Follower consensus of multi-agent systems with antagonistic leaders

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Abstract: In this paper, we study follower consensus of multi-agent systems with antagonistic leaders. An antagonistic leader can be an exogenous attack or a malicious agent who sends misinformation to others. We consider a model of multi-agent system consisting of antagonistic leaders and followers. In the process of state updating, the agents interact with each other by employing a gossip-like algorithm. Based on graph theory and matrix theory, we obtain some criteria for solving follower consensus. Moreover, we show that the follower consensus in different senses is equivalent. Simulation example are provided to illustrate the effectiveness of our theoretical results.

Key Words: Follower consensus, Multi-agent systems, Antagonistic leader

1 Introduction

Over the past decade, distributed coordination of multi-agent systems has attracted a major attention of researchers from various disciplines, including control theory, mathematics, biology, sociology, etc. This is mainly due to its broad applications in modeling many complex phenomena involving information exchange between agents, such as formation control of unmanned air vehicles, target tracking of robots, congestion control in communication networks, flocking in biology, etc [1]-[4].

As a challenging research topic in coordination of multi-agent systems, consensus concerns on the design of interaction protocols to ensure that the agents agree on some values of interest in an appropriate sense. Vicsek et al. [5] presented a compelling model for a group of self-driven agents update their headings using the nearest neighbor rule. Jadbabaie et al. [6] provided a theoretical explanation for Vicsek's simulation results. They showed that consensus can be achieved if the union of interaction graph is connected frequently enough. Olfati-Saber et al. [7] considered the consensus problem of continuous-time multi-agent systems with switching topologies and time-delays. Ren and Beard [8] relaxed the results of [7] for solving consensus problem under switching topologies. Some further issues about consensus were also investigated, such as asynchronous consensus [9], optimal consensus [10], consensus under random networks [11]-[13] and so on. On the other hand, owing to its significant robustness and efficiency, gossip algorithms become more and more popular in designing consensus protocols. Boyd et al. [14] considered the gossip algorithm under arbitrary connected networks for the asynchronous and synchronous time models. Acemoglu et al. [15] used a gossip algorithm to describe the spread of misinformation induced by forceful update in social networks and investigated the relation between the consensus value and the structure of network.

Leader-follower structure widely exists in multi-agent systems. In multi-agent systems, the leader is defined as an agent whose dynamical behavior is independent from others, that is, the leader sends information to other agents but never receives others' information. Leader-follower structure in multi-agent systems was considered in [16]-[19] for both single leader and multiple leaders. Hong et al. [17] studied tracking control of multi-agent systems with an active leader. Ma et al. [16] investigated the optimal topology for the leader-following consensus in the presence of single leader. Zheng et al. [18] considered containment control of heterogenous multi-agent systems in the presences of multiple leaders. Wang and Slotine [19] presented a theoretical study of the consensus conditions for the existence or coexistence of different types of leaders.

However, the aforementioned works always supposed that the influence of leaders on followers was positive. Altafini [20] considered the bipartite consensus problem on networks with antagonistic interactions, i.e., the weights of the communication graph are not constrained to be nonnegative. To our best knowledge, there is no literature researching consensus of multi-agent systems with antagonistic leaders. An antagonistic leader can be an exogenous attack or a faulty agent. It sends misinformation to followers for purpose of disturbing the actions of followers. This type of leader exists in practice widely, such as the spy and traitor in business or polity and the malicious servers on the internet. In this paper, follower consensus of multi-agent systems with antagonistic leaders is considered. We propose a multi-agent system consisting of followers and antagonistic leaders. The agents in the network interact with each other employing a gossip-like algorithm. Leaders send their opposite state information to others. Based on graph theory and matrix theory, some criteria for follower consensus are proposed. One of the challenges is that the negative elements in iteration matrix prevent the application of the theory for stochastic matrix. The main contribution of this paper is twofold. Firstly, we show that the follower consensus can be achieved under some restrictions for the agent pair selection probability. Secondly, we show that the follower consensus in different senses of probability is equivalent.

The rest of this paper is organized as follows. In Sec-

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tion 2, we introduce some notions in the graph theory and propose the system model. In Section 3, we give the main results. In Section 4, numerical simulations are given to illustrate the effectiveness of theoretical results. Finally, some conclusions are drawn in Section 5.

Notation: We denote the set of real number by $\mathbb{R}$. $\mathbb{R}^n$ denotes the n-dimensional real vector space. $\mathbf{1}_n$ is a vector with elements being all ones. e_i is a vector with the ith component is one and all others are zero. I_n is the $n \times n$ identity matrix. For a given vector or matrix A, A^T denotes its transpose. A^{-1} denotes the inverse matrix of A. $\mathbf{0}$ denotes an all-zero vector or matrix with compatible dimension.

2 Problem Statement

In this section, some basic concepts concerning on graph theory are introduced firstly. Following comes a brief formulation of the system model.

2.1 Graph theory

In this subsection, we shall briefly introduce some concepts about graph theory.

A weighted directed graph (digraph) $\mathcal{G}$ of order n is a triple $\mathcal{G}(\mathcal{A}) = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ which consists of a vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$, an edge set $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\}$ and a weighted adjacency matrix $\mathcal{A} = [a_{ij}]_{n\times n}$ with entries $a_{ij} \geq 0$. Let $\sum_{j=1}^{n} a_{ij} = 1$ for any $i \in \mathcal{V}$. The adjacency element a_{ij} characterizes the interaction between agent i and agent j. $a_{ij} > 0$ implies that agent i can pick agent j to exchange information, i.e. $(i, j) \in \mathcal{E}$, otherwise, $a_{ij} = 0$. In this paper, we assume that $a_{ii} = 0$ for all $i \in \mathcal{V}$. For more details about graph theory, please refer to [25].

2.2 System model

In this subsection, we present the system model and the definitions of follower consensus in different senses of probability.

Consider n ($n > 2$) agents labeled from 1 to n consisting of m ($n > m \geq 1$) antagonistic leaders and $n - m$ followers. Without loss of generality, we denote the set of leaders as $\mathcal{V}_1 = \{1, 2, \ldots, m\}$. Suppose that the agents interact with others using a gossip algorithm. At each update time t_k, agent i is selected with probability $\frac{1}{n}$. Then the agent i picks agent j to exchange information with probability a_{ij}. We assume that the leaders send their opposite state information to others and never change their own states.

If agent i and agent j are picked out to share information and update their states at time t_{k+1}, there are three update rules corresponding to three different possibilities:
(i) If $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_1$:

$$x_i(k+1) = x_i(k) \ \ and \ \ x_j(k+1) = x_j(k). \tag{1}$$

(ii) If $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$:

$$\begin{aligned} x_i(k+1) &= x_i(k), \\ x_j(k+1) &= x_j(k) + \lambda(-x_i(k) - x_j(k)). \end{aligned} \tag{2}$$

(iii) If $i \in \mathcal{V}_2$ and $j \in \mathcal{V}_2$:

$$\begin{aligned} x_i(k+1) &= x_i(k) + \lambda(x_j(k) - x_i(k)), \\ x_j(k+1) &= x_j(k) + \lambda(x_i(k) - x_j(k)), \end{aligned} \tag{3}$$

where $x_i(k)$ represents the state of agent i at time t_k, $0 < \lambda \leq 1$ is the average weight.

The update rules (1)-(3) are obtained naturally from the assumption that leaders send their opposite states to others. This phenomenon is also common in the decision making of animal groups. The interest conflicts between group members may cause individuals to lie about their information in order to bias the consensus, even though widely shared information speed up the process of decision making [21].

Let $x(k) = (x_1(k), x_2(k), \ldots, x_n(k))^T \in \mathbb{R}^n$ denote the state vector of the system at time t_k. We can also express the state update rules (1)-(3) as the following compact form.

$$x(k+1) = W^{ij}(k)x(k), \ \ i, j \in \mathcal{V}, \tag{4}$$

where $W^{ij}(k)$ is an $n \times n$ random iteration matrix given by

$$W^{ij}(k) = \begin{cases} I & i \in \mathcal{V}_1, j \in \mathcal{V}_1, \\ I - \lambda e_j(e_i + e_j)^T & i \in \mathcal{V}_1, j \in \mathcal{V}_2, \\ I - \lambda(e_i - e_j)(e_i - e_j)^T & i \in \mathcal{V}_2, j \in \mathcal{V}_2. \end{cases} \tag{5}$$

Based on above assumptions, we have the following definitions for follower consensus in the sense of probability.

Definition 1 *Multi-agent system (4) is said to achieve follower consensus*
(i) in expectation if for any $i, j \in \mathcal{V}_2$ and any initial states $x(0)$, there holds

$$\lim_{k\to\infty} E\left(x_i(k) - x_j(k)\right) = 0.$$

(ii) almost surely if for any $i, j \in \mathcal{V}_2$ and any initial states $x(0)$, there holds

$$P\left(\lim_{k\to\infty}\left(x_i(k) - x_j(k)\right) = 0\right) = 1.$$

(iii) in probability if for any $i, j \in \mathcal{V}_2$ and any initial states $x(0)$, there holds

$$\lim_{k\to\infty} P(|x_i(k) - x_j(k)| \geq \eta) = 0, \quad \forall \eta > 0.$$

3 Main results

In this section, we present our results for solving the follower consensus of multi-agent system (4).

From equation (5), $W^{ij}(k)$ is independent and identically distributed (*i.i.d.*) over all $k \geq 0$. Let $\bar{W} = E\left(W^{ij}(k)\right)$ represent the expectation of random matrix $W^{ij}(k)$.

Lemma 1 *Let $\mathcal{A}^* = \mathcal{A} + \mathcal{A}^T = [a^*_{ij}]_{n\times n}$, then we have*

$$\bar{W} = I - \frac{\lambda}{n}\left(\sum_{i=m+1}^{n}\sum_{j=1}^{m} a^*_{ij}e_ie_j^T + \sum_{i=m+1}^{m}\sum_{j=1}^{n} a^*_{ij}e_ie_i^T - \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} a^*_{ij}e_ie_j^T\right).$$

Proof. From the process of agent pair selection and equation

(5), we obtain

$$
\begin{aligned}
\bar{W} &= \frac{1}{n}\sum_{(i,j)} a_{ij}W^{ij}(k)\\
&= \frac{1}{n}(\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}I + \sum_{i=1}^{m}\sum_{j=m+1}^{n} a_{ij}(I-\lambda e_j(e_i+e_j)^T)\\
&+ \sum_{i=m+1}^{n}\sum_{j=1}^{m} a_{ij}(I-\lambda e_i(e_i+e_j)^T)\\
&+ \sum_{i=m+1}^{m}\sum_{j=m+1}^{n} a_{ij}(I-\lambda(e_i-e_j)(e_i-e_j)^T))\\
&= I - \frac{\lambda}{n}(\sum_{i=1}^{m}\sum_{j=m+1}^{n}(a_{ij}+a_{ji})e_j(e_i+e_j)^T\\
&+ \sum_{i=m+1}^{m}\sum_{j=m+1}^{n}(a_{ij}+a_{ji})e_ie_i^T - \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}(a_{ij}+a_{ji})e_ie_j^T)\\
&= I - \frac{\lambda}{n}(\sum_{i=m+1}^{n}\sum_{j=1}^{m}(a_{ij}+a_{ji})e_ie_j^T + \sum_{i=m+1}^{m}\sum_{j=1}^{n}(a_{ij}+a_{ji})e_ie_i^T\\
&- \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}(a_{ij}+a_{ji})e_ie_j^T)\\
&= I - \frac{\lambda}{n}(\sum_{i=m+1}^{n}\sum_{j=1}^{m}(a_{ij}^*e_ie_j^T + \sum_{i=m+1}^{m}\sum_{j=1}^{n} a_{ij}^*e_ie_i^T\\
&- \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} a_{ij}^*e_ie_j^T),
\end{aligned}
$$

where ${e_i e_j}^T$ is an n by n matrix whose elements are all zero except the one in the ith row, the jth column is unit. ■

Lemma 2 *Let $M = [m_{ij}]_{n\times n}$ be a matrix with uniform row sums m, then the row sums of M^{-1} are equal to $\frac{1}{m}$.*

Proof. Let $M^{-1} = [b_{ij}]_{n\times n}$, we have $\sum_{s=1}^{n} b_{is}m_{sj} = 1$ for any $i \neq j$ and $\sum_{s=1}^{n} b_{is}m_{si} = 0$. Thus,

$$
\begin{aligned}
1 &= \sum_{j=1}^{n}\sum_{s=1}^{n} b_{is}m_{sj} = \sum_{s=1}^{n}\sum_{j=1}^{n} b_{is}m_{sj}\\
&= \sum_{s=1}^{n} b_{is}\sum_{j=1}^{n} m_{sj} = m\sum_{s=1}^{n} b_{is},
\end{aligned}
$$

which means $\sum_{s=1}^{n} b_{is} = \frac{1}{m}$. ■

Theorem 1 *Suppose that $a_{ij}^* = a_{lj}^*$ and $\sum_{j=1}^{m} a_{ij}^* > 0$ for any $i, l \in \mathcal{V}_2, j \in \mathcal{V}_1$. Then, follower consensus can be achieved in expectation, i.e.,*

$$\lim_{k\to\infty} E\left(x_i(k) - x_j(k)\right) = 0 \quad \textit{for any } i, j \in \mathcal{V}_2.$$

Proof. Let $y(k) = E(x(k))$, then

$$y(k+1) = E(x(k+1)) = E\left(W^{ij}(k)x(k)\right) = \bar{W}y(k). \tag{6}$$

The last equality follows from the independence of $W^{ij}(k)$ and $x(k)$. We can write $\mathcal{A}^*$ as the following block form:

$$\mathcal{A}^* = \begin{bmatrix} A_{m\times m}^* & A_{fl}^* \\ A_{lf}^* & A_f^* \end{bmatrix}. \tag{7}$$

From Lemma 1, we have

$$\bar{W} = I - \frac{\lambda}{n}(\sum_{i=m+1}^{n}\sum_{j=1}^{m}(a_{ij}^*e_ie_j^T + \sum_{i=m+1}^{m}\sum_{j=1}^{n} a_{ij}^*e_ie_i^T - \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} a_{ij}^*e_ie_j^T).$$

Let $d_i = \sum_{j=1}^{n} a_{ij}^*$ for any $i \in \mathcal{V}_2$. Combining with equation (7), we rewrite $\bar{W}$ as the form of block matrix:

$$\bar{W} = \begin{bmatrix} I_{m\times m} & \mathbf{0} \\ \bar{W}_{lf} & \bar{W}_f \end{bmatrix}, \tag{8}$$

with

$$
\begin{aligned}
\bar{W}_{lf} = -\frac{\lambda}{n}A_{lf}^*, \quad \bar{W}_f = I - \frac{\lambda}{n}(D - A_f^*),\\
D = diag(d_{m+1}, d_{m+2}, \ldots, d_n).
\end{aligned} \tag{9}
$$

Hence, the row sum of the ith row of $\bar{W}_f$ is equal to $1 - \frac{\lambda}{n}\sum_{j=1}^{m} a_{m+ij}^*$. Since $\sum_{j=1}^{m} a_{ij}^* > 0$ for any $i \in \mathcal{V}_2, j \in \mathcal{V}_1$, the row sums of $\bar{W}_f$ are less than one. Thus, $\bar{W}_f$ is a convergent matrix. From equation (6) we have

$$
\begin{aligned}
y(k) &= \begin{bmatrix} I_{m\times m} & \mathbf{0} \\ \bar{W}_{lf} & \bar{W}_f \end{bmatrix}^k y(0)\\
&= \begin{bmatrix} I_{m\times m} & \mathbf{0} \\ (\sum_{t=0}^{k-1}\bar{W}_f^t)\bar{W}_{lf} & \bar{W}_f^k \end{bmatrix} y(0).
\end{aligned} \tag{10}
$$

Since $\bar{W}_f$ is convergent, we obtain $\bar{W}_f^k \to \mathbf{0}$ and $\sum_{t=0}^{k}\bar{W}_f^t \to (I-\bar{W}_f)^{-1}$ as $k\to\infty$. Thus,

$$\lim_{k\to\infty}\bar{W}^k = \begin{bmatrix} I & \mathbf{0} \\ (I-\bar{W}_f)^{-1}\bar{W}_{lf} & \mathbf{0} \end{bmatrix}. \tag{11}$$

Because $a_{ij}^* = a_{lj}^*$ for any $i, l \in \mathcal{V}_2, j \in \mathcal{V}_1$, then $\sum_{j=1}^{m} a_{ij}^* = \alpha$ for any $i \in \mathcal{V}_2$. Thus the row sums of $\bar{W}_f$ are equal to $1 - \frac{\lambda}{n}\alpha$. From Lemma 2, the row sums of $(I-\bar{W}_f)^{-1}$ are equal to $\frac{n}{\lambda\alpha}$. Hence,

$$(I-\bar{W}_f)^{-1}\bar{W}_{lf} = \mathbf{1}_{n-m}c^T,$$

where $c \in \mathbb{R}^m$ and $c = -\frac{1}{\alpha}(a_{i1}^*, a_{i2}^*, \ldots, a_{im}^*)^T, i \in \mathcal{V}_2$. Thus, for any $i \in \mathcal{V}_2$, we have $y_i(k) \to -\frac{1}{\alpha}\sum_{j=1}^{m} a_{ij}^*x_j(0)$ as $k\to\infty$. This implies that

$$\lim_{k\to\infty} E(x_i(k) - x_j(k)) = 0$$

for any $i, j \in \mathcal{V}_2$, which means that the states of followers reach consensus in expectation. ■

Now we establish the equivalency for follower consensus in different senses of probability. Before proposing our equivalence property, we need the following lemma.

Lemma 3 *Let $\{\xi(k)\}_0^\infty$ be a sequence of random variables, assume that $\xi(k)$ is non-increasing, i.e. $\xi(k+1) \leq \xi(k)$ for all $k \geq 0$. If $\xi(k) \to \xi^*$ with $k\to\infty$ in probability, then $\xi(k) \to \xi^*$ a.s. with $k\to\infty$.*

Proof. Because $\xi(k) \to \xi^*$ in probability, following from Riesz Theorem (Theorem 2.5.3, [27]), there exists a subsequence of $\{\xi(k)\}$ such that $\xi(k_l) \to \xi^*$ almost surely. That is to say, for any $\epsilon > 0$, there exists $L > 0$ such that for any $l \geq L$, $|\xi(k_l) - \xi^*| < \epsilon$ holds almost surely. Since $\xi(k)$ is non-increasing and $\xi(k_l) \to \xi^*$ a.s., we have that $\xi(k_l) \geq \xi^*$ almost surely, thus $0 \leq \xi(k_l) - \xi^* < \epsilon$ almost surely. Since for any

$k \geq k_L$, there exists $k_{L'} \geq k$, hence $\xi^* \leq \xi(k_{L'}) \leq \xi(k) \leq \xi(k_L)$. Then we obtain that $0 \leq \xi(k_{L'}) - \xi^* \leq \xi(k) - \xi^* \leq \xi(k_L) - \xi^* < \epsilon$ almost surely. In other words, for aforementioned $\epsilon > 0$, $\exists\, k_L > 0$ such that for any $k \geq k_L$ there holds $0 \leq \xi(k) - \xi^* < \epsilon$ almost surely, which means $\xi(k) \to \xi^*$ almost surely. ■

The next theorem states the equivalency of consensus in different senses of probability.

Theorem 2 *Suppose that $a^*_{ij} = a^*_{lj}$ and $\sum_{j=1}^{m} a^*_{ij} > 0$ for any $i, l \in \mathcal{V}_2$, $j \in \mathcal{V}_1$. If $x_i(k) = x_j(k)$ for any $i, j \in \mathcal{V}_1$, then the following statements are equivalent.*
(i) Multi-agent system (4) achieves follower consensus in expectation.
(ii) Multi-agent system (4) achieves follower consensus in probability.
(iii) Multi-agent system (4) achieves follower consensus almost surely.

Proof. Define $\chi(k) = max_{i,j \in \mathcal{V}_2} |x_i(k) - x_j(k)|$ to represent consensus metrics. Cause $x_i(k) = x_j(k)$ for any $i, j \in \mathcal{V}_1$, equation (1-3) suggest that $\chi(k)$ is non-increasing in k.

Consensus in expectation $\Longrightarrow$ consensus in probability. From $\lim_{k\to\infty} E(\chi(k)) = 0$ and $\chi(k)$ is non-increasing, we have

$$\lim_{k\to\infty} E\left(\chi^2(k)\right) \leq \lim_{k\to\infty} E\left(\chi(k)\chi(0)\right) \leq \chi(0) \lim_{k\to\infty} E\left(\chi(k)\right) = 0.$$

Hence, we obtain

$$\lim_{k\to\infty} E\left(\chi^2(k)\right) = 0.$$

From Chebyshev's inequality [26], we have that

$$P(|\chi(k)| \geq \epsilon) \leq \frac{E(\chi^2(k))}{\epsilon^2}$$

holds for any $\epsilon > 0$. Since $E(\chi^2(k)) \to 0$ as $k \to \infty$, we can obtain that

$$\lim_{k\to\infty} P\left(|\chi(k)| \geq \epsilon\right) = 0$$

for any $\epsilon > 0$, which implies that followers reach consensus in probability.

Consensus in probability $\Longrightarrow$ almost surely consensus. Since $\chi(k)$ is non-increasing in k, from Lemma 3 we obtain

$$P(\lim_{k\to\infty} \chi(k) = 0) = 1,$$

which implies that followers reach consensus almost surely.

Almost surely consensus $\Longrightarrow$ consensus in expectation. Because followers achieve consensus almost surely, we have $\chi(k) \to 0$ almost surely. Since $\chi(k)$ is nonnegative and non-increasing, employing the Monotone Convergence Theorem [26], we obtain that $E\chi(k) \to 0$ as $k \to \infty$. ■

4 Simulations

In this section, we provide a simulation to illustrate the effectiveness of the theoretical results proposed in this paper.

Example Suppose there are five agents in the communication network labeled from 1 to 5. Agent 1 and agent 2 are leaders, that is, $n = 5, m = 2$. $x(t)$ is initialized as $x(0) = [0.5, 0.7, 0.1, -0.5, 0.2]$. Moreover, let $\lambda = 0.5$ in the process of state update. Let $a^*_{j1} = 0.3, a^*_{j2} = 0.5$ for $5 \geq j \geq 3$. Fig. 1 shows that the follower consensus is achieved in the sense of expectation. We can find that the agreement value is -0.6, which equals to the opposite of the weighted average of leaders' initial states.

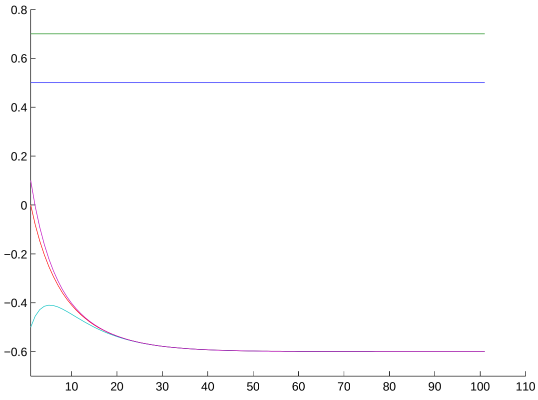

Fig. 1: The trajectories of $E(x(t))$.

5 Conclusions

In this paper, follower consensus of multi-agent systems with antagonistic leaders has been studied. The leaders were stationary and they sent their opposite states to others. The interactions of agents were modeled by a gossip algorithm. We showed that if the influence of leaders on each follower was equal, the follower consensus could be achieved in expectation. Moreover, we established an equivalency for the follower consensus under different senses of probability. The future work will focus on the internal cooperation existing between followers and leaders.

References

[1] R. Olfati-Saber, Flocking for multi-agent dynamics systems: algorithms and theory, IEEE Transactions on Automatic Control, 51(3): 401–420, 2006.

[2] T. Chu, L. Wang, T. Chen, S. Mu, Complex emergent dynamics of anisotropic swarms: convergence vs oscillation, Chaos Solitons and Fractals, 30(4): 875–885, 2006.

[3] F. Xiao, L. Wang, J. Chen, Y. Gao, Finite-time formation control for multi-agent systems, Automatica, 45(11): 2605-2611, 2009.

[4] Z. Ji, Z. Wang, H. Lin, Z. Wang, Interconnection topologies for multi-agent coordination under leader-follower framework, Automatica, 45(12): 2857–2863, 2009.

[5] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, O. Schochet, Novel type of phase transition in a system of self-driven particles, Physical Review Letters, 75(6): 1226–1229, 1995.

[6] A. Jadbabaie, J. Lin, A. S. Morse, Coordination of groups of mobile autonomous agents using neaest neighbor rules, IEEE Transactions on Automatic Control, 48(6): 988–1001, 2003.

[7] R. Olfati-Saber, R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Transactions on Automatic Control, 49(9): 1520–1533, 2004.

[8] W. Ren, R. W. Beard, Consensus seeking in multiagent systems under dynamically changing interaction topologies, IEEE Transactions on Automatic Control, 50(5): 655–661, 2005.

[9] F. Xiao, L. Wang, Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays, IEEE Transactions on Automatic Control, 53(8): 1804–1816, 2008.

[10] A. Nedić, A. Ozdaglar, P. A. Parrilo, Constrained consensus

and optimization in multi-agent networks, IEEE Transactions on Automatic Control, 55(4): 922–938, 2010.

[11] Y. Hatano, M. Mesbahi, Agreement over random networks, IEEE Transactions on Automatic Control, 50(11): 1867–1872, 2005.

[12] A. Tahbaz-Salehi, A. Jadbabaie, A necessary and sufficient condition for consensus over random networks, IEEE Transactions on Automatic Control, 53(3): 791–795, 2008.

[13] I. Matei, J. S. Baras, C. Somarakis, Convergence results for the linear consensus problem under markovian random graphs, SIAM Journal on Control and Optimization, 51(2): 1574–1591, 2013.

[14] S. Boyd, A. Ghosh, B. Prabhakar, D. Shah, Randomized gossip algorithms, IEEE Transactions on Information Theory, 52(6): 2508–2530, 2006.

[15] D. Acemoglu, A. Ozdaglar, A. ParandehGheibi, Spread of (mis) information in social networks, Games and Economic Behavior, 70(2): 194–227, 2010.

[16] J. Ma, Y. Zheng, L. Wang, LQR-based optimal topology of leader-following consensus, International Journal of Robust and Nonlinear Control, 25(17): 3404–3421, 2014.

[17] Y. Hong, J. Hu, L. Gao, Tracking control for multi-agent consensus with an active leader and variable topology, Automatica, 42(7): 1177–1182, 2006.

[18] Y. Zheng, L. Wang, Containment control of heterogeneous multi-agent systems, International Journal of Control, 87(1): 1–8, 2014.

[19] W. Wang, J. J. E. Slotine, A theoretical study of different leader roles in networks, IEEE Transactions on Automatic Control, 51(7): 1156–1161, 2006.

[20] C. Altafini, Consensus problems on networks with antagonistic interactions, IEEE Transactions on Automatic Control, 58(4): 935–946, 2013.

[21] N. R. Franks, A. Dornhaus, J. P. Fitzsimmons, M. Stevens, Speed versus accuracy in collective decision making, Proceedings of the Royal Society of London B: Biological Sciences, 270(1532): 2457–2463, 2003.

[22] L. Conradt, T. J. Roper, Group decision-making in animals, Nature, 421(6919): 155–158, 2003.

[23] D. Bindel, J. Kleinberg, S. Oren, How bad is forming your own opinion?, Proceedings of the 52nd Annual IEEE Symposium on Foundations of Computer Science, 57–66, 2011.

[24] M. J. Fischer, N. A. Lynch, M. S. Paterson, Impossibility of distributed consensus with one faulty process, Journal of the Association for Computing Machinery, 32(2): 374–382, 1985.

[25] C. Godsil, G. Royal, Algebraic graph theory, Springer-Verlag, New York, 2001.

[26] R. Durrett, Probability: theory and examples, Cambridge University Press, New York, 2010.

[27] R. B. Ash, C. Doleans-Dade, Probability and measure theory, Academic Press, London, 2000.