Solved Problems in Quantum Information Assignment

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April 20, 2024

Problem 1

For a bipartite state ρ^{AB} the geometric entanglement is defined as

$$E_g(\rho) = 1 - \max_{\sigma \in \mathcal{S}} F(\rho, \sigma) \tag{1}$$

with fidelity $F(\rho, \sigma) = (Tr[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}])^2$, and $\mathcal{S}$ denotes the set of separable states. The relative entropy of entanglement is defined as

$$E_r(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho||\sigma) \tag{2}$$

with the quantum relative entropy $S(\rho||\sigma) = Tr[\rho \log_2 \rho] - Tr[\rho \log_2 \sigma]$. Entanglement of formation is defined as

$$E_f(\rho) = \min \sum_i p_i S(\psi_i^A), \tag{3}$$

where the minimum is taken over all pure state decompositions of ρ such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|^{AB}$, and $S(\psi_i^A) = -Tr(\psi_i^A \log_2 \psi_i^A)$ is the von Neumann entropy.

(a) For a pure state $|\psi\rangle^{AB}$ show that the geometric entanglement can be expressed as

$$E_g(|\psi\rangle) = 1 - \max_{|\phi\rangle \in \mathcal{S}} |\langle\phi|\psi\rangle|^2. \tag{4}$$

(b) For a general bipartite pure state $|\psi\rangle^{AB}$ determine the geometric entanglement as a function of Schmidt coefficients of the state.

(c) Using results from scientific literature, plot the geometric entanglement, the relative entropy of entanglement and the entanglement of formation for two-qubit Werner states. In the solution, put references to the articles used to solve this part of the problem.

Solution 1

(a) Let's define:

$|\psi\rangle\langle\psi| = \rho$, (since $|\psi\rangle$ is a pure state), and,
$|\phi\rangle\langle\phi| = \sigma$, ($|\phi\rangle$ is arbitrary separable state in $\mathcal{S}$)
Also, using the properties of density matrices of pure state, i.e., $\rho^2 = \rho$ and $\rho \geq 0$, we can write $\sqrt{\rho} = \rho$.

Now, from given expression of fidelity:

$$\begin{aligned}
F(\rho, \sigma) &= (Tr[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}])^2 \\
&= (Tr[\sqrt{\rho\sigma\rho}])^2, \text{ since } \sqrt{\rho} = \rho \\
&= (Tr[\sqrt{|\psi\rangle\langle\psi|\phi\rangle\langle\phi|\psi\rangle\langle\psi|}])^2 \\
&= (Tr[\sqrt{(\langle\phi|\psi\rangle)^2(|\psi\rangle\langle\psi|)}])
\end{aligned} \tag{5}$$

We can take out the inner product as it is just a scalar.

$$\begin{aligned}
&= |\langle\phi|\psi\rangle|^2 (Tr[\sqrt{\rho}])^2, \\
&= |\langle\phi|\psi\rangle|^2 (Tr[\rho])^2
\end{aligned}$$

Finally, since $Tr[\rho] = 1$. We get, $F(\rho, \sigma) = |\langle\phi|\psi\rangle|^2$ for pure state ρ and separable state ϕ

Now using this fact and of geometric entanglement we can directly write,

$$E_g(|\psi\rangle) = 1 - \max_{|\phi\rangle \in \mathcal{S}} |\langle\phi|\psi\rangle|^2$$

(b) Since, the state in question is a bipartite pure state, therefore the state $|\psi\rangle$ can be written in its Schmidt decomposition as [1]:

$$|\psi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i}|i\rangle_A|i\rangle_B \qquad (6)$$

where, $\lambda_i s$ are Schmidt coefficients such that $\sum_i \lambda_i = 1$ and $|\phi\rangle = |\alpha\rangle_A \otimes |\beta\rangle_B$ is an arbitrary separable state, such that $\langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle = 1$.

To get geometric entanglement let's compute the distance measure (angle) between $|\psi\rangle$ and $|\phi\rangle$, (which we need to maximize).

$$|\langle\phi|\psi\rangle|^2 = (\sum_{ii'} \sqrt{\lambda_i}\langle\alpha|i\rangle_A.\langle\beta|i'\rangle_B)^2 \qquad (7)$$

Now from Cauchy-Schwarz inequality for both subsystems A and B :

$$|\langle\alpha|i\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle i|i\rangle, \text{ and } |\langle\beta|i'\rangle|^2 \leq \langle\beta|\beta\rangle\langle i'|i'\rangle. \qquad (8)$$

We can make the choice of closest separable state such that the above inequality is maximized; also, since $|\psi\rangle$ is a pure state, there is only one set of possible $\lambda_i s$ (Schmidt Number is 1) we make the choice of $\lambda_i = \Lambda_{\max} = \lambda_0$ (if the eigenvalues $\lambda_i \geq \lambda_{i+1} \geq \cdots \geq 0$), correspondingly. Therefore,

$$\max_{|\phi\rangle \in \mathcal{S}} |\langle\phi|\psi\rangle|^2 = \Lambda_{max}.$$

So, the geometric entanglement as a function of Schmidt coefficients of the state is given as:

$$E_G(|\psi\rangle) = 1 - \Lambda_{max} \qquad (9)$$

Remark: Much more formal and sophisticated ways of proving this result is given in Ref.[2] using set of entanglement witnesses and in Ref.[3] using Lagrange multiplier, but since I wasn't able to fully understand the proof, I followed the proof with above arguments. Some attempts towards these proofs are presented in Appendix 0.1.

(c) For the plots of geometric entanglement, entanglement of formation and relative entropy of entanglement for two-qubit Werner state, I have guessed that derivation of the relations (used from papers) is not the scope of this assignment, therefore I have used the expressions directly from the literatures to produce the graphs. The code used to generate the plot can be found here, Github. I have also provided some additional information in the Appendix0.2.

(i) **Geometric entanglement vs argument(f) of Werner state plot for 2 qubit Werner State**

For this plot I have used the definitions and analytical solution of geometric entanglement given in Refs. [2, 4].
Following the Refs.[2, 4], the definition of:

- **Werner State:**
$$\rho_W(f) = \frac{d^2 - fd}{d^4 - d^2}\mathbb{I}_4 + \frac{fd^2 - d}{d^4 - d^2}\mathbb{F}_4, \qquad (10)$$

  subject to conditions $f = Tr(\rho_W F)$ and $Tr(\rho_W) = 1$, where $\mathbb{I}_4$ is identity operator for two qubits and $\mathbb{F}$ is an operator that flips the system states. For bipartite system, $\mathbb{F} = \sum_{i,j}|ij\rangle\langle ji|$.
- **The analytical solution for Geometry of Entanglement for two-qubit Werner state:**

$$E_G(\rho_W(f)) = \begin{cases} \frac{1}{2}(1 - \sqrt{1-f^2}) & \text{, for } f \leq 0 \\ 0 & \text{, otherwise} \end{cases} \qquad (11)$$
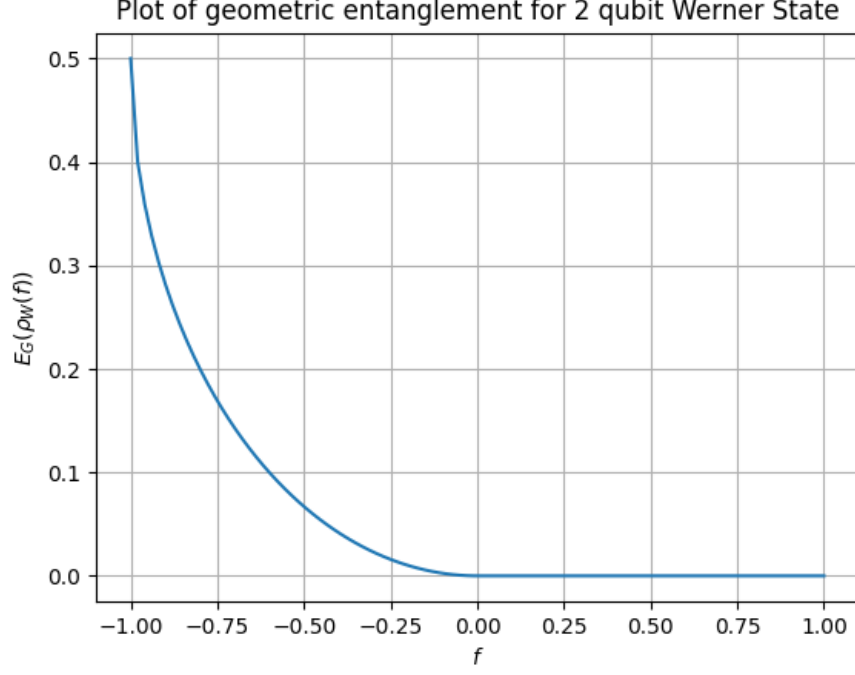
Figure 1: Plot of geometric entanglement for two-qubit Werner State

Apart from this numerical solutions have also been proposed in Refs. [4–6].

(ii) **Entanglement of Formation vs argument(f) of Werner state plot for two-qubit Werner State**

For this plot I have used the definitions and analytical solution of EoF given in Refs. [7, 8]. I have mainly followed the Ref. [8] as it is simpler and it uses the work from the former Ref. [7].Following the Refs. [7, 8], the definition of:

- **Werner State for two-qubit:**

$$\rho_W(f) = \frac{1-f}{4}\mathbb{I}_4 + \frac{f}{2}\mathbb{F}_4 \tag{12}$$

where $\mathbb{I}_4$ and $\mathbb{F}$ has same meaning as above and $f \in [-1, 1/3]$.

- **The analytical solution for entanglement of formation for two-qubit Werner state:**

$$EoF_W(\rho(f)) = \begin{cases} H_2\left(\frac{2-\sqrt{4-(3p+1)^2}}{4}\right) & \text{, for, } -1 \le f \le -\frac{1}{3} \\ 0 & \text{, for, } \frac{-1}{3} \le f \le \frac{1}{3} \end{cases} \tag{13}$$

where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the Shannon binary entropy function
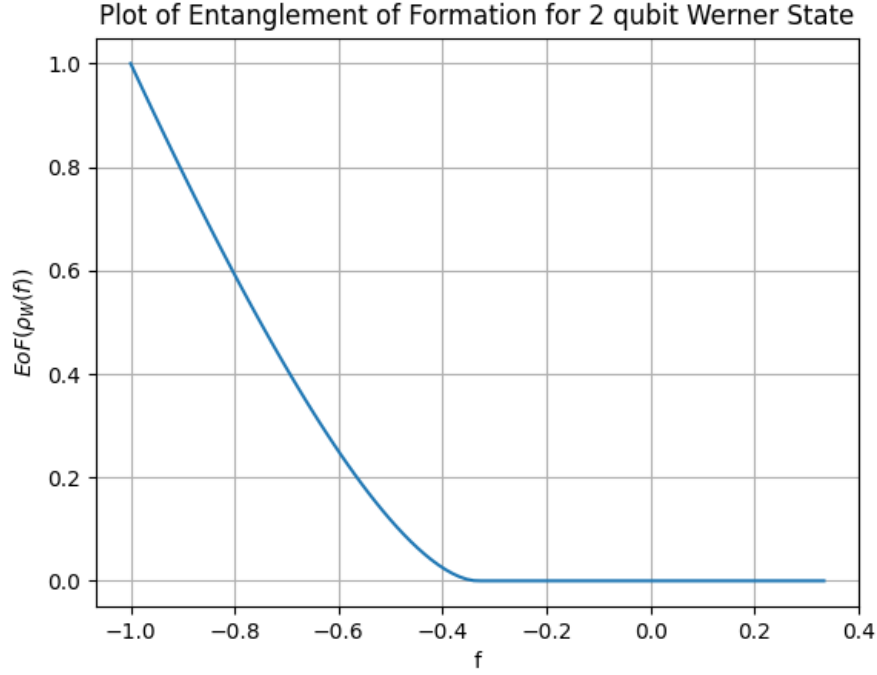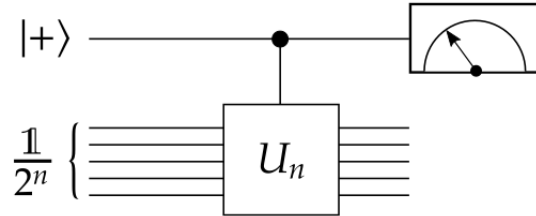
Figure 2: Plot of Entanglement of Formation for two-qubit Werner State

Problem 2

Consider the following quantum algorithm: a quantum computer has $n + 1$ qubit registers, where the first qubit is initialized in the state

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{14}$$

and n remaining qubits are initialized in the maximally mixed state $\mathbb{1}/2^n$. The overall state of the computer undergoes a controlled unitary operation U_n with the first qubit acting as the control qubit (see the above figure). After this process, a measurement on the first qubit is performed.
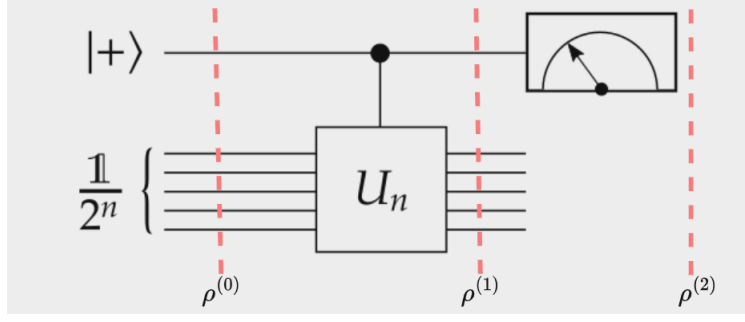
Evaluate the expectation values $\langle \sigma_x \rangle$ and $\langle \sigma_y \rangle$ of the first qubit as a function of U_n. What is this quantum algorithm evaluating?

Solution 2

I would like to track the density matrix of the circuit at each stage and then calculate the required quantities. For this I have:

$$\rho^{(0)} = |+\rangle\langle+| \otimes \frac{\mathbb{1}_n}{2^n}, \text{ and}$$

$$\mathcal{O} = |0\rangle\langle0| \otimes \mathbb{1}_n + |1\rangle\langle1| \otimes U_n$$

Here, $\mathbb{1}_n$ is n dimensional identity matrix and $\mathcal{O}$ is the controlled unitary operator.
After application of the controlled unitary the state becomes:

$$\rho^{(1)} = \mathcal{O}\rho\mathcal{O}^\dagger = \mathcal{O}[|+\rangle\langle+| \otimes \mathbb{1}_n/2^n][|0\rangle\langle0| \otimes \mathbb{1}_n + |1\rangle\langle1| \otimes U_n^\dagger]$$

$$= \frac{1}{2^n}\mathcal{O}[|+\rangle\langle+|0\rangle\langle0| \otimes \mathbb{1}_n + |+\rangle\langle+|1\rangle\langle1| \otimes U_n]$$

$$= \frac{1}{2^{n+1}}[|0\rangle\langle0| \otimes \mathbb{1}_n + |1\rangle\langle1| \otimes U_n][(|0\rangle\langle0| + |1\rangle\langle0|) \otimes \mathbb{1}_n + (|0\rangle\langle1| + |1\rangle\langle1|) \otimes U_n^\dagger]$$

$$= \frac{1}{2^{n+1}}[|0\rangle\langle0|0\rangle\langle0| \otimes \mathbb{1}_n + |0\rangle\langle0|1\rangle\langle0| \otimes \mathbb{1}_n + |0\rangle\langle0|0\rangle\langle1| \otimes U_n^\dagger + |0\rangle\langle0|1\rangle\langle1| \otimes U_n^\dagger + \tag{15}$$

$$|1\rangle\langle1|0\rangle\langle0| \otimes U_n + |1\rangle\langle1|1\rangle\langle0| \otimes U_n + |1\rangle\langle1|0\rangle\langle1| \otimes U_n U_n^\dagger + |1\rangle\langle1|1\rangle\langle1| \otimes U_n U_n^\dagger]$$

$$= \frac{1}{2^{n+1}}[|0\rangle\langle0| \otimes \mathbb{1}_n + |0\rangle\langle1| \otimes U_n^\dagger + |1\rangle\langle0| \otimes U_n + |1\rangle\langle1| \otimes U_n U_n^\dagger]$$

$$= \frac{1}{2^{n+1}}[|0\rangle\langle0| \otimes \mathbb{1}_n + |0\rangle\langle1| \otimes U_n^\dagger + |1\rangle\langle0| \otimes U_n + |1\rangle\langle1| \otimes \mathbb{1}_n]$$

Next, we take only first qubit in consideration by taking the partial trace:

$$\rho^{(2)} = Tr_B(\rho^{(1)}) = \frac{1}{2^{n+1}}[|0\rangle\langle0|(Tr(\mathbb{1}_n)) + |0\rangle\langle1|(Tr(U_n^\dagger)) + |1\rangle\langle0|(Tr(U_n^\dagger)) + |1\rangle\langle1|(Tr(\mathbb{1}_n))]$$

$$= \frac{1}{2^{n+1}}[n|0\rangle\langle0| + |0\rangle\langle1|(\sum_i \lambda_i^*) + |1\rangle\langle0|(\sum_i \lambda_i) + n|1\rangle\langle1|], \text{ where } \lambda_i \text{ are eigenvalues of } U. \tag{16}$$

Now, since $\langle\sigma_x\rangle = Tr(\rho\sigma_x)$,

$$\implies \langle\sigma_x\rangle = Tr(\rho^{(2)}\sigma_x) = Tr(\frac{1}{2^{n+1}}[n|0\rangle\langle0| + |0\rangle\langle1|(\sum_i \lambda_i^*) + |1\rangle\langle0|(\sum_i \lambda_i) + n|1\rangle\langle1|][|0\rangle\langle1| + |1\rangle\langle0|])$$

$$= \frac{1}{2^{n+1}}Tr([n|0\rangle\langle1| + |0\rangle\langle0|(\sum_i \lambda_i^*) + |1\rangle\langle1|(\sum_i \lambda_i) + n|1\rangle\langle0|]) \tag{17}$$

$$= \frac{1}{2^n} \times \sum_i \text{Re}(\lambda_i)$$

Similarly,

$$\implies \langle\sigma_y\rangle = Tr(\rho^{(2)}\sigma_y) = Tr(\frac{1}{2^{n+1}}[n|0\rangle\langle0| + |0\rangle\langle1|(\sum_i \lambda_i^*) + |1\rangle\langle0|(\sum_i \lambda_i) + n|1\rangle\langle1|][-i|0\rangle\langle1| + i|1\rangle\langle0|])$$

$$= \frac{1}{2^{n+1}}Tr([n(-i|0\rangle\langle1|) + i|0\rangle\langle0|(\sum_i \lambda_i^*) - i|1\rangle\langle1|(\sum_i \lambda_i) + n(i|1\rangle\langle0|)])$$

$$= \frac{1}{2^n} \times \sum_i \text{Im}(\lambda_i)$$

$$\tag{18}$$

The structure of circuit itself the algorithms most likely are[GeminiAI]:

- **Phase Estimation:** If U_n is a rotation operator, the expectation values might be used to extract information about the rotation angle.

- **Quantum Oracles:** Controlled unitary gates are used in certain quantum oracle constructions. The expectation values might reveal properties of the function the oracle represents.

From the expectation values of σ_x and σ_y which is computing the sum of eigenvalues of the oracle, very much similar to phase estimation but here we are not picking out a single real part of the eigen value (no QFT part). Therefore, if U is evolution operator corresponding to a Hamiltonian then the algorithm computes the total binding energy of all the states and energy of resonance states.

Appendix

0.1 Geometric entanglement as a function of Schmidt coefficients of the state

$|\psi\rangle = \sum_{ij} \chi_{ij} |i\rangle \otimes |j\rangle$, $\langle \psi | \psi \rangle = 1 \Rightarrow$

bipartite pure state $\qquad \sum_{ij} \chi_{ij}^* \chi_{ij} = 1$

$|\phi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B$, $|\phi\rangle_A = \sum_i a_i |i\rangle$,

Separable state $\qquad |\phi\rangle_B = \sum_j b_j |j\rangle$

S.t: $\langle \phi | \phi \rangle = 1 \Rightarrow \sum_{ij} (a_i^* a_i)(b_j^* b_j) = 1$

$\rightarrow$ <u>Distance</u>

$D^2 = \langle (\phi - \psi) | (\phi - \psi) \rangle = \sum_{ij} (a_i^* b_j^* - \chi_{ij}^*)(a_i b_j - \chi_{ij}) = \sum_{ij} a_i^* b_j^* a_i b_j - a_i^* b_j^* \chi_{ij} - a_i b_j \chi_{ij}^* + \chi_{ij}^* \chi_{ij}$

$\rightarrow$ Extrema of distance

$\frac{\partial D^2}{\partial a_i} = 0 \Rightarrow a_i^* b_j^* b_j = b_j \chi_{ij}^*$ $\qquad \left| \frac{\partial D^2}{\partial b_j} = 0 \Rightarrow a_i^\dagger a_i b_j^* = a_i \chi_{ij}^* \not\Rightarrow \sum (a_i^* a_i)(b_j^* b_j) = \right.$

$\Rightarrow \sum a_i b_j \chi_{ij}^*$

Distance extreme , $D_{ex}^2 = \sum_{ij} \left(a_i b_j \chi_{ij}^* - a_i^* b_j^* \chi_{ij} - a_i b_j \chi_{ij}^* + 1 \right)$

$= 1 - \sum_{ij} a_i^* b_j^* \chi_{ij} = 1 - \sum_{ij} a_i b_j \chi_{ij}^*$

$1 - \cos^2 \theta_{ex} = D_{ex}^2 \Rightarrow \cos^2 \theta_{ex} = \sum_{ij} a_i b_j \chi_{ij}^*$

$\rho = |\psi\rangle \langle \psi|$, $\rho_A = \sum_{ij} \sum_{i'} \chi_{ij}^* \chi_{i'j} |i\rangle \langle i|$ or $(\rho_A)_{ii'} = \sum_j \chi_{i'j}^* \chi_{ij}$

$\chi_{ij} b_j^* = a_i b_j b_j^*$

$\Rightarrow \frac{\chi_{ij} b_j^*}{b_j^* b_j} = a_i$

$\mathcal{Ref:}$ https://arxiv.org/pdf/1003.4755.pdf

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0.2 Geometric entanglement and entanglement of formation for two-qubit Werner state

- In numerical solutions for geometric entanglement[4–6], they :

  (i) take the Werner state defined in Eq.10 and perform the SIC-POVM measurement to recreate the Werner state ρ', (quantum state tomography).

  (ii) After that they obtain the Fidelity using $F(\rho, \rho_W(f)) = [Tr(\sqrt{\sqrt{\rho'}\rho_W\sqrt{\rho'}})]^2$.

  (iii) Finally, they compute the geometric entanglement.

- In Ref. [7] the EoF is proposed as :

  (i) Compute the eigenvalues($\lambda_i s$) of matrix $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ in decreasing order as $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, where $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. ρ is the two-qubit Werner state and ρ^* is the complex conjugate of ρ.

  (ii) After that they obtain the concurrence $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ and $\mathcal{E}(C) = H_2(\frac{1+\sqrt{(1-C^2)}}{2})$ where H_2 is again the same Shannon binary entropy function.

  (iii) Finally, they compute the $EoF(\rho) = (E)(C)$.

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