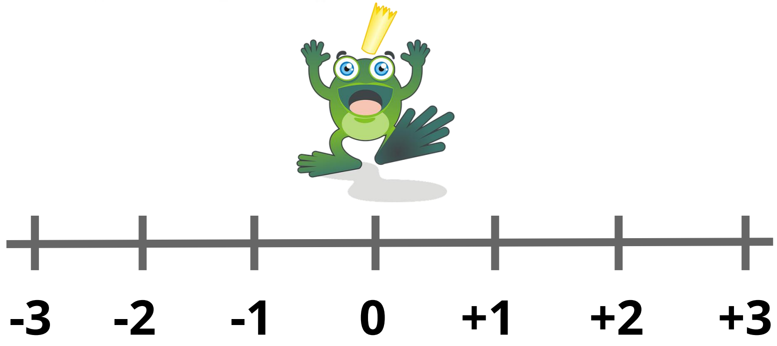

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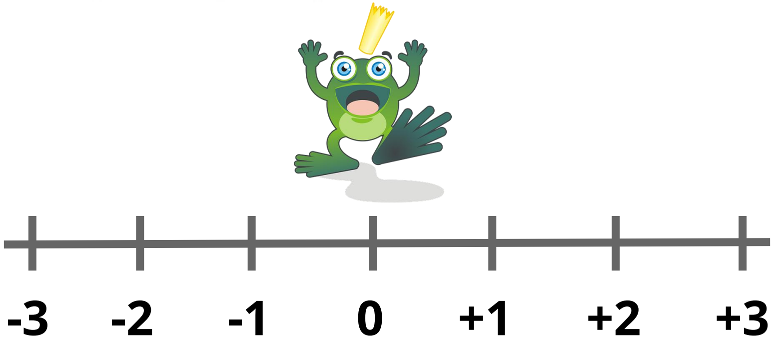

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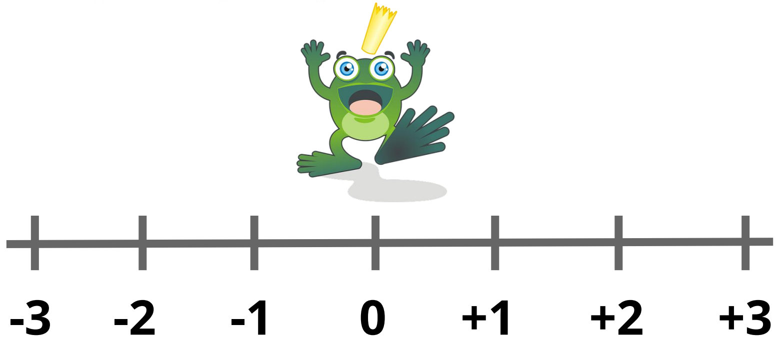

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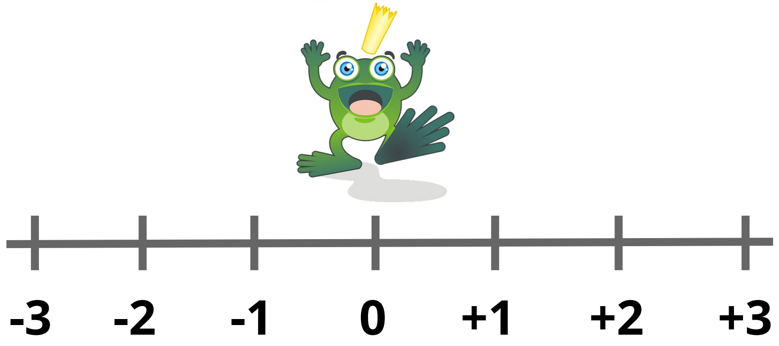

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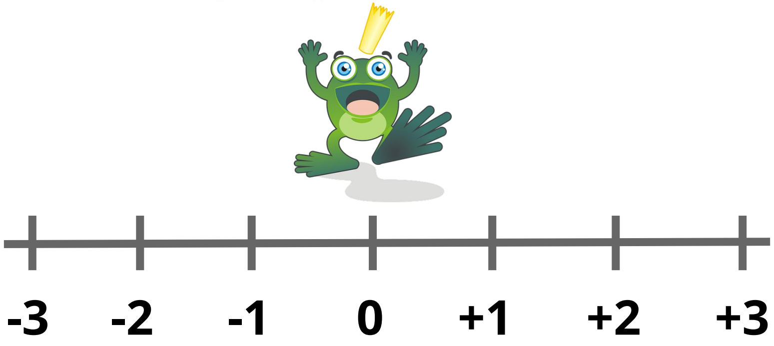

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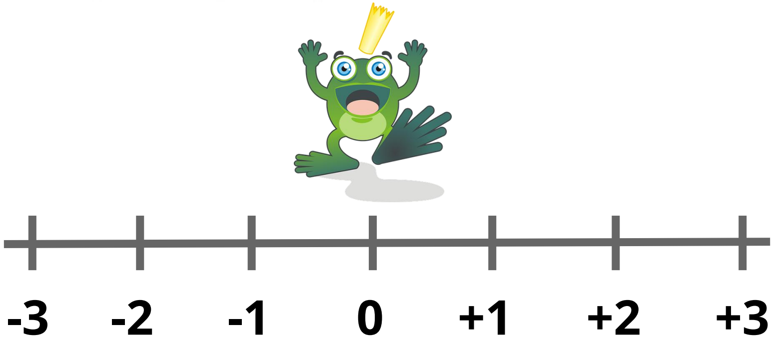

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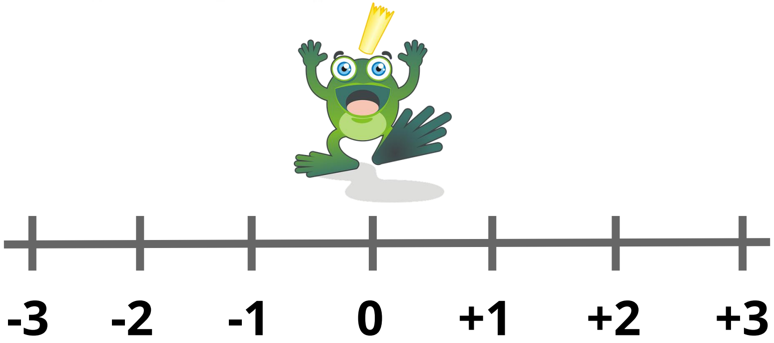

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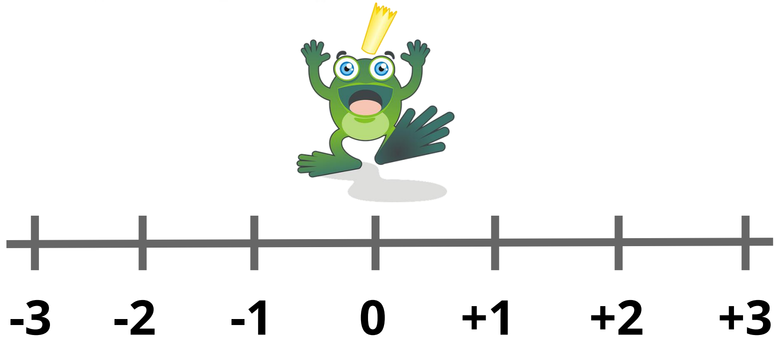

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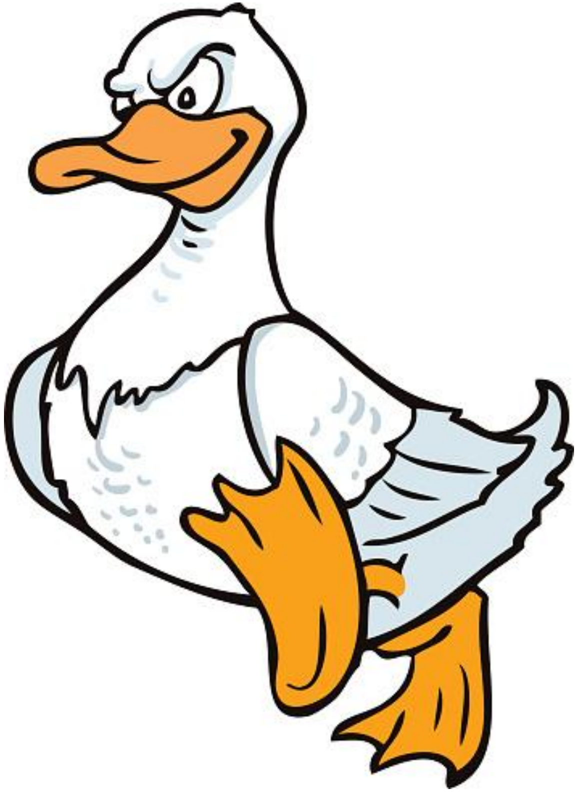
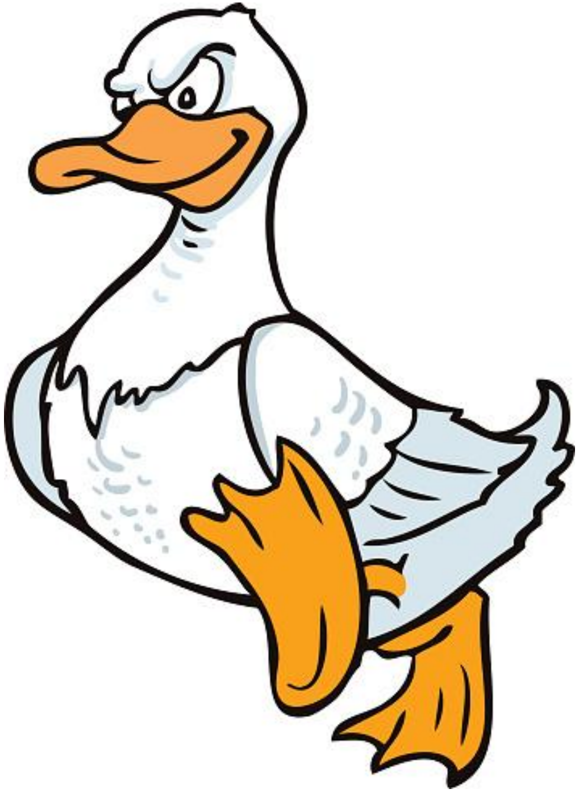
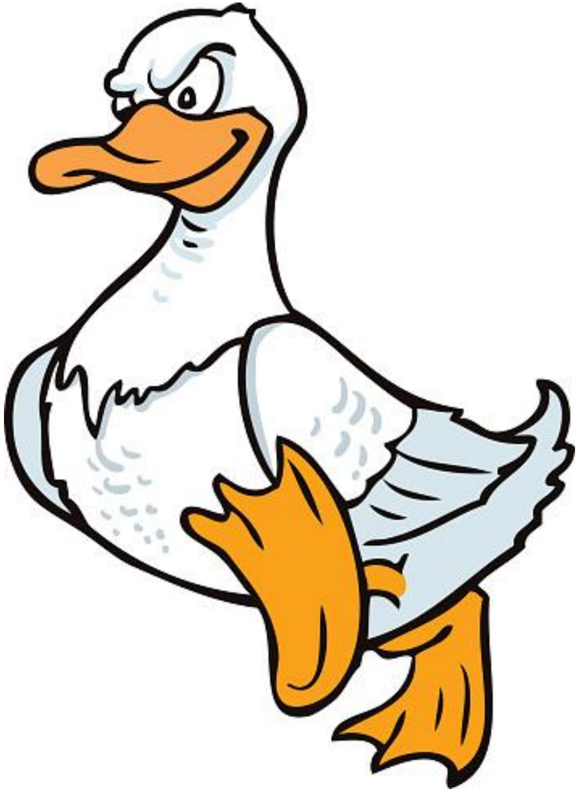
$$E[X] = (-2) \cdot p_L^2 + (-1) \cdot 2p_L p_S + \cdots = 2(p_R - p_L)$$

Random Picture

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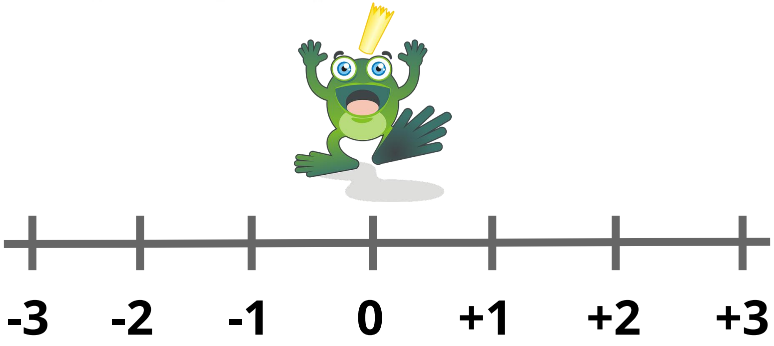

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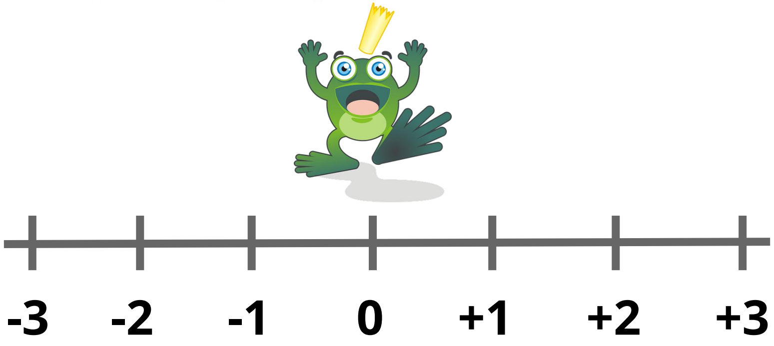

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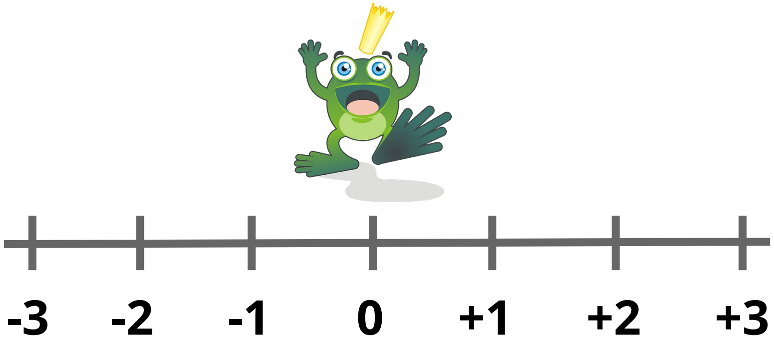

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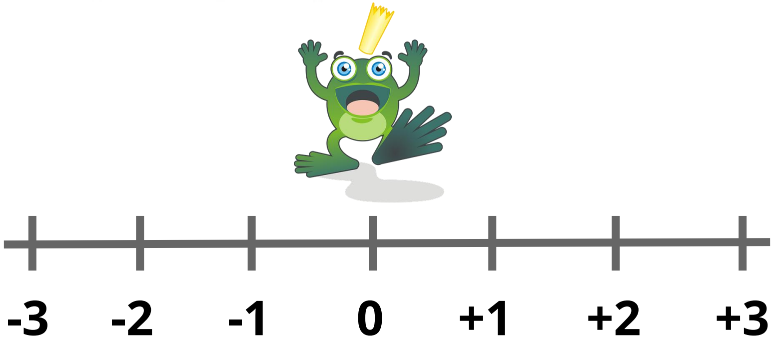

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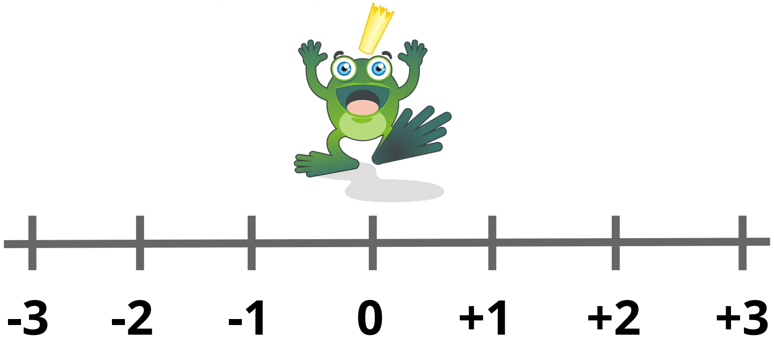

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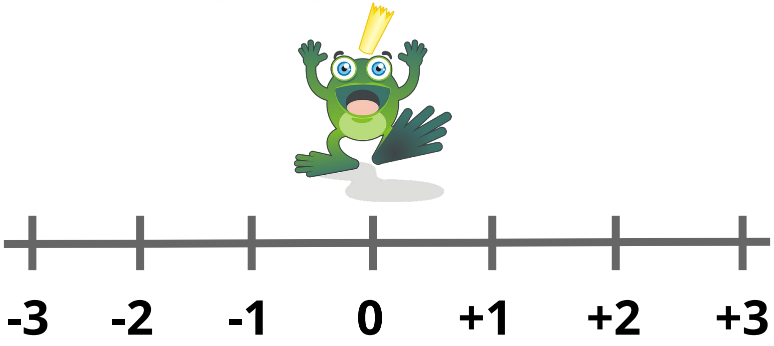

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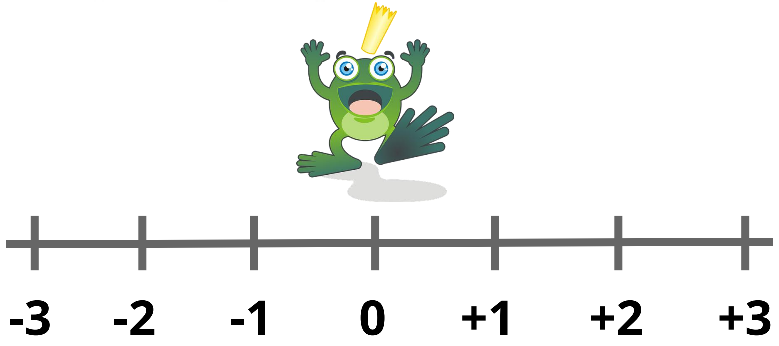

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By linearity of expectation,

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But what if we changed the number of time steps to 100 or 1000?

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But what if we changed the number of time steps to 100 or 1000?

the first method would be completely infeasible, but the second (LoE) would be basically the same.

Flipping 2 coins again

Let X be the number of heads in two independent flips of a fair coin. Recall

$$\Omega_X = \{0,1,2\}$$

$$p_X(d) = \begin{cases} 1/4, & d = 0 \\ 1/2, & d = 1 \\ 1/4, & d = 2 \end{cases}$$

$$E[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

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$$E[X^3] = E[Y] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 8 \cdot \frac{1}{4} = 2.5$$

Flipping 2 coins again

$$E[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1 \qquad E[X^3] = E[Y] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 8 \cdot \frac{1}{4} = 2.5$$

See an easier way to compute $E[X^3] = E[Y]$ without going through the trouble of writing out p_Y?

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$$E[X^3] = \sum_{b \in \Omega_X} b^3 p_X(b)$$

$$E[g(X)] = \sum_{b \in \Omega_X} g(b) p_X(b)$$

Flipping 2 coins again

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$$E[X^3] = \sum_{b \in \Omega_X} b^3 p_X(b)$$

$$E[g(X)] = \sum_{b \in \Omega_X} g(b) p_X(b)$$

Notice $2.5 = E[X^3] \neq E[X]^3 = 1$.

Law of the Unconscious Statistician (LOTUS)

Law of the Unconscious Statistician (LOTUS): Let X be a discrete random variable with range Ω_X and $g: D \to \mathbb{R}$ be a function defined at least over Ω_X ($\Omega_X \subseteq D$). Then,

$$E[g(X)] = \sum_{b \in \Omega_X} g(b) p_X(b)$$

Note that in general, $E[g(X)] \neq g(E[X])$. For example, $E[X^2] \neq (E[X])^2$, or $E[\log(X)] \neq \log(E[X])$.

Probability

3.3 Variance

Alex Tsun

Agenda

- LoE with Indicator RVs
- Variance and Standard Deviation (SD)

Red Hair

There are 7 mermaids in the sea. Below is a table of each mermaid with their hair color. How many mermaids have red hair?

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1 / 0	1	0	0	1	0	0	1

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Mermaid	1	2	3	4	5	6	7
Color	RED	BLUE	PURPLE	RED	BLACK	YELLOW	RED
1 / 0	1	0	0	1	0	0	1

Sum the bottom row of "indicator variables" to get 3.

Hat Check

n people go to a party and leave their hat with a hat-check person. At the end of the party, they return hats randomly since they don't care about their job. Let X be the number of people who got their original hat back. What is $E[X]$?

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Brute Force: $\Omega_X = \{0,1,2, \ldots, n-2, n\}$.

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$$p_X(n) = \frac{1}{n!}$$

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$$p_X(0) = ???$$

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$$p_X(n) = \frac{1}{n!}$$

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Too hard → Use linearity!

Hat Check

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Quick question: does it matter where you are in line?

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Quick question: does it matter where you are in line?

If first in line, $P(\text{get hat back}) = \frac{1}{n}$, because there are n in total.

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Quick question: does it matter where you are in line?

If first in line, $P(\text{get hat back}) = \frac{1}{n}$, because there are n in total.
If last in line, $P(\text{get hat back}) = \frac{1}{n}$, because there is 1 left, and the chance it is yours is $\frac{1}{n}$.

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For $i = 1, \dots, n$, let $X_i = \begin{cases} 1, & \text{if } i^{th} \text{ person got hat back} \\ 0, & \text{otherwise} \end{cases}$. Then $X = \sum_{i=1}^{n} X_i$.

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We will use linearity of expectation.

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NOT "INDEPENDENT" RVS

We will use linearity of expectation.

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Linearity of Expectation with Indicators

<u>**Linearity of Expectation with Indicators:**</u> If asked only about the expectation of a RV X (and not its PMF), then you may be able to write X as a sum of possibly "dependent" indicator (1/0) random variables, and apply LoE.

For an indicator RV X_i,

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(X_i = 1)$$

Random Picture

Variance (Intuition)

Which game would you rather play? We flip a fair coin.

Variance (Intuition)

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Game 1:

- If heads, You pay me $1.
- If Tails, I pay you $1.

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Which game would you rather play? We flip a fair coin.

Game 1:

- If heads, You pay me \$1.
- If Tails, I pay you \$1.

Game 2:

- If Heads, you pay me \$1000.
- If Tails, I pay you \$1000.

Variance (Intuition)

Which game would you rather play? We flip a fair coin.

Game 1:

- If heads, You pay me \$1.
- If Tails, I pay you \$1.

Game 2:

- If Heads, you pay me \$1000.
- If Tails, I pay you \$1000.

Both games are fair.

$$E[G_1] = -1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = 0$$

$$E[G_2] = -1000\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = 0$$

Variance (Intuition)

How far is a random variable from its mean, on average?

$$X - E[X]$$

Variance (Intuition)

HOW FAR IS A RANDOM VARIABLE FROM ITS MEAN, ON AVERAGE?

$$\lvert X - E[X] \rvert$$

Variance (Intuition)

how far is a random variable from its mean, on average?

$$|X - E[X]|$$

$$(X - E[X])^2$$

VARIANCE (INTUITION)

HOW FAR IS A RANDOM VARIABLE FROM ITS MEAN, ON AVERAGE?

$$E[\,|X - E[X]|\,]$$

$$E[\,(X - E[X])^2\,]$$

Variance and Standard Deviation (SD)

Variance: The variance of a random variable X is

$$Var(X) = E[(X - E[X])^2]$$

The variance is always nonnegative since we take an expectation of a nonnegative random variable $(X - E[X])^2$. We can also show that for any scalars $a, b \in \mathbb{R}$,

$$Var(aX + b) = a^2\ Var(X)$$

Variance and Standard Deviation (SD)

<u>Variance:</u> The variance of a random variable X is

More Useful

$$Var(X) = E[(X - E[X])^2] \boxed{= E[X^2] - (E[X])^2}$$

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<u>**Standard Deviation (SD):**</u> The standard deviation of a random variable X is

$$\sigma_X = \sqrt{Var(X)}$$

We want this because the units of variance are squared in terms of the original variable X, and this "undo's" our squaring, returning the units to the same as X.

Variance (Property)

$$Var(aX + b) = a^2Var(X)$$

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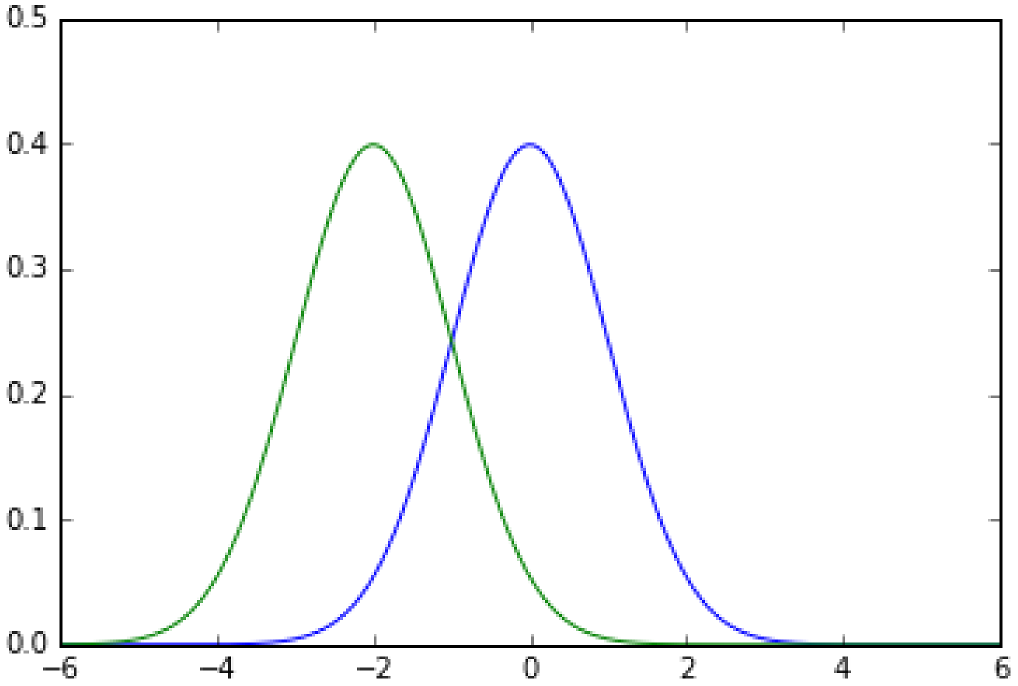
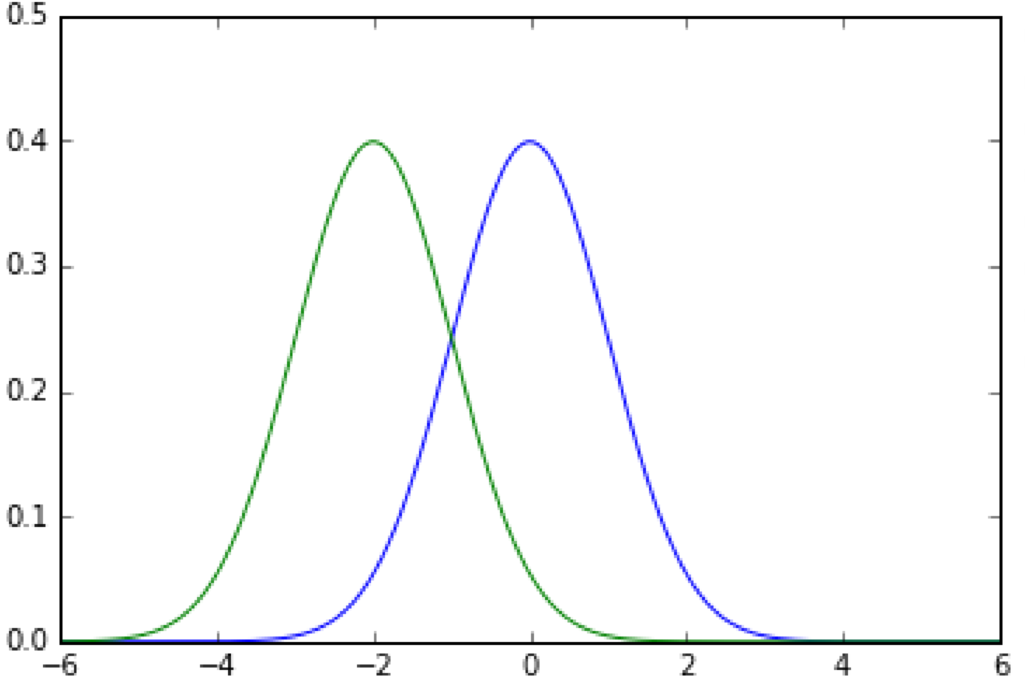


Variance (Property)

$$Var(aX + b) = a^2Var(X)$$

$$Var(X + b) = Var(X)$$



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$Var(aX)$

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$$Var(aX + b) = a^2 Var(X)$$

$$Var(aX) = E[(aX)^2] - (E[aX])^2 = E[a^2X^2] - (aE[X])^2$$

$$= a^2E[X^2] - a^2(E[X])^2 = a^2(E[X^2] - E[X]^2) = a^2Var(X)$$

Variance (Example)

Let X be the outcome of a fair 6-sided die roll. What is $Var(X)$?

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$$Var(X) = E[X^2] - E[X]^2$$

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = 3.5$$

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$$E[X^2] = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + \cdots + 6^2 \cdot \frac{1}{6} = \frac{91}{6}$$

LOTUS

Variance (Example)

Let X be the outcome of a fair 6-sided die roll. What is $Var(X)$?

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = 3.5$$

$$E[X^2] = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + \cdots + 6^2 \cdot \frac{1}{6} = \frac{91}{6}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$

Bài 3.1. Cho biến ngẫu nhiên rời rạc X có bảng phân phối xác suất sau:

X	-2	-1	0	1	2
$\mathbb{P}$	$1/8$	$2/8$	$2/8$	$2/8$	$1/8$

(a) Tìm hàm phân phối xác suất $F(x)$.

(b) Tính $\mathbb{P}(-1 \leq X \leq 1)$ và $\mathbb{P}\big(X \leq -1$ hoặc $X = 2\big)$.

(c) Lập bảng phân phối xác suất của biến ngẫu nhiên $Y = X^2$.

Đáp án. (b) 6/8, 4/8. (c)

Y	0	1	4
$\mathbb{P}$	$2/8$	$4/8$	$2/8$

Bài 3.2. Biến ngẫu nhiên rời rạc X có hàm xác suất cho bởi

$$f(x) = \frac{2x+1}{25}, \quad x = 0, 1, 2, 3, 4$$

(a) Lập bảng phân phối xác suất của X.

(b) Tính $\mathbb{P}(2 \leq X < 4)$ và $\mathbb{P}(X > -10)$.

Đáp án. (b) 12/25, 1.

Probability

3.4 Zoo of Discrete RV's Part I

Alex Tsun

Agenda

- Independence
- The Bernoulli Process
- The Bernoulli/indicator RV
- The Binomial RV

Independence

Independence: Random variables X and Y are independent, denoted $X \perp Y$, if for all $x \in \Omega_X$ and $y \in \Omega_Y$, any of the three equivalent properties holds:

1. $P(X = x|Y = y) = P(X = x)$
2. $P(Y = y|X = x) = P(Y = y)$
3. $P(X = x, Y = y) = P(X = x)P(Y = y)$

Independence

<u>Independence:</u> Random variables X and Y are independent, denoted $X \perp Y$, if for all $x \in \Omega_X$ and $y \in \Omega_Y$, any of the three equivalent properties holds:

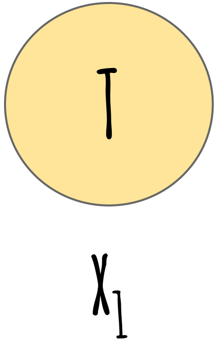
1. $P(X = x | Y = y) = P(X = x)$
2. $P(Y = y | X = x) = P(Y = y)$
3. $P(X = x, Y = y) = P(X = x)P(Y = y)$

This is the same as the event definition, but it must hold for all events $\{X = x\}$ and $\{Y = y\}$. We'll discuss this further, and prove this useful fact later as well: If $X \perp Y$, then

$$Var(X + Y) = Var(X) + Var(Y)$$

The Bernoulli Process

<u>Bernoulli Process:</u> A Bernoulli process with parameter p is a sequence of independent coin flips $X_1, X_2, X_3, \ldots$ where $P(\text{head}) = p$. If flip i is heads, then we encode $X_i = 1$; otherwise if flip i is tails, then $X_i = 0$.



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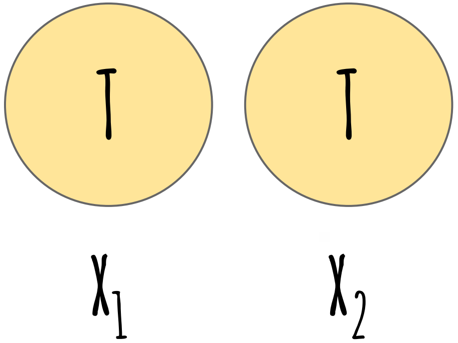

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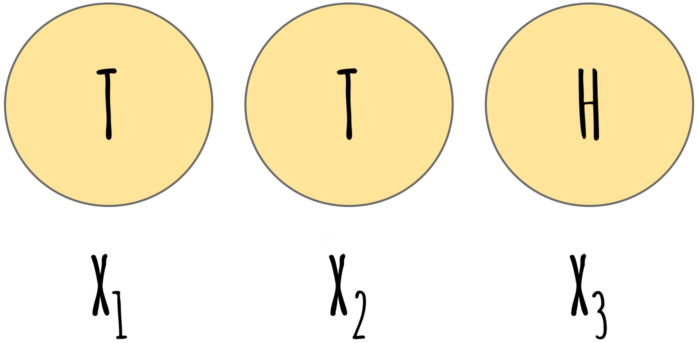

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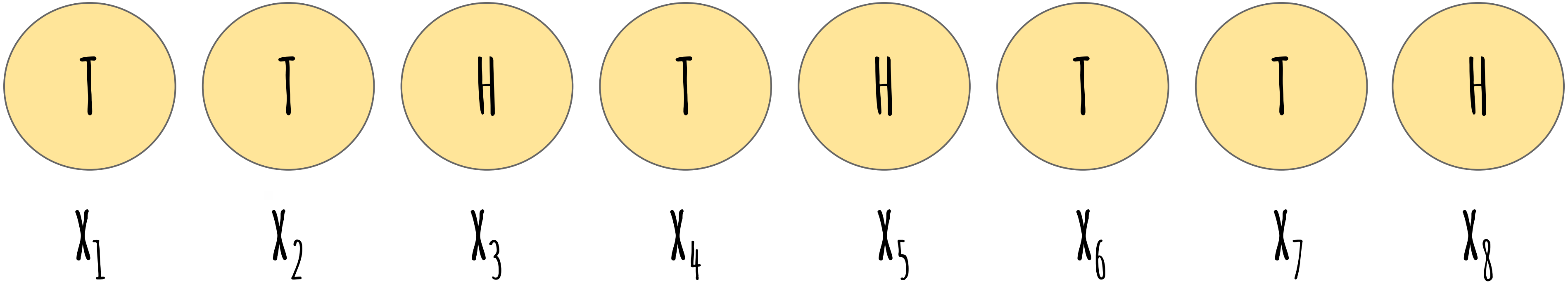

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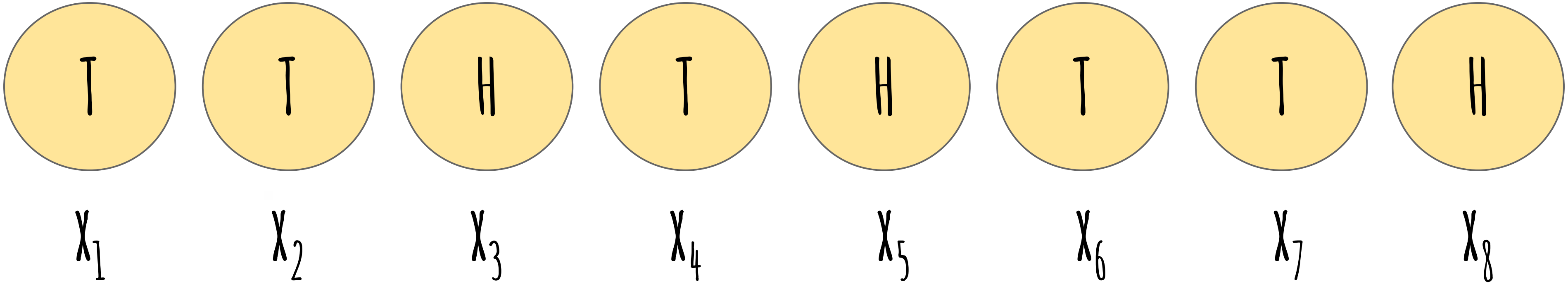

From this process, we can measure several interesting things.

The Bernoulli RV

Ber(p)

T

X_1

The Bernoulli RV

Ber(p) Ber(p)

T T

X_1 X_2

The Bernoulli RV

Ber(p)	Ber(p)	Ber(p)
T	T	H
X_1	X_2	X_3

The Bernoulli RV

Ber(p) Ber(p) Ber(p) Ber(p) ...

T	T	H	T	H	T	T	H
X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8

Bernoulli RV Properties

$$E[X] =$$

Bernoulli RV Properties

$$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

Bernoulli RV Properties

$$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

$$E[X^2] =$$

Bernoulli RV Properties

$$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1 - p) = p$$ LOTUS

Bernoulli RV Properties

$$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1 - p) = p$$ LOTUS

$$Var(X) = E[X^2] - (E[X])^2 =$$

Bernoulli RV Properties

$$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1 - p) = p$$ LOTUS

$$Var(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1 - p)$$

The Bernoulli/indicator RV

Bernoulli RV: $X \sim Ber(p)$ if and only if X has the following pmf:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1 - p, & k = 0 \end{cases}$$

The Bernoulli/indicator RV

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Each X_i in the Bernoulli process with parameter p is a Bernoulli/indicator rv with parameter p. It simply represents a binary outcome, like a coin flip.

$$E[X] = p \qquad Var(X) = p(1-p)$$

The Bernoulli/indicator RV

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Each X_i in the Bernoulli process with parameter p is a Bernoulli/indicator rv with parameter p. It simply represents a binary outcome, like a coin flip.

$$E[X] = p \qquad Var(X) = p(1 - p)$$

By a clever trick, we can write the PMF as

$$p_X(k) = p^k(1 - p)^{1-k}, \quad k = 0,1$$

Random Picture

The Binomial RV

$\text{Bin}(3, p) = 1$

T	T	H
X_1	X_2	X_3

The Binomial RV

$\text{Bin}(3, p) = 1$ $\quad$ $\text{Bin}(1, p) = 0$

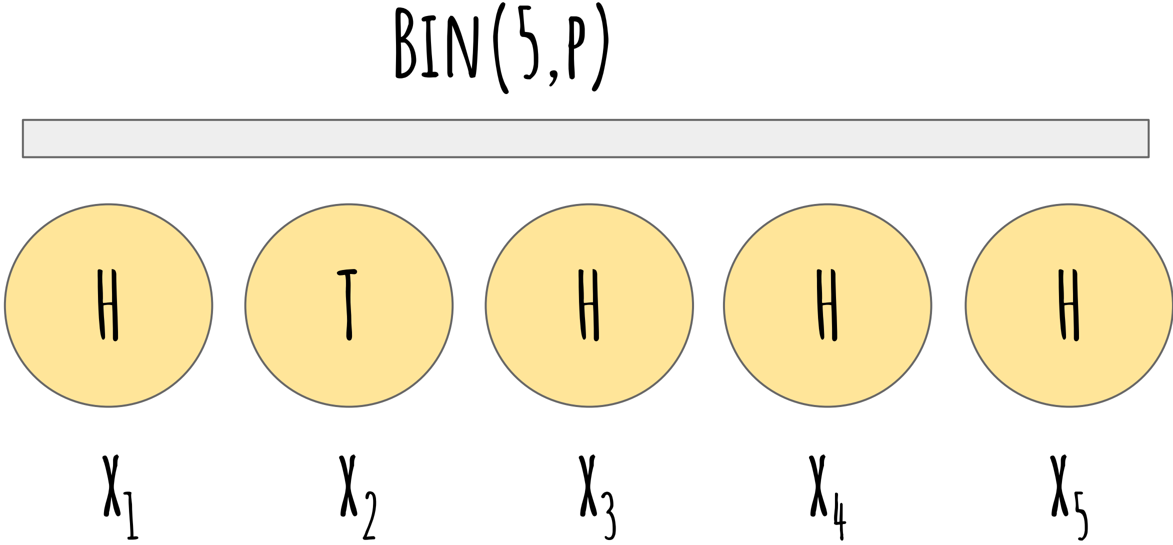
T	T	H	T
X_1	X_2	X_3	X_4

The Binomial RV

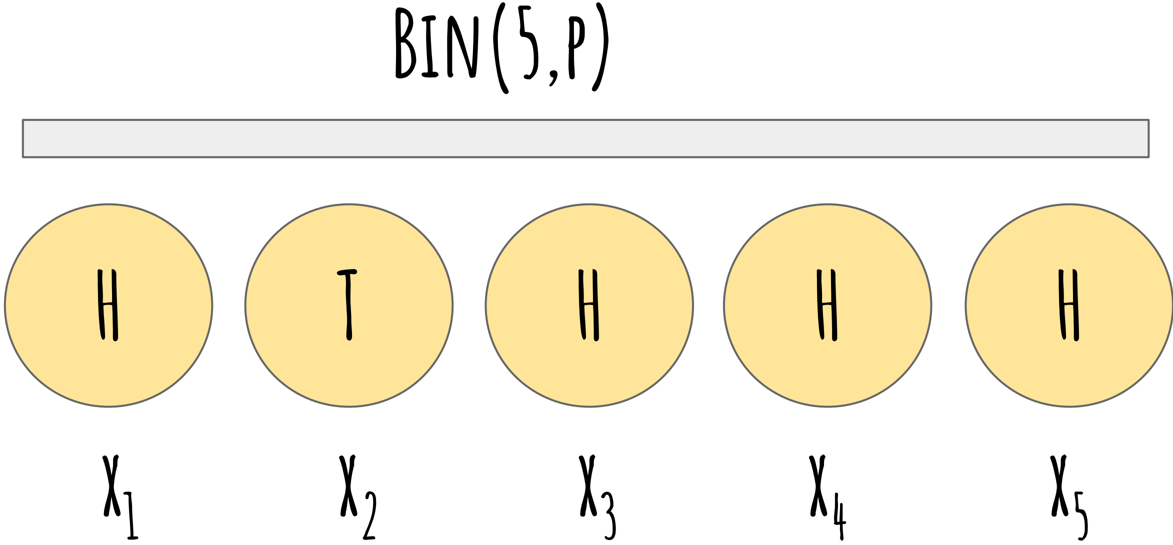
$\text{Bin}(3,p) = 1$ $\text{Bin}(1,p) = 0$ $\text{Bin}(4,p) = 2$

T	T	H	T	H	T	T	H
X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8

Binomial RV PMF

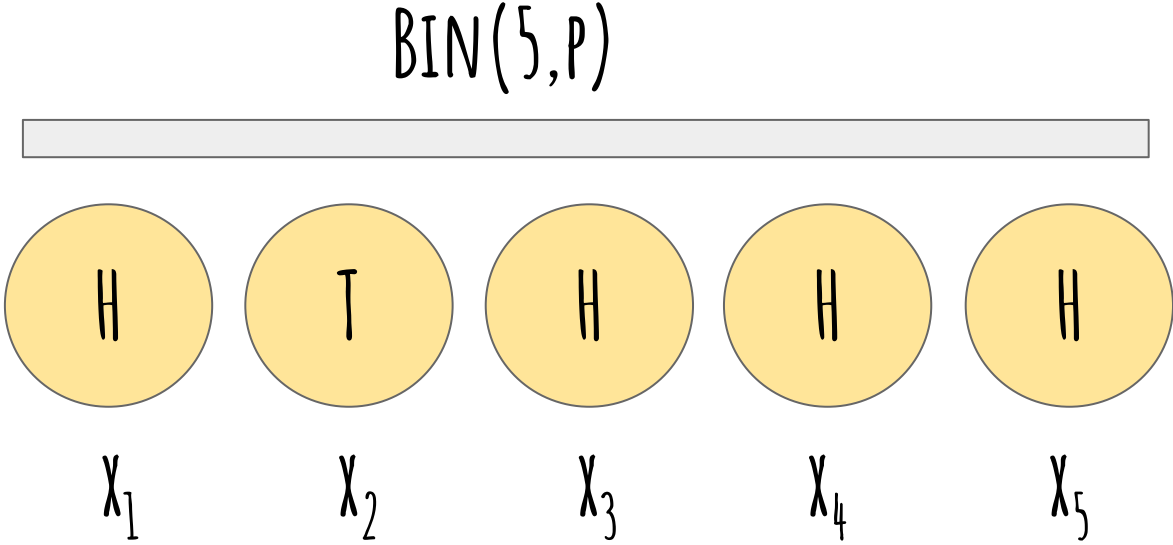


Binomial RV PMF



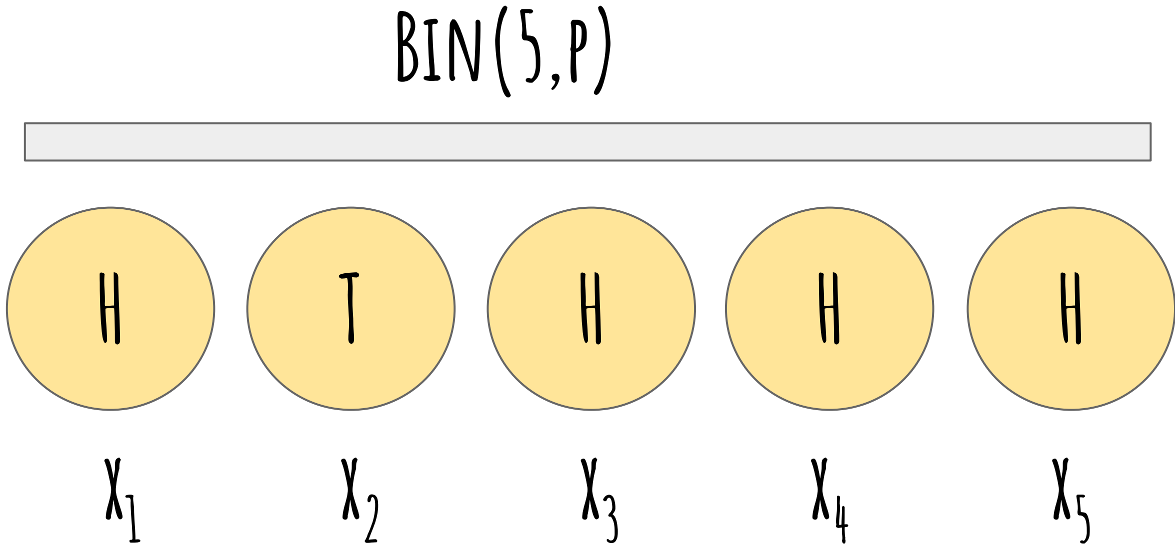
$$P(HTHHH) =$$

Binomial RV PMF



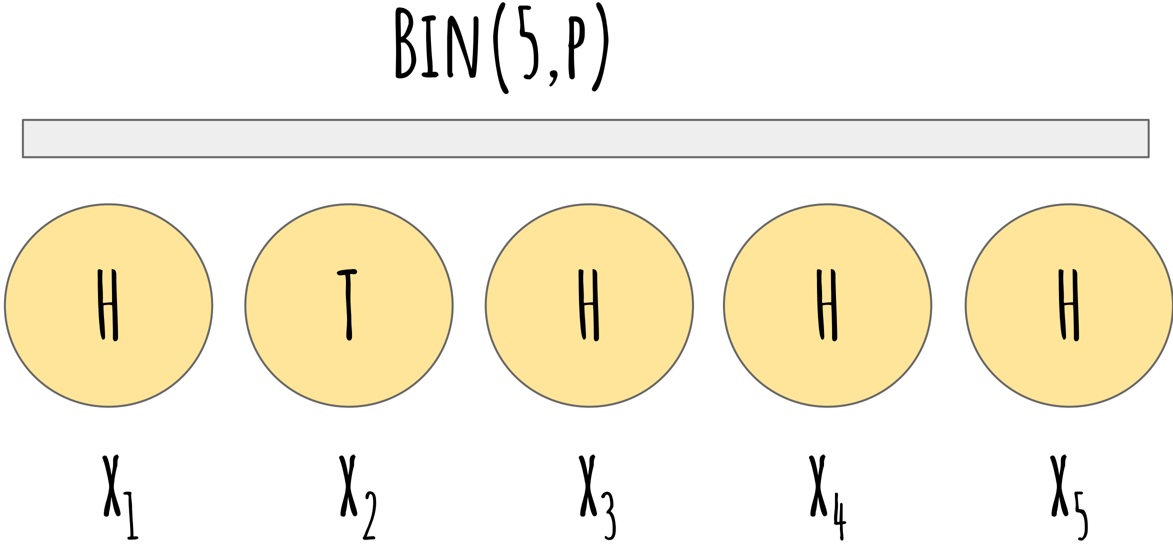
$$P(HTHHH) = p$$

Binomial RV PMF



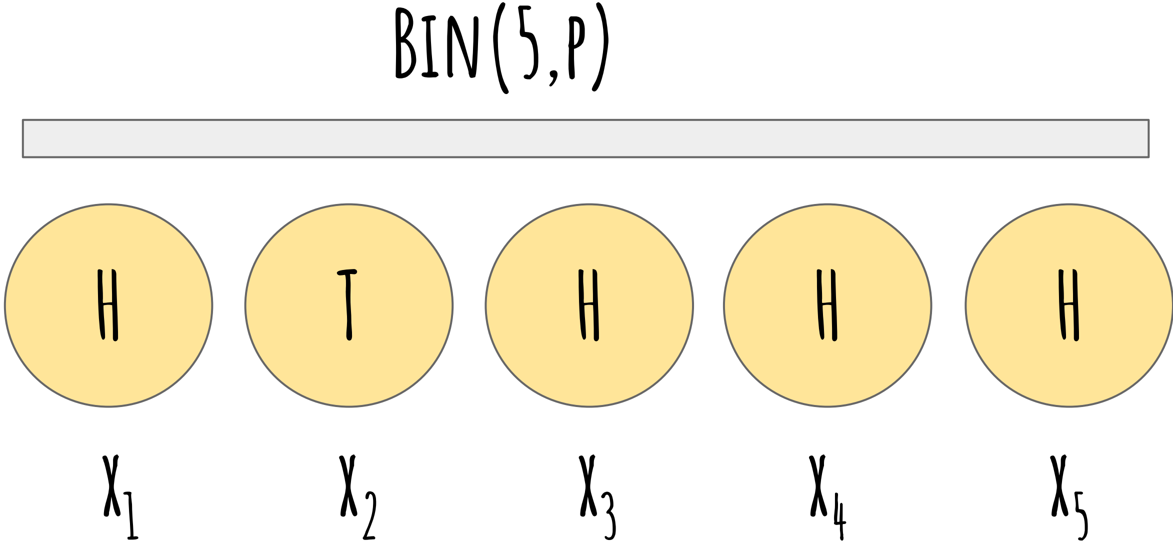
$$P(HTHHH) = p(1-p)$$

Binomial RV PMF



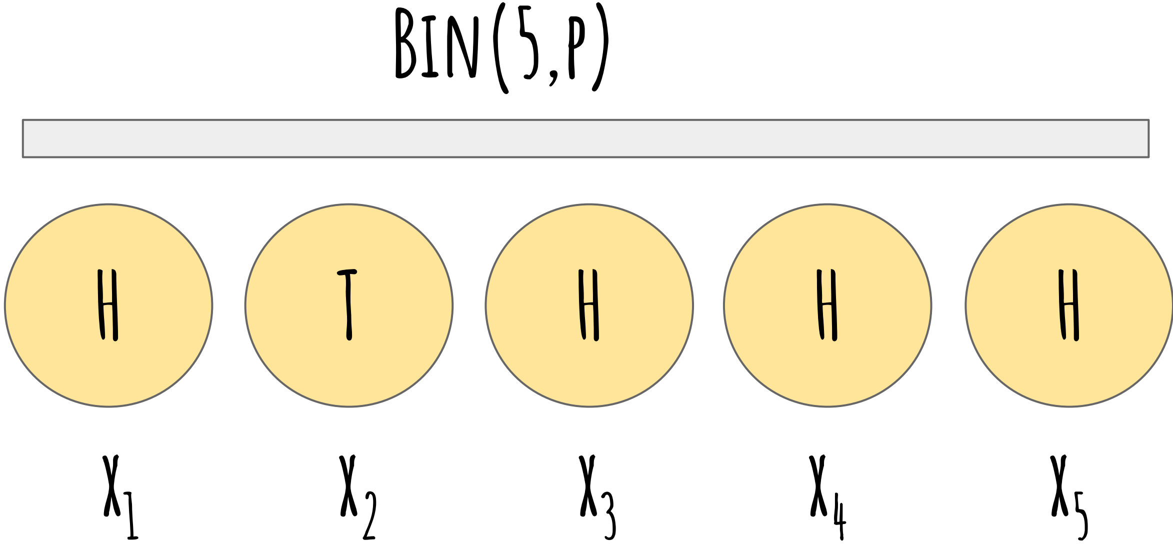
$$P(HTHHH) = p(1-p)ppp =$$

Binomial RV PMF



$$P(HTHHH) = p(1-p)ppp = p^4(1-p)^1$$

Binomial RV PMF



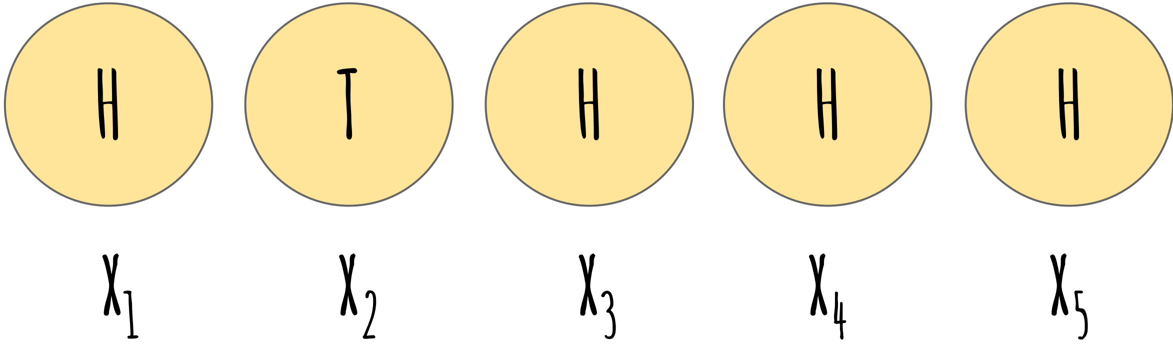
$$P(HTHHH) = p(1-p)ppp = p^4(1-p)^1$$

$$P(X = 4) =$$

Binomial RV PMF

HHHHT

Bin(5,p)



$$P(HTHHH) = p(1-p)ppp = p^4(1-p)^1$$

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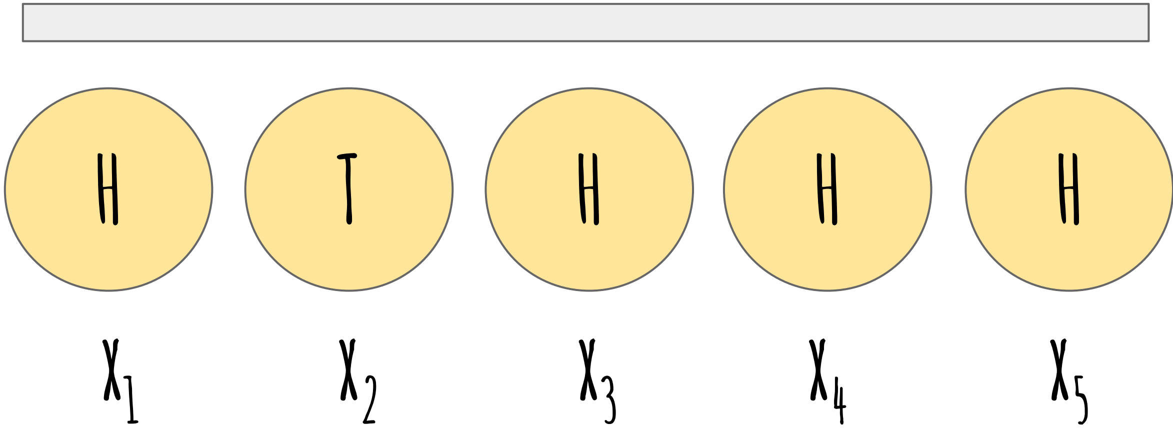
Binomial RV PMF

HHHHT

HHHTH

$\mathrm{Bin}(5, p)$



$$P(HTHHH) = p(1-p)ppp = p^4(1-p)^1$$

$$P(X = 4) =$$

Binomial RV PMF

$\text{Bin}(5, p)$

HHHHT
HHHTH
HHTHH
HTHHH
THHHH

H	T	H	H	H
X_1	X_2	X_3	X_4	X_5

$$P(HTHHH) = p(1-p)ppp = p^4(1-p)^1$$

$$P(X = 4) =$$

Binomial RV PMF

HHHHT
HHHTH
HHTHH
HTHHH
THHHH

$\binom{5}{4}$ ways to get 4 heads out of 5

Bin(5,p)

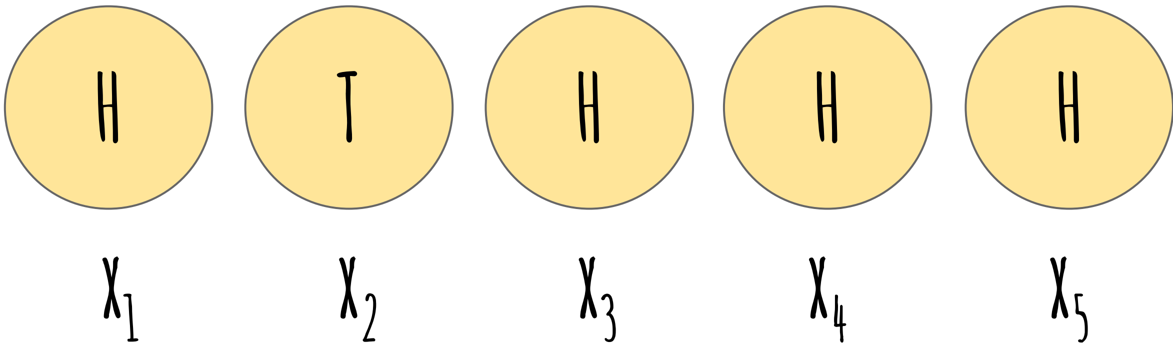
H	T	H	H	H
X_1	X_2	X_3	X_4	X_5

$$P(HTHHH) = p(1-p)ppp = p^4(1-p)^1$$

$$P(X=4) =$$

Binomial RV PMF

BIN(5,p)



HHHHT
HHHTH
HHTHH
HTHHH
THHHH

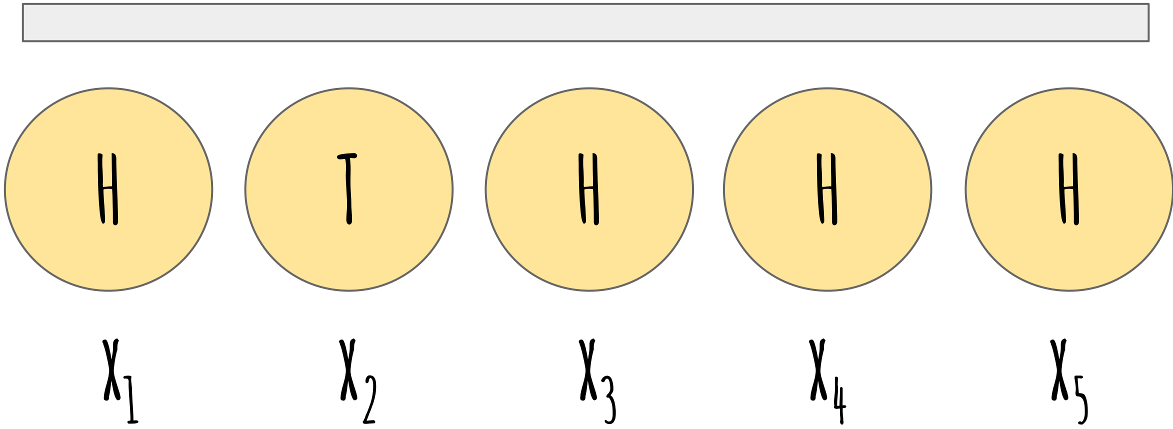
$$P(HTHHH) = p(1-p)ppp = p^4(1-p)^1$$

$\binom{5}{4}$ ways to get 4 heads out of 5

$$P(X=4) = \binom{5}{4} p^4 (1-p)^{5-4}$$

Binomial RV PMF

BIN(5,p)



HHHHT
HHHTH
HHTHH
HTHHH
THHHH

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$\binom{5}{4}$ ways to get 4 heads out of 5

$$P(X=4) = \binom{5}{4} p^4 (1-p)^{5-4}$$

Binomial RV PMF sums to 1?

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

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$$\sum_{k \in \Omega_X} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \left(p + (1-p)\right)^n = 1^n = 1$$

Binomial RV Properties

$$E[X] = \sum_{k \in \Omega_X} k p_X(k) =$$

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$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] =$$

Binomial RV Properties

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$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] =$$

LoE regardless of independence

Binomial RV Properties

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$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

LoE regardless of independence

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LoE regardless of independence

Accept variance for now... But if X, Y are "independent",
$Var(X+Y) = Var(X) + Var(Y)$. Recall if $X_i \sim Ber(p)$, $Var(X_i) = p(1-p)$.

Binomial RV Properties

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$Var(X+Y) = Var(X) + Var(Y)$. Recall if $X_i \sim Ber(p)$, $Var(X_i) = p(1-p)$.

$$Var(X) = np(1-p)$$

The Binomial RV

Binomial RV: $X \sim Bin(n, p)$ if and only if X has the following pmf:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n$$

X is the sum of n independent $Ber(p)$ random variables.

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X is the sum of n independent $Ber(p)$ random variables.

$$E[X] = np \qquad Var(X) = np(1-p)$$

For example, X is the number of heads in n independent coin flips with $P(\text{head}) = p$. Notice that $Bin(1, p) \equiv Ber(p)$. By definition, if X, Y are independent with $X \sim Bin(n, p)$ and $Y \sim Bin(m, p)$, then $X + Y \sim Bin(n + m, p)$.

Binomial RV Example

A factory produces 100 cars per day, but a car is defective with probability 0.02. What's the probability the factory produces 2 or more defective cars?

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Let X be the number of defective cars it produces. $X \sim Bin(100, 0.02)$

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$$P(X \geq 2) =$$

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$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 0) - P(X = 1)$$
$$= 1 - \binom{100}{0}(0.02)^0(1 - 0.02)^{100} - \binom{100}{1}(0.02)^1(1 - 0.02)^{99}$$

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$$= 1 - \binom{100}{0}(0.02)^0(1-0.02)^{100} - \binom{100}{1}(0.02)^1(1-0.02)^{99}$$
$$\approx 0.5967$$

Probability

3.5 Zoo of Discrete RV's Part II

Alex Tsun

Agenda

- The Uniform RV
- The Geometric RV
- The Negative Binomial RV

The Uniform RV

Uniform (Discrete) RV: $X \sim Unif(a, b)$ where $a < b$ are integers, if and only if X has the following pmf:

$$p_X(k) = \begin{cases} \dfrac{1}{b - a + 1}, & k \in \{a, a + 1, \dots, b\} \\ 0, & \text{otherwise} \end{cases}$$

The Uniform RV

Uniform (Discrete) RV: $X \sim Unif(a, b)$ where $a < b$ are integers, if and only if X has the following pmf:

$$p_X(k) = \begin{cases} \dfrac{1}{b - a + 1}, & k \in \{a, a + 1, \dots, b\} \\ 0, & \text{otherwise} \end{cases}$$

X is equally likely to take on any value in $\Omega_X = \{a, a + 1, \dots, b\}$ of size $b - a + 1$.

$$E[X] = \frac{a + b}{2} \qquad Var(X) = \frac{(b - a)(b - a + 2)}{12}$$

For example, a roll of a fair 6-sided die is $Unif(1,6)$.

The Geometric RV

$\text{Geo}(p) = 3$

T T H

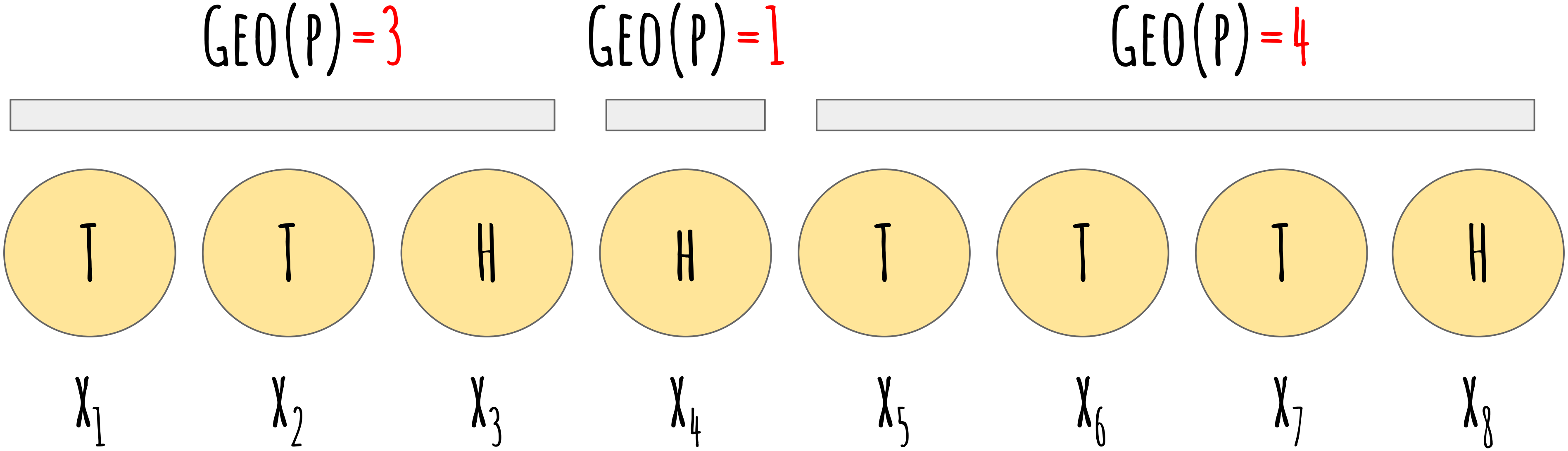
X_1 X_2 X_3

The Geometric RV

$Geo(p) = 3$ $Geo(p) = 1$

T	T	H	H
X_1	X_2	X_3	X_4

The Geometric RV



Geometric RV PMF

Geo(p) = 4

T T T H

$P(X = 4) =$

Geometric RV PMF

Geo(p) = 4

T T T H

$$P(X = 4) = P(TTTH) =$$

Geometric RV PMF

Geo(p) = 4

T T T H

$$P(X = 4) = P(TTTH) = (1 - p)$$

Geometric RV PMF

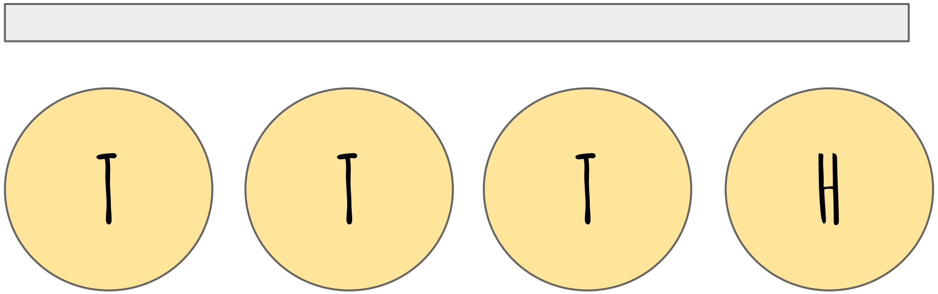
Geo(p) = 4

T T T H

$$P(X = 4) = P(TTTH) = (1 - p)(1 - p)$$

Geometric RV PMF

Geo(p) = 4



$$P(X = 4) = P(TTTH) = (1 - p)(1 - p)(1 - p)p =$$

Geometric RV PMF

Geo(p) = 4

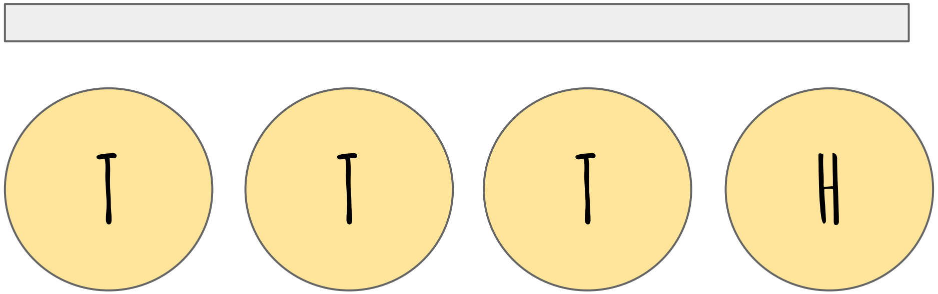
T T T H

$$P(X=4) = P(TTTH) = (1-p)(1-p)(1-p)p = (1-p)^{4-1}p$$

Geometric RV PMF

$$p_X(k) = (1-p)^{k-1}p$$

Geo(p) = 4



$$P(X=4) = P(TTTH) = (1-p)(1-p)(1-p)p = (1-p)^{4-1}p$$

Geometric RV PMF sums to 1?

$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$$

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$$= p \sum_{k=1}^{\infty} (1-p)^{k-1}$$

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$$= p \sum_{k=0}^{\infty} (1-p)^{k}$$

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$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$$

$$= p \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$$= p \sum_{k=0}^{\infty} (1-p)^{k}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{for } -1 < r < 1$$

Geometric RV PMF sums to 1?

$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$$

$$= p \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$$= p \sum_{k=0}^{\infty} (1-p)^{k}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{for } -1 < r < 1$$

$$= p \cdot \frac{1}{1-(1-p)}$$

Geometric RV PMF sums to 1?

$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$$

$$= p \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$$= p \sum_{k=0}^{\infty} (1-p)^{k}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{for } -1 < r < 1$$

$$= p \cdot \frac{1}{1-(1-p)}$$

$$= \frac{p}{p} = 1$$

Geometric RV Properties

Suppose $X \sim Geo(p)$. What do you think $E[X]$ is if $P(head) =$

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- 1/2? $\rightarrow E[X] = 2$

Geometric RV Properties

Suppose $X \sim Geo(p)$. What do you think $E[X]$ is if $P(head) =$

- $1/2$? $\rightarrow E[X] = 2$
- $1/10$?

Geometric RV Properties

Suppose $X \sim Geo(p)$. What do you think $E[X]$ is if $P(head) =$

- 1/2? $\rightarrow E[X] = 2$
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Geometric RV Properties

Suppose $X \sim Geo(p)$. What do you think $E[X]$ is if $P(head) =$

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- $1/7$?

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Intuitively, $E[X] = \frac{1}{p}$. We'll prove this later in an elegant way. See the next slide for brute force (requires calculus).

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Intuitively, $E[X] = \frac{1}{p}$. We'll prove this later in an elegant way. See the next slide for brute force (requires calculus).

$Var(X) = \frac{1-p}{p^2}$ by messy algebra.

Geometric RV Expectation (ugly way)

$$E[X] = \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = \sum_{k=0}^{\infty} k(1-p)^{k-1}p$$

$$= p\sum_{k=0}^{\infty} k(1-p)^{k-1} = p\sum_{k=0}^{\infty} \frac{d}{dp}[-(1-p)^k] = -p \cdot \frac{d}{dp}\left[\sum_{k=0}^{\infty}(1-p)^k\right]$$

$$= -p \cdot \frac{d}{dp}\left[\sum_{k=0}^{\infty}(1-p)^k\right] = -p \cdot \frac{d}{dp}\left[\frac{1}{1-(1-p)}\right] = -p \cdot \frac{d}{dp}\left[\frac{1}{p}\right]$$

$$= -p\left[-\frac{1}{p^2}\right] = \frac{1}{p}$$

The Geometric RV

<u>Geometric RV:</u> $X \sim Geo(p)$ if and only if X has the following pmf:

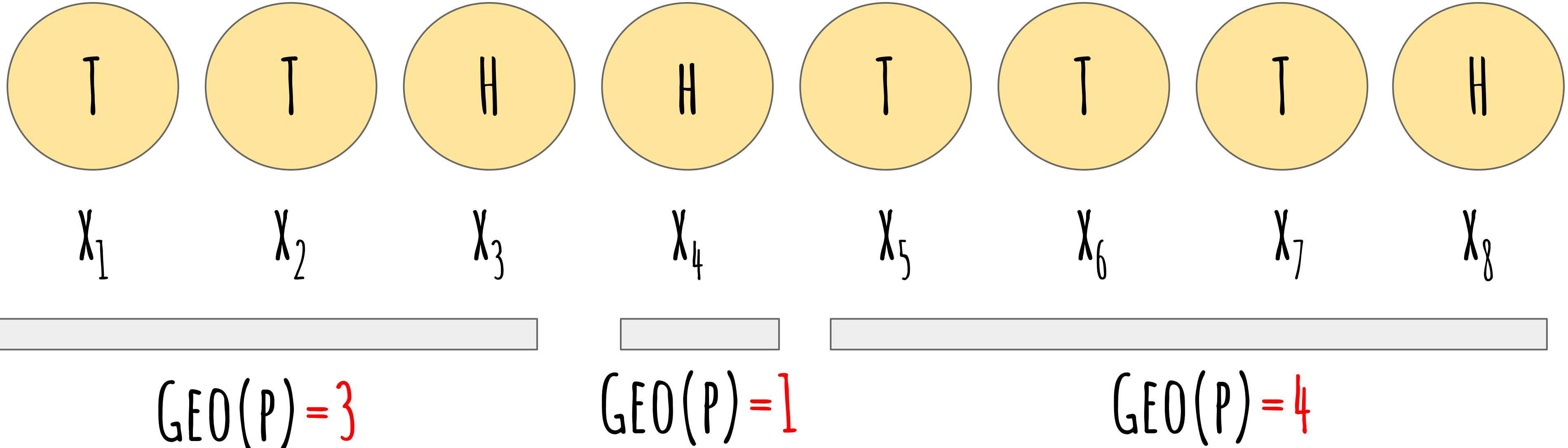
$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1,2,3,\ldots$$

$$E[X] = \frac{1}{p} \qquad Var(X) = \frac{1-p}{p^2}$$

For example, X is the number of independent coin flips with $P(\text{head}) = p$ up to and including the first head.

Random Picture

The Negative binomial RV



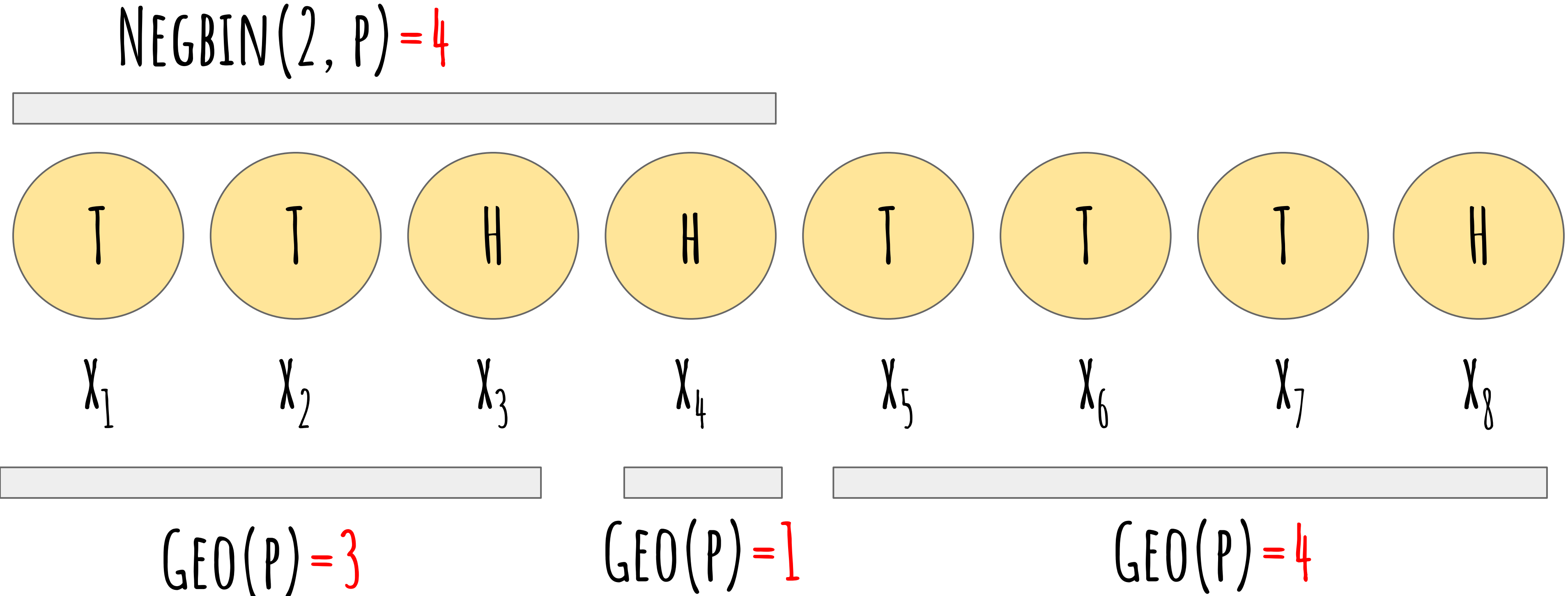
The Negative binomial RV

$\text{NegBin}(1, p) = 3$

T	T	H	H	T	T	T	H
X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8

$\text{Geo}(p) = 3$ $\quad$ $\text{Geo}(p) = 1$ $\quad$ $\text{Geo}(p) = 4$

The Negative binomial RV



The Negative Binomial RV

$\text{NegBin}(3, p) = 8$

T	T	H	H	T	T	T	H
X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8

$\text{Geo}(p) = 3$ $\text{Geo}(p) = 1$ $\text{Geo}(p) = 4$

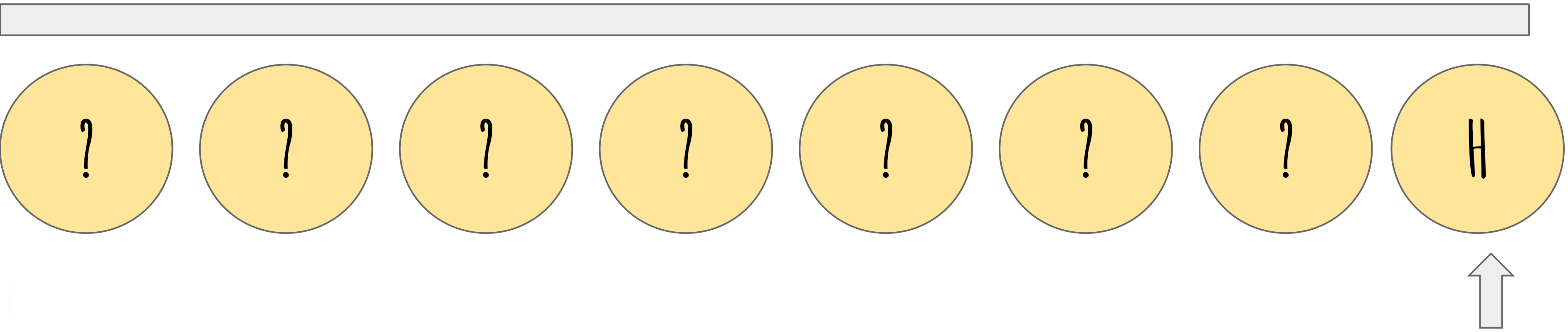
Negative Binomial RV PMF

$$X \sim NegBin(r = 3, p)$$

? ? ? ? ? ? ? ?

Negative Binomial RV PMF

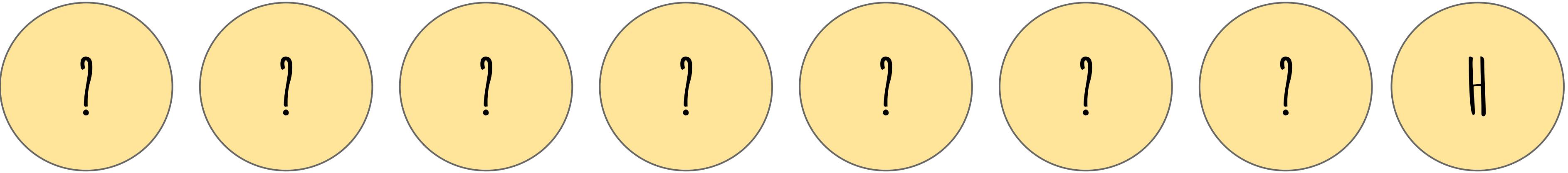
$$X \sim NegBin(r = 3, p)$$



Must be H

Negative Binomial RV PMF

$$X \sim NegBin(r = 3, p)$$



Exactly 2 $(r - 1)$ of the first 7 $(k - 1)$ must be heads.

Negative Binomial RV PMF

$X \sim NegBin(r = 3, p)$

? ? ? ? ? ? ? H

Exactly 2 ($r - 1$) of the first 7 ($k - 1$) must be heads.

$$P(X = 8) = \binom{8-1}{3-1}$$

Negative Binomial RV PMF

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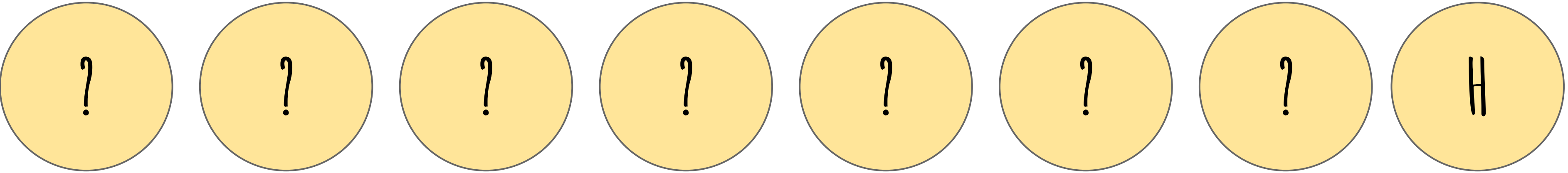

Exactly 2 ($r - 1$) of the first 7 ($k - 1$) must be heads.

$$P(X = 8) = \binom{8-1}{3-1} p^{3-1} (1-p)^5$$

Negative Binomial RV PMF

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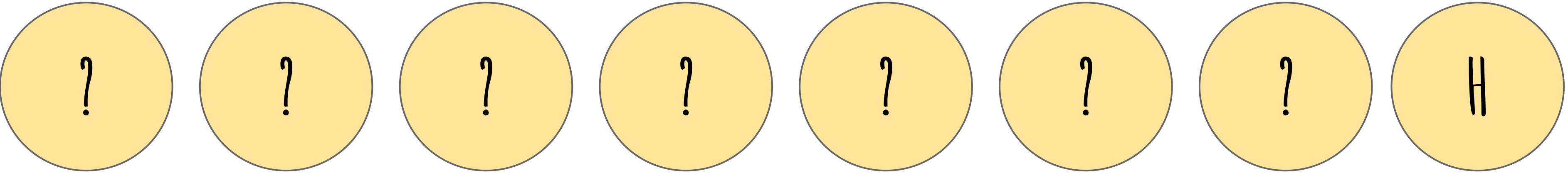

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Exactly 2 $(r-1)$ of the first 7 $(k-1)$ must be heads.

$$P(X = 8) = \binom{8-1}{3-1} p^{3-1} (1-p)^5 \cdot p$$

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Negative Binomial RV Properties

$$E[X] = \sum_{k \in \Omega_X} k p_X(k) = \sum_{k=r}^{\infty} k \binom{k-1}{r-1} p^k (1-p)^{k-r} = ???$$

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$$E[X] = E\left[\sum_{i=1}^{r} X_i\right] = \sum_{i=1}^{r} E[X_i] = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$

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Accept variance for now… But if X, Y are "independent",
$Var(X+Y) = Var(X) + Var(Y)$. Recall if $X_i \sim Geo(p)$, $Var(X_i) = \frac{1-p}{p^2}$.

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Accept variance for now… But if X, Y are "independent",
$Var(X+Y) = Var(X) + Var(Y)$. Recall if $X_i \sim Geo(p)$, $Var(X_i) = \frac{1-p}{p^2}$.

$$Var(X) = \frac{r(1-p)}{p^2}$$

The Negative Binomial RV

Negative Binomial RV: $X \sim NegBin(r, p)$ if and only if X has the following pmf:

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \qquad k = r, r+1, \ldots$$

X is the sum of r independent $Geo(p)$ random variables.

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X is the sum of r independent $Geo(p)$ random variables.

$$E[X] = \frac{r}{p} \qquad Var(X) = \frac{r(1-p)}{p^2}$$

For example, X is the number of independent coin flips with $P(\text{head}) = p$ up to and including the r^{th} head. Notice that $NegBin(1,p) \equiv Geo(p)$. By definition, if X, Y are independent with $X \sim NegBin(r,p)$ and $Y \sim NegBin(s,p)$, then $X + Y \sim NegBin(r+s,p)$.

Geometric RV Example

You gamble by flipping a fair coin independently up to and including the first head. If it takes k tries, you earn $\$2^k$ (i.e., if your first head was the third flip, you would earn $\$8$). How much would you pay to play this game?

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$$X \sim Geo\left(\frac{1}{2}\right)$$

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So theoretically, any finite amount since your expected winnings is infinite!

Probability

3.6 Zoo of Discrete RV's Part III

Alex Tsun

Agenda

- The Poisson RV
- The Poisson Process
- The Hypergeometric RV

The Poisson RV (Motivation)

No random variable so far for events in a unit time:

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No random variable so far for events in a unit time:

- How many babies born in the next minute?

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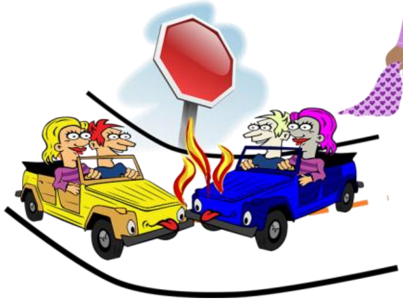
- How many car crashes happen per hour?



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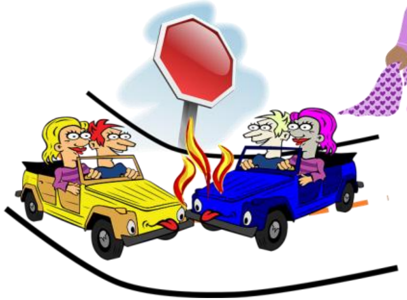

- How many babies born in the next minute?
- How many car crashes happen per hour?

If use binomial, what to choose for n? There is no upper bound!

The Poisson RV (Idea)

Let's say we want to model babies born in the next minute, if the historical average is 2 babies/min.

One Unit of Time

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____ One Unit of Time

_______ ________ _______ ________ _______ $Bin(n = \;\; 5, \;\; p = \qquad)$

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One Unit of Time

$Bin(n = \;\; 5, \;\; p = 2/5)$

$Bin(n = 10, \;\; p = \qquad)$

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One Unit of Time

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Let's say we want to model babies born in the next minute, if the historical average is 2 babies/min.

One Unit of Time

$Bin(n = \;\; 5, \;\; p = 2/5)$

$Bin(n = 10, \;\; p = 2/10)$

$Bin(n = 70, \;\; p = \qquad)$

The Poisson RV (Idea)

Let's say we want to model babies born in the next minute, if the historical average is 2 babies/min.

One Unit of Time

$Bin(n = 5, \; p = 2/5)$

$Bin(n = 10, \; p = 2/10)$

$Bin(n = 70, \; p = 2/70)$

The Poisson RV PMF

Let λ be the historical average number of events per unit of time.

Send $n \to \infty$ in such a way that $np = \lambda$ is fixed (i.e., $p = \lambda/n$).

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$$p_Y(k) = \lim_{n\to\infty} p_{X_n}(k) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \lim_{n\to\infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty} \frac{n!}{k!\,(n-k)!} \frac{\lambda^k}{n^k} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} = \frac{\lambda^k}{k!} \cdot \lim_{n\to\infty} \frac{n!}{(n-k)!} \frac{1}{n^k} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

$$= \frac{\lambda^k}{k!} \cdot \lim_{n\to\infty} \left(\frac{n}{n} \cdot \frac{n-1}{n} \cdot \ldots \cdot \frac{n-k+1}{n}\right) \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} = \frac{\lambda^k}{k!} \cdot \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} e^{-\lambda}$$

Poisson RV PMF sums to 1?

Recall Taylor series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

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Taylor Series

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Taylor Series

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Let λ be the historical average number of events per unit of time.
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<u>**Poisson RV:**</u> $X \sim Poi(\lambda)$ if and only if X has the following pmf:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0,1,2,\ldots$$

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$$E[X] = \lambda \qquad Var(X) = \lambda$$

For example, X is the number of babies born in a minute if on average λ babies are born per minute. By definition, if X, Y are independent with $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$, then $X + Y \sim Poi(\lambda + \mu)$.

Random Picture

Poisson RV Example

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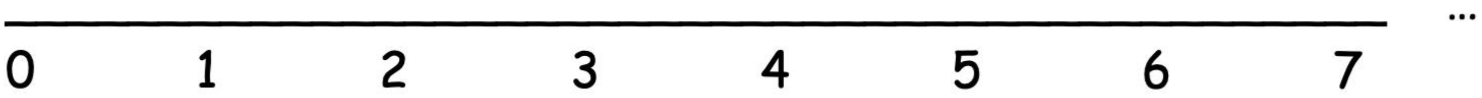
Poisson Process: A Poisson process with rate $\lambda > 0$ per unit of time, is a continuous-time process indexed by $t \in [0, \infty)$, so that $X(t)$ is the number of events that happened in the interval $[0, t]$. Notice that if $t_1 < t_2$, then $X(t_2) - X(t_1)$ is the number of events in $(t_1, t_2]$. The process has three properties:



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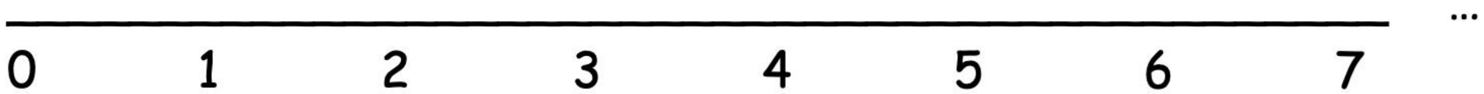

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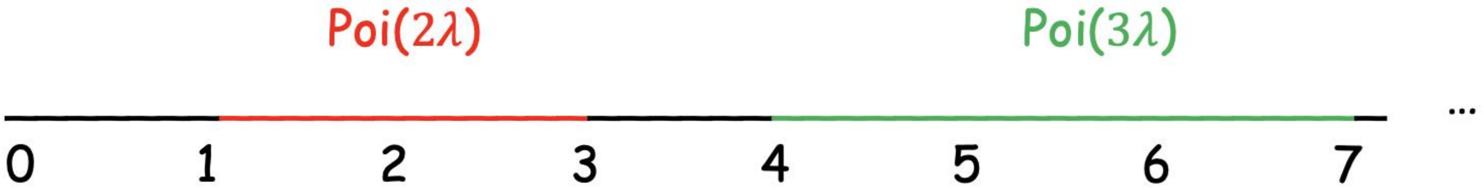

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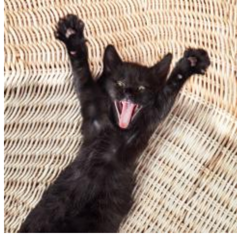
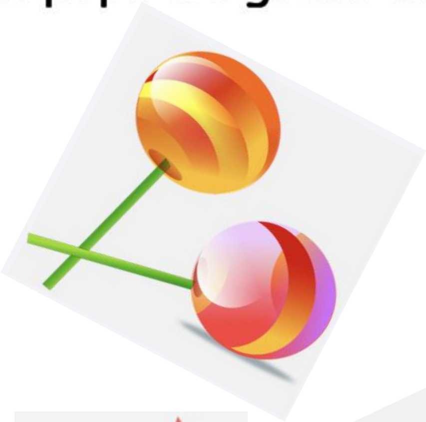
- The number of events in any time interval $[t_1, t_2]$ is $Poi(\lambda(t_2 - t_1))$.

Hypergeometric RV PMF

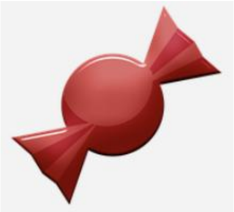


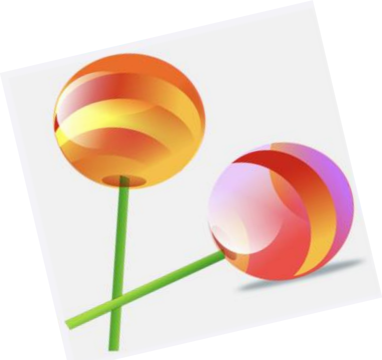
Suppose there is a candy bag of $N = 9$ total candies, $K = 4$ of which are lollipops. Our parents allow us grab $n = 3$ of them. Let X be the number of lollipops we grab. What is the probability that we get exactly 2 lollipops?

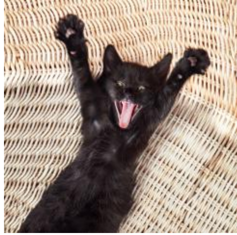


$$P(X = 2) =$$

$$p_X(k) =$$







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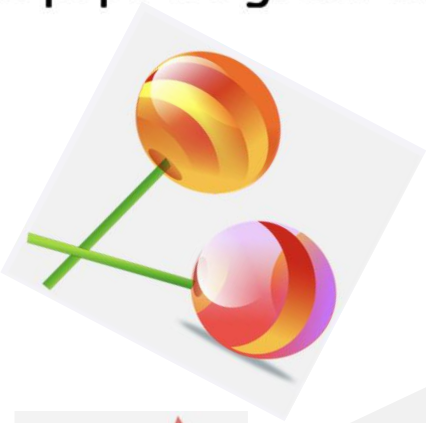

$$P(X = 2) = \frac{\quad}{\binom{9}{3}}$$

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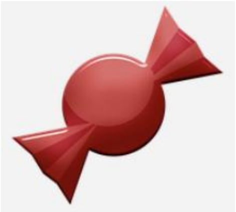

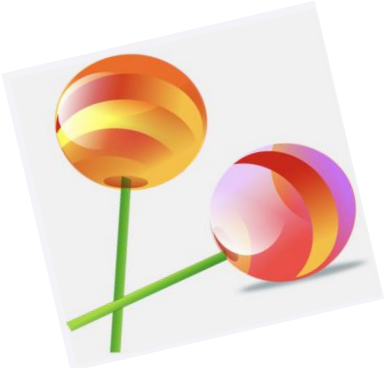

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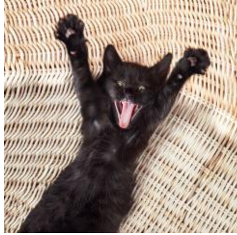

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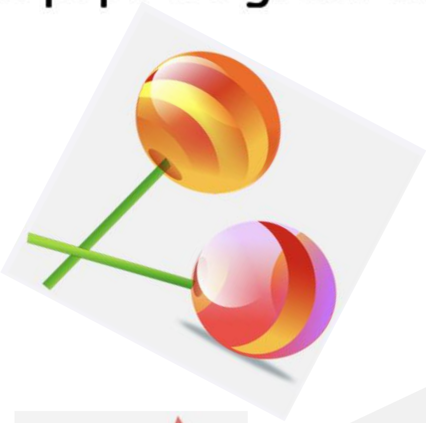

$$P(X = 2) = \frac{\binom{4}{2}}{\binom{9}{3}}$$

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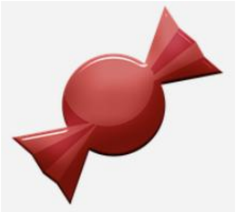

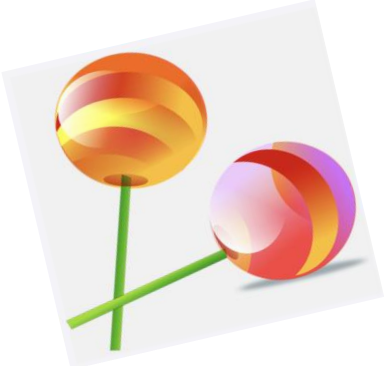

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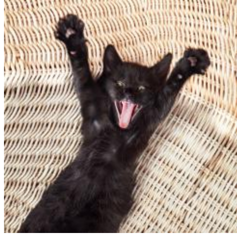

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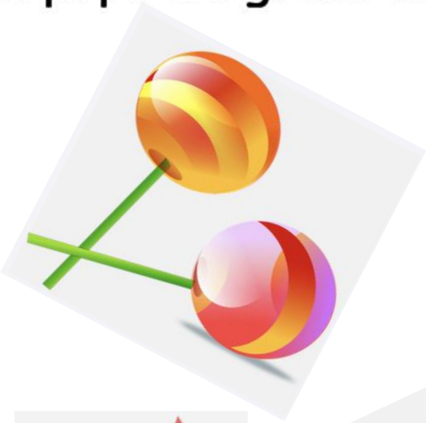

$$P(X = 2) = \frac{\binom{4}{2}\binom{5}{1}}{\binom{9}{3}}$$

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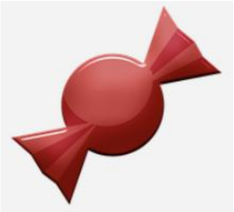

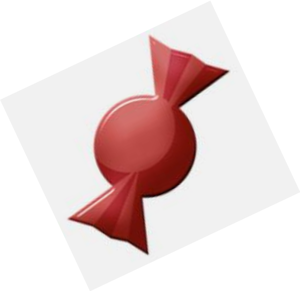

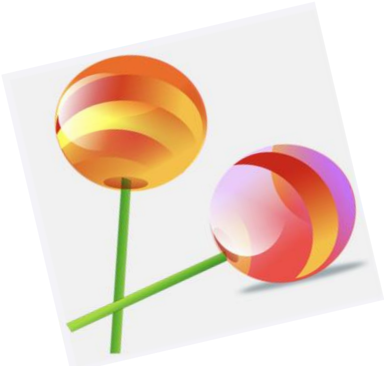

The Hypergeometric RV

Hypergeometric RV: $X \sim HypGeo(N, K, n)$ if and only if X has the following pmf:

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \qquad k = \max\{0, n+K-N\}, \ldots, \min\{K, n\}$$

For example, X is the number of successes when drawing n items without replacement from a bag containing N items, K of which are successes (hence $N - K$ failures).

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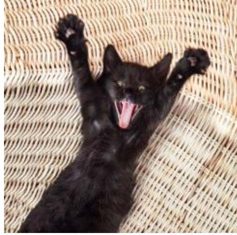
For example, X is the number of successes when drawing n items without replacement from a bag containing N items, K of which are successes (hence $N - K$ failures).

$$E[X] = n\frac{K}{N} \qquad Var(X) = \text{un-important}$$

If we drew with replacement, then we would model this situation using $Bin\left(n, \frac{K}{N}\right)$.

Hypergeometric RV Properties



Suppose $X \sim HypGeo(N, K, n)$.

Let $X_1, \ldots, X_n$ be indicator RV's (not independent) so that $X_i = 1$ if we got a lollipop on the i^{th} draw, and 0 otherwise. So $X = \sum_{i=1}^{n} X_i$.

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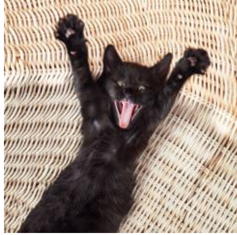

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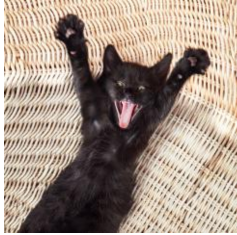

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$$= \frac{K-1}{N-1} \cdot \frac{K}{N} + \frac{K}{N-1} \cdot \frac{N-K}{N} = \frac{K(N-1)}{N(N-1)} = \frac{K}{N}$$

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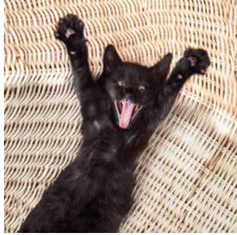

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$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = \frac{K}{N}$$

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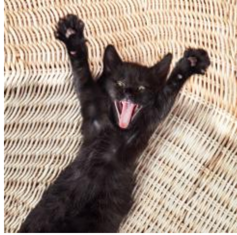

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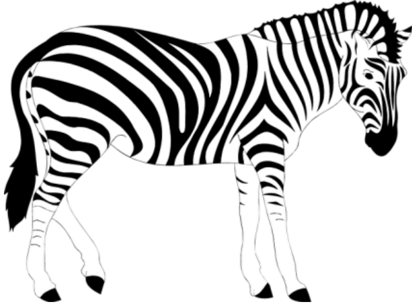
$$P(X_2 = 1) = P(X_2 = 1 | X_1 = 1)P(X_1 = 1) + P(X_2 = 1 | X_1 = 0)P(X_1 = 0)$$
$$= \frac{K-1}{N-1} \cdot \frac{K}{N} + \frac{K}{N-1} \cdot \frac{N-K}{N} = \frac{K(N-1)}{N(N-1)} = \frac{K}{N}$$

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = \frac{K}{N}$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{K}{N} = n\frac{K}{N}$$

The Zoo of Discrete RV's



- The Bernoulli RV
- The Binomial RV
- The Geometric RV
- The Negative Binomial RV
- The Uniform RV
- The Poisson RV
- The Hypergeometric RV