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Optimizing the wisdom of crowds in influence networks with sparse interpersonal influence structures

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Abstract: In this paper, we study the problem of optimally aggregating individuals' private knowledge in social influence networks by designing their interpersonal influence structures based on their opinions. Individuals interact their opinions on an unknown truth in round by round discussion with their initial opinions in each round being independent random variables associated with their expertise. We propose a mechanism to design the relative interaction matrix with sparse property according to their expressed opinions. We prove that with this mechanism, individuals' opinions achieve consensus and the collective opinion achieves its optimum as the number of the discussion rounds increases.

Keywords: social networks, opinion dynamics, sparse networks, wisdom of crowds

1. INTRODUCTION

It is a long-standing question that what is the best way of utilizing knowledge initially dispersed among people [5]? Empirical experiments suggest that simply pooling many independent opinions on an unknown truth yields collective opinions essentially close to the truth, known as the effect of the wisdom of crowds [3]. A social influence process is a natural system that aggregates individuals' respective opinions through the interaction among them. It has long been debated that whether social influence improves the wisdom of crowds or undermines it [1, 6]. In [7], theoretical results are provided to explain how social influence improves, undermines and optimizes the wisdom of crowds based on the influence system theory. However, it is still unclear that what is the best interpersonal influence structure to optimize the collective intelligence in influence networks.

In this paper, we aim to design the interpersonal influence structure of a social influence process according to individuals' opinions to optimize the wisdom of crowds. A group of individuals evolve their opinions on an unknown truth in an influence network round by round. In each single round, each individual's initial opinion is an independent random variable whose expectation is the truth and variance indicates its accuracy. Individuals update their opinions according to the DeGroot opinion dynamics in a naïve learning way [2]. A moderator is observing individuals' opinions and designing the relative interaction matrix for the discussion of next round. We propose a mechanism which allows the moderator to estimate individuals' accuracy based on the observed opinions and to design the relative interaction matrices with sparse property accordingly. We prove that with the proposed mechanism, individuals opinions achieve consensus in each round and the variance of the collective estimate converges to its minimum as the number of the discussion rounds increases.

The rest of the paper is organized as follows: Section 2 formulates the problem, Section 3 proposes our main results and Section 4 concludes the paper.

† Ye Tian is the presenter of this paper.

2. PROBLEM STATEMENT

We formulate the problem based on the naïve learning setting [4]. Consider that a group of n individuals interact their opinions on an unknown state of the world with truth μ. Suppose that these individuals are all experts in the sense that their independent opinions on the state are unbiased random variables whose variances depend on their expertise. Specifically, individual i's independent opinion, denoted by y_i, is a random variable with expectation $\mathbb{E}[y_i] = \mu$ and variance $\mathrm{Var}[y_i] = \sigma_i^2$. Assume that individuals evolve their opinions round by round. In each round, individual adopts its independent opinion as its initial opinion and update its opinions according to the DeGroot opinion dynamics:

$$
\begin{aligned}
y_i(r, k+1) = {} & \gamma_i(r) y_i(r, k) \\
& + (1 - \gamma_i(r)) \sum_{j=1}^{n} C_{ij}(r) y_j(r, k), \quad (1)
\end{aligned}
$$

where $y_i(r, k)$ is i's opinion in round r at time k, $\gamma_i(r) \in (0, 1)$ and $C_{ij}(r)$ indicate its level of confidence and the relative influence weight of j to i, respectively. Let $\gamma(r) = (\gamma_1(r), \dots, \gamma_n(r))^\top$ and $C(r) = [C_{ij}(r)]^{n\times n}$, system (1) can be written as:

$$
y(r, k+1) = W(r) y(r, k), \quad (2)
$$

where $W(r) = \mathrm{diag}(\gamma(r)) + (I_n - \mathrm{diag}(\gamma(r)))C(r)$ and $C(r)$ is the row-stochastic and zero-diagonal relative interaction matrix.

If $W(r)$ is primitive, individuals' opinions achieve consensus, that is

$$
\lim_{k\to\infty} y(r, k) = \omega^\top(r) y(r, 0) \mathbf{1}_n,
$$

where $\omega(r)$ is the left dominant eigenvector of $W(r)$ and indicates the social power allocation of the influence process in round r. Let $y_{\mathrm{col}}(r) = \omega^\top(r) y(r, 0)$ be the collective opinion in round r. Since $y_i(r, 0)$ is independent random variable with $\mathbb{E}[y_i(r,0)] = \mu$ and $\mathrm{Var}[y_i(r,0)] = \sigma_i^2$, we have

$$\mathbb{E}[y_{\mathrm{col}}(r)] = \mu \text{ and } \mathrm{Var}[y_{\mathrm{col}}(r)] = \sum_{i=1}^{n} \omega_i^2(r)\sigma_i^2,$$

which imply that the collective opinion in each round is unbiased and its variance is depended on the social power allocation of the opinion dynamics. By the Chebyshev's inequality,

$$\mathbb{P}[|y_{\mathrm{col}}(r) - \mu| > \epsilon] \le \frac{\mathrm{Var}[y_{\mathrm{col}}(r)]}{\epsilon^2}$$

for any $\epsilon > 0$. Therefore, a smaller variance means that the collective opinion is closer to the truth.

Problem 1: Suppose that individuals interact their opinions on the unknown truth according to influence process (1). Each individual's initial opinion in each round is an independent random variable with expectation μ and variance σ_i^2. Find the optimal relative interaction matrix $C(r)$ according to individuals' expressed opinions $y(r,k)$ such that

$$\mathrm{Var}[y_{\mathrm{col}}(r)] = \min_{x\in\Delta_n} \sum_{i=1}^{n} x_i^2\sigma_i^2, \tag{3}$$

where $\Delta_n = \{x \in \mathbb{R}^n | x \ge 0, x^\top \mathbf{1}_n = 1\}$ is the n-simplex.

3. MAIN RESULTS

Assume that in each round, individuals' confidence $\gamma_i(r)$ is stationary and the relative interaction matrix $C(r)$ is design by a moderator. Then, according to (1),

$$\begin{aligned} y_i(r,k+1) - \sum_{j=1}^{n} C_{ij}(r)y_j(r,k) \\ = \gamma_i(r)(y_i(r,k) - \sum_{j=1}^{n} C_{ij}(r)y_j(r,k)). \end{aligned}$$

Hence, the moderator can compute each individual's confidence based on individuals' expressed opinions and the relative interaction matrix by

$$\gamma(r) = \mathrm{diag}((I - C(r))y(r,k))^{-1}(y(r,k+1) - y(r,k)) \tag{4}$$

as long as $y_i(r,k) - \sum_{j=1}^{n} C_{ij}(r)y_j(r,k) \ne 0$ for all i.

Lemma 1: Suppose that $c(r) \in \Delta_n$ is the left dominant eigenvector of $C(r)$. Then $C(r)$ is the optimal relative interaction matrix if and only if

$$\frac{c_i(r)}{c_j(r)} = \frac{1-\gamma_i(r)}{1-\gamma_j(r)}\frac{\sigma_j^2}{\sigma_i^2}.$$

Proof: By (3), the optimal influence matrix $W(r)$ satisfies

$$\frac{\omega_i(r)}{\omega_j(r)} = \frac{\sigma_j^2}{\sigma_i^2},$$

where $\omega(r)$ is the left dominant eigenvector of $W(r)$. Since $W(r) = \mathrm{diag}(\gamma(r)) + (I_n - \mathrm{diag}(\gamma(r)))C(r)$, we have $\omega^\top(r)W(r) = \omega^\top(r)\,\mathrm{diag}(\gamma(r)) + \omega^\top(r)(I_n - \mathrm{diag}(\gamma(r)))C(r) = \omega^\top(r)$, that is, $\omega^\top(r)(I_n - \mathrm{diag}(\gamma(r)))C(r) = \omega^\top(r)(I_n - \mathrm{diag}(\gamma(r)))$, which implies that $\omega^\top(r)(I_n - \mathrm{diag}(\gamma(r)))$ is a left eigenvector of $C(r)$ associated with 1. By the uniqueness of the left dominant eigenvector, we obtain

$$\frac{c_i(r)}{c_j(r)} = \frac{\omega_i(r)}{\omega_j(r)}\frac{1-\gamma_i(r)}{1-\gamma_j(r)} = \frac{\sigma_j^2}{\sigma_i^2}\frac{1-\gamma_i(r)}{1-\gamma_j(r)}.$$

■

By Lemma 1, the key point for designing the relative interaction matrix is to estimate the variance of each individual's independent opinions based on their expressed opinions. Note that the initial opinions in each round are unbiased, independent and identically distributed. In round r, the moderator establishes an estimation for the truth by

$$\hat{\mu}(r) = \frac{r}{r+1}\hat{\mu}(r-1) + \frac{1}{n(r+1)}\sum_{i=1}^{n} y_i(r,0) \tag{5}$$

with $\hat{\mu}(0) = \frac{1}{n}\sum_{i=1}^{n} y_i(0,0)$. Then, the variance of individual i can be estimated by

$$\hat{\sigma}_i^2(r) = \frac{1}{r}\sum_{t=0}^{r}(y_i(t,0) - \hat{\mu}(r))^2. \tag{6}$$

Let $\hat{\sigma}^2(r) = (\hat{\sigma}_1^2(r), \ldots, \hat{\sigma}_n^2(r))^\top$. Based on these estimations, the relative interaction matrix for the next round is designed by solving

$$\begin{aligned} C(r+1) = \mathrm{argmin}_{C\in\mathbb{R}^{n\times n}} & \parallel C \parallel_1 \\ \text{s.t. } & C \ge 0, \\ & \mathrm{diag}(C) = 0, \\ & C\mathbf{1}_n = \mathbf{1}_n, \\ & (I_n + C)^{n+1} > 0, \\ & (C^\top - I)\,\mathrm{diag}(\hat{\sigma}^2(r))^{-1}(\mathbf{1}_n - \gamma(r)) = 0. \end{aligned} \tag{7}$$

Theorem 1: Suppose that the moderator designs the relative interaction matrix according to (5)-(7). Then,

$$\lim_{r\to\infty} \mathrm{Var}[y_{\mathrm{col}}(r)] = \min_{x\in\Delta_n}\sum_{i=1}^{n} x_i^2\sigma_i^2.$$

Proof: Since $y_1(r,0), \ldots, y_n(r,0)$ are independent and identically distributed random variables with respect to r, then (5) and (6) are unbiased estimations. Therefore, $\lim_{r\to\infty}\hat{\mu}(r) = \mu$ and $\lim_{r\to\infty}\hat{\sigma}_i^2(r) = \sigma_i^2$. Combining Lemma 1 and (7), $\lim_{r\to\infty} C(r)$ is the optimal relative interaction matrix, which implies that $\mathrm{Var}[y_{\mathrm{col}}(r)]$ converges to its minimum as $r \to \infty$. ■

4. CONCLUSIONS

This paper has studied the problem of optimizing individuals' collective opinions for an unknown truth in influence networks. A mechanism was proposed to design sparse interpersonal influence structures based on individuals' expressed opinions. We proved that our mechanism yields sparse relative interaction matrices with which individuals' collective opinion on the unknown truth achieve its optimum as the number of the discussion rounds increases. Future work will focus on the scenarios where individuals' initial opinions are non-independent and the influence networks are not strongly connected.

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