# Solved Problems in Quantum Information Assignment

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**Note:** This document contains solutions (hopefully correct) by the author (Aman Gupta) for the problem set given by Prof. Alexander Streltsov, Institute of Fundamental Technological Research, Polish Academy of Sciences, as part of application for PhD. position in quantum information under his supervision.

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**Problem 1**

For a bipartite state $\rho^{AB}$ the geometric entanglement is defined as

$$E_g(\rho) = 1 - \max_{\sigma \in \mathcal{S}} F(\rho, \sigma) \tag{1}$$

with fidelity $F(\rho,\sigma) = (Tr[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}])^2$, and $\mathcal{S}$ denotes the set of separable states. The relative entropy of entanglement is defined as

$$E_r(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho||\sigma) \tag{2}$$

with the quantum relative entropy $S(\rho||\sigma) = Tr[\rho \log_2 \rho] - Tr[\rho \log_2 \sigma]$. Entanglement of formation is defined as

$$E_f(\rho) = \min \sum_i p_i S(\psi_i^A), \tag{3}$$

where the minimum is taken over all pure state decompositions of $\rho$ such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|^{AB}$, and $S(\psi_i^A) = -Tr(\psi_i^A \log_2 \psi_i^A)$ is the von Neumann entropy.

(a) For a pure state $|\psi\rangle^{AB}$ show that the geometric entanglement can be expressed as

$$E_g(|\psi\rangle) = 1 - \max_{|\phi\rangle \in \mathcal{S}} |\langle\phi|\psi\rangle|^2. \tag{4}$$

(b) For a general bipartite pure state $|\psi\rangle^{AB}$ determine the geometric entanglement as a function of Schmidt coefficients of the state.

(c) Using results from scientific literature, plot the geometric entanglement, the relative entropy of entanglement and the entanglement of formation for two-qubit Werner states. In the solution, put references to the articles used to solve this part of the problem.

**Solution 1**

(a) Let's define:

$|\psi\rangle\langle\psi| = \rho$, (since $|\psi\rangle$ is a pure state), and,
$|\phi\rangle\langle\phi| = \sigma$, ($|\phi\rangle$ is arbitrary separable state in $\mathcal{S}$)
Also, using the properties of density matrices of pure state, i.e., $\rho^2 = \rho$ and $\rho \geq 0$, I can write $\sqrt{\rho} = \rho$.
Now, from given expression of fidelity:

$$\begin{aligned} F(\rho,\sigma) &= (Tr[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}])^2 \\ &= (Tr[\sqrt{\rho\sigma\rho}])^2, \text{ since } \sqrt{\rho} = \rho \\ &= (Tr[\sqrt{|\psi\rangle\langle\psi|\phi\rangle\langle\phi|\psi\rangle\langle\psi|}])^2 \\ &= (Tr[\sqrt{(\langle\phi|\psi\rangle)^2(|\psi\rangle\langle\psi|)}])^2 \\ &\text{I can take out the inner product as it is just a scalar.} \\ &= |\langle\phi|\psi\rangle|^2 (Tr[\sqrt{\rho}])^2, \\ &= |\langle\phi|\psi\rangle|^2 (Tr[\rho])^2 \end{aligned} \tag{5}$$

Finally, since $Tr[\rho] = 1$. I get, $F(\rho,\sigma) = |\langle\phi|\psi\rangle|^2$ for pure state $\psi$ and separable state $\phi$

Now using this fact, I can directly write the geometric entanglement as,

$$E_g(|\psi\rangle) = 1 - \max_{|\phi\rangle \in \mathcal{S}} |\langle\phi|\psi\rangle|^2$$

(b) Since, the state in question is a bipartite pure state, therefore the state $|\psi\rangle$ can be written in its Schmidt decomposition as [1]:

$$|\psi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i}|i\rangle_A|i\rangle_B \tag{6}$$

where, $\lambda_i s$ are Schmidt coefficients such that $\sum_i \lambda_i = 1$ and $|\phi\rangle = |\alpha\rangle_A \otimes |\beta\rangle_B$ is an arbitrary separable state, such that $\langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle = 1$.
To get geometric entanglement let's compute the distance measure (angle) between $|\psi\rangle$ and $|\phi\rangle$, (which I need to maximize).

$$|\langle\phi|\psi\rangle|^2 = (\sum_{ii'} \sqrt{\lambda_i}\langle\alpha|i\rangle_A.\langle\beta|i'\rangle_B)^2 \tag{7}$$

Now from Cauchy-Schwarz inequality for both subsystems A and B :

$$|\langle\alpha|i\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle i|i\rangle, \text{ and } |\langle\beta|i'\rangle|^2 \leq \langle\beta|\beta\rangle\langle i'|i'\rangle. \tag{8}$$

I can make the choice of closest separable state such that the above inequality is maximized; also, since $|\psi\rangle$ is a pure state, there is only one set of possible $\lambda_i s$ (Schmidt Number is 1), correspondingly, I have the choice of as $\lambda_i = \Lambda_{\max} = \lambda_0$ (if the eigenvalues $\lambda_i \geq \lambda_{i+1} \geq \cdots \geq 0$). Therefore,

$$\max_{|\phi\rangle\in\mathcal{S}} |\langle\phi|\psi\rangle|^2 = \Lambda_{max}.$$

So, the geometric entanglement as a function of Schmidt coefficients of the state is given as:

$$E_g(|\psi\rangle) = 1 - \Lambda_{max} \tag{9}$$

**Remark:** Much more formal and sophisticated ways of proving this result is given in Ref.[2] using set of entanglement witnesses and in Ref.[3] using Lagrange multiplier, but since I wasn't able to fully understand the proof, I followed the proof with above arguments. Some attempts towards these proofs are presented in Appendix 0.1.

(c) For the plots of geometric entanglement, entanglement of formation and relative entropy of entanglement for two-qubit Werner state, I have guessed that derivation of the relations (used from papers) is not the scope of this assignment, therefore I have used the expressions directly from the literatures to produce the graphs. The code used to generate the plot can be found here, Github. I have also provided some additional information in the Appendix0.2.

(i) **Geometric entanglement vs argument($f$) of Werner state plot for 2 qubit Werner State**

For this plot I have used the definitions and analytical solution of geometric entanglement given in Refs. [2, 4].
Following the Refs.[2, 4], the definition of:

- **Werner State:**

$$\rho_W(f) = \frac{d^2 - fd}{d^4 - d^2}\mathbb{I}_4 + \frac{fd^2 - d}{d^4 - d^2}\mathbb{F}_4, \tag{10}$$

subject to conditions $f = Tr(\rho_W F)$ and $Tr(\rho_W) = 1$, where $\mathbb{I}_4$ is identity operator for two qubits and $\mathbb{F}$ is an operator that flips the system states. For bipartite system, $\mathbb{F} = \sum_{i}, j|ij\rangle\langle ji|$.

- **The analytical solution for Geometric Entanglement for two-qubit Werner state:**

$$E_G(\rho_W(f)) = \begin{cases} \frac{1}{2}(1 - \sqrt{1-f^2}) & \text{, for } f \leq 0 \\ 0 & \text{, otherwise} \end{cases} \tag{11}$$

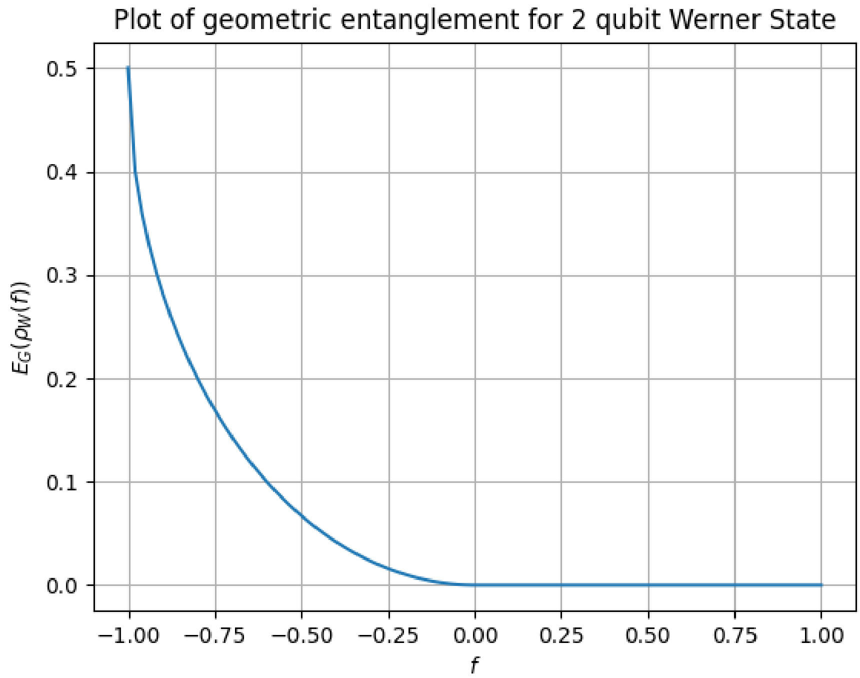



Figure 1: Plot of geometric entanglement for two-qubit Werner State

Apart from the above analytical solution, numerical solutions have also been proposed in Refs. [4–6].

(ii) **Entanglement of Formation vs argument($f$) of Werner state plot for two-qubit Werner State**

For this plot I have used the definitions and analytical solution of EoF given in Refs. [7, 8]. I have mainly followed the Ref. [8] as it is simpler and it uses the work from the former Ref. [7].Following the Refs. [7, 8], the definition of:

- **Werner State for two-qubit:**

$$\rho_W(f) = \frac{1-f}{4}\mathbb{I}_4 + \frac{f}{2}\mathbb{F}_4 \tag{12}$$

where $\mathbb{I}_4$ and $\mathbb{F}$ has same meaning as above and $f \in [-1, 1/3]$.

- **The analytical solution for entanglement of formation for two-qubit Werner state:**

$$EoF_W(\rho(f)) = \begin{cases} H_2(\frac{2-\sqrt{4-(3p+1)^2}}{4}) & \text{, for, } -1 \leq f \leq -\frac{1}{3} \\ 0 & \text{, for, } \frac{-1}{3} \leq f \leq \frac{1}{3} \end{cases} \tag{13}$$

where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the Shannon binary entropy function

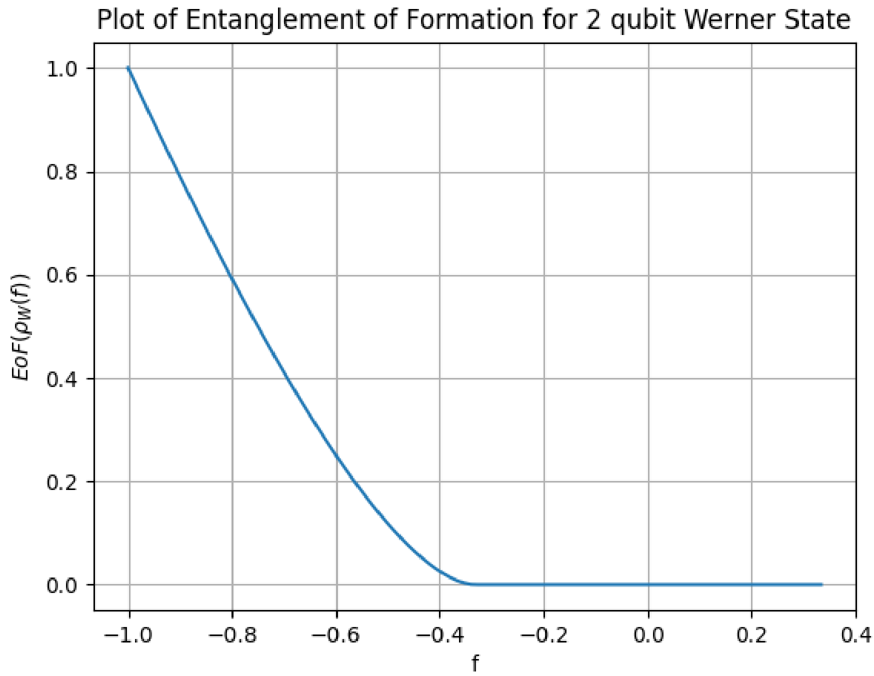



Figure 2: Plot of Entanglement of Formation for two-qubit Werner State

**Fact :** Incidentally, at $p = -\frac{1}{3}$ the Werner state, $\rho_W(-\frac{1}{3}) = |\phi_-\rangle\langle\phi_-|$ is a pure state, where $|\phi_-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is a Bell state.

(iii) **Relative entropy of entanglement vs argument($f$) of Werner state plot for two-qubit Werner State**

- The relative entropy of entanglement is related to the minimum quantum relative entropy with respect to an arbitrary separable state closest to the target (two-qubit Werner state).
- According to statement in Refs. [9–12], there is no closed formula for entanglement of formation (EoF) and relative entropy of entanglement (REE) for arbitrary qubit state. Wootters[7], in 1998, gave a closed formula for EoF for arbitrary two qubit state, however for REE there is no closed formula even for a two qubit state.
- The entanglement measures, like EoF and geometric entanglement for two qubit Werner state is related to its Concurrence ($C$), but there does not seem to be any such relation of concurrence with REE in [13] and it is not obvious to me how the necessity to obtain closest separable state (CSS) was replaced with concurrence for these measures.
- Currently numerical methods exist for REE in Ref. [14] where they find the REE for $f = 2/3$; this work [12] attempts to find the CSS by training an Ansatz through neural networks, and asymptotic formula for REE has been given in Refs.[11] where they compute the REE to be

$$E_r^\infty(\sigma) = \lim_{n\to\infty} \frac{E_r(\sigma^{\otimes n})}{n},$$

  where n is number of copies of the state, while bound for REE has been given in Ref. [15]. Finally, there are some highly symmetrical states for which analytical formula for REE has been calculated Ref.[9, 10, 13].
- I have attempted to form CSS and thereby derive the analytical relation for REE for two-qubit Werner state (see Appendix 0.3 for complete derivation) by closely following the works in Refs. [9, 10]. *(If this work 0.3 seem to be in in the right direction it could be a small conference publication).*

- For the plot I have decided to follow the work in Ref.[9] which gives the form of CSS and REE for a state that has the same density matrix structure as two-qubit Werner state with some conditions, which incidentally, also matches with the form of CSS for two-qubit "X" states given in Ref. [10]
- After derivation the REE is :

$$E_r(\rho_W) = \Omega_1 - \Omega_2, \tag{14}$$

where

$$\begin{aligned}\Omega_1 &= 3A_1 \log_2 A_1 + (A_2 - D)\log_2(A_2 - D) \\ \Omega_2 &= 2A_1\Big(\log_2 A_1 - \log_2\Big(\frac{3+f}{4}\Big)\Big) + A_2\Big(\log_2\Big(\frac{3+f}{4}\Big) - 2\log_2\Big(\frac{3+f}{4}\Big)\Big) \\ &\quad - D\log_2\Big(\frac{1+3f}{4}\Big),\end{aligned} \tag{15}$$

valid for range $-2 < f < 1$.

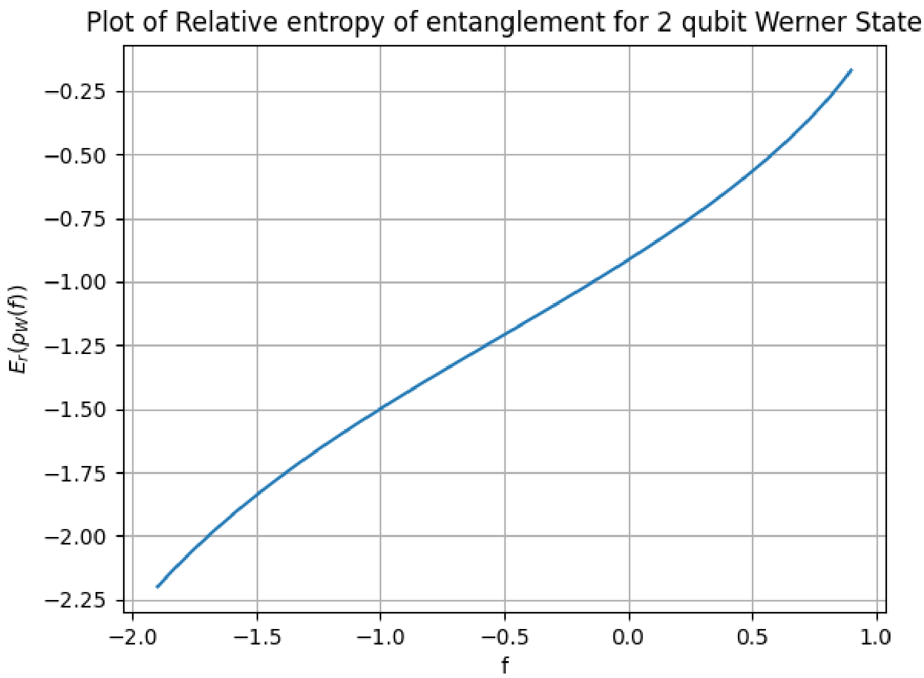
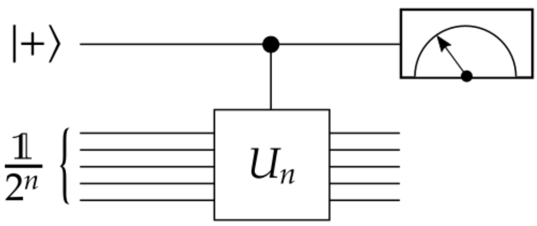



Figure 3: Plot of relative entropy of entanglement for two-qubit Werner State

**Problem 2**



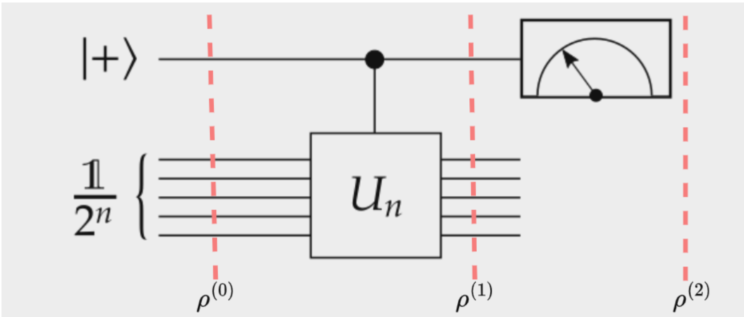

Consider the following quantum algorithm: a quantum computer has $n+1$ qubit registers, where the first qubit is initialized in the state

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{16}$$

and $n$ remaining qubits are initialized in the maximally mixed state $\mathbb{1}/2^n$. The overall state of the computer undergoes a controlled unitary operation $U_n$ with the first qubit acting as the control qubit (see the above figure). After this process, a measurement on the first qubit is performed.
Evaluate the expectation values $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ of the first qubit as a function of $U_n$. What is this quantum algorithm evaluating?

**Solution 2**

I would like to track the density matrix of the circuit at each stage and then calculate the required quantities.




For this I have:

$$\rho^{(0)} = |+\rangle\langle+| \otimes \frac{\mathbb{1}_n}{2^n}, \text{ and}$$

$$\mathcal{O} = |0\rangle\langle 0| \otimes \mathbb{1}_n + |1\rangle\langle 1| \otimes U_n$$

Here, $\mathbb{1}_n$ is $n$ dimensional identity matrix and $\mathcal{O}$ is the controlled unitary operator.
After application of the controlled unitary the state becomes:

$$\begin{aligned}
\rho^{(1)} &= \mathcal{O}\rho\mathcal{O}^\dagger = \mathcal{O}[|+\rangle\langle+| \otimes \mathbb{1}_n/2^n][|0\rangle\langle 0| \otimes \mathbb{1}_n + |1\rangle\langle 1| \otimes U_n^\dagger] \\
&= \frac{1}{2^n}\mathcal{O}[|+\rangle\langle+|0\rangle\langle 0| \otimes \mathbb{1}_n + |+\rangle\langle+|1\rangle\langle 1| \otimes U_n] \\
&= \frac{1}{2^{n+1}}[|0\rangle\langle 0| \otimes \mathbb{1}_n + |1\rangle\langle 1| \otimes U_n][(|0\rangle\langle 0| + |1\rangle\langle 0|) \otimes \mathbb{1}_n + (|0\rangle\langle 1| + |1\rangle\langle 1|) \otimes U_n^\dagger] \\
&= \frac{1}{2^{n+1}}[|0\rangle\langle 0|0\rangle\langle 0| \otimes \mathbb{1}_n + |0\rangle\langle 0|1\rangle\langle 0| \otimes \mathbb{1}_n + |0\rangle\langle 0|0\rangle\langle 1| \otimes U_n^\dagger + |0\rangle\langle 0|1\rangle\langle 1| \otimes U_n^\dagger + \\
&\quad |1\rangle\langle 1|0\rangle\langle 0| \otimes U_n + |1\rangle\langle 1|1\rangle\langle 0| \otimes U_n + |1\rangle\langle 1|0\rangle\langle 1| \otimes U_n U_n^\dagger + |1\rangle\langle 1|1\rangle\langle 1| \otimes U_n U_n^\dagger] \\
&= \frac{1}{2^{n+1}}[|0\rangle\langle 0| \otimes \mathbb{1}_n + |0\rangle\langle 1| \otimes U_n^\dagger + |1\rangle\langle 0| \otimes U_n + |1\rangle\langle 1| \otimes U_n U_n^\dagger] \\
&= \frac{1}{2^{n+1}}[|0\rangle\langle 0| \otimes \mathbb{1}_n + |0\rangle\langle 1| \otimes U_n^\dagger + |1\rangle\langle 0| \otimes U_n + |1\rangle\langle 1| \otimes \mathbb{1}_n]
\end{aligned} \tag{17}$$

Next, I take only first qubit in consideration by taking the partial trace:

$$\begin{aligned}
\rho^{(2)} &= Tr_B(\rho^{(1)}) = \frac{1}{2^{n+1}}[|0\rangle\langle 0|(Tr(\mathbb{1}_n)) + |0\rangle\langle 1|(Tr(U_n^\dagger)) + |1\rangle\langle 0|(Tr(U_n^\dagger)) + |1\rangle\langle 1|(Tr(\mathbb{1}_n))] \\
&= \frac{1}{2^{n+1}}[n|0\rangle\langle 0| + |0\rangle\langle 1|(\sum_i \lambda_i^*) + |1\rangle\langle 0|(\sum_i \lambda_i) + n|1\rangle\langle 1|], \text{ where } \lambda_i \text{ are eigenvalues of } U.
\end{aligned} \tag{18}$$

Now, since $\langle\sigma_x\rangle = Tr(\rho\sigma_x)$,

$$\Longrightarrow \langle\sigma_x\rangle = Tr(\rho^{(2)}\sigma_x) = Tr(\frac{1}{2^{n+1}}[n|0\rangle\langle 0| + |0\rangle\langle 1|(\sum_i \lambda_i^*) + |1\rangle\langle 0|(\sum_i \lambda_i) + n|1\rangle\langle 1|][|0\rangle\langle 1| + |1\rangle\langle 0|])$$
$$= \frac{1}{2^{n+1}} Tr([n|0\rangle\langle 1| + |0\rangle\langle 0|(\sum_i \lambda_i^*) + |1\rangle\langle 1|(\sum_i \lambda_i) + n|1\rangle\langle 0|]) \tag{19}$$
$$= \frac{1}{2^n} \times \sum_i \mathrm{Re}(\lambda_i)$$

Similarly,

$$\Longrightarrow \langle\sigma_y\rangle = Tr(\rho^{(2)}\sigma_y) = Tr(\frac{1}{2^{n+1}}[n|0\rangle\langle 0| + |0\rangle\langle 1|(\sum_i \lambda_i^*) + |1\rangle\langle 0|(\sum_i \lambda_i) + n|1\rangle\langle 1|][-i|0\rangle\langle 1| + i|1\rangle\langle 0|])$$
$$= \frac{1}{2^{n+1}} Tr([n(-i|0\rangle\langle 1|) + i|0\rangle\langle 0|(\sum_i \lambda_i^*) - i|1\rangle\langle 1|(\sum_i \lambda_i) + n(i|1\rangle\langle 0|)])$$
$$= \frac{1}{2^n} \times \sum_i \mathrm{Im}(\lambda_i) \tag{20}$$

The structure of circuit itself the algorithms most likely are[GeminiAI]:

- **Phase Estimation:** If $U_n$ is a rotation operator, the expectation values might be used to extract information about the rotation angle.
- **Quantum Oracles:** Controlled unitary gates are used in certain quantum oracle constructions. The expectation values might reveal properties of the function the oracle represents.

From the expectation values of $\sigma_x$ and $\sigma_y$ which is computing the sum of eigenvalues of the oracle, very much similar to phase estimation but here I are not picking out a single real part of the eigen value (no QFT part). Therefore, if $U$ is evolution operator corresponding to a Hamiltonian then the algorithm computes the total binding energy of all the states and energy of resonance states.

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# Appendix

## 0.1 Geometric entanglement as a function of Schmidt coefficients of the state

$$|\psi\rangle = \sum_{ij} \chi_{ij} |i\rangle \otimes |j\rangle, \quad \langle\psi|\psi\rangle = 1 \Rightarrow \sum_{ij} \chi_{ij}^* \chi_{ij} = 1$$

bipartite pure state

$$|\phi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B, \quad |\phi\rangle_A = \sum_i a_i |i\rangle, \quad |\phi\rangle_B = \sum_j b_j |j\rangle$$

Separable State

$$\text{s.t.: } \langle\phi|\phi\rangle = 1 \Rightarrow \sum_{ij} (a_i^* a_i)(b_j^* b_j) = 1$$

→ Distance

$$D^2 = \langle(\phi - \psi)|(\phi - \psi)\rangle = \sum_{ij} (a_i^* b_j^* - \chi_{ij}^*)(a_i b_j - \chi_{ij}) = \sum_{ij} a_i^* b_j^* a_i b_j - a_i^* b_j^* \chi_{ij} - a_i b_j \chi_{ij}^* + \chi_{ij}^* \chi_{ij}$$

→ Extrema of distance

$$\frac{\partial D^2}{\partial a_i} = 0 \Rightarrow a_i^* b_j^* b_j = b_j \chi_{ij}^*$$

$$\frac{\partial D^2}{\partial b_j} = 0 \Rightarrow a_i^* a_i b_j^* = a_i \chi_{ij}^* \Rightarrow \sum (a_i^* a_i)(b_j^* b_j) = \sum a_i b_j \chi_{ij}^*$$

$$\text{Distance extreme}, D_{ex}^2 = \sum_{ij} \left( a_i b_j \cancel{\chi_{ij}^*} - a_i^* b_j^* \chi_{ij} - a_i b_j \cancel{\chi_{ij}^*} + 1 \right)$$

$$= 1 - \sum_{ij} a_i^* b_j^* \chi_{ij} = 1 - \sum_{ij} a_i b_j \chi_{ij}^*$$

$$1 - \cos^2\theta_{ex} = D_{ex}^2 \Rightarrow \cos^2\theta_{ex} = \sum_{ij} a_i b_j \chi_{ij}^*$$

$$\rho = |\psi\rangle\langle\psi|, \quad \rho_A = \sum_{ij} \sum_{i'} \chi_{i'j}^* \chi_{ij} |i\rangle\langle i'| \quad \text{or} \quad (\rho_A)_{ii'} = \sum_j \chi_{i'j}^* \chi_{ij}$$

$$\chi_{ij} b_j^* = a_i b_j b_j^*$$

$$\Rightarrow \frac{\chi_{ij} b_j^*}{b_j^* b_j} = a_i$$

Ref: https://arxiv.org/pdf/1003.4755.pdf

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## 0.2 Geometric entanglement and entanglement of formation for two-qubit Werner state

- In numerical solutions for geometric entanglement[4–6], they :
  - (i) take the Werner state defined in Eq.10 and perform the SIC-POVM measurement to recreate the Werner state $\rho'$, (quantum state tomography).
  - (ii) After that they obtain the Fidelity using $F(\rho, \rho_W(f)) = [Tr(\sqrt{\sqrt{\rho'}\rho_W\sqrt{\rho'}})]^2$.
  - (iii) Finally, they compute the geometric entanglement.
- In Ref. [7] the EoF is proposed as :
  - (i) Compute the eigenvalues($\lambda_i s$) of matrix $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ in decreasing order as $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, where $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. $\rho$ is the two-qubit Werner state and $\rho^*$ is the complex conjugate of $\rho$.
  - (ii) After that they obtain the concurrence $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ and $\mathcal{E}(C) = H_2(\frac{1+\sqrt{(1-C^2)}}{2})$ where $H_2$ is again the same Shannon binary entropy function.
  - (iii) Finally, they compute the $EoF(\rho) = (E)(C)$.

## 0.3 Derivation of relative entropy of entanglement for two-qubit Werner state

Following the method in Ref.[9] and with two qubit Werner state as:

$$W_{AB}^{(2)}(f) = f|\psi_-\rangle\langle\psi_-| + \frac{1-f}{4}\mathbb{I}_{AB}, \text{ where } |\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Hence, the form of Werner state is:

$$\rho_W = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & D & 0 \\ 0 & D & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix},$$

where $A_1 = A_4 = \frac{1-f}{4}, A_2 = A_3 = \frac{1+f}{4}$ and $D = \frac{-f}{2}$.
Correspondingly the CSS will be of form:

$$\sigma = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & y & 0 \\ 0 & y & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix}, \text{ with condition } D^2 > A_1A_4.$$

Since $A_2 = A_3$ and $A_1 = A_4$, by symmetry I can say, $r_1 = r_4$ and $r_2 = r_3$. Now according to their [9] solution for the case: $A_2 = A_3$,

$$E_r(\rho) = \Omega_1 - \Omega_2 \tag{21}$$

where,

$$\begin{aligned} \Omega_1 &= A_1 \log_2 A_1 + A_4 \log_2 A_4 + (A_2 + D)\log_2(A_2 + D) + (A_2 - D)\log_2(A_2 - D) \\ \Omega_2 &= A_1 \log_2 r_1 + A_4 \log_2 r_4 + A_2 \log_2(r_2^2 - r_1 r_2) + D\log_2(\frac{r_2 + r_1}{r_2 - r_1}), \end{aligned} \tag{22}$$

and the values of $r_1, r_2, r_3, r_4$ and $y$ are as follows:

$$\begin{aligned} r_1 &= \frac{1}{F}[2A_1(A_1 + A_2)(A_1 + A_2 + A_4) - D^2(A_1 - A_4) + \Delta] \\ r_4 &= \frac{1}{F}[2A_4(A_2 + A_4)(A_1 + A_2 + A_4) + D^2(A_1 - A_4) + \Delta] \\ r_2 = r_3 &= \frac{1}{F}[2(A_1 + A_2)(A_2 + A_4)(A_1 + A_2 + A_4) - D^2(A_1 + 2A_2 + A_4) - \Delta] \\ y &= \sqrt{r_1 r_4}, \end{aligned} \tag{23}$$

where,

$$F = 2(A_1 + A_2 + A_4 + D)(A_1 + A_2 + A_4 - D)$$
$$\Delta = D\sqrt{D^2(A_1 - A_4)^2 + 4A_1A_4(A_1 + A_2)(A_2 + A_4)}$$

These equations can be solved for this case by putting the facts, $A_1 = A_4$ and $A_1 + A_2 = \frac{1}{2}$ and simplified for $F$ and $\Delta$ to get:

$$\begin{aligned} r_1 = r_4 = y &= \frac{1}{3(1/2 + A_1 - D)}[(\frac{1}{2} + A_1) + DA_1] \\ r_2 = r_3 &= \frac{1}{3A_1(1/2 + A_1 - D)}[\frac{1}{2}(\frac{1}{2} + A_1) - D^2 - DA_1] \end{aligned} \tag{24}$$

Now, putting these values and the fact that $\frac{1}{2} - A_1 - D = A_2 - D = \frac{1+3f}{4}$, $\frac{1}{2} + A_1 + D = \frac{1}{4} + A_1 + A_1^2 - D^2 = 3A_1 = 3\left(\frac{1-f}{4}\right)$ and $\frac{1}{2} + A_1 - D = \frac{3-f}{4}$ in Eq.22 I get:

$$\begin{aligned} \Omega_1 &= 3A_1 \log_2 A_1 + (A_2 - D)\log_2(A_2 - D) \\ \Omega_2 &= 2A_1\Big(\log_2 A_1 - \log_2\Big(\frac{3+f}{4}\Big)\Big) + A_2\Big(\log_2\Big(\frac{3+f}{4}\Big) - 2\log_2\Big(\frac{3+f}{4}\Big)\Big) \\ &\quad - D\log_2\Big(\frac{1+3f}{4}\Big), \end{aligned} \tag{25}$$

Finally, plugging the obtained $\Omega_1$ and $\Omega_2$ in Eq. 21, I get:

$$E_r(\rho) = \frac{(1-f)\log_2(1-f)}{4} + \frac{f-5}{2} + \log_2(3+f), \tag{26}$$

valid in range $-2 < f < 1$.

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## References


[1] Xian Shi. "Entanglement polygon inequalities for pure states in qudit systems". In: *The European Physical Journal Plus* 138.8 (2023), pp. 1–10.

[2] Tzu-Chieh Wei and Paul M Goldbart. "Geometric measure of entanglement and applications to bipartite and multipartite quantum states". In: *Physical Review A* 68.4 (2003), p. 042307.

[3] ME Carrington et al. "Geometric measures of entanglement and the Schmidt decomposition". In: *Journal of Physics A: Mathematical and Theoretical* 43.31 (2010), p. 315302.

[4] Zhou Zhang et al. "Numerical and analytical results for geometric measure of coherence and geometric measure of entanglement". In: *Scientific Reports* 10.1 (2020), p. 12122.

[5] Haigang Wang and Kan He. "Quantum Tomography of Two-Qutrit Werner States". In: *Photonics for Solar Energy Systems IX* 9.10 (2022), p. 741.

[6] Artur Czerwinski. "Quantifying entanglement of two-qubit Werner states". In: *Communications in Theoretical Physics* 73.8 (2021), p. 085101.

[7] William K Wootters. "Entanglement of formation of an arbitrary state of two qubits". In: *Physical Review Letters* 80.10 (1998), p. 2245.

[8] S Díaz-Solórzano and E Castro. "Exact analytical solution of Entanglement of Formation and Quantum Discord for Werner state and Generalized Werner-Like states". In: *arXiv e-prints* (2018), arXiv–1802.

[9] DaeKil Park. "Relative entropy of entanglement for two-qubit state with z-directional Bloch vectors". In: *International Journal of Quantum Information* 8.05 (2010), pp. 869–879.

[10] Jie-Hui Huang et al. "Relative entropy of entanglement of two-qubit 'X'states". In: *Chinese Physics B* 19.11 (2010), p. 110307.

[11] Koenraad Audenaert et al. “Asymptotic relative entropy of entanglement for orthogonally invariant states”. In: *Physical Review A* 66.3 (2002), p. 032310.

[12] Antoine Girardin, Nicolas Brunner, and Tamás Kriváchy. “Building separable approximations for quantum states via neural networks”. In: *Physical Review Research* 4.2 (2022), p. 023238.

[13] Adam Miranowicz and Andrzej Grudka. “A comparative study of relative entropy of entanglement, concurrence and negativity”. In: *Journal of Optics B: Quantum and Semiclassical Optics* 6.12 (2004), p. 542.

[14] Yuriy Zinchenko, Shmuel Friedland, and Gilad Gour. “Numerical estimation of the relative entropy of entanglement”. In: *Physical Review A* 82.5 (2010), p. 052336.

[15] Vlatko Vedral and Martin B Plenio. “Entanglement measures and purification procedures”. In: *Physical Review A* 57.3 (1998), p. 1619.