Susceptibility optimization and the wisdom of crowds in influence networks ⋆

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Abstract: This paper studies how to maximize the wisdom of crowds in social networks by shaping individual susceptibilities. The problem is formulated as a nonlinear, nonconvex optimization over the Friedkin-Johnsen opinion dynamics, in which individuals' susceptibilities are tuned to minimize the collective estimation bias. Exploiting the intrinsic social power structure of the opinion dynamics, we propose the sequential social power method, which first computes an optimal social power allocation via a quadratic program, and then recovers all susceptibility vectors that yield this allocation. We prove that, if the influence network is strongly connected, these optimal susceptibility vectors can be obtained in closed form using the Laplacian pseudoinverse and the dominant left eigenvector of the influence matrix. Numerical simulations and application examples illustrate our theoretical results.

Keywords: Wisdom of crowds, Social networks, Opinion dynamics, Social learning, Nonconvex optimization, Laplacian pseudoinverse

1. INTRODUCTION

A central question in collective decision-making is how to harness the knowledge initially dispersed among individuals (Hayek, 1945). Empirical studies indicate that simply pooling many independent estimates on an unknown quantity can yield a collective decision remarkably close to the truth, known as the wisdom of crowds (WoC) (Galton, 1907). Since its inception, this concept of collective intelligence has attracted considerable attention (Lorenz et al., 2011; Becker et al., 2017).

Building on the WoC, the Delphi method structures anonymity, iteration, and controlled feedback to systematically aggregate experts' opinions (Dalkey and Helmer, 1963). In a typical Delphi inquiry, participants remain anonymous while a moderator controls the flow of information: the moderator determines which answers each participant sees and invites round-by-round revisions. This process mitigates the biases associated with face-to-face interaction while retaining the benefits of WoC. Extensive experimental evidence shows that the Delphi method outperforms uncontrolled group discussion across a range of domains (Madirolas and de Polavieja, 2015; Becker et al., 2017).

Both WoC and the Delphi method are situated within the broader field of social influence and collective decision-making (Centola, 2022). The study of opinion dynamics, also known as influence system theory, provides a theoretical foundation for understanding how individual interactions shape group outcomes (DeGroot, 1974; Friedkin and Johnsen, 1999). The recent literature features an increasing body of work explaining how social influence affects WoC using opinion dynamics models, e.g., Golub and Jackson (2010); Tian et al. (2023); see Tian and Wang (2023) for further details.

One of the most influential models of opinion dynamics is the Friedkin-Johnsen (FJ) model, which introduces individual susceptibilities to capture heterogeneity in resistance to social influence (Friedkin and Johnsen, 1999). A closely related notion, *susceptibility to persuasion*, has been examined in developmental psychology, social psychology, and political science (Steinberg and Monahan, 2007). Evidence from these domains suggests that susceptibility is not fixed: it can be shaped by contextual framings, targeted strategies, and situational cues (McGuire, 1964). Parallel work in human-computer interaction has given rise to persuasive technologies, interactive systems explicitly designed to alter people's attitudes or behaviors (Fogg, 2002). This line of research shows both that individuals differ systematically in their willingness to be persuaded, and that technological interventions can be tailored to amplify or attenuate this susceptibility. In Abebe et al. (2021), the FJ model is employed to study inter-

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ventions on susceptibility aimed at minimizing the sum of individuals' final opinions.

In this paper, we study how to maximize WoC in influence networks by intervening on individuals' susceptibilities. We formulate the problem based on the FJ model in a naïve learning setting. For an unknown state of nature, individuals only possess private knowledge but do not know the exact value. At the beginning of the influence process, each individual proposes an estimate as its initial opinion based on its private knowledge, described by a random variable. As the influence process converges, the collective estimate, defined as the average of all individuals' final opinions, is determined by the initial opinions and the social power allocation of the FJ model.

Our aim is to minimize the difference between the mean of the collective estimate and the truth. We show that this optimization problem is highly nonlinear and is neither convex nor concave. We analyze the structural properties of the social power of the FJ model. Leveraging this social power structure, we propose the *sequential social power method* (SSPM) to address the problem.

Our approach first relaxes the problem to a quadratic program (QP) over the probability simplex, from which we compute an optimal social power. For mixed-sign bias vectors, we provide a constructive procedure to obtain this optimal social power. We then prove that optimal susceptibility vectors yielding such an optimal social power can be recovered using the *Laplacian pseudoinverse* and the *dominant left eigenvector* of the influence matrix, provided the influence network is strongly connected. In other words, SSPM decomposes the original nonconvex optimization into one convex QP for the optimal social power and two strongly convex QPs for susceptibility recovery, which can be solved in polynomial time.

Organization. In Section 2 we formulate the problem. Section 3 analyzes the structural properties of social power. Our main results are presented in Section 4. Application and simulation examples are provided in Section 5. Section 6 concludes the paper and discusses future directions.

Notation. Let $\mathbf{1}_n$ and I_n denote the $n \times 1$ all-ones vector and the $n \times n$ identity matrix, respectively. 0 denotes a zero scalar, vector, or matrix depending on the context. $\mathbf{e}_i$ denotes the i-th standard basis vector with proper dimension. All inequalities and equalities involving vectors or matrices are interpreted elementwise. Given $\boldsymbol{x} \in \mathbb{R}^n$, $[\boldsymbol{x}]$ denotes a diagonal matrix with diagonal elements $x_1, \ldots, x_n$. The n-simplex is denoted by $\Delta_n = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq 0, \mathbf{1}_n^\top \boldsymbol{x} = 1\}$; its interior is denoted by $\operatorname{int} \Delta_n$. For a square matrix $W \in \mathbb{R}^{n \times n}$, its spectral radius is denoted by $\rho(W)$. W is irreducible if there is no permutation matrix P such that $P^\top W P$ is block upper triangular with a zero block below the diagonal. W is a Metzler matrix if all its off-diagonal entries are nonnegative. A nonnegative matrix is row-stochastic if its row sums are 1. If W is irreducible and row-stochastic, its dominant left eigenvector $\boldsymbol{\omega}$ is guaranteed unique, satisfying $\boldsymbol{\omega}^\top W = \boldsymbol{\omega}^\top$ and $\boldsymbol{\omega} \in \operatorname{int} \Delta_n$.

The weighted digraph $\mathcal{G}(W)$ associated with a nonnegative matrix W is defined as: the node set is $\mathcal{V} = \{1, \ldots, n\}$; there is a directed edge (i, j) from nodes i to j if and only if $W_{ij} > 0$. A directed path of length m from nodes i_0 to i_m consists of an ordered sequence of m directed edges: $\{(i_l, i_l{+}1)\}_{l=0}^{m-1}$. $\mathcal{G}(W)$ is strongly connected if there exists a directed path between every pair of distinct nodes i and j; equivalently, W is irreducible. A strongly connected component (SCC) of $\mathcal{G}(W)$ is a maximal strongly connected subgraph. An SCC is a sink SCC if there exists no directed edge from this SCC to others.

2. PROBLEM FORMULATION

We formulate the problem based on the naïve learning setting introduced in Golub and Jackson (2010). Consider n individuals whose opinions on an unknown state of nature $\theta \in \mathbb{R}$ evolve over an influence network $\mathcal{G}(W)$. While individuals do not know the exact value, each possesses partial knowledge or clues about it.

At time step $k = 0$, each individual forms its initial opinion based on its private knowledge: $y_i(0)$ is a random variable with mean $\mu_i \neq \theta$ and variance $\sigma_i^2 > 0$. We assume that the initial opinions $\{y_i(0)\}_{i=1}^n$ are mutually independent. Upon exposure to others' opinions, each individual updates its own opinions according to the FJ opinion dynamics (Friedkin and Johnsen, 1999):

$$y_i(k+1) = a_i \sum_{j=1}^{n} W_{ij} y_j(k) + (1 - a_i) y_i(0), \qquad (1)$$

where a_i is i's susceptibility and W is the row-stochastic adjacency matrix of $\mathcal{G}(W)$. For brevity, we classify agents as *non-stubborn* if $a_i = 1$, *fully stubborn* if $a_i = 0$, and *partially stubborn* if $0 < a_i < 1$. Let $\boldsymbol{a} = (a_1, \ldots, a_n)^\top$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$. The following assumption ensures the convergence of the FJ model, as shown in Lemma 1.

Assumption 1. There is no sink SCC in $\mathcal{G}(W)$ that consists only of non-stubborn individuals.

Lemma 1. (Tian and Wang, 2023, Lemma 4) For the FJ model (1),

$$\lim_{k \to \infty} \boldsymbol{y}(k) = (I_n - [\boldsymbol{a}]W)^{-1}(I_n - [\boldsymbol{a}])\boldsymbol{y}(0) \qquad (2)$$

if and only if Assumption 1 holds.

Let $V = (I_n - [\boldsymbol{a}]W)^{-1}(I_n - [\boldsymbol{a}])$ and define

$$y_{\text{ave}} = \lim_{k \to \infty} \frac{1}{n} \sum_{i=1}^{n} y_i(k)$$

as the collective estimate. Then, we have

$$y_{\text{ave}} = \frac{1}{n} \mathbf{1}_n^\top (I_n - [\boldsymbol{a}]W)^{-1}(I_n - [\boldsymbol{a}])\boldsymbol{y}(0) = \boldsymbol{x}^\top \boldsymbol{y}(0),$$

where $\boldsymbol{x} = V^\top \mathbf{1}_n / n$ is the social power allocation of the FJ model (Tian and Wang, 2023). In an influence system, social power measures the relative control of individuals' initial states over others' final states. As $\{y_i(0)\}_{i=1}^n$ are independent random variables, y_{ave} is also a random variable, with

$$\mathbb{E}[y_{\text{ave}}] = \sum_{i=1}^{n} x_i \mu_i, \quad \text{and} \quad \operatorname{Var}[y_{\text{ave}}] = \sum_{i=1}^{n} x_i^2 \sigma_i^2. \qquad (3)$$

The difference between the collective estimate y_{ave} and the truth θ measures the deviation of the collective estimate from the truth, thus indicating the level of collective intelligence. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^\top$ and $\boldsymbol{\delta} = \boldsymbol{\mu} - \theta \mathbf{1}_n$. We define the squared collective bias $f(\boldsymbol{a}) : [0,1]^n \to \mathbb{R}$ as

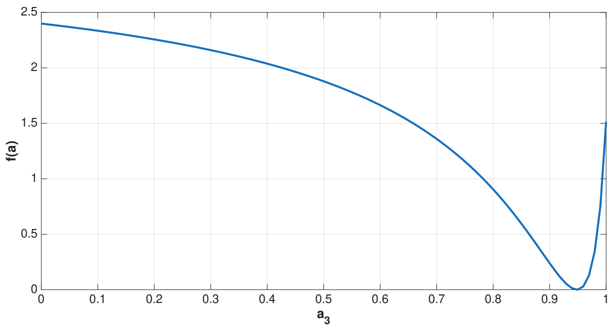

Fig. 1. Plot of the squared collective bias $f(\boldsymbol{a})$ versus susceptibility a_3, with a_1 and a_2 fixed.

$$f(\boldsymbol{a}) = (\mathbb{E}[y_{\text{ave}}] - \theta)^2 = \left(\sum_{i=1}^{n} x_i \mu_i - \theta\right)^2 = (\boldsymbol{x}^\top \boldsymbol{\delta})^2. \quad (4)$$

For a given influence network $\mathcal{G}(W)$ and bias vector $\boldsymbol{\delta}$, our objective is to minimize the squared collective bias:

$$\min_{\boldsymbol{a}\in[0,1]^n} \quad f(\boldsymbol{a}). \quad (5)$$

In practice, individuals' susceptibilities $\boldsymbol{a}$ can be modulated, either through external interventions or via self-adaptation (McGuire, 1964; Fogg, 2002). The objective function $f(\boldsymbol{a})$ is highly nonlinear and exhibits neither convexity nor concavity; see the following example.

Example 1. (Nonconvexity of the objective) Consider an influence network with $n = 3$, $\theta = 0$,

$$W = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0 & 0.5 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\mu} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Fixing $a_1 = 0.7$ and $a_2 = 0.3$, we vary $a_3 \in [0, 1]$. Fig. 1 shows the behavior of $f(\boldsymbol{a})$ as a function of a_3. The curve is evidently neither convex nor concave. ◁

Remark 1. (Measures of collective wisdom) In the literature, several measures of WoC have been proposed for both theoretical and empirical analysis. In general, the *mean square error* (MSE), defined as the mean of the squared difference between y_{ave} and θ, has been used to quantify the overall accuracy of the collective estimate:

$$\begin{aligned} \text{MSE}[y_{\text{ave}}] &= \mathbb{E}[y_{\text{ave}} - \theta]^2 \\ &= (\mathbb{E}[y_{\text{ave}}] - \theta)^2 + \mathbb{E}[y_{\text{ave}} - \mathbb{E}[y_{\text{ave}}]]^2 \\ &= f(\boldsymbol{a}) + \text{Var}[y_{\text{ave}}]. \end{aligned}$$

Most existing theoretical studies assume that individual estimates are unbiased (Tian et al., 2023). Under this assumption, $f(\boldsymbol{a}) \equiv 0$, and the collective intelligence is evaluated solely through $\text{Var}[y_{\text{ave}}]$. In contrast, empirical studies typically consider biased individual estimates, and employ $f(\boldsymbol{a})$ as the measure of collective wisdom (Lorenz et al., 2011; Becker et al., 2017).

In settings where $f(\boldsymbol{a}) \neq 0$, however, minimizing the MSE does not guarantee an accurate collective estimate: even tightly clustered collective estimates around an incorrect mean yield poor collective performance. Therefore, minimizing the systematic bias $f(\boldsymbol{a})$ is the more relevant first step. A natural subsequent problem is

$$\begin{aligned} \min_{\boldsymbol{a}\in[0,1]^n} \quad & \text{Var}[y_{\text{ave}}] \\ \text{s.t.} \quad & \boldsymbol{a} = \operatorname*{argmin}_{\boldsymbol{a}'\in[0,1]^n} f(\boldsymbol{a}'). \end{aligned} \quad (6)$$

◁

3. PROPERTIES OF SOCIAL POWER

By (4), $f(\boldsymbol{a})$ is determined directly by the social power $\boldsymbol{x}$. In this section, we analyze properties of $\boldsymbol{x}$. We first derive upper bounds on the components of $\boldsymbol{x}$. Then, we prove that $\boldsymbol{x}$ is the dominant left eigenvector of an irreducible Metzler matrix associated with the FJ model.

3.1 Upper bounds on social power

Let $\mathcal{V}_{\text{f}}$, $\mathcal{V}_{\text{p}}$ and $\mathcal{V}_{\text{n}}$ denote the sets of fully stubborn, partially stubborn, and non-stubborn individuals, respectively. The following lemma proceeds along lines similar to those of Tian and Wang (2023, Lemma 5) and Tian et al. (2025, Lemma A.5).

Lemma 2. Suppose that Assumption 1 holds. For $V = (I_n - [\boldsymbol{a}]W)^{-1}(I_n - [\boldsymbol{a}])$, the following statements hold:

(i) V is row-stochastic;
(ii) $V_{ij} = 0$ for all $i \in \mathcal{V}$ and $j \in \mathcal{V}_{\text{n}}$;
(iii) $V_{ii} > 0$ for all $i \in \mathcal{V}_{\text{f}} \cup \mathcal{V}_{\text{p}}$;
(iv) $V_{ii} > V_{ji}$ for all $i \in \mathcal{V}_{\text{f}} \cup \mathcal{V}_{\text{p}}$ and $j \neq i$;

Lemma 2 (ii) and (iii) imply that, under Assumption 1, $x_i = 0$ if and only if $i \in \mathcal{V}_{\text{n}}$. Thus, a non-stubborn individual has no influence on the collective estimate. This motivates the following assumption.

Assumption 2. $a_i < 1$ for all $i \in \mathcal{V}$.

Proposition 1. Suppose that Assumption 1 holds. Then,

(i) $\boldsymbol{x} \in \Delta_n$;
(ii) $\boldsymbol{x} \in \operatorname{int} \Delta_n$ if and only if Assumption 2 holds;
(iii) $x_i < \frac{1}{n(1-a_i W_{ii})}(1 + \sum_{j\neq i} a_j)$ for all $i \in \mathcal{V}_{\text{f}} \cup \mathcal{V}_{\text{p}}$.

Proof. Statements (i) and (ii) directly follow from Lemma 2. Regarding (iii), by $(I_n - [\boldsymbol{a}]W)V = I_n - [\boldsymbol{a}]$ we have

$$V_{ii} = \frac{1-a_i}{1-a_i W_{ii}} + \frac{a_i}{1-a_i W_{ii}} \sum_{l\neq i} W_{il} V_{li}, \; i \in \mathcal{V}_{\text{f}} \cup \mathcal{V}_{\text{p}}, \quad (7)$$

and

$$V_{ji} = \frac{a_j}{1-a_j W_{jj}} \sum_{l\neq j} W_{jl} V_{li}, \; j \neq i.$$

Therefore, we obtain

$$\begin{aligned} n x_i = \sum_{j=1}^{n} V_{ji} =& \frac{1-a_i}{1-a_i W_{ii}} + \sum_{j=1}^{n} \frac{a_j}{1-a_j W_{jj}} \sum_{l\neq j} W_{jl} V_{li} \\ <& \frac{1-a_i}{1-a_i W_{ii}} + V_{ii} \sum_{j=1}^{n} \frac{a_j}{1-a_j W_{jj}} (1 - W_{jj}) \\ \leq& \frac{1-a_i}{1-a_i W_{ii}} + V_{ii} \sum_{j=1}^{n} a_j, \end{aligned}$$

where the first inequality is implied by Lemma 2 (iv) and the last one follows from $1 - a_j W_{jj} \geq 1 - W_{jj}$. Similarly, by (7) we have

$$\begin{aligned} V_{ii} < \frac{1-a_i}{1-a_i W_{ii}} + V_{ii} \frac{a_i}{1-a_i W_{ii}} (1 - W_{ii}) \\ \leq \frac{1-a_i}{1-a_i W_{ii}} + a_i V_{ii}, \end{aligned}$$

which implies $V_{ii} < \frac{1}{1-a_iW_{ii}}$. As a result, we obtain

$$x_i < \frac{1}{n}(\frac{1-a_i}{1-a_iW_{ii}} + \frac{1}{1-a_iW_{ii}}\sum_{j=1}^{n} a_j) = \frac{1}{n(1-a_iW_{ii})}(1+\sum_{j\neq i} a_j).$$

□

3.2 Spectral characterization of social power

We now prove that $\boldsymbol{x}$ is the dominant left eigenvector of an irreducible Metzler matrix.

Proposition 2. Define $F(\boldsymbol{a}) : [0,1)^n \to \mathbb{R}^{n\times n}$ as

$$F(\boldsymbol{a}) = \mathbf{1}_n\mathbf{1}_n^\top \frac{1}{n} - [\boldsymbol{a}](I_n - [\boldsymbol{a}])^{-1}(I_n - W).$$

For any $\boldsymbol{a} \in [0,1)^n$, the following statements hold:

(i) $F(\boldsymbol{a})$ is an irreducible Metzler matrix with $\rho(F(\boldsymbol{a})) = 1$;
(ii) $\boldsymbol{x}$ is the social power of the FJ model (1) if and only if it is the dominant left eigenvector of $F(\boldsymbol{a})$ associated with eigenvalue 1, i.e., $\boldsymbol{x}^\top F(\boldsymbol{a}) = \boldsymbol{x}^\top$.

Proof. By the definition of $F(\boldsymbol{a})$, we have

$$\begin{cases} F_{ij}(\boldsymbol{a}) = \dfrac{1}{n} + \dfrac{a_i}{1-a_i}W_{ij}, & i \neq j, \\ F_{ii}(\boldsymbol{a}) = \dfrac{1}{n} - \dfrac{a_i}{1-a_i}(1-W_{ii}), & i \in \mathcal{V}. \end{cases} \tag{8}$$

Regarding (i), since $F_{ij} \geq 1/n > 0$ for all $\boldsymbol{a}$ and W, $F(\boldsymbol{a})$ has all positive off-diagonal entries. Hence, $F(\boldsymbol{a})$ is an irreducible Metzler matrix. Moreover, it follows from

$$\left(\mathbf{1}_n\mathbf{1}_n^\top \frac{1}{n} - [\boldsymbol{a}](I_n - [\boldsymbol{a}])^{-1}(I_n - W)\right)\mathbf{1}_n = \mathbf{1}_n$$

that 1 is an eigenvalue of $F(\boldsymbol{a})$. Moreover, by (8) and the Geršgorin disks theorem (Bullo, 2020, Theorem 2.8), any eigenvalue λ of $F(\boldsymbol{a})$ satisfies

$$\left|\lambda - \left(\frac{1}{n} - \frac{a_i}{1-a_i}(1-W_{ii})\right)\right| \leq \frac{1}{n} + \frac{a_i}{1-a_i}(1-W_{ii}),$$

which implies that its real part satisfies $\mathcal{R}(\lambda) \leq 1$.

On the other hand, since $F(\boldsymbol{a})$ is irreducible, the Perron-Frobenius theorem for Metzler matrix (Bullo, 2020, Theorem 9.4) implies that $\rho(F(\boldsymbol{a}))$ is a positive real simple eigenvalue. Hence, $\rho(F(\boldsymbol{a})) = 1$.

Regarding (ii), by the definition of $\boldsymbol{x}$, we have

$$\begin{aligned} &\boldsymbol{x}^\top F(\boldsymbol{a}) \\ =&\boldsymbol{x}^\top \left(\mathbf{1}_n\mathbf{1}_n^\top \frac{1}{n} - [\boldsymbol{a}](I_n - [\boldsymbol{a}])^{-1}(I_n - W)\right) \\ =&\frac{1}{n}\mathbf{1}_n^\top - \boldsymbol{x}^\top[\boldsymbol{a}](I_n - [\boldsymbol{a}])^{-1}(I_n - W) \\ =&\frac{1}{n}\mathbf{1}_n^\top + \boldsymbol{x}^\top - \boldsymbol{x}^\top(I_n - [\boldsymbol{a}])^{-1}(I_n - [\boldsymbol{a}]W) = \boldsymbol{x}^\top, \end{aligned}$$

where the second equation is implied by $\boldsymbol{x}^\top\mathbf{1}_n = 1$, and the subsequent equations follow from

$$[\boldsymbol{a}](I_n - W) = [\boldsymbol{a}] - I_n + I_n - [\boldsymbol{a}]W$$

and $\boldsymbol{x}^\top(I_n - [\boldsymbol{a}])^{-1}(I_n - [\boldsymbol{a}]W) = \mathbf{1}^\top/n$. □

4. THE SEQUENTIAL SOCIAL POWER METHOD

Since $f(\boldsymbol{a})$ is differentiable on $[0,1]^n$ under Assumption 1, one can employ a *projected gradient descent* (PGD) method to interatively update $\boldsymbol{a}$. However, due to the nonconvexity and highly nonlinear dependence of f on $\boldsymbol{a}$, PGD can be computationally costly, and does not guarantee convergence to a global minimum.

To overcome these limitations, we instead exploit the intrinsic structure of the FJ model and develop a more efficient and interpretable approach. Specifically, we show that the problem admits a sequential decomposition: we first determine the optimal social power by solving a QP (Section 4.1), and then recover the corresponding optimal susceptibilities (Section 4.2). This two-step procedure is referred to as the SSPM.

4.1 Optimal social power

Define $g : \Delta_n \to \mathbb{R}$ as $g(\boldsymbol{x}) = (\boldsymbol{x}^\top\boldsymbol{\delta})^2$. We first relax problem (5) to

$$\min_{\boldsymbol{x}\in\Delta_n} g(\boldsymbol{x}), \tag{9}$$

which is a QP over the simplex. Since $\boldsymbol{\delta}\boldsymbol{\delta}^\top$ is positive semidefinite for all $\boldsymbol{\delta} \in \mathbb{R}^n$, problem (9) admits optimal solutions. Intuitively, if the components of $\boldsymbol{\delta}$ are all negative or are all positive, (9) is solved by $\boldsymbol{x}^* = \boldsymbol{e}_r$ with $r = \operatorname{argmin}_i |\delta_i|$. In the following, we focus on the case that individual biases δ_i have mixed signs.

Assumption 3. $\min_i \delta_i < 0 < \max_i \delta_i$.

Theorem 1. Suppose that Assumptions 1 and 3 hold. Then there exists an $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ that solves problem (9).

Proof. For problem (9), we introduce the Lagrange multiplier $\nu \in \mathbb{R}$ and the Lagrangian function

$$\mathcal{L}(\boldsymbol{x}, \lambda) = \boldsymbol{x}^\top\boldsymbol{\delta}\boldsymbol{\delta}^\top\boldsymbol{x} + \nu(\boldsymbol{x}^\top\mathbf{1}_n - 1).$$

By the KKT conditions, the optimal solution $\boldsymbol{x}^*$ satisfies

$$2\boldsymbol{\delta}^\top\boldsymbol{x}^*\delta_i + \nu = 0,\ \forall i \in \mathcal{V}.$$

Since the components of $\boldsymbol{\delta}$ are nonuniform, we obtain $\boldsymbol{\delta}^\top\boldsymbol{x}^* = 0$. Hence, $\boldsymbol{x}^* \in \{\boldsymbol{x} \in \Delta_n \mid \boldsymbol{x}^\top\boldsymbol{\delta} = 0\}$.

Under Assumption 3, for any $i \in \mathcal{V}$, there exists a $q(i) \in \mathcal{V}$ such that $\delta_i\delta_{q(i)} < 0$. Thus, there exists $\lambda_i \in (0,1)$ such that $\boldsymbol{\delta}^\top(\lambda_i\boldsymbol{e}_i + (1-\lambda_i)\boldsymbol{e}_{q(i)}) = 0$, which means $(\lambda_i\boldsymbol{e}_i + (1-\lambda_i)\boldsymbol{e}_{q(i)}) \in \{\boldsymbol{x} \in \Delta_n \mid \boldsymbol{x}^\top\boldsymbol{\delta} = 0\}$. Consequently, for any $\boldsymbol{z} \in \operatorname{int}\Delta_n$,

$$\boldsymbol{x}^* = \sum_{i=1}^{n} z_i(\lambda_i\boldsymbol{e}_i + (1-\lambda_i)\boldsymbol{e}_{q(i)}) \in \{\boldsymbol{x} \in \operatorname{int}\Delta_n \mid \boldsymbol{x}^\top\boldsymbol{\delta} = 0\}$$

follows from that $\{\boldsymbol{x} \in \Delta_n \mid \boldsymbol{x}^\top\boldsymbol{\delta} = 0\}$ is convex. □

Theorem 1 not only establishes the existence of the optimal social power, but also provides a constructive characterization within the simplex. In particular, the proof provides a direct procedure to construct an $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ satisfying $\boldsymbol{\delta}^\top\boldsymbol{x}^* = 0$, leveraging convexity properties and thus avoiding the need to solve the QP.

4.2 Optimal susceptibilities

Theorem 1 constructs an optimal social power $\boldsymbol{x}^*$ that minimizes the squared collective bias $g(\boldsymbol{x})$. In this subsection, we recover the optimal susceptibility $\boldsymbol{a}^*$ that yields $\boldsymbol{x}^*$.

Proposition 3. Suppose that Assumption 2 holds, and $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ is a solution of problem (9). Let $\gamma_i = a_i/(1-a_i)$. Then, $\boldsymbol{a}^*$ is a solution of problem (5) if and only if $\boldsymbol{\gamma}^* = \boldsymbol{a}^*(I_n - [\boldsymbol{a}^*])^{-1}$ satisfies

$$L^\top[\boldsymbol{x}^*]\boldsymbol{\gamma}^* = \frac{\mathbf{1}_n}{n} - \boldsymbol{x}^*, \tag{10}$$

where $L = I_n - W$ is the Laplacian of $\mathcal{G}(W)$.

Proof. By Proposition 2, $\boldsymbol{x}^*$ is the social power of the FJ model with susceptibilities $\boldsymbol{a}^*$ if and only if it is the dominant left eigenvector of $F(\boldsymbol{a}^*)$, which is equivalent to

$$\begin{aligned}\boldsymbol{x}^* =& \frac{\mathbf{1}_n}{n}\mathbf{1}_n^\top\boldsymbol{x}^* - (I_n - W^\top)[\boldsymbol{a}^*](I_n - [\boldsymbol{a}^*])^{-1}\boldsymbol{x}^*\\ =& \frac{\mathbf{1}_n}{n} - L^\top[\boldsymbol{\gamma}^*]\boldsymbol{x}^* = \frac{\mathbf{1}_n}{n} - L^\top[\boldsymbol{x}^*]\boldsymbol{\gamma}^*.\end{aligned}$$

Hence, we obtain (10). □

Since L is singular, and $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ under Assumption 2, we can approximate $\boldsymbol{\gamma}^*$ by $\boldsymbol{\gamma}^* \approx [\boldsymbol{x}^*]^{-1}(L^\top)^\dagger(\mathbf{1}_n/n - \boldsymbol{x}^*)$, where $(L^\top)^\dagger(\mathbf{1}_n/n - \boldsymbol{x}^*)$ is the minimum-norm solution of

$$\min_{\boldsymbol{z}\in\mathbb{R}^n} \quad ||L^\top\boldsymbol{z} - (\frac{\mathbf{1}_n}{n} - \boldsymbol{x}^*)||^2. \tag{11}$$

In general, solving (11) does not guarantee an exact solution for (10), as the minimum residual may be strictly positive, and the solution of (11) is not guaranteed nonnegative. However, if the influence network $\mathcal{G}(W)$ is strongly connected, we can ensure that (10) admits an exact solution. In this case, the structure of the Laplacian gives rise to an explicit family of solutions.

Theorem 2. Suppose that Assumptions 1 and 3 hold, $\mathcal{G}(W)$ is strongly connected, and $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ is a solution of problem (9). Then problem (5) is solved by $\boldsymbol{a}^* = (I_n + [\boldsymbol{\gamma}^*])^{-1}\boldsymbol{\gamma}^*$ with

$$\boldsymbol{\gamma}^* = [\boldsymbol{x}^*]^{-1}(L^\top)^\dagger(\frac{\mathbf{1}_n}{n} - \boldsymbol{x}^*) + \beta[\boldsymbol{x}^*]^{-1}\boldsymbol{\omega}, \tag{12}$$

where $\boldsymbol{\omega} \in \operatorname{int}\Delta_n$ is the dominant left eigenvector of W, $\beta \in \mathbb{R}$ is any constant that ensures $\boldsymbol{\gamma}^* \geq 0$.

Proof. Since $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$, $[\boldsymbol{x}^*]^{-1}$ exists. Let $\boldsymbol{z} = [\boldsymbol{x}^*]\boldsymbol{\gamma}$ and $\boldsymbol{b} = \mathbf{1}_n/n - \boldsymbol{x}^*$. Proposition 3 implies that problem (5) has a solution for $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ if

$$L^\top\boldsymbol{z} = \boldsymbol{b} \tag{13}$$

has a nonnegative solution.

Since $\mathcal{G}(W)$ is strongly connected, $-L$ is an irreducible Metzler matrix. The Perron-Frobenius theorem thus implies that $\mathbf{Null}(L) = \mathbf{Null}(-L) = \mathbf{span}\{\mathbf{1}_n\}$. Note that $\boldsymbol{b}^\top\mathbf{1}_n = 0$ for any $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$, which is equivalent to

$$\boldsymbol{b} \in \mathbf{Range}(L^\top) = \mathbf{Null}(L)^\perp.$$

Hence, (13) has at least one solution. By Campbell and Meyer (2009, Theorem 2.1.2), all the solutions of (13) have the form $\boldsymbol{z} = (L^\top)^\dagger\boldsymbol{b} + \boldsymbol{z}'$, where $\boldsymbol{z}' \in \mathbf{Null}(L^\top)$. As $\mathcal{G}(W)$ is strongly connected, $\mathbf{Null}(L^\top) = \mathbf{span}\{\boldsymbol{\omega}\}$, where $\boldsymbol{\omega} \in \operatorname{int}\Delta_n$ is the dominant left eigenvector of W. Therefore, there exists a constant $\beta \in \mathbb{R}$ such that

$$\boldsymbol{z}^* = (L^\top)^\dagger\boldsymbol{b} + \beta\boldsymbol{\omega} \geq 0. \tag{14}$$

As a result, we obtain

$$\boldsymbol{\gamma}^* = [\boldsymbol{x}^*]^{-1}(L^\top)^\dagger(\frac{\mathbf{1}_n}{n} - \boldsymbol{x}^*) + \beta[\boldsymbol{x}^*]^{-1}\boldsymbol{\omega} \geq 0.$$

Hence, $\boldsymbol{a}^* = (I_n + [\boldsymbol{\gamma}^*])^{-1}\boldsymbol{\gamma}^*$ solves problem (5). □

Theorem 2 proves that for any optimal social power $\boldsymbol{x}^* \in \operatorname{int}\Delta_n$ obtained by solving the QP (9), there exists an optimal susceptibility vector $\boldsymbol{a}^*$ solving problem (5), provided that $\mathcal{G}(W)$ is strongly connected. In particular, it establishes an equivalence between problem (5) and the computation of $(L^\top)^\dagger$ and $\boldsymbol{\omega}$. By Campbell and Meyer (2009, Theorem 2.1.1), $(L^\top)^\dagger(\mathbf{1}_n/n - \boldsymbol{x}^*)$ is the unique solution to the strongly convex QP:

$$\begin{aligned}\min_{\boldsymbol{z}\in\mathbb{R}^n} \quad & \boldsymbol{z}^\top\boldsymbol{z}\\ \text{s.t.} \quad & L^\top\boldsymbol{z} = \boldsymbol{b}.\end{aligned}$$

Similarly, since W is irreducible, $\boldsymbol{\omega}$ is the unique solution to the following strongly convex QP:

$$\min_{\boldsymbol{z}\in\Delta_n} \quad \boldsymbol{z}^\top LL^\top\boldsymbol{z},$$

where $\boldsymbol{z}^\top LL^\top\boldsymbol{z}$ is strongly convex on Δ_n. Consequently, Theorems 1 and 2 show that, although problem (5) is highly nonconvex, SSPM reduces it to one convex QP followed by two strongly convex QPs, all of which are solvable in polynomial time.

5. APPLICATIONS AND SIMULATIONS

In this section, we provide examples to illustrate the applicability of our results. We first show how the susceptibility optimization problem arises in the setting of distributed optimization. We then present a simulation example that demonstrates the performance of SSPM.

Example 2. (Applications in distributed optimization) Consider n agents trying to minimize a global objective

$$\min_{y\in\mathbb{R}} \quad \sum_{i=1}^n h_i(y) = \frac{1}{2}\sum_{i=1}^n (y - c_i)^2$$

where each c_i is only known to agent i. Then, a standard distributed gradient descent algorithm is

$$\boldsymbol{y}(k+1) = M\boldsymbol{y}(k) - \alpha_i\nabla G(\boldsymbol{y}(k)), \tag{15}$$

where M is a row-stochastic matrix corresponding to the communication network, α_i is the step size or learning rate, and $G(\boldsymbol{y}(k)) = \boldsymbol{y}(k) - \boldsymbol{c}$ is the gradient vector. If $\alpha_i = 1 - a_i$ and $M = [\boldsymbol{a}]W + I_n - [\boldsymbol{a}]$, then (15) can be written as

$$\boldsymbol{y}(k+1) = [\boldsymbol{a}]W\boldsymbol{y}(k) + (I_n - [\boldsymbol{a}])\boldsymbol{c},$$

which is exactly the FJ model (1) with $\boldsymbol{y}(0) = \boldsymbol{c}$. Therefore, problem (5) is equivalent to finding the optimal learning rates in a scenario where agents have only partial information on their local objectives c_i. ◁

Example 3. (Performance of SSPM) Consider an influence network consisting of 5 individuals with adjacency matrix

$$W = \begin{bmatrix} 0.1 & 0.6 & 0.3 & 0 & 0\\ 0 & 0.2 & 0.5 & 0.3 & 0\\ 0 & 0 & 0.3 & 0.4 & 0.3\\ 0.2 & 0 & 0 & 0.2 & 0.6\\ 0.5 & 0.2 & 0 & 0 & 0.3 \end{bmatrix}.$$

Let $\theta = 0$ and $\boldsymbol{\delta} = \boldsymbol{\mu} = (-2, 5, 3, -6, 1)^\top$. We construct an optimal social power using the procedure in Theorem 1. Choosing $q(1) = 2$, $q(2) = 1$, $q(3) = 4$, $q(4) = 3$ and $q(5) = 1$, we obtain 3 different vectors in $\{\boldsymbol{x} \in \Delta_5 \mid \boldsymbol{x}^\top\boldsymbol{\delta} = 0\}$:

$$(\frac{5}{7},\ \frac{2}{7},\ 0,\ 0,\ 0)^\top,\ (0,\ 0,\ \frac{2}{3},\ \frac{1}{3},\ 0)^\top,\ (\frac{1}{3},\ 0,\ 0,\ 0,\ \frac{2}{3})^\top.$$

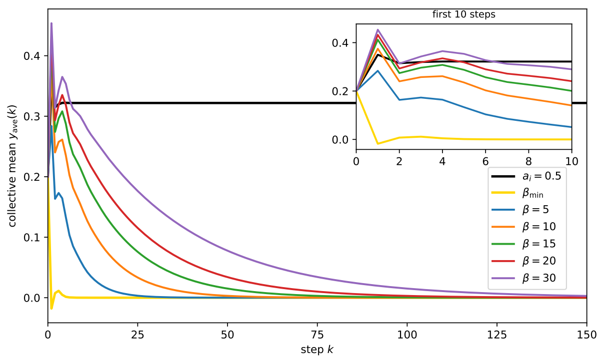

Fig. 2. Trajectories of $y_{\mathrm{ave}}(k)$ with $\boldsymbol{y}(0) = \boldsymbol{\mu}$ under different susceptibility vectors.

Picking uniform weights $\frac{1}{3}\mathbf{1}_3$ yields

$$\boldsymbol{x}^* = (\frac{22}{63}, \frac{2}{21}, \frac{2}{9}, \frac{1}{9}, \frac{2}{9})^\top.$$

Next, we compute the Laplacian pseudoinverse and the dominant left eigenvector of W:

$$(L^\top)^\dagger \approx \begin{bmatrix} 0.6111 & -0.3717 & -0.2556 & -0.0150 & 0.0313 \\ 0.1020 & 0.6533 & -0.4504 & -0.2362 & -0.0688 \\ 0.0491 & 0.0217 & 0.7116 & -0.4609 & -0.3214 \\ -0.1872 & 0.0058 & -0.0631 & 0.6810 & -0.4365 \\ -0.4251 & -0.2714 & -0.0349 & 0.1005 & 0.6309 \end{bmatrix},$$

$$\boldsymbol{\omega} \approx (0.1748,\ 0.1919,\ 0.2120,\ 0.1780,\ 0.2434)^\top.$$

By Theorem 2, setting the minimal $\beta_{\min} \approx 0.7237$ that ensures $\boldsymbol{\gamma}^* \geq 0$ gives

$$\boldsymbol{a}^* \approx (0,\ 0.6573,\ 0.3076,\ 0.6733,\ 0.4821)^\top.$$

Moreover, any $\beta > \beta_{\min}$ also yields an optimal susceptibility vector with the same $\boldsymbol{x}^*$. For instance, with $\beta = 1$ we obtain

$$\boldsymbol{a}^* \approx (0.1215,\ 0.7122,\ 0.4145,\ 0.7146,\ 0.5522)^\top.$$

Fig. 2 plots the trajectories of $y_{\mathrm{ave}}(k)$ under different susceptibility vectors. The baseline susceptibility $\boldsymbol{a} = \frac{1}{2}\mathbf{1}_5$ is not optimal, thus the corresponding trajectory converges to a value biased away from 0. Consistent with our theoretical results, all trajectories with optimal susceptibilities converge to 0, while the convergence becomes slower as β increases. $\triangleleft$

6. CONCLUSION

We investigated how to maximize collective intelligence in influence networks, building on the WoC effect and persuasion technologies. The problem was formulated as a nonlinear, nonconvex optimization over individual susceptibilities in the FJ model. By analyzing the social power structure of the FJ model, we developed a sequential method, *SSPM*, that (i) reformulates the original nonconvex objective to a QP on the probability simplex to obtain an optimal social power, and (ii) recovers optimal susceptibility vectors from the Laplacian pseudoinverse and the dominant left eigenvector of the influence matrix in strongly connected networks. This decomposition converts the initial nonconvex problem into sequential convex QPs, enabling scalable computation. Future work includes distributed constructions of optimal social power and distributed recovery of optimal susceptibilities.

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