Probability

4.1 Continuous random Variables Basics

Alex Tsun

Agenda

- Probability Density Functions (PDFs)
- Cumulative Distribution Functions (CDFs)
- From Discrete to Continuous

The Need For PDFs

What if we want a random number that was equally likely in [0, 10]?

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The Need For PDFs

PDF

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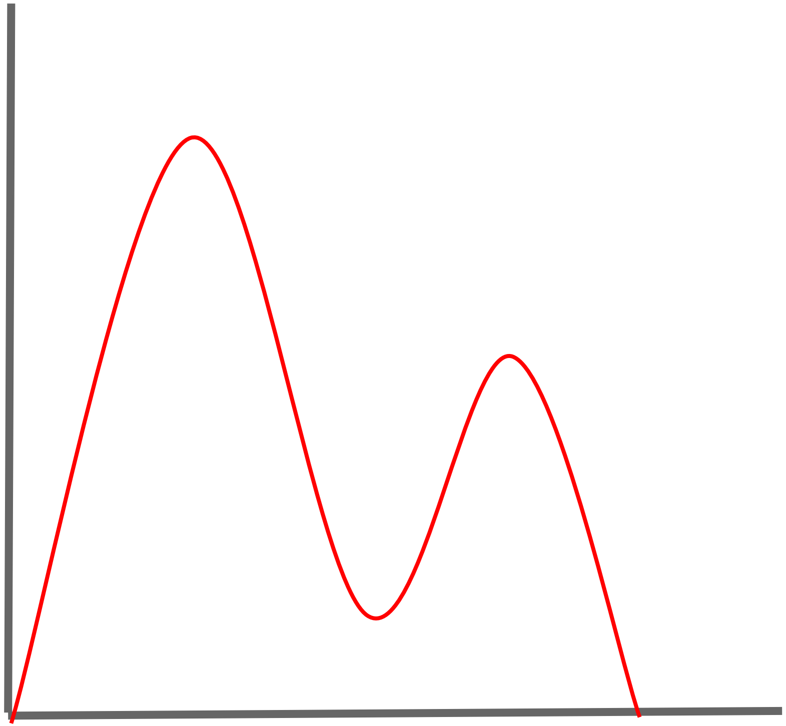
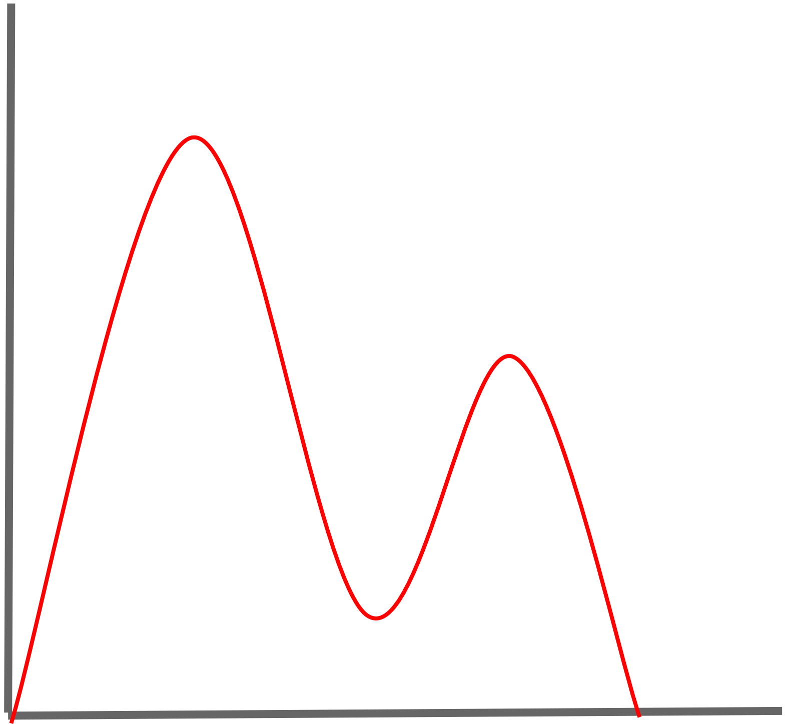
PDF Intuition



PDF Intuition

$f_X(z) \geq 0$ for all $z \in \mathbb{R}$

PDF



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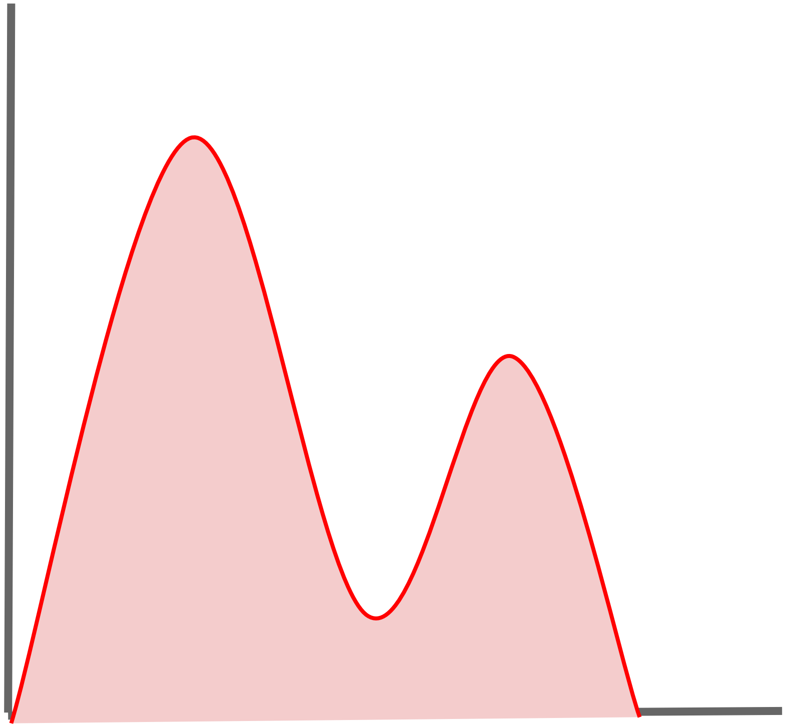

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PDF

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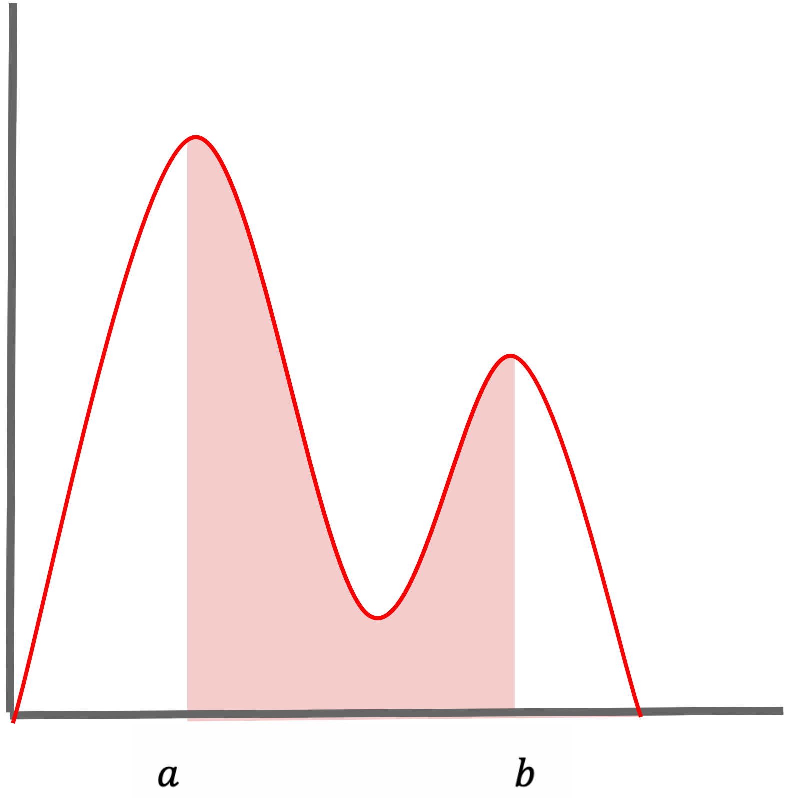

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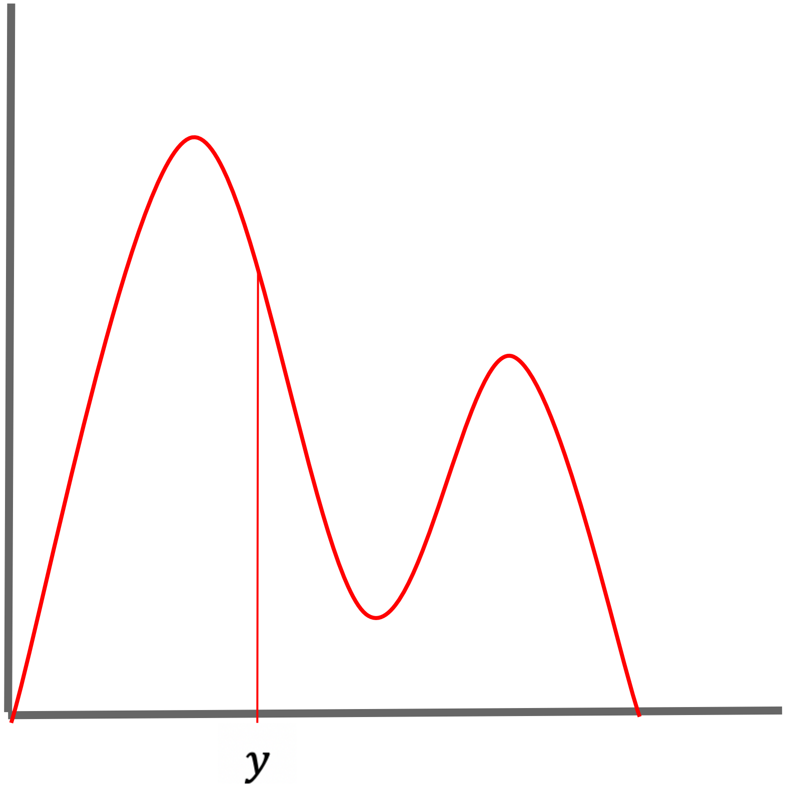

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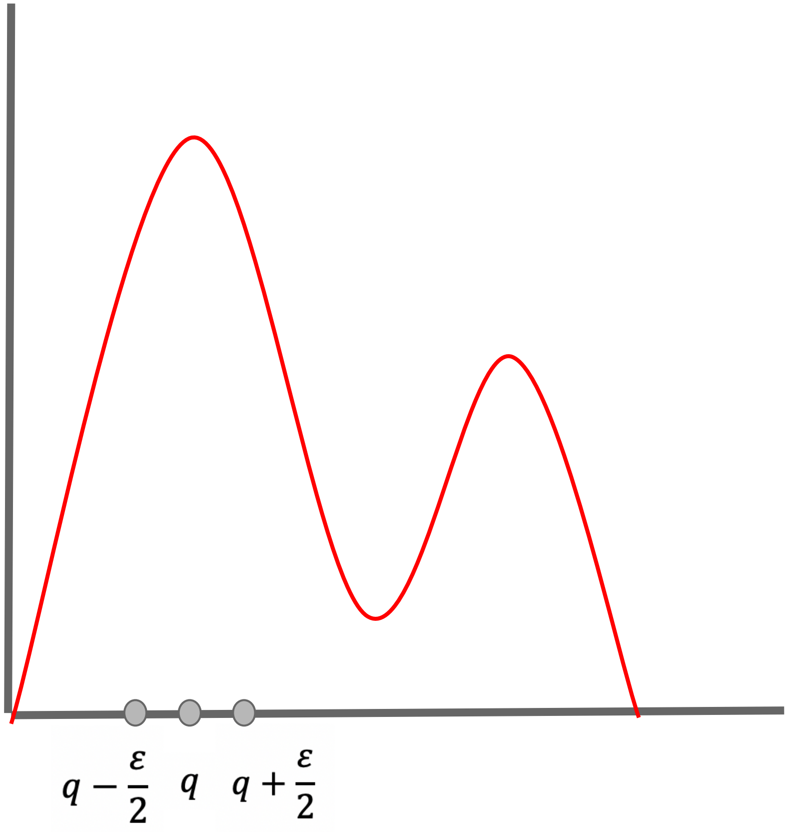

PDF Intuition

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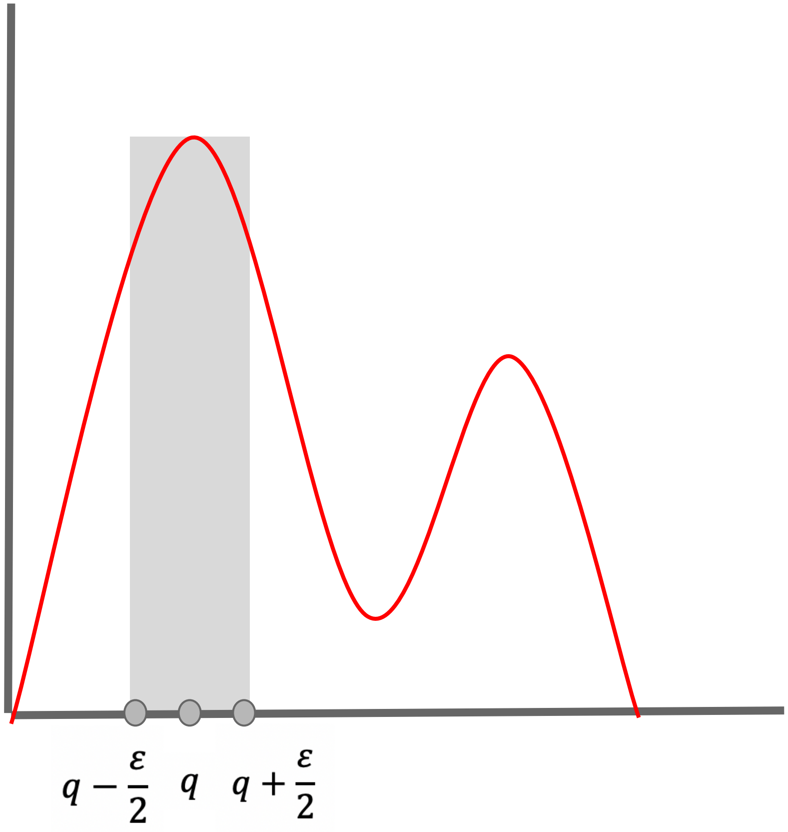

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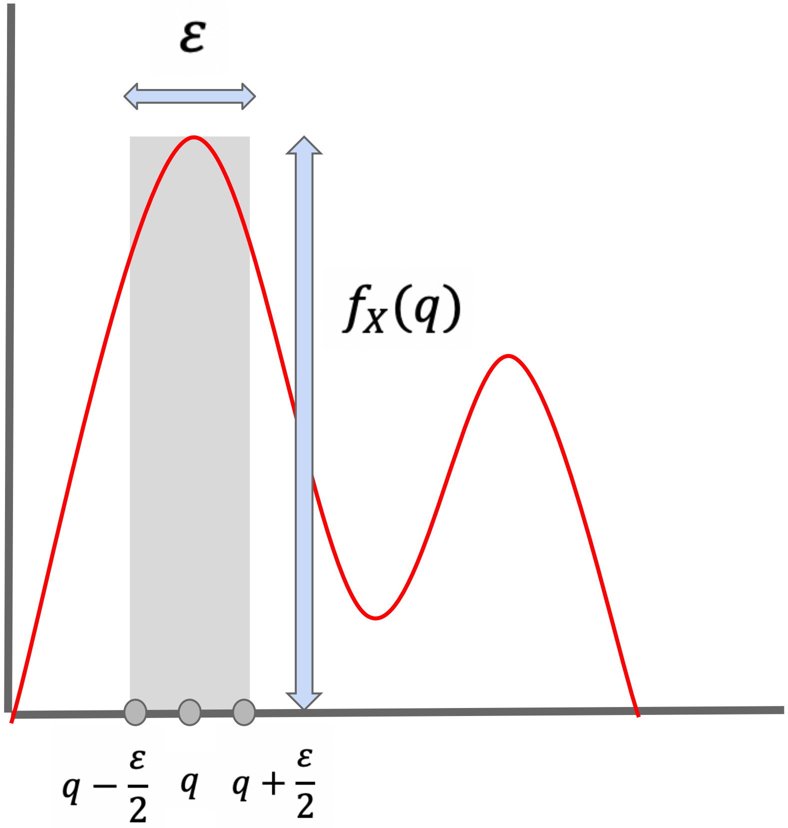

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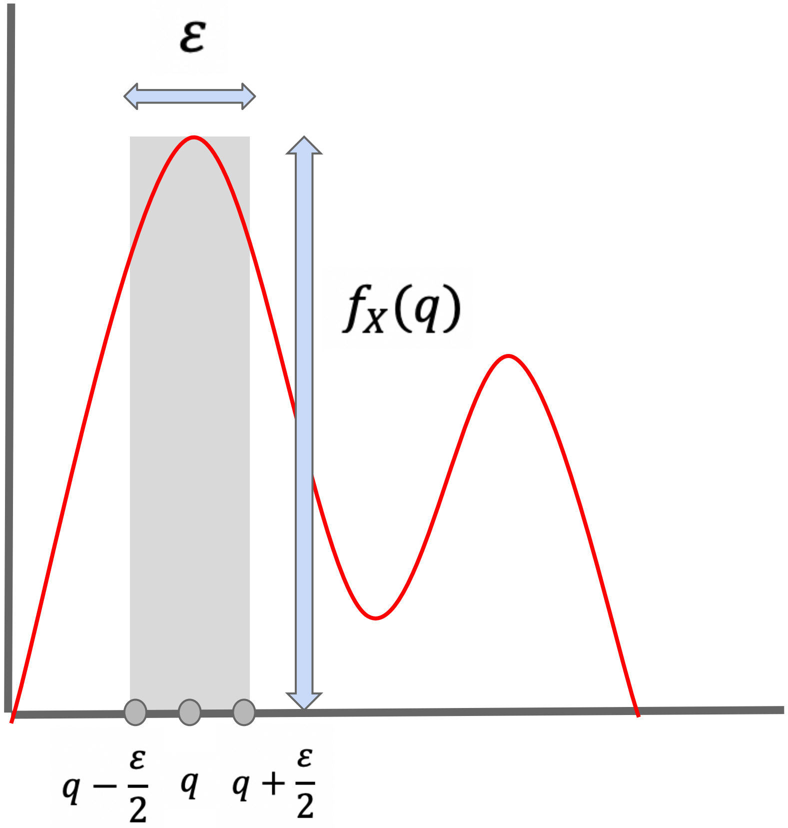

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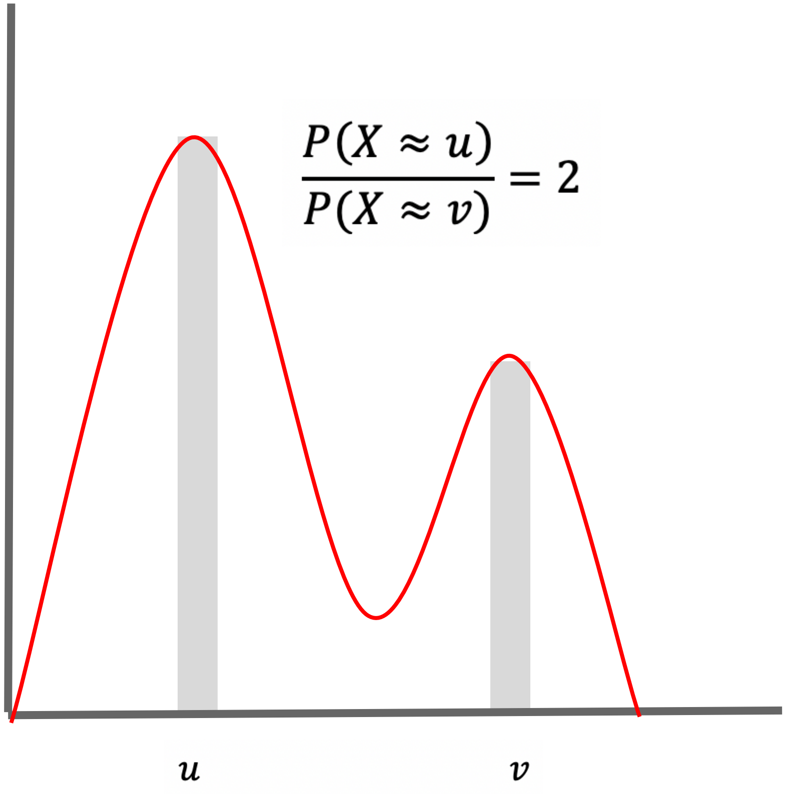

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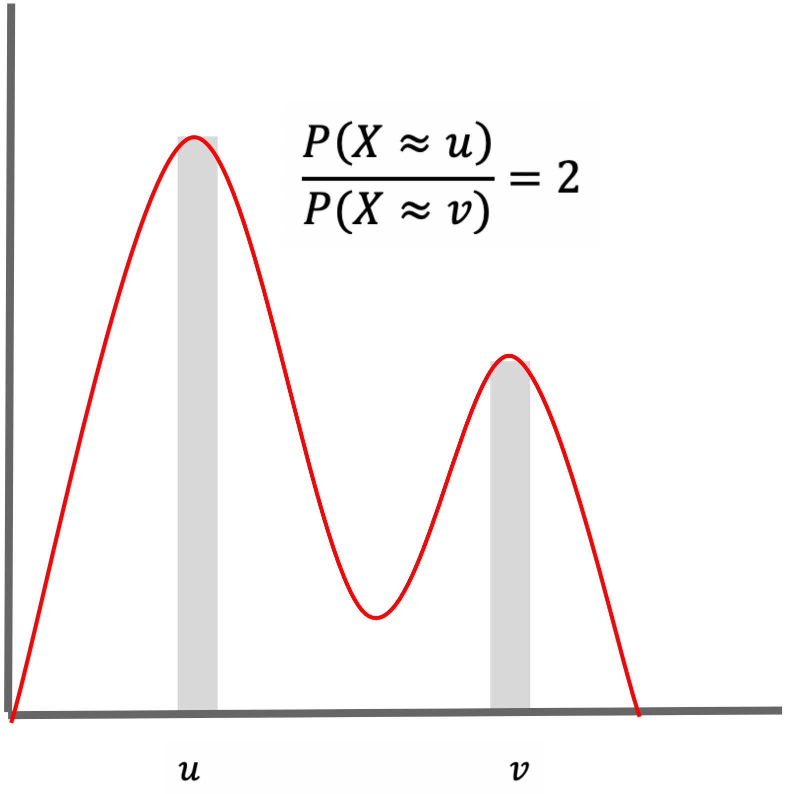
PDF Intuition



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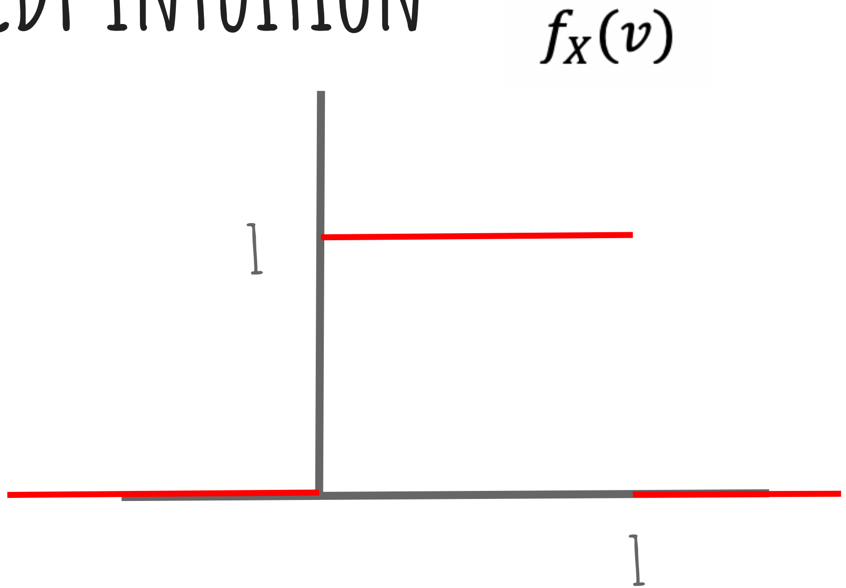
Probability Density Functions (PDFs)

<u>Probability Density Function (PDF):</u> Let X be a continuous rv (one whose range is typically an interval or union of intervals). The probability density function (PDF) of X is the function $f_X: \mathbb{R} \to \mathbb{R}$ such that

- $f_X(z) \geq 0$ for all $z \in \mathbb{R}$.
- $\int_{-\infty}^{\infty} f_X(t)dt = 1$.
- $P(a \leq X \leq b) = \int_a^b f_X(w)dw$.
- $P(X = y) = 0$ for any $y \in \mathbb{R}$.
- The probability that X is close to q is proportional to $f_X(q)$: $P(X \approx q) \approx P\left(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}\right) \approx \varepsilon f_X(q)$.
- Ratios of probabilities of being near points are maintained: $\frac{P(X \approx u)}{P(X \approx v)} = \frac{\varepsilon f_X(u)}{\varepsilon f_X(v)} = \frac{f_X(u)}{f_X(v)}$.

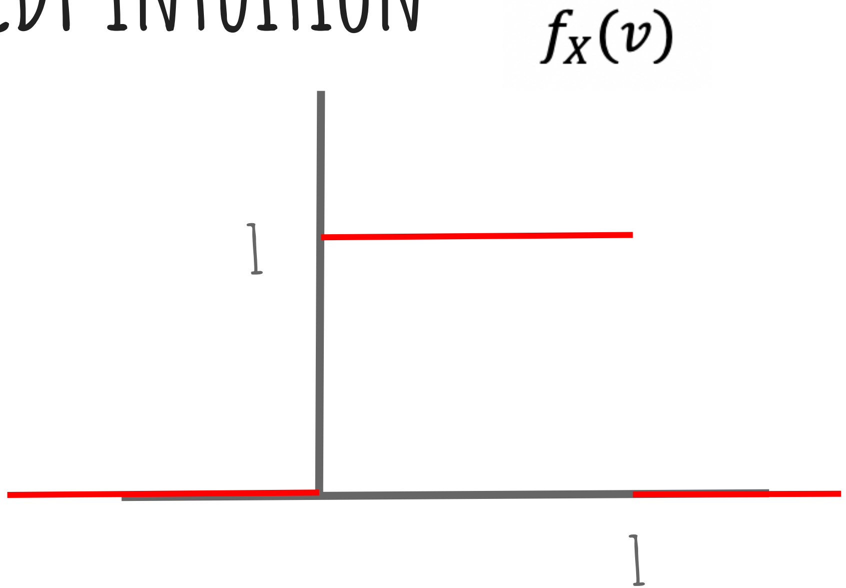
Random Picture

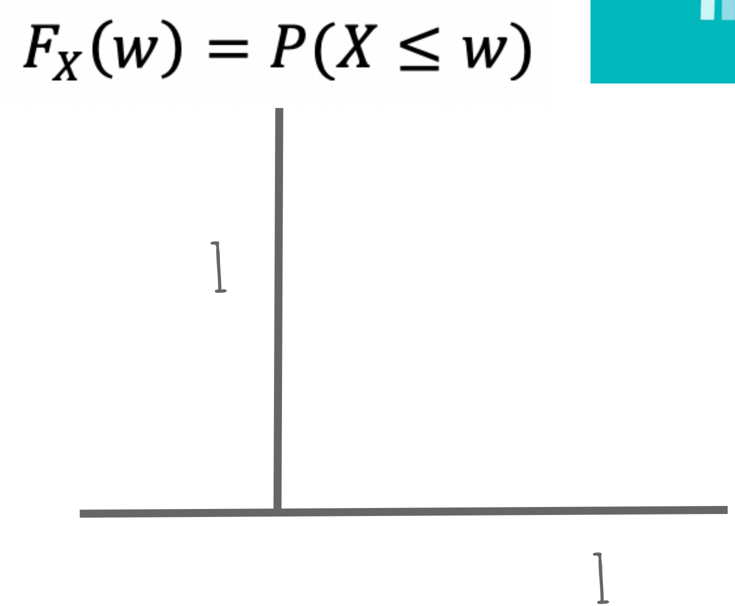
CDF Intuition



$$f_X(v) = \begin{cases} 1, & 0 \le v \le 1 \\ 0, & \text{otherwise} \end{cases}$$

CDF Intuition





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CDF Intuition

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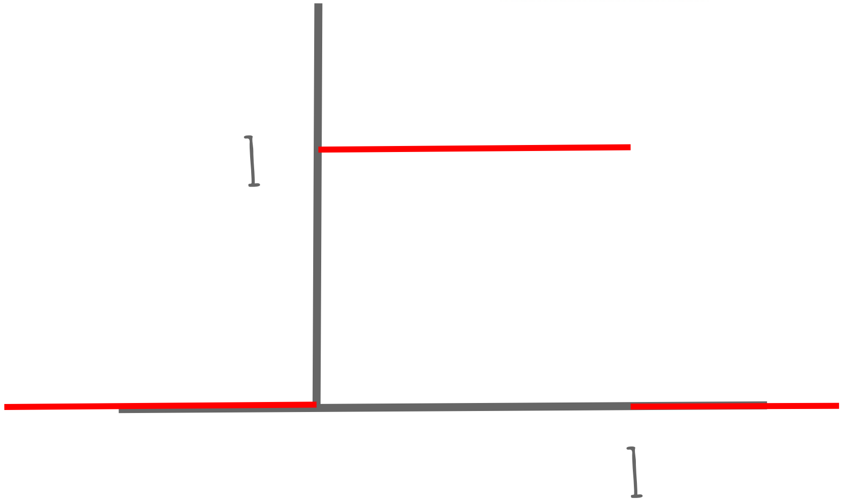
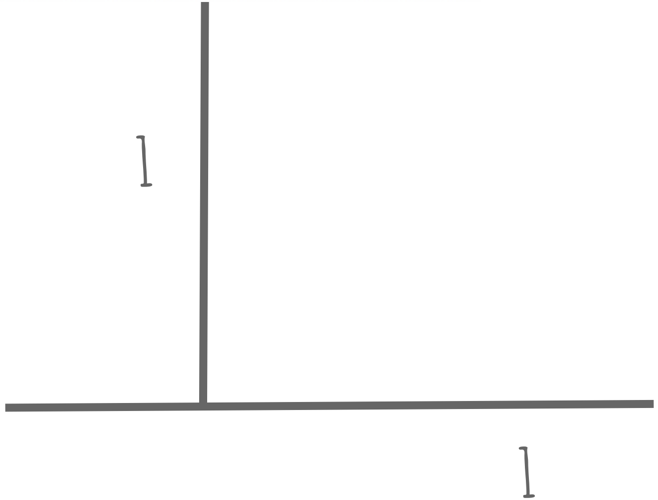


$F_X(w) = P(X \leq w)$



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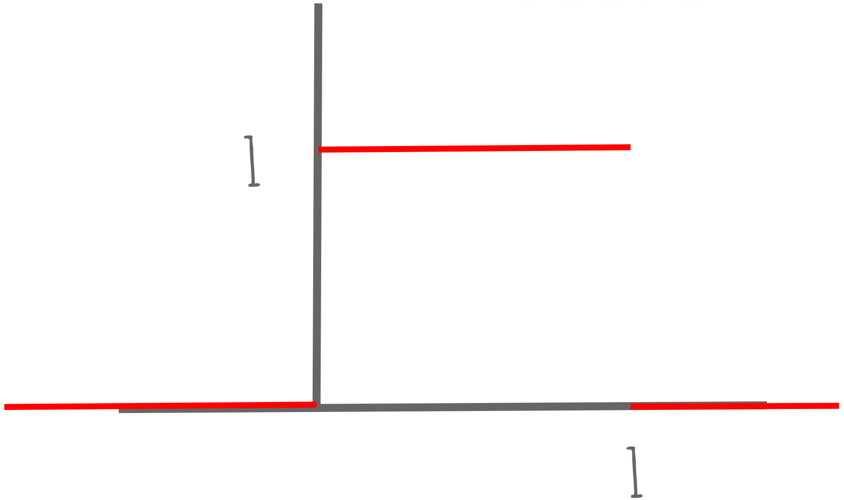

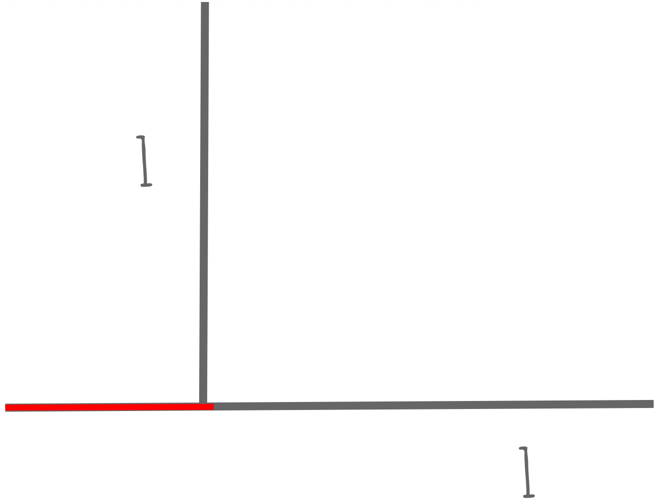

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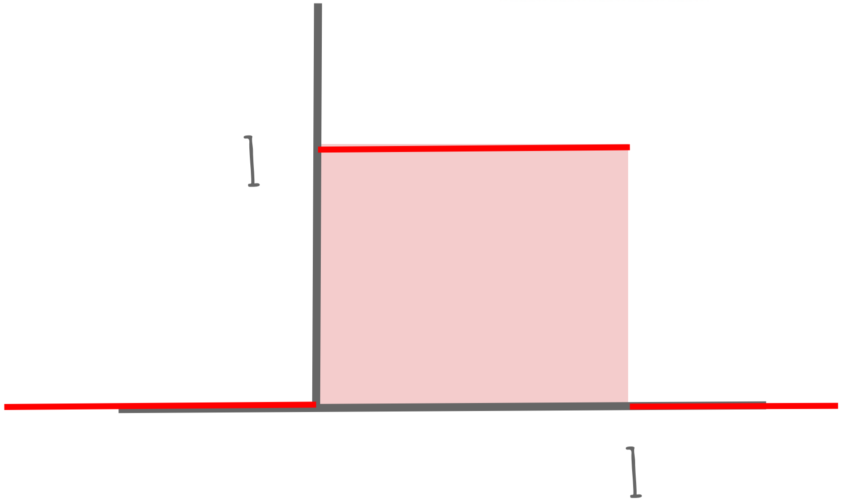

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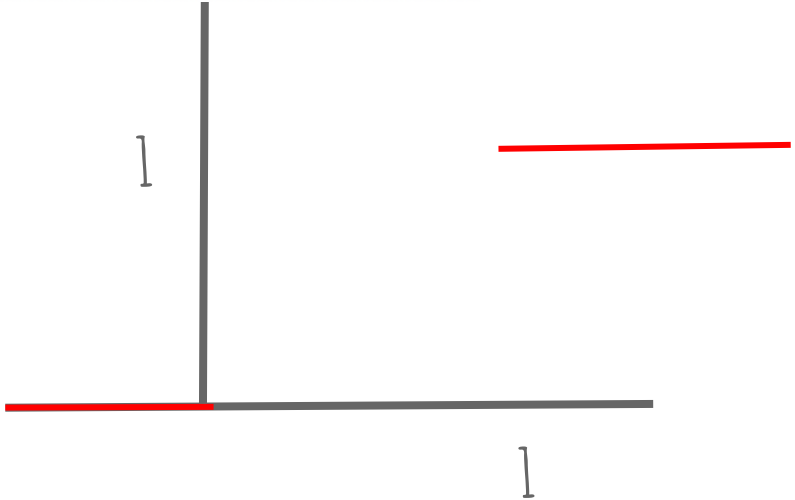

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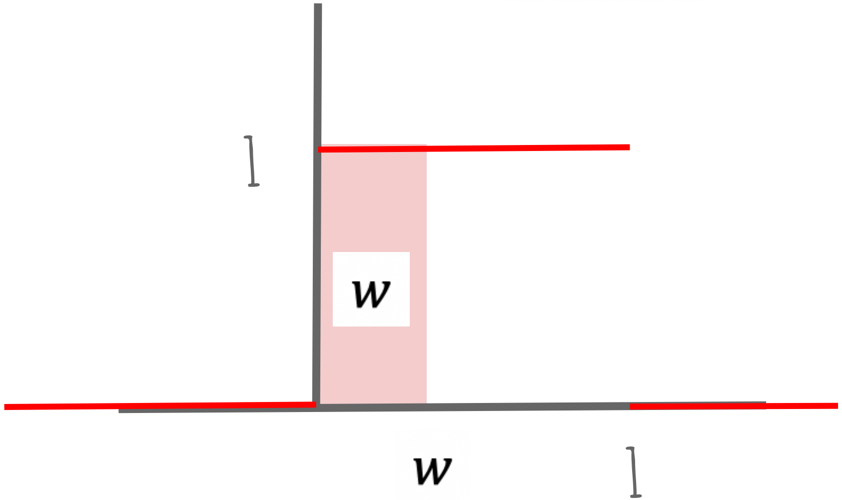

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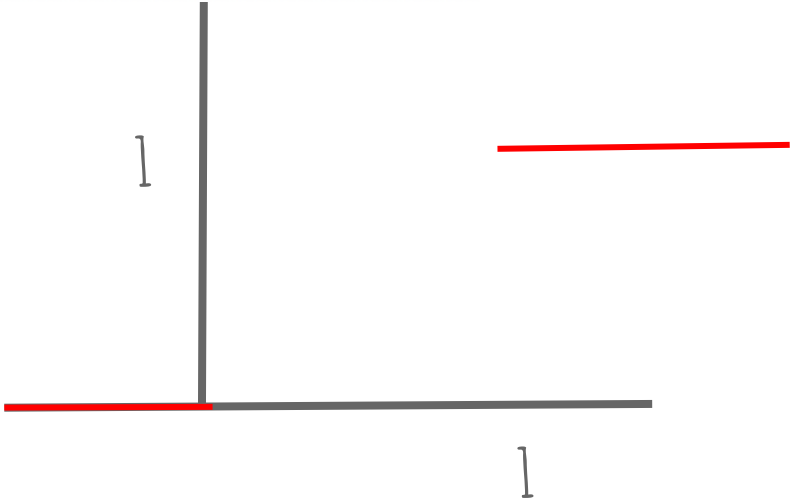

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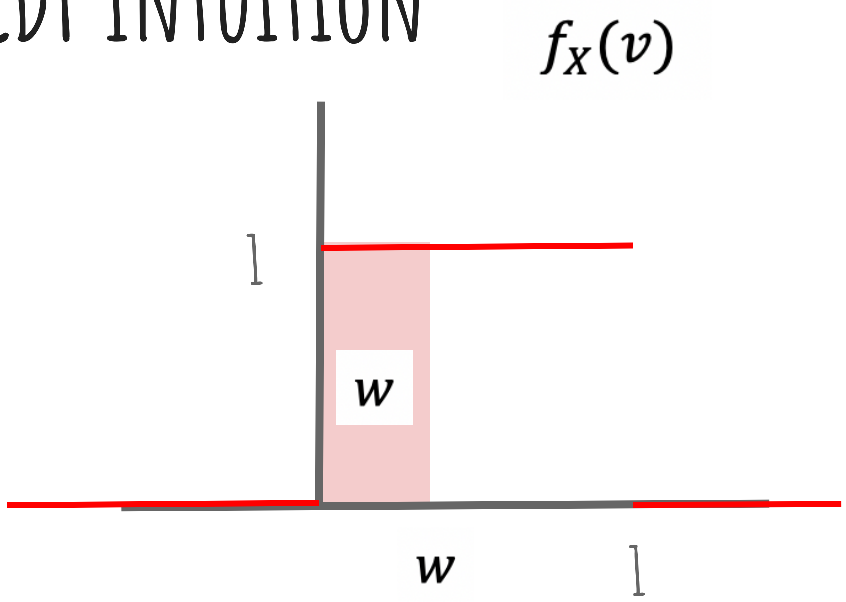

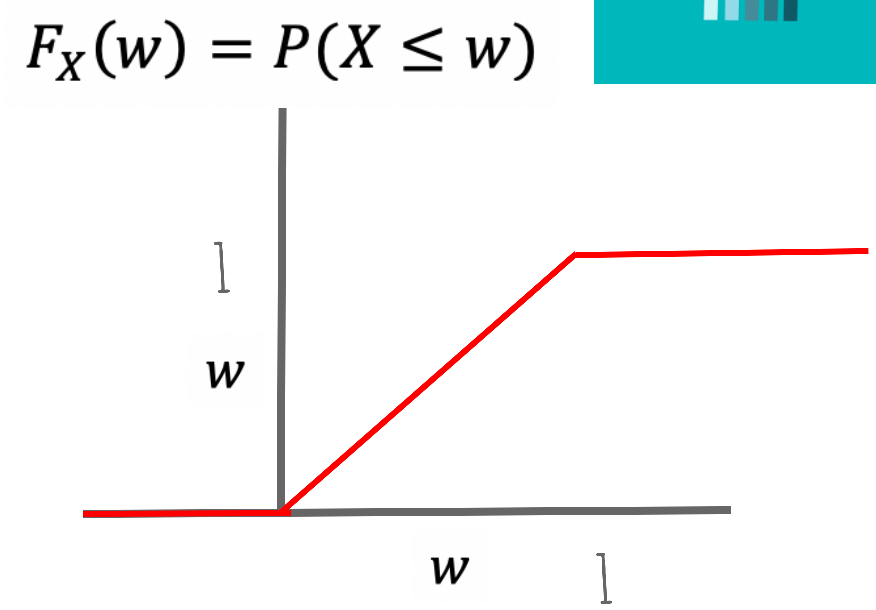

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CDF Intuition

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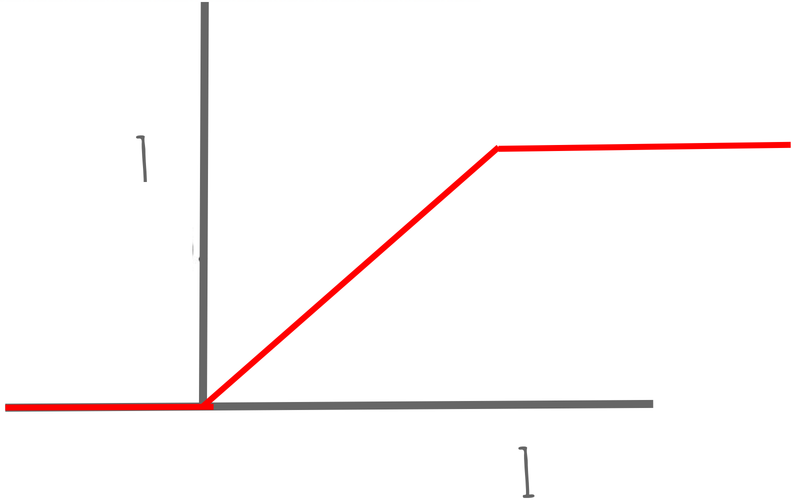

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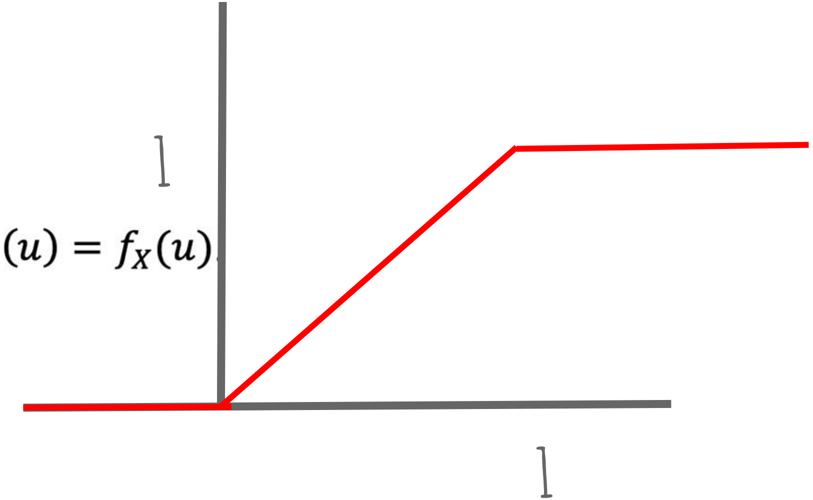

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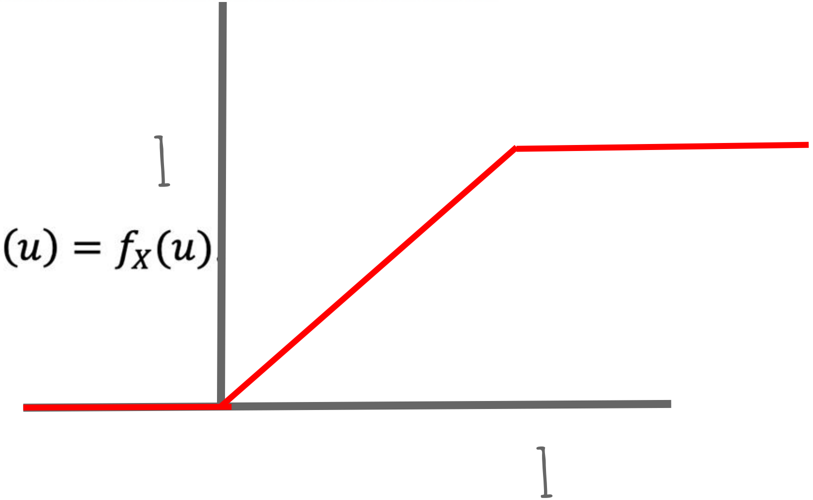

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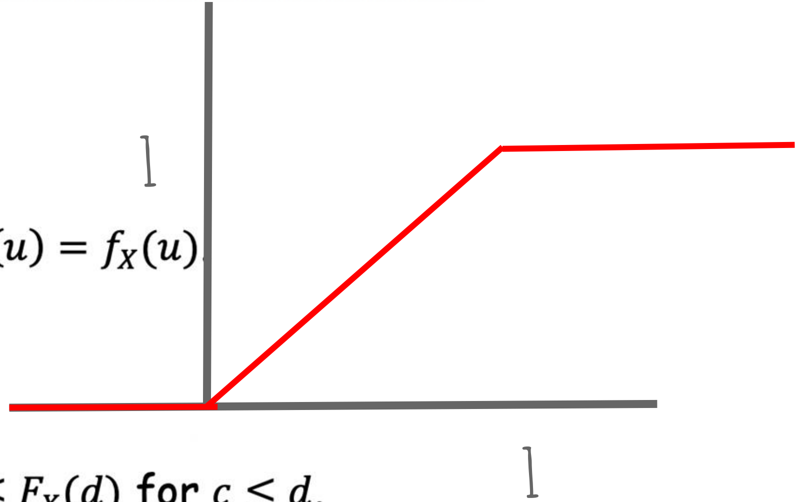

$P(a \leq X \leq b) = F_X(b) - F_X(a)$.

F_X is monotone increasing, since $f_X \geq 0$. That is, $F_X(c) \leq F_X(d)$ for $c \leq d$.

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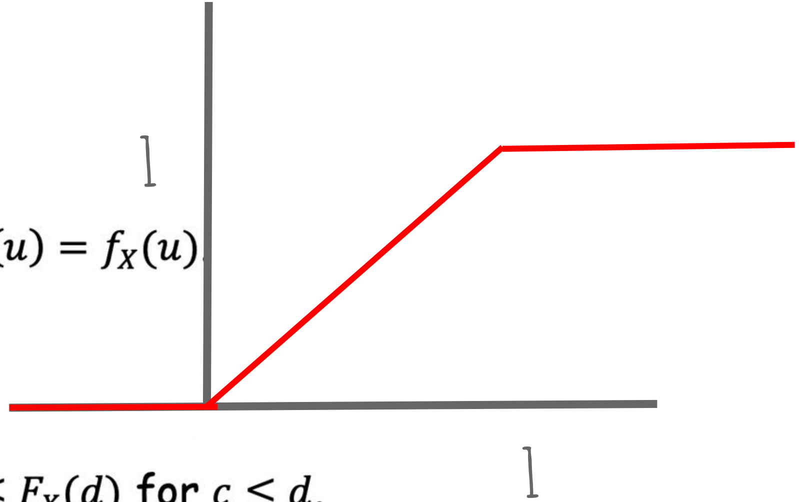

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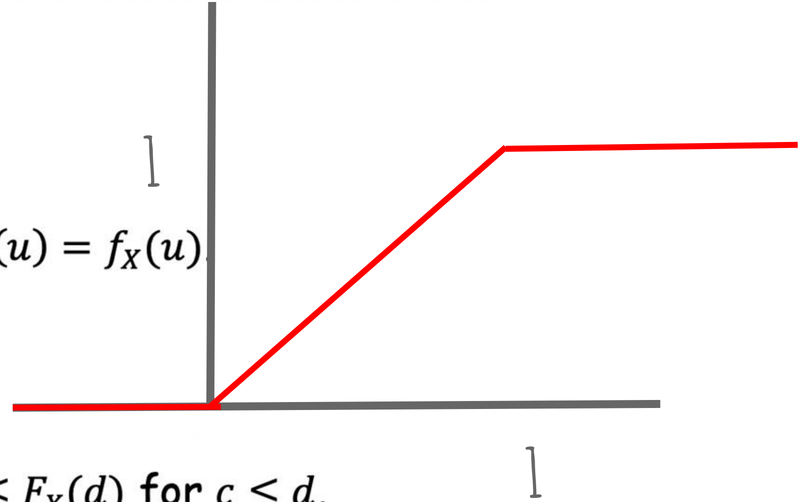

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Cumulative Distribution Functions (CDFs)

Cumulative Distribution Function (CDF): Let X be a continuous rv (one whose range is typically an interval or union of intervals). The cumulative distribution function (CDF) of X is the function $F_X: \mathbb{R} \to \mathbb{R}$ such that

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- Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du}F_X(u) = f_X(u)$.
- $P(a \leq X \leq b) = F_X(b) - F_X(a)$.
- F_X is monotone increasing, since $f_X \geq 0$. That is, $F_X(c) \leq F_X(d)$ for $c \leq d$.
- $\lim_{v \to -\infty} F_X(v) = P(X \leq -\infty) = 0$.
- $\lim_{v \to +\infty} F_X(v) = P(X \leq +\infty) = 1$.

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$

From Discrete to Continuous

	Discrete	**Continuous**
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

Probability

4.2 Zoo of Continuous RVs

Alex Tsun

Agenda

- The (Continuous) Uniform RV
- The Exponential RV
- Memorylessness
- The Gamma RV

The (Continuous) Uniform RV

Uniform (Continuous) RV: $X \sim Unif(a, b)$ where $a < b$ are real numbers, if and only if X has the following pdf:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

X is equally likely to take on any value in $[a, b]$.

The Uniform (Continuous) RV

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X is equally likely to take on any value in $[a, b]$.

$$E[X] = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

The cdf is

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$$

The Exponential PDF/CDF

Recall the Poisson Process with parameter $\lambda > 0$ has events happening at average rate of λ per unit of time forever. The exponential RV measures the time until the first occurrence of an event, so is a continuous RV with range $[0, \infty)$ (unlike the Poisson RV, which counts the number of occurrences in a unit of time, with range $\{0,1,2, \dots\}$.)

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The Exponential RV Properties

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx =$$

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$$Var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

The Exponential RV

Exponential RV: $X \sim Exp(\lambda)$, if and only if X has the following pdf:

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X is the waiting time until the first occurrence of an event in a Poisson process with parameter λ.

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$$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

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$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Random Picture

Memorylessness (Intuition)

A random variable X is **memoryless** if for all $s, t \geq 0$,

$$P(X > s + t \mid X > s) = P(X > t)$$

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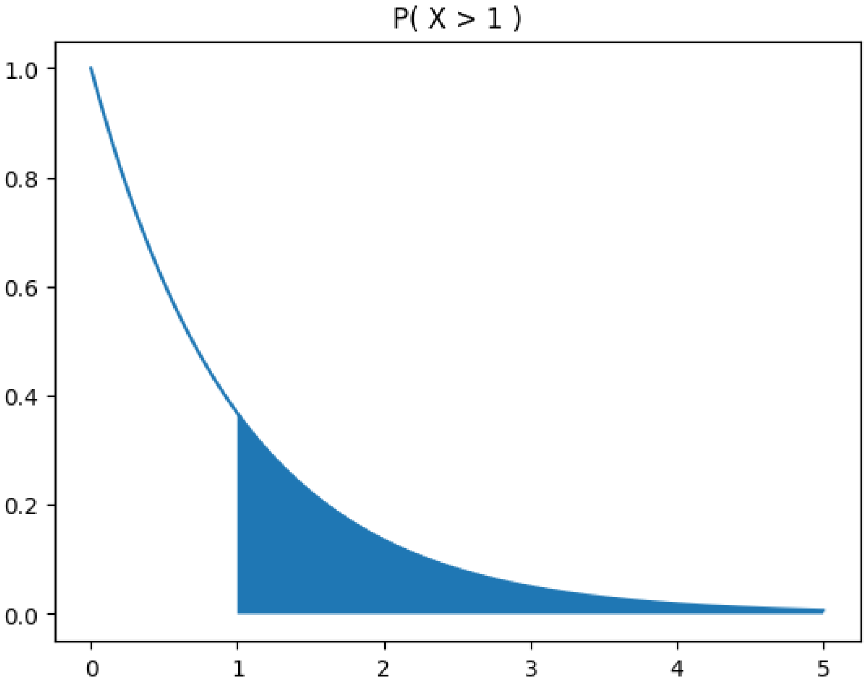
The only memoryless RVs are the **Geometric** (discrete) and **Exponential** (continuous)!

Memorylessness (Intuition)



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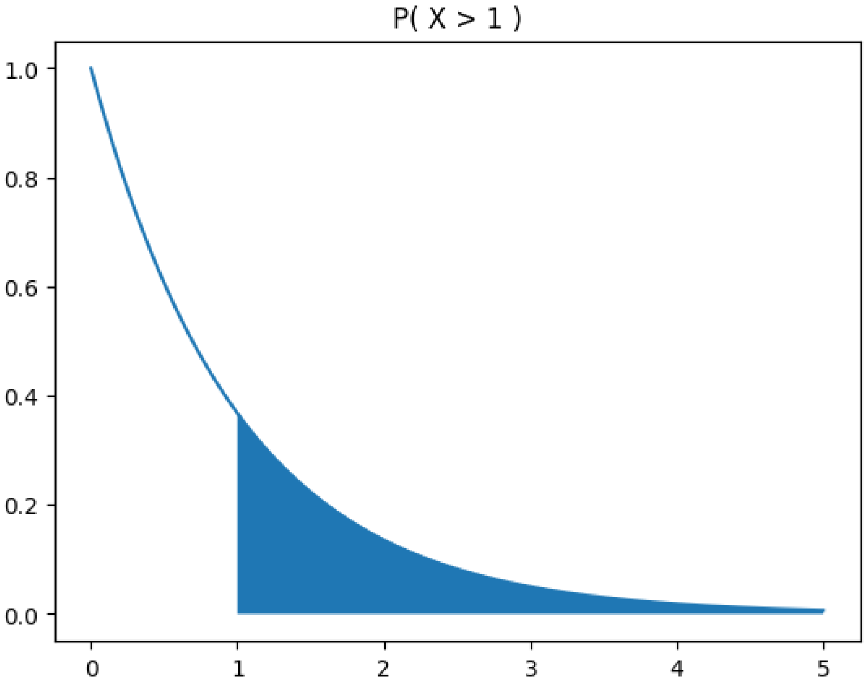

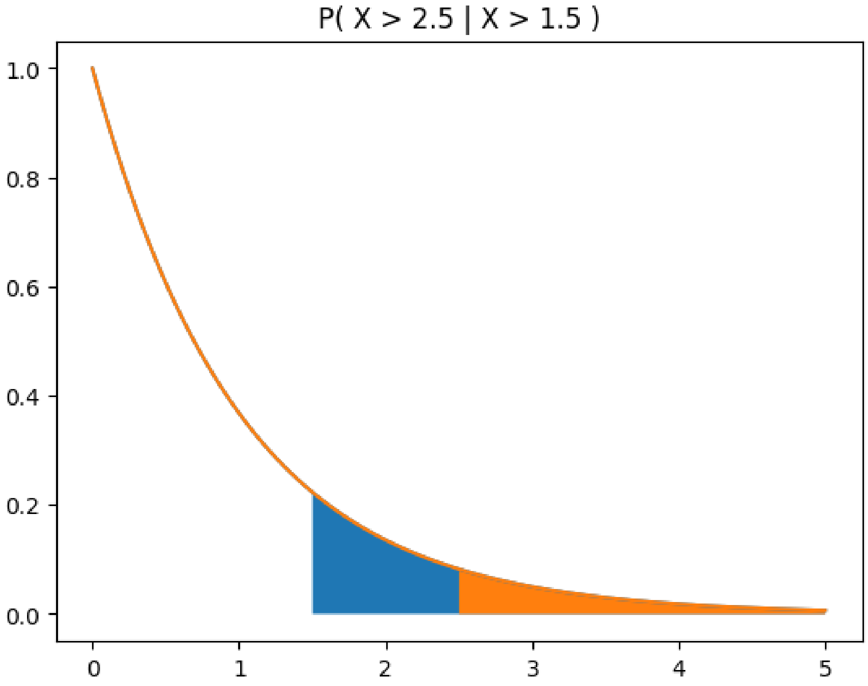

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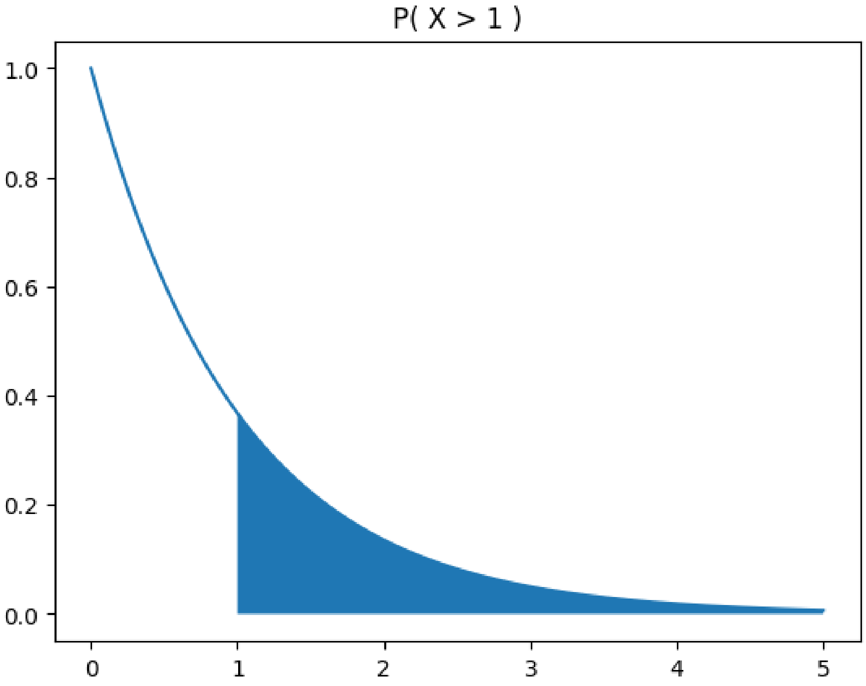

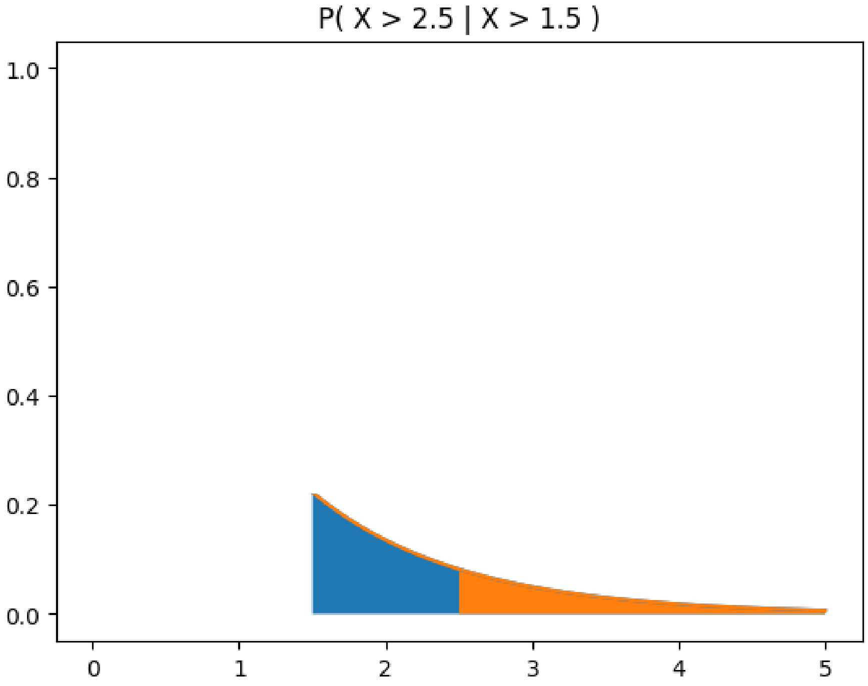

Memorylessness of Exponential (Proof)

If $X \sim Exp(\lambda)$ and $x \geq 0$, then recall

$$P(X > x) = 1 - F_X(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

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$$P(X > x) = 1 - F_X(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

$$P(X > s + t \mid X > s) = \frac{P(X > s \mid X > s + t)P(X > s + t)}{P(X > s)}$$

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The Gamma RV

Gamma RV: $X \sim Gamma(r, \lambda)$ if and only if X has the following pdf:

$$f_X(x) = \begin{cases} \dfrac{\lambda^r}{(r-1)!} x^{r-1} e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

X is the sum of r independent $Exp(\lambda)$ random variables.

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X is the sum of r independent $Exp(\lambda)$ random variables.

$$E[X] = \frac{r}{\lambda} \qquad Var(X) = \frac{r}{\lambda^2}$$

X is the waiting time until the r^{th} occurrence of an event in a Poisson process with parameter λ. Notice that $Gamma(1, \lambda) \equiv Exp(\lambda)$. By definition, if X, Y are independent with $X \sim Gamma(r, \lambda)$ and $Y \sim Gamma(s, \lambda)$, then $X + Y \sim Gamma(r + s, \lambda)$.

Probability

4.3 The Normal/Gaussian Random Variable

Joshua Fan
Alex Tsun

Agenda

- Standardizing RVs
- The Normal/Gaussian RV
- Closure properties of the Normal RV
- The standard normal CDF

Standardizing RVs (Intuition)

On your history test, you got a 90% when the mean was 70% and SD was 10%.

On your math test, you got a 50% when the mean was a 35% and the SD was 5%.

Which test did you do "better on", in terms of standard deviations above the mean?

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$$\frac{90-70}{10}=2$$

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$$\frac{50-35}{5}=3$$

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$$\frac{90 - 70}{10} = 2$$

On your math test, you got a 50% when the mean was a 35% and the SD was 5%.

$$\frac{50 - 35}{5} = 3$$

Which test did you do "better on", in terms of standard deviations above the mean?

You computed $\frac{X - \mu}{\sigma}$. Did better on **math.**

Standardized for fair comparison!

Standardizing RVs

Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Then,

Standardizing RVs

Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Then,

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Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Then,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

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In particular, we call $\frac{X-\mu}{\sigma}$ a standardized version of X, as it measures how many standard deviations above the mean a point is.

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In particular, we call $\frac{X-\mu}{\sigma}$ a standardized version of X, as it measures how many standard deviations above the mean a point is.

$$E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}(E[X] - \mu) = 0$$

Standardizing RVs

Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Then,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2Var(X) = a^2\sigma^2$$

In particular, we call $\frac{X-\mu}{\sigma}$ a standardized version of X, as it measures how many standard deviations above the mean a point is.

$$E\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma}(E[X] - \mu) = 0$$

$$Var\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(X - \mu) = \frac{1}{\sigma^2}\sigma^2 = 1$$

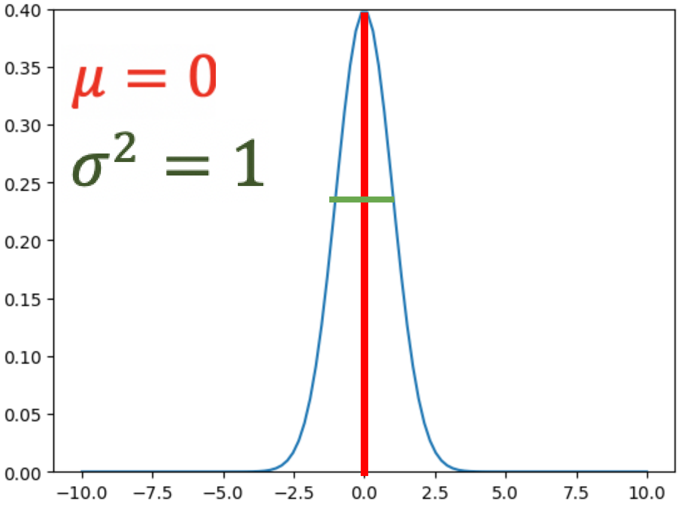
The Normal/Gaussian RV

Normal (Gaussian, "bell curve") Distribution: $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if X has the following pdf:

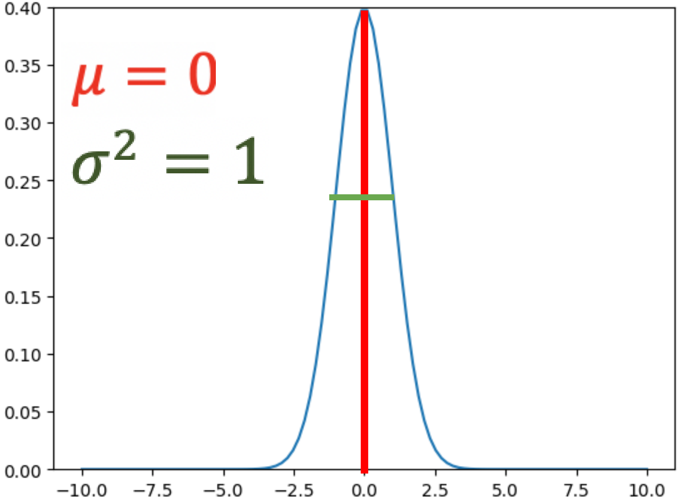
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

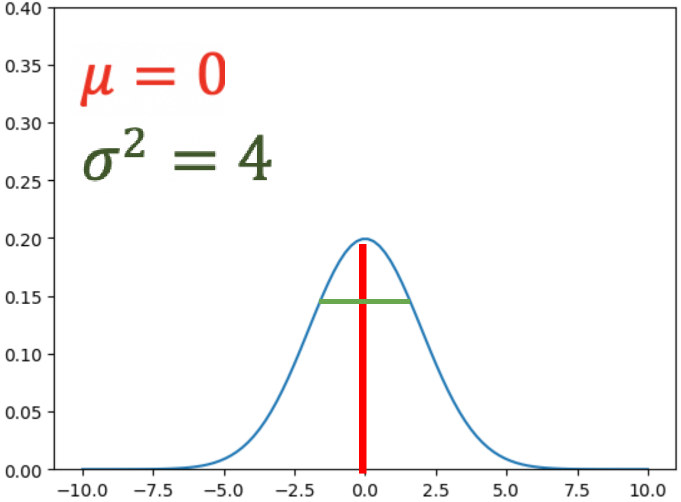
$$E[X] = \mu \qquad\qquad Var(X) = \sigma^2$$

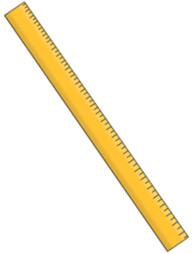
THE NORMAL PDF



THE NORMAL PDF







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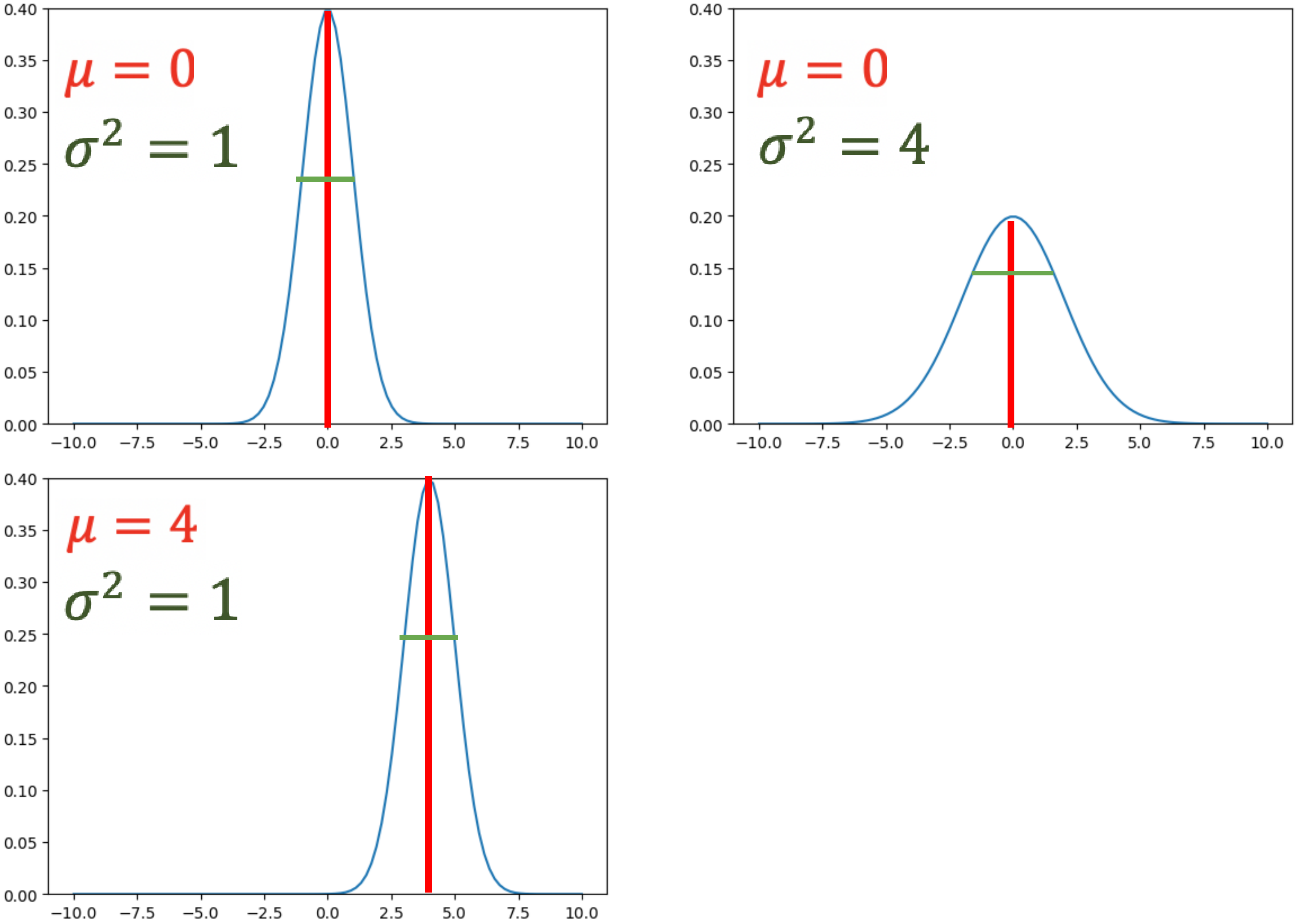

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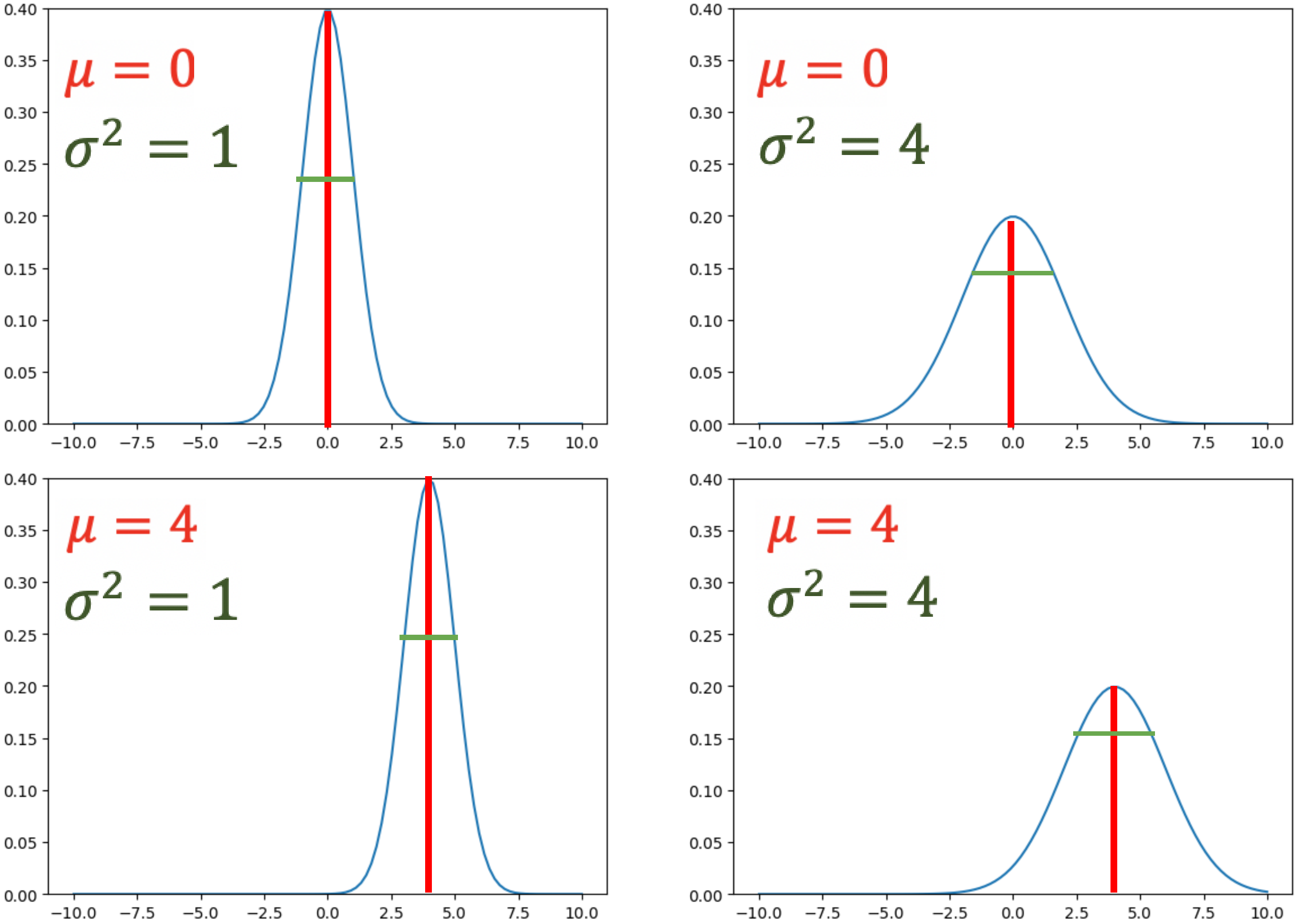

Random Picture

Closure of the Normal (Under Scale+Shift)

Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Recall,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

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$$Var(aX + b) = a^2Var(X) = a^2\sigma^2$$

But if $X \sim \mathcal{N}(\mu, \sigma^2)$ (a Normal rv), then

$$aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

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In particular,

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

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$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

Note the "special" thing here is that the transformed RV remains a Normal rv - the mean and variance are no surprise.

Closure of the normal (under addition)

Let X, Y be **ANY independent** random variables (discrete or continuous) with $E[X] = \mu_X$, $E[Y] = \mu_Y$, $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$, and $a, b, c \in \mathbb{R}$. Recall,

$$E[aX + bY + c] = aE[X] + bE[Y] + c = a\mu_X + b\mu_Y + c$$

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y) = a^2\sigma_X^2 + b^2\sigma_Y^2$$

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But if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent Normal rvs), then

$$aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$$

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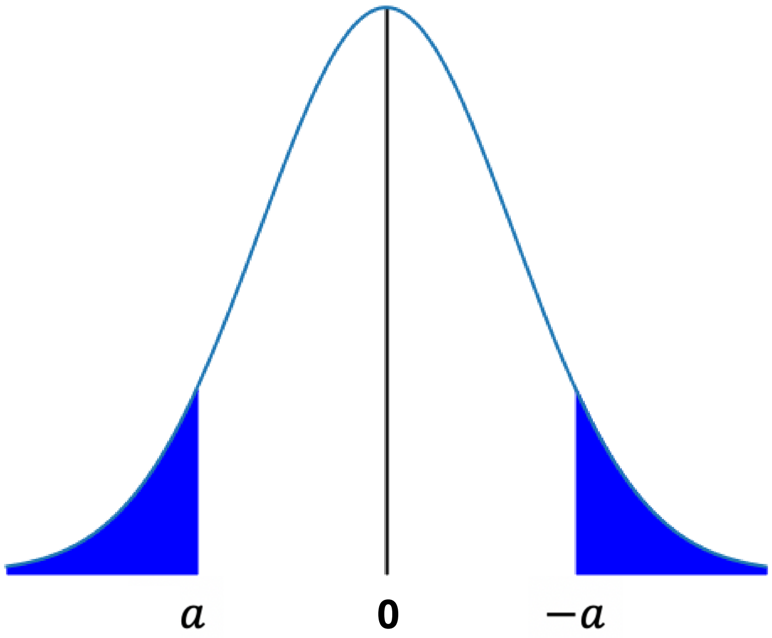
$$aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$$

Note the "special" thing here is that the sum remains a Normal rv – the mean and variance are no surprise.

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.



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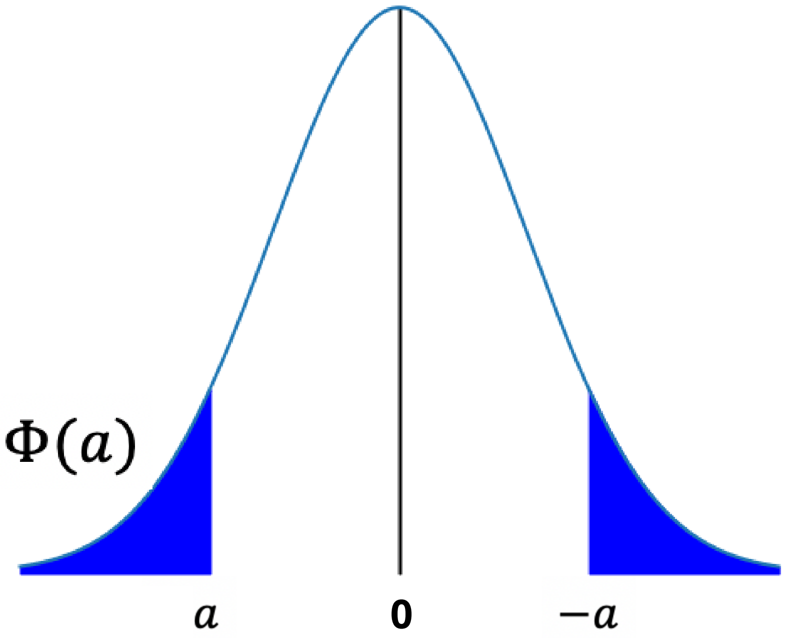

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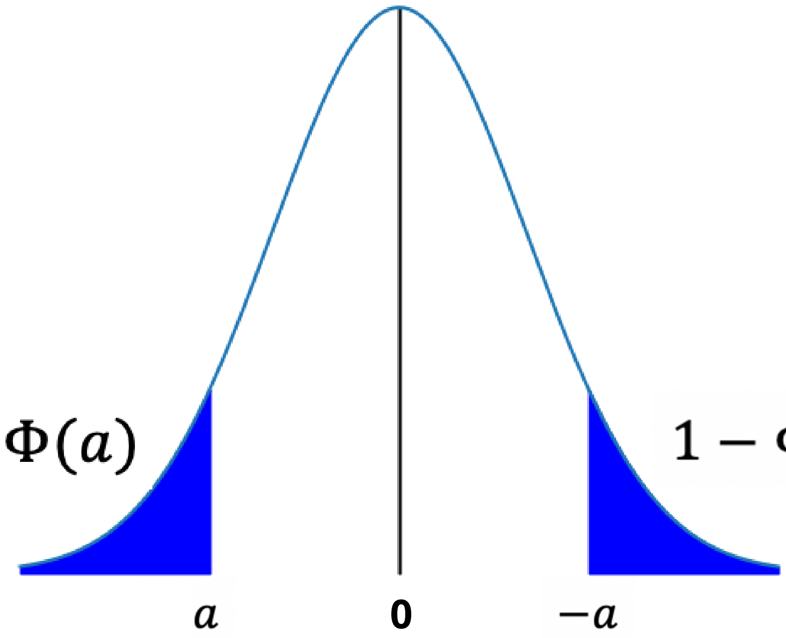

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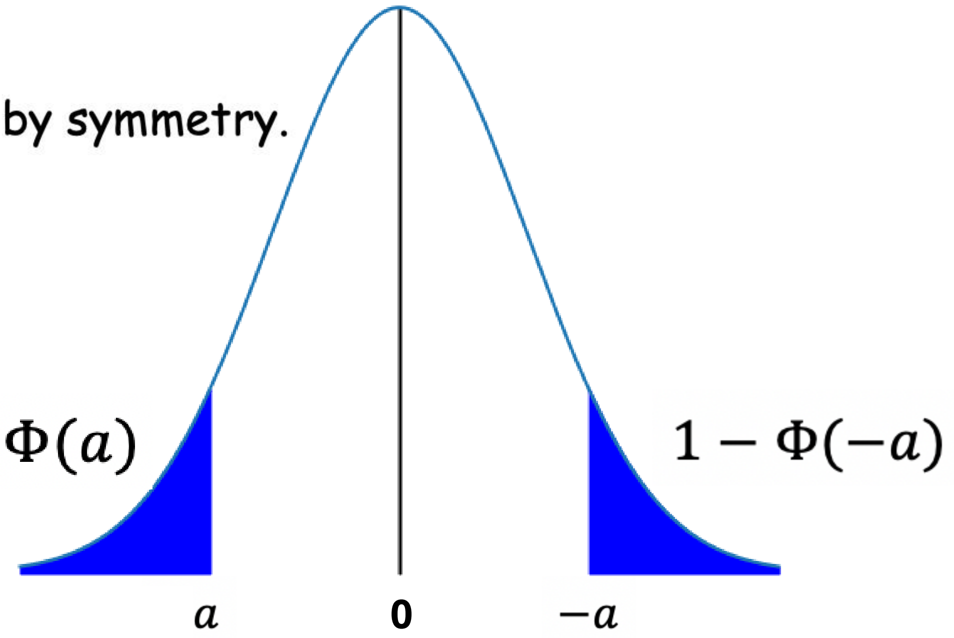
Notice $\Phi(a) = 1 - \Phi(-a)$ by symmetry.



The Standard Normal CDF

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

The Standard Normal CDF

$$P(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$$

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
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2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

The Standard Normal CDF

$$P(Z \le 1.09) = \Phi(1.09) \approx 0.8621$$

Usually only has positive numbers, so use the trick $\Phi(a) = 1 - \Phi(-a)$.

Φ Table: $\mathbb{P}(Z \le z)$ when $Z \sim \mathcal{N}(0,1)$

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1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

DEF. OF CDF

$$F_X(y) = P(X \leq y) =$$

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

Standardize Both Sides

$$F_X(y) = P(X \leq y) = P\left(\underbrace{\boxed{\frac{X-\mu}{\sigma}}}_{\sim N(0,1)} \leq \frac{y-\mu}{\sigma}\right) =$$

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$F_X(y) = P(X \leq y) = P\left(\boxed{\frac{X-\mu}{\sigma}} \leq \frac{y-\mu}{\sigma}\right) = P\left(Z \leq \frac{y-\mu}{\sigma}\right) =$$

$\sim N(0,1)$

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

Def. of phi (Φ)

$$F_X(y) = P(X \leq y) = P\left(\frac{X-\mu}{\sigma} \leq \frac{y-\mu}{\sigma}\right) = P\left(Z \leq \frac{y-\mu}{\sigma}\right) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$F_X(y) = P(X \leq y) = P\left(\frac{X-\mu}{\sigma} \leq \frac{y-\mu}{\sigma}\right) = P\left(Z \leq \frac{y-\mu}{\sigma}\right) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

$$P(a \leq X \leq b) =$$

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$F_X(y) = P(X \leq y) = P\left(\frac{X-\mu}{\sigma} \leq \frac{y-\mu}{\sigma}\right) = P\left(Z \leq \frac{y-\mu}{\sigma}\right) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

$$P(a \leq X \leq b) = F_X(b) - F_X(a) =$$

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$F_X(y) = P(X \leq y) = P\left(\frac{X-\mu}{\sigma} \leq \frac{y-\mu}{\sigma}\right) = P\left(Z \leq \frac{y-\mu}{\sigma}\right) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

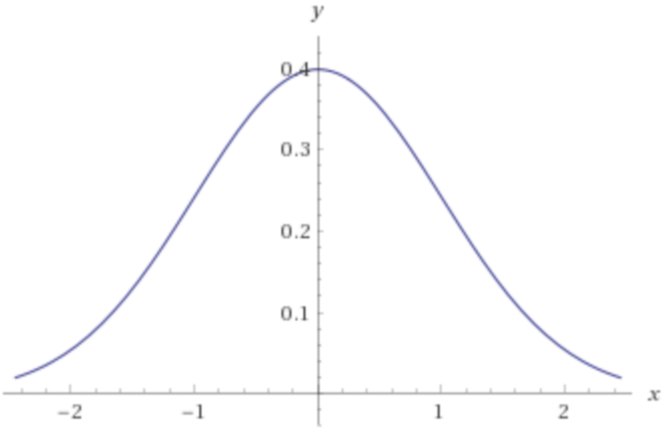
Summary: The Normal/Gaussian RV

<u>Normal (Gaussian, "bell curve") Distribution:</u> $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if X has the following pdf:

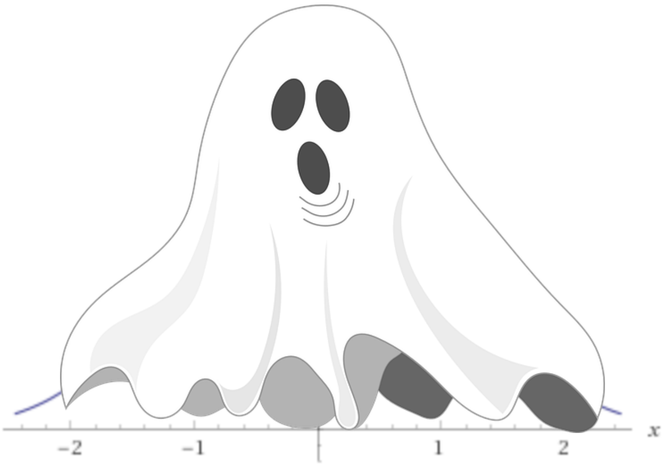
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[X] = \mu \qquad\qquad Var(X) = \sigma^2$$

The "standard normal" random variable is typically denoted Z and has mean 0 and variance 1. By the closure property of normals, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. The CDF has no closed form, but we denote the CDF of the standard normal by $\Phi(a) = F_Z(a) = P(Z \leq a)$. Note that by symmetry of the density about 0, $\Phi(-a) = 1 - \Phi(a)$.



Normal Distribution



Paranormal Distribution

Probability

4.4 Transforming Continuous RVs

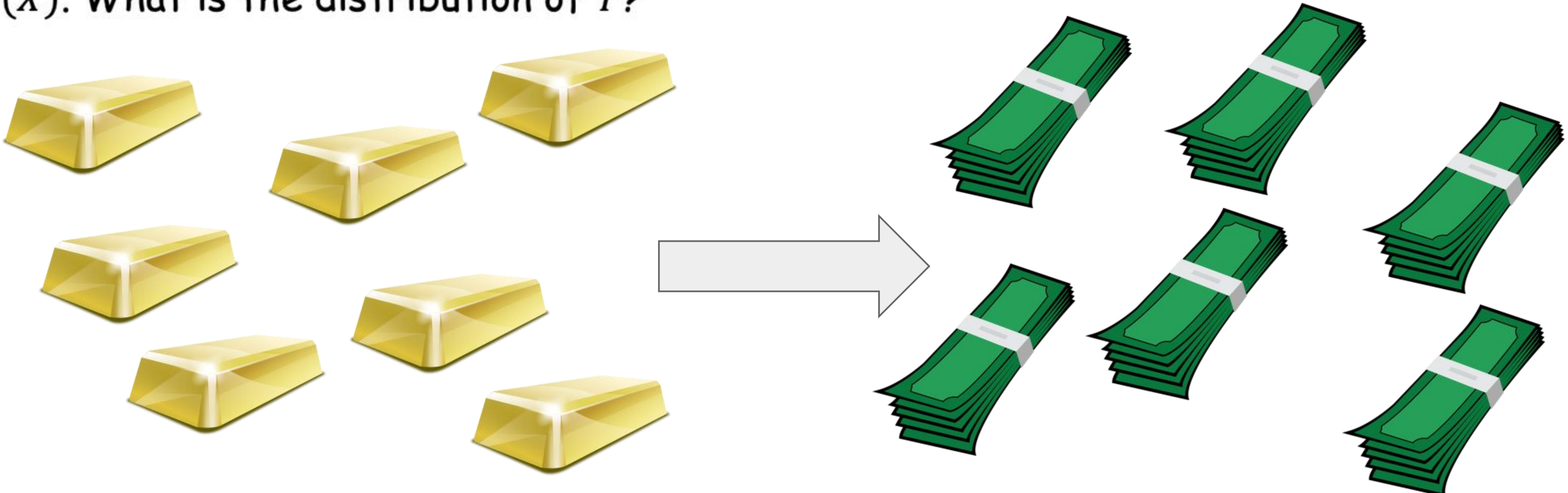
Joshua Fan
Alex Tsun

Agenda

- Transforming 1-D RVs via CDF
- Transforming 1-D RVs via Formula
- Transforming Multidimensional RVs via Formula

Transforming RVs (Motivation)

Suppose the amount of gold a company can mine is X tons per year, and you have some distribution to model this. Your earning is a function of the amount of product, $Y = g(X)$. What is the distribution of Y?



Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x/9, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

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$$f_X(x) = \begin{cases} 1/9\,, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0\,, & x < 0 \\ x/9\,, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

$$\Omega_Y = [0,3]$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9\,, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0\,, & x < 0 \\ x/9\,, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

$$\Omega_Y = [0,3]$$

$$f_Y(y) = \begin{cases} 1/\sqrt{9}\,, & 0 \le y \le \sqrt{9} \\ 0, & \text{otherwise} \end{cases} \;???$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9\,, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0\,, & x < 0 \\ x/9\,, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

$$\Omega_Y = [0,3]$$

$$f_Y(y) = \begin{cases} 1/\sqrt{9}\,, & 0 \le y \le \sqrt{9} \\ 0, & \text{otherwise} \end{cases} \; ???$$

$$[0,1] \rightarrow [0,1]$$
$$[1,4] \rightarrow [1,2]$$
$$[4,9] \rightarrow [2,3]$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x/9, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

$$\Omega_Y = [0,3]$$

$$f_Y(y) = \begin{cases} 1/\sqrt{9}, & 0 \le y \le \sqrt{9} \\ 0, & \text{otherwise} \end{cases} ???$$

$[0,1] \to [0,1]$

$[1,4] \to [1,2]$

$[4,9] \to [2,3]$

Can't be uniform!

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x/9, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

$$\Omega_Y = [0,3]$$

$$\cancel{f_Y(y) = \begin{cases} 1/\ldots & \ldots \le y \le \sqrt{9} \\ 0, & \text{ot}\ldots \end{cases}} \quad ???$$

We'll compute the CDF F_Y and differentiate to get f_Y. For $y \in [0,3]$,

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

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$$\Omega_Y = [0,3]$$

$$f_Y(y) = \begin{cases} \ldots & \ldots \le y \le \sqrt{9} \\ 0, & \ldots \end{cases} \quad ???$$

We'll compute the CDF F_Y and differentiate to get f_Y. For $y \in [0,3]$,

$$F_Y(y) = P(Y \le y)$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

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$$\Omega_Y = [0,3]$$

$$\cancel{f_Y(y) = \begin{cases} \ldots & \ldots y \le \sqrt{9} \\ 0, & \ldots \end{cases} \;???}$$

We'll compute the CDF F_Y and differentiate to get f_Y. For $y \in [0,3]$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P\left(\sqrt{X} \le y\right) \end{aligned}$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x/9, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

$$\Omega_Y = [0,3]$$

$$f_Y(y) = \begin{cases} 1/\ldots & \ldots \le y \le \sqrt{9} \\ 0, & \text{ot}\ldots \end{cases} \quad ???$$

We'll compute the CDF F_Y and differentiate to get f_Y. For $y \in [0,3]$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(\sqrt{X} \le y) \\ &= P(X \le y^2) \end{aligned}$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

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$$\Omega_Y = [0,3]$$

$$\cancel{f_Y(y) = \begin{cases} 1/\ldots & \ldots \le y \le \sqrt{9} \\ 0, & \text{ot}\ldots \end{cases}} \quad ???$$

We'll compute the CDF F_Y and differentiate to get f_Y. For $y \in [0,3]$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(\sqrt{X} \le y) \\ &= P(X \le y^2) \\ &= F_X(y^2) \end{aligned}$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

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$$\cancel{f_Y(y) = \begin{cases} \ldots & \ldots \le y \le \sqrt{9} \\ 0, & \ldots \end{cases}} \quad ???$$

We'll compute the CDF F_Y and differentiate to get f_Y. For $y \in [0,3]$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(\sqrt{X} \le y) \\ &= P(X \le y^2) \\ &= F_X(y^2) \\ &= y^2/9 \end{aligned}$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

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$$\cancel{f_Y(y) = \begin{cases} 1/ & \quad \le y \le \sqrt{9} \\ 0, & \text{ot} \end{cases}} \quad ???$$

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$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P\left(\sqrt{X} \le y\right) \\ &= P(X \le y^2) \\ &= F_X(y^2) \\ &= y^2/9 \end{aligned}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) =$$

Transforming 1-D RVs via CDF (Example)

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

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$$\Omega_Y = [0,3]$$

$$\cancel{f_Y(y) = \begin{cases} \ldots \le y \le \sqrt{9} \\ 0, & \ldots \end{cases}} \quad ???$$

We'll compute the CDF F_Y and differentiate to get f_Y. For $y \in [0,3]$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P\left(\sqrt{X} \le y\right) \\ &= P(X \le y^2) \\ &= F_X(y^2) \\ &= y^2/9 \end{aligned}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2y}{9}, \qquad y \in [0,3]$$

Random Picture

Transforming 1-D RVs via CDF

Steps to get PDF of $Y = g(X)$ from X (via CDF).

Transforming 1-D RVs via CDF

Steps to get PDF of $Y = g(X)$ from X (via CDF).

1. Write down the range Ω_X, PDF f_X, and CDF F_X.

Transforming 1-D RVs via CDF

Steps to get PDF of $Y = g(X)$ from X (via CDF).

1. Write down the range Ω_X, PDF f_X, and CDF F_X.

2. Compute the range $\Omega_Y = \{g(x): x \in \Omega_X\}$.

Transforming 1-D RVs via CDF

Steps to get PDF of $Y = g(X)$ from X (via CDF).

1. Write down the range Ω_X, PDF f_X, and CDF F_X.

2. Compute the range $\Omega_Y = \{g(x): x \in \Omega_X\}$.

3. Start computing the CDF of Y on Ω_Y, $F_Y(y) = P(g(X) \le y)$, in terms of F_X.

Transforming 1-D RVs via CDF

Steps to get PDF of $Y = g(X)$ from X (via CDF).

1. Write down the range Ω_X, PDF f_X, and CDF F_X.

2. Compute the range $\Omega_Y = \{g(x) : x \in \Omega_X\}$.

3. Start computing the CDF of Y on Ω_Y, $F_Y(y) = P(g(X) \leq y)$, in terms of F_X.

4. Differentiate the CDF $F_Y(y)$ to get the PDF $f_Y(y)$ on Ω_Y. f_Y is 0 outside Ω_Y.

Transforming 1-D RVs via Formula

Formula to get PDF of $Y = g(X)$ from X.

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = \begin{cases} f_X\big(h(y)\big)\,|h'(y)|, & y \in \Omega_Y \\ 0, & \text{otherwise} \end{cases}$$

Exercise: Prove this yourself using the CDF method we just learned!

Note: Not as general as the previous method, since g must satisfy monotonicity and invertibility.

Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = f_X\big(h(y)\big)|h'(y)|$$

Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = f_X\big(h(y)\big)|h'(y)|$$

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = f_X\big(h(y)\big)|h'(y)|$$

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x/9, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = f_X(h(y))|h'(y)|$$

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$g(t) = \sqrt{t}$ is monotone on $[0,9]$.

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

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Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

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$$\Omega_X = [0,9]$$

$g(t) = \sqrt{t}$ is monotone on $[0,9]$.

$h(y) = g^{-1}(y) = y^2$.

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Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = f_X\big(h(y)\big)|h'(y)|$$

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$g(t) = \sqrt{t}$ is monotone on $[0,9]$.

$h(y) = g^{-1}(y) = y^2$.

$h'(y) = 2y$.

Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = f_X\big(h(y)\big)|h'(y)|$$

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9\,, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0\,, & x < 0 \\ x/9\,, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

$g(t) = \sqrt{t}$ is monotone on $[0,9]$.
$h(y) = g^{-1}(y) = y^2$.
$h'(y) = 2y$.

For $y \in [0,3]$,

$$f_Y(y) = f_X\big(h(y)\big)|h'(y)|$$

Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g: \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = f_X\big(h(y)\big)|h'(y)|$$

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9\,, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

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$g(t) = \sqrt{t}$ is monotone on $[0,9]$.

$h(y) = g^{-1}(y) = y^2$.

$h'(y) = 2y$.

For $y \in [0,3]$,

$$\begin{aligned} f_Y(y) &= f_X\big(h(y)\big)|h'(y)| \\ &= \frac{1}{9}|2y| \end{aligned}$$

Transforming 1-D RVs via Formula (Example)

If $Y = g(X)$ and $g\colon \Omega_X \to \Omega_Y$ is **strictly monotone** and **invertible** with inverse $X = g^{-1}(Y) = h(Y)$, then

$$f_Y(y) = f_X\big(h(y)\big)\left|h'(y)\right|$$

Suppose you know $X \sim Unif(0,9)$ (continuous). What is the PDF of $Y = \sqrt{X}$?

$$\Omega_X = [0,9]$$

$$f_X(x) = \begin{cases} 1/9\,, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0\,, & x < 0 \\ x/9\,, & 0 \le x \le 9 \\ 1, & x > 9 \end{cases}$$

$g(t) = \sqrt{t}$ is monotone on $[0,9]$.

$h(y) = g^{-1}(y) = y^2$.

$h'(y) = 2y$.

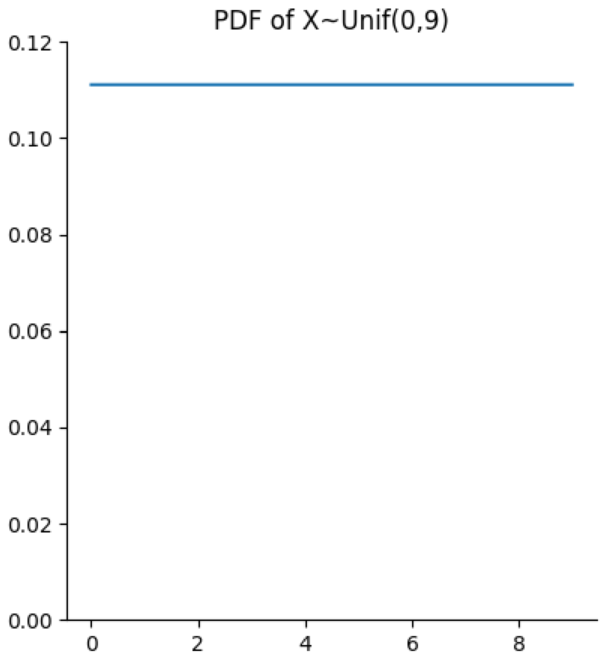
For $y \in [0,3]$,

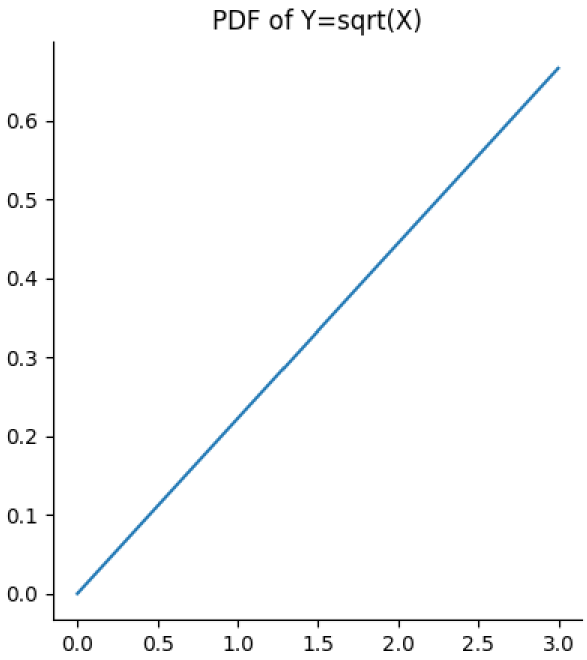
$$\begin{aligned} f_Y(y) &= f_X\big(h(y)\big)\left|h'(y)\right| \\ &= \frac{1}{9}|2y| = \frac{2}{9}y \end{aligned}$$

Transforming 1-D RVs via Formula (Example)

$$f_X(x) = \begin{cases} 1/9, & 0 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y/9, & 0 \le y \le 3 \\ 0, & \text{otherwise} \end{cases}$$





Transforming Multidimensional RVs via Formula

Formula to get PDF of $Y = g(X)$ from X (Multidimensional Case).

Let $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n)$ be continuous random **vectors** (each component is a continuous rv) in the same dimension n (so $\Omega_X, \Omega_Y \subseteq \mathbb{R}^n$), and $Y = g(X)$ where $g: \Omega_X \to \Omega_Y$ is invertible and differentiable, with differentiable inverse g^{-1}. Then,

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left|\det\left(\frac{\partial g^{-1}(y)}{\partial y}\right)\right|$$

where $\frac{\partial g^{-1}(y)}{\partial y} \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of partial derivatives of g^{-1}, with

$$\left(\frac{\partial g^{-1}(y)}{\partial y}\right)_{ij} = \frac{\partial\left(g^{-1}(y)\right)_i}{\partial y_j}$$