Lecture Notes in Calculus I

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These lecture notes are based on the first few chapters of Robert A. Adams's *Calculus: A Complete Course*, [Ad13], wherein recommended exercises are also found. All the material covered can be found in there, though the exposition might sometimes be altered. Occasionally there will be small sections marked as 'exercise.' These are examples or simpler proofs which are left for the student to think about on their own.

Throughout this document, $\square$ signifies end proof, and $\blacktriangle$ signifies end of example.

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Lecture 0 Distance and Functions[1]

0.1 The Absolute Value and the Triangle Inequality

Calculus is the art of using distances to measure changes in things (functions, typically). Since we so far in our mathematical careers live on the real number line (which we denote $\mathbb{R}$), the distance of choice is the absolute value:

Definition 0.1.1 (Absolute value)**.** Given a real number x, its ***absolute value***, denoted $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0, \end{cases}$$

i.e. x without its sign.

In a very real sense Calculus (at least in the modern treatment of the subject) always boils down to manipulating this distance in sufficiently clever ways. To this end, we will spend a bit of time recalling some basic properties of the absolute value.

The absolute value behaves well under multiplication and division. In particular, we have that for all real x

$$|-x| = |x|,$$

for all real x and y we have

$$|xy| = |x||y|,$$

and for all real x and y, where $y \neq 0$, we have

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

Exercise 0.1.2. Prove the three properties listed above.

On the other hand, the absolute value does not behave quite as well under addition and subtraction. Indeed, we can't guarantee that we maintain equality anymore!

Theorem 0.1.3 (The triangle inequality)**.** *Let x and y be real numbers. Then*

$$|x + y| \leq |x| + |y|.$$

There are many, many ways to prove this. This is one of them:

Proof. Consider the following equalities:

$$|x + y|^2 = (x + y)^2 = x^2 + 2xy + y^2.$$

Now suppose that we replace x with $|x|$ and y with $|y|$. Clearly the squares are unchanged since they're both nonnegative regardless of the sign of x or y, but the middle term might change. If one of x and y is negative, we've made

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the sum bigger by replacing them with their absolute values, and in every other case we've changed nothing. Therefore

$$|x+y|^2 = x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2.$$

But this last expression we recognise as $(|x|+|y|)^2$, whence $|x+y|^2 \leq (|x|+|y|)^2$. By taking (positive) square roots, we get the desired result. □

Note that $|x-y| \leq |x| + |y|$ holds by almost exactly the same argument, which might come in handy later on.

We will find that the triangle inequality is *the* main tool in all of this Calculus course, which is why we bother with repeating this.

As an exercise in using the triangle inequality, let us consider the following related version of it:

Example 0.1.4 (The reverse/inverse triangle inequality)**.** If x and y be real numbers, then

$$|x-y| \geq \big||x| - |y|\big|.$$

To see this, consider first the following consequence of the basic properties we discussed above:

$$|x-y| = |-(y-x)| = |-1| \cdot |y-x| = |y-x|.$$

Therefore by adding and subtracting the same thing (a trick that will appear again and again throughout this course) and using the triangle inequality we have

$$|x| = |(x-y)+y| \leq |x-y| + |y|,$$

which if we subtract $|y|$ from both sides becomes $|x| - |y| \leq |x-y|$. Similarly, if we start with $|y|$, we get $|y| - |x| \leq |y-x|$, which if we multiply both sides by -1 gives us $|x| - |y| \geq -|x-y|$.

Combining these two we have $-|x-y| \leq |x| - |y| \leq |x-y|$, which if we take absolute values everywhere means that $\big||x| - |y|\big| \leq |x-y|$. ▲

0.2 Functions and Some of Their Properties

Since we will spend the next few weeks concerning ourselves with how functions change and what the area underneath their graphs are and so on, it behooves us to define, once and for all, what we mean by a function.

Definition 0.2.1 (Function, domain, codomain, and range)**.** A ***function*** f on a set X into a set Y is a rule that assigns exactly one element $y \in Y$ to each $x \in X$. We use the notation $f\colon X \to Y$ for the function together with the two sets. We use $y = f(x)$ to denote this unique y corresponding to x.

The set X is called the ***domain*** of the function and the set S is called its ***codomain***. By ***range*** or ***image*** we mean the set $\{\, y = f(x) \mid x \in X \,\} \subseteq Y$ containing all elements of Y that we may reach using the function f on its domain D.

Note that a function strictly speaking depends on its domain and codomain, not just the formula expressing how we translate an element in the domain to an element in the codomain.

For example, the functions $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $g(x) = x^2$ (where we by $\mathbb{R}_{\geq 0}$ mean the nonnegative real numbers) are identical for $x \geq 0$, but $f(-1) = 1$, whereas $g(-1)$ is undefined.

If we do not specify the domain of a function, we implicitly give the function its ***natural domain***, by which we mean all $x \in \mathbb{R}$ such that $f(x) \in \mathbb{R}$, i.e. as big a subset of $\mathbb{R}$ as possible. Similarly when we don't specify the codomain we take it to be the range of the function.

As an example it is then understood that the function $h(x) = \sqrt{x}$ has the set $\mathbb{R}_{\geq 0}$ both as its domain and its codomain.

For two additional properties of functions, suppose that we have a function $f\colon X \to Y$ such that $-x \in X$ whenever $x \in X$.

We call such a function ***even*** if $f(-x) = f(x)$ for all x in X, and we call it ***odd*** if $f(-x) = -f(x)$ for all x in X.

Some odd functions are, for instance, $f_1(x) = x$ and $f_2(x) = \sin(x)$. For even functions, consider perhaps $f_3(x) = |x|$ or $f_4(x) = \cos(x)$. Note that most functions are neither even nor odd, say for example $f_5(x) = 2x + 3$.

Lecture 1 Limits[2]

1.1 Informal Introduction

Consider a function such as

$$f(x) = \frac{2x+5}{7-3x},$$

defined for all $x \neq 7/3$. What happens if we plug in *big* values of x? It is then (perhaps) natural to consider $2x$ to be the bigger term of the numerator, and similarly $-3x$ the bigger term in the denominator. Thus

$$f(x) \approx \frac{2x}{-3x} = \frac{2}{-3} = -\frac{2}{3}$$

for *big* values of x.

On the other hand for x near 0, the dominating terms are 5 and 7 in the numerator and denominator respectively, whereby

$$f(x) \approx \frac{5}{7}$$

for x near 0.

To describe these two circumstances we introduce the notation

$$\lim_{x\to\infty} f(x) = -\frac{2}{3} \qquad \text{and} \qquad \lim_{x\to 0} f(x) = \frac{5}{7},$$

read as "the ***limit*** as x goes to infinity of $f(x)$," and similarly for the second one. We call ∞ and 0 ***limit points***.

Of course the "dominating" argument isn't rigorous... We need proper definition and rules of computation. For instance,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x+5}{7-3x} = \lim_{x\to\infty} \frac{x(2+5/x)}{x(7/x-3)} = \lim_{x\to\infty} \frac{2+5/x}{7/x-3} = \frac{2+0}{0-3} = -\frac{2}{3}.$$

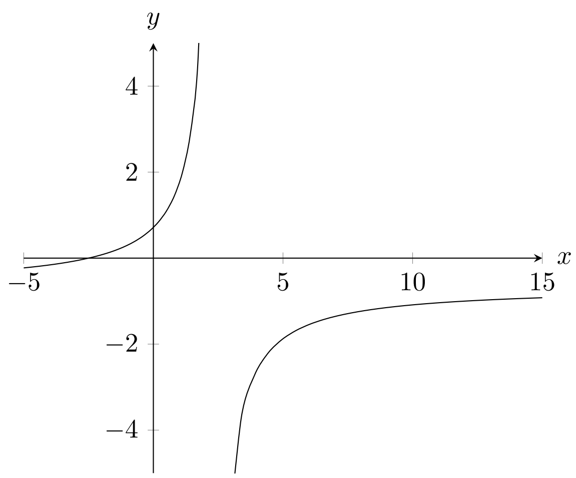


Figure 1.1.1: Plot of $y = f(x)$.

Moreover, by studying the plot of the function in Figure 1.1.1, we might observe two more interesting limits, namely

$$\lim_{x \to 7/3^+} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 7/3^+} f(x) = -\infty,$$

where by the superscript $-$ and $+$ mean that the values of x approach $7/3$ either from below or from above. We call the first limit a ***left limit*** and the second a ***right limit***.

We will spend the remainder of this lecture formalising what we mean by these limits and creating computational rules for them, informed by the intuition of what we do above.

1.2 Punctured or Deleted Neighbourhoods

If we wish to study what happens to a function close to some point (or off at infinity), it is first of all important that our function is defined around this point. More specifically, we require (at least in this course) that the function is defined *for all* x in *some* ***deleted*** or ***punctured neighbourhood*** of the limit point.

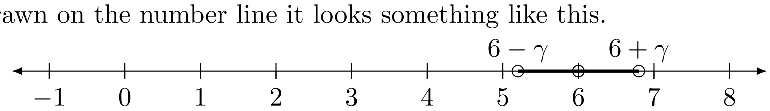
We will explain what we mean by this by means of examples.

Example 1.2.1. For all real $\gamma > 0$ the set

$$\{\, x \in \mathbb{R} \mid 0 < |x - 6| < \gamma \,\}$$

is a ***two-sided deleted neighbourhood*** of $x = 6$. Deleted naturally means that the point $x = 6$ itself is excluded.

Drawn on the number line it looks something like this.



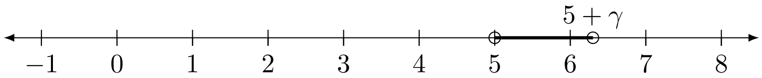
▲

[2]Date: January 17, 2017.

Example 1.2.2. For right limits we instead consider ***deleted right neighbourhoods***, like

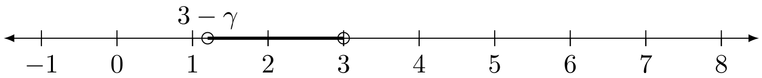
$$\{ x \in \mathbb{R} \mid 0 < x - 5 < \gamma \} = \{ x \in \mathbb{R} \mid 5 < x < 5 + \gamma \} =]5, 5 + \gamma[,$$

around the point $x = 5$ for all real $\gamma > 0$.



Similarly for left limits we use ***deleted left neighbourhoods*** such as the following for $x = 3$:

$$\{ x \in \mathbb{R} \mid 0 < 3 - x < \gamma \} = \{ x \in \mathbb{R} \mid 3 - \gamma < x < 3 \} =]3 - \gamma, 3[.$$

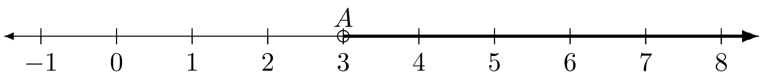


▲

For limits at infinity (or negative infinity) we also need a deleted neighbourhood to work with, but they looks rather different.

Example 1.2.3. A ***deleted neighbourhood of infinity*** consists of all x bigger than some fixed $A \in \mathbb{R}$,

$$\{ x \in \mathbb{R} \mid x > A \} =]A, \infty[.$$



For a ***deleted neighbourhood of negative infinity***, we instead take all x smaller than some fixed $A \in \mathbb{R}$. ▲

1.3 Definition of Limit

We will now take the final steps toward a good definition of a limit.

(i) First of all we need the function we are studying to be defined *around* the point of interest, i.e. the limit point. This is the reason we are interested in deleted neighbourhoods around points! In other words, for some function $f\colon X \to Y$ we need there to exist some $\gamma > 0$ such that the deleted neighbourhood

$$\{ x \in \mathbb{R} \mid 0 < |x - a| < \gamma \} \subseteq X,$$

meaning that the function is defined somewhere around the limit point a.

Remark 1.3.1. Note that the function needn't be defined at the limit point itself for us to study limits at this point.

Secondly we want the y in $y = f(x)$ to get *arbitrarily* close to the limit, let's call it L.

What we mean by this is that we should be able to pick *any* neighbourhood around L,

$$|y - L| < \varepsilon$$

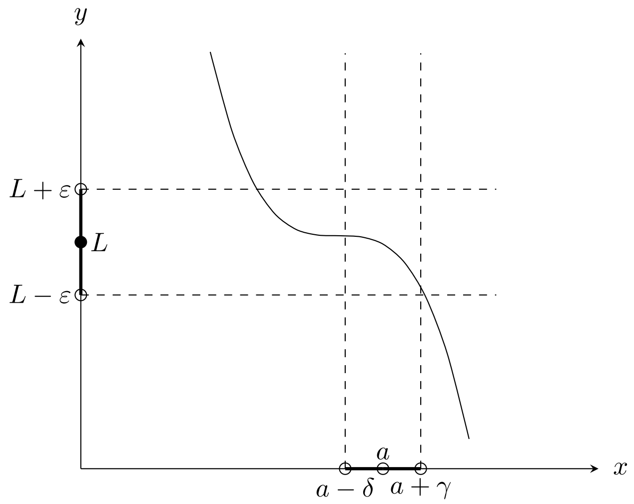


Figure 1.3.1: Constructing neighbourhoods around limit and limit point.

for any $\varepsilon > 0$, and regardless of what ε we choose, we should be able to arrange for a deleted neighbourhood around a,

$$\{\, x \in \mathbb{R} \mid 0 < |x - a| < \delta \,\},$$

such that for all x in this deleted neighbourhood, $y = f(x)$ is in the above neighbourhood of L. We present a sketch of what this looks like in Figure 1.3.1.

Therefore,

(ii) For each $\varepsilon > 0$ there exists some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

In other words, if x is within δ distance away from a, then $f(x)$ is within ε distance away from L.

Combining these two we have a proper definition of a limit at a point:

Definition 1.3.2 (Limit at a point)**.** We say that the function $f\colon X \to Y$ has the ***limit*** L as x approaches a if and only if

- (i) there exists some $\gamma > 0$ such that $\{\, x \in \mathbb{R} \mid 0 < |x - a| < \gamma \,\} \subseteq X$, and
- (ii) for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that for each $x \in X$ with $0 < |x - a| < \delta$ we have $|f(x) - L| < \varepsilon$.

We call this an "epsilon-delta" definition.

In order to construct definitions for limits at infinity, left or right limits, or ***improper limits*** (meaning that $f(x)$ approaches positive or negative infinity), we simply replace our various neighbourhoods in the above definition by the corresponding types of neighbourhoods. For example:

Definition 1.3.3 (Limit at negative infinity)**.** We say that $f\colon X \to Y$ has the limit L as x approaches $-\infty$ if and only if

- (i) there exists some $A \in \mathbb{R}$ such that $\{\, x \in \mathbb{R} \mid x < A \,\} \subseteq X$, and (*ii*) for each $\varepsilon > 0$ there exists a number $B \in \mathbb{R}$ such that for each $x \in X$ with $x < B$ we have $|f(x) - L| < \varepsilon$.

Exercise 1.3.4. Feel free to construct definitions for the remaining types of limits. Alternatively, try modify the wording of the original definition by replacing all neighbourhoods with expressions like "there exists a deleted neighbourhood around the point", etc, whereby we may acquire a completely general definition (albeit maybe slightly less practical).

The idea that limits concern themselves with deleted neighbourhoods around the limit point is important. That is to say, we don't care about the function's value in the limit point, or even if it is defined there. We illustrate this by example:

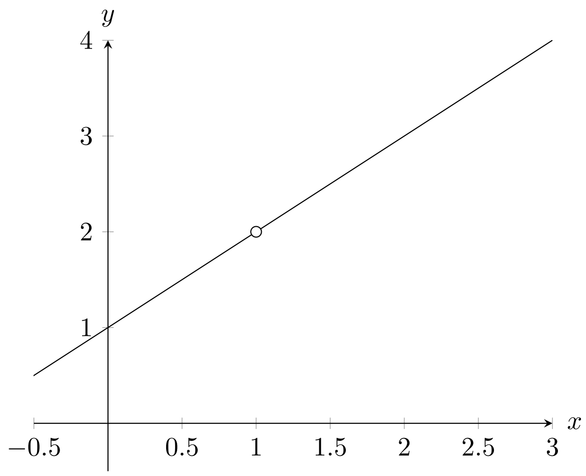
Example 1.3.5. Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1},$$

defined for all $x \neq 1$. For all such x we have

$$f(x) = \frac{x^2 - 1^2}{x - 1} = \frac{(x+1)(x-1)}{x - 1} = x + 1,$$

i.e. a straight line passing through $y = 1$, except it's undefined at $x = 1$, meaning that there's a hole there.



Even though $f(x)$ is *undefined* at $x = 1$, it is clear that we can make it as close to $y = 2$ as we like by taking x sufficiently close to $x = 1$, i.e.

$$\lim_{x \to 1} f(x) = 2.$$

Formally, for any $\varepsilon > 0$ we want

$$|f(x) - 2| = \left| \frac{x^2 - 1}{x - 1} - 2 \right| = \left| \frac{(x-1)(x+1) - 2(x-1)}{x - 1} \right| = |x - 1| < \varepsilon$$

whenever $|x - 1| < \delta$, so by taking $\delta = \varepsilon$ we have proved formally that the limit is what we expect! ▲

Example 1.3.6. To further emphasise how the limit at a point doesn't care about the value of the function at that point, consider the following function, closely related to the one in the last example:

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1. \end{cases}$$

This function *is* defined at $x = 1$, but even so, by the exact same computation as above, the limit at that point is still 2. ▲

Finally in this section we will prove something quite important, that is sometimes overlooked:

Theorem 1.3.7. *Limits are unique.*

Proof. We will prove this in the case of a two-sided limit at a point, but the remaining types of limits follow by similar arguments.

Suppose

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} f(x) = M.$$

We want to prove that $L = M$.

By definition we have that for any $\varepsilon > 0$, or, equivalently, for any $\varepsilon/2 > 0$, there exists some δ_1 such that whenever $|x - a| < \delta_1$, it implies that $|f(x) - L| < \varepsilon/2$, and similarly there exists some δ_2 such that if $|x - a| < \delta_2$, then $|f(x) - M| < \varepsilon/2$. If we now let $\delta = \min\{\, \delta_1, \delta_2 \,\}$, then when $|x - a| < \delta$, both of the above conditions are satisfied, so by adding and subtracting the same thing and using the triangle inequality we get

$$\begin{aligned} |L - M| &= |L - f(x) + f(x) - M| \leq |L - f(x)| + |f(x) - M| \\ &= |f(x) - L| + |f(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily small, and $|L - M|$ is smaller than every one of them, we must have that $|L - M| = 0$, which means that $L = M$. □

1.4 Computing Limits

We have used epsilon-delta explicitly to compute a limit in Example 1.3.5, but we will now use the same in the abstract to establish several general rules for how limits may be computed and manipulated.

Theorem 1.4.1 (Computational rules for limits)**.** *Let f and g be some functions such that*

$$\lim_{x\to a} f(x) = L \qquad \textit{and} \qquad \lim_{x\to a} g(x) = M.$$

Moreover let $k \in \mathbb{R}$ be a constant. Then

(i) $\lim_{x\to a} (f(x) + g(x)) = L + M$,

(ii) $\lim_{x\to a} (f(x) - g(x)) = L - M$,

(iii) $\lim_{x\to a} (kf(x)) = kL$,

(*iv*) $\lim_{x \to a} (f(x)g(x)) = LM$,

(*v*) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$,

(*vi*) $\lim_{x \to a} (f(x))^{m/n} = L^{m/n}$ *for* $m, n \in \mathbb{Z}$, $n > 0$, $L > 0$ *if* n *is even, and* $L \neq 0$ *if* $m < 0$, *and finally*

(*vii*) *if* $f(x) \leq g(x)$ *in a deleted neighbourhood around* a, *then* $L \leq M$.

Proof. We will prove some of them. First of all note that, since both f and g must be defined in some punctured neighbourhood of a, since they have limits there, the combined functions must as well. For example, if f is defined for $0 < |a - x| < \gamma_f$, and g is defined for $0 < |a - x| < \gamma_g$, then at the very least $f + g$ must be defined for $0 < |a - x| < \min\{\,\gamma_f, \gamma_g\,\}$. By this argument (and similar for the other limits), we find that we are allowed to use our limit definition on these new functions in the first place.

The proof technique is invariably the same, in the sense that it boils down to choosing appropriate δ. By definition, for all $\varepsilon_1, \varepsilon_2 > 0$ there exist some $\delta_1, \delta_2 > 0$ such that we have $|f(x) - L| < \varepsilon_1$ whenever $|x - a| < \delta_1$ and $|g(x) - M| < \varepsilon_2$ whenever $|x - a| < \delta_2$.

(*i*) In order to verify that $L + M$ is the limit of $f(x) + g(x)$ as x approaches a, we simply compute the distance between these two quantities:

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| < \varepsilon_1 + \varepsilon_2. \end{aligned}$$

Note the use of the triangle inequality. Taking $\varepsilon_1 = \varepsilon_2 = \varepsilon/2$, with $\varepsilon > 0$ being arbitrary, and also taking $\delta = \min\{\,\delta_1, \delta_2\,\}$, we have that whenever $|x - a| < \delta$, we must have $|(f(x) - g(x)) - (L - M)| < \varepsilon$, so by definition

$$\lim_{x \to a} (f(x) + g(x)) = L + M.$$

(*iv*) First, the relation between ε_2 and δ_2 is that for *any* ε_2, there must exist a corresponding δ, whereby it must in particular work if we take $\varepsilon_2 = 1$. Thus there exists a δ_3 such that if $|x - a| < \delta_3$, then $|g(x) - M| < 1$. This means that

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

when $|x - a| < \delta_3$.

We now turn to the distance between $f(x)g(x)$ and LM,

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\leq |(f(x) - L)g(x)| + |L(g(x) - M)| \\ &= |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M|, \end{aligned}$$

so if take let $|x - a| < \min\{\,\delta_1, \delta_2, \delta_3\,\}$ and take

$$\varepsilon_1 = \frac{\varepsilon}{2(1 + |M|)} \qquad \text{and} \qquad \varepsilon_2 = \frac{\varepsilon}{2|L|},$$

then the above quantity is less than

$$(1 + |M|)\,\frac{\varepsilon}{2(1 + |M|)} + |L|\,\frac{\varepsilon}{2|L|} = \varepsilon,$$

meaning that by definition

$$\lim_{x \to a} (f(x)g(x)) = LM. \qquad \square$$

Finally we present one more useful theoretical result before we use these computational rules in a few examples.

Theorem 1.4.2 (Squeeze theorem)**.** *Suppose $f(x) \le g(x) \le h(x)$ holds for all x in some deleted interval around a point a, and suppose further that*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then

$$\lim_{x \to a} g(x) = L$$

as well.

Proof. We have that for all $\varepsilon_1, \varepsilon_2 > 0$ there exist some $\delta_1, \delta_2 > 0$ such that $|x - a| < \delta_1$ implies $|f(x) - L| < \varepsilon_1$ and $|x - a| < \delta_2$ implies $|f(x) - L| < \varepsilon_2$.

Let us now try to compute the distance between $g(x)$ and L:

$$|g(x) - L| = |g(x) - f(x) + f(x) - L| \le |g(x) - f(x)| + |f(x) - L|.$$

Now since $g(x) \le h(x)$ sufficiently close to a, this is bounded by

$$\begin{aligned} |g(x) - L| &\le |g(x) - f(x)| + |f(x) - L| \\ &\le |h(x) - f(x)| + |f(x) - L| \\ &= |h(x) - L - f(x) + L| + |f(x) - L| \\ &= |(h(x) - L) - (f(x) - L)| + |f(x) - L| \\ &\le |h(x) - L| + |f(x) - L| + |f(x) - L|. \end{aligned}$$

If we then take $\varepsilon_1 = \varepsilon_2 = \varepsilon/3$, the last line above is equal to ε so long as $|x - a| < \min\{\,\delta_1, \delta_2\,\}$, which is what we wanted to prove. $\square$

Remark 1.4.3. Just like the uniqueness of limits, these last two theorems also hold for the other types of limits, with similar proofs.

Example 1.4.4. Suppose that $3 - x^2 \le f(x) \le 3 + x^2$ holds for all $x \ne 0$. Find the limit of $f(x)$ as x tends to 0.

By our computational rules we have that

$$\lim_{x \to 0} (3 - x^2) = \lim_{x \to 0} 3 - \lim_{x \to 0} x^2 = \lim_{x \to 0} 3 - \big(\lim_{x \to 0} x\big)^2 = 3 - 0^2 = 3,$$

and by almost identical computation $\lim_{x \to 0} (3 + x^2) = 3$ as well. Therefore the Squeeze theorem applies, and thus $\lim_{x \to 0} f(x) = 3$ as well. ▲

Example 1.4.5. Show that if $\lim_{x \to a} |f(x)| = 0$, then $\lim_{x \to a} f(x) = 0$.

To this end we observe that $-|f(x)| \le f(x) \le |f(x)|$, and $\lim_{x \to a} |f(x)| = 0$ implies that

$$\lim_{x \to a} -|f(x)| = -\lim_{x \to a} |f(x)| = -0 = 0,$$

so by the Squeeze theorem

$$\lim_{x \to a} f(x) = 0. \qquad ▲$$

Lecture 2 Continuity[3]

An important type of functions (their importance will be justified throughout the course) are ones that, in some sense, don't jump. That is to say, when we draw their graph on a piece of paper, we (sort of) never need to lift the pen.

However it isn't entirely straight forward how to define this. In order to move toward a good definition, we'll discuss two important special cases that cover most, but not all, of our needs.

2.1 Special Cases

The domain is an open interval

If a function f has the domain

$$X = \{\, x \in \mathbb{R} \mid b < x < c \,\},$$

for some fixed b and c, then f is said to be ***continuous*** if and only if for all $a \in X$ we have

$$\lim_{x \to a} f(x) = f(a).$$

In other words, the function agrees with its limit at every point.

The domain is a closed interval

If a function f has the domain

$$X = \{\, x \in \mathbb{R} \mid b \leq x \leq c \,\},$$

where b and c are some constants, then f is said to be ***continuous*** if and only if the following are satisfied:

(i) for all interior points, i.e. $b < a < c$, we have $\lim\limits_{x \to a} f(x) = f(a)$,

(ii) for the left endpoint b, we have $\lim\limits_{x \to b^+} f(x) = f(b)$, and

(iii) for the right endpoint c, we have $\lim\limits_{x \to c^-} f(x) = f(c)$.

The reason for this distinction is hopefully intuitively clear: if we are at an endpoint of the closed interval, we can't hope to find a deleted two-sided neighbourhood around this point for which the function is defined, and therefore by definition it cannot have a two-sided limit at this point. Even so we still want to have a notion of continuity at such a point, so effectively what we do is the following:

We investigate the continuity of a function point by point, in the way that is reasonable depending on the point and its surrounding:

- If the function is defined in a two-sided neighbourhood around a point, we require that the two-sided limit exists and agrees with the function value.

[3]Date: January 24, 2017.

- If the function is defined only in a right neighbourhood of the point, we (obviously?) require only that the right limit exists and equals the function value.
- Likewise is the function is defined only in a left neighbourhood, we require a left limit equal to the value of the function.

Of course mathematicians don't like special cases if they can avoid them, so we attempt to construct a general definition motivated by this discussion.

2.2 Definition of Continuity

Definition 2.2.1 (Continuity in a point)**.** We say that the function $f\colon X \to Y$ is ***continuous in the point*** $a \in \mathbb{R}$ if and only if the following two conditions are satisfied:

(i) $a \in X$, and

(ii) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $x \in X$, we have that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

Remark 2.2.2. There is a crucial difference in this epsilon-delta construction compared to that of our limit definition, namely that we don't actually demand that the function is defined anywhere except for the point a itself. However if the function happens to be defined outside of this point a, then we impose some restrictions on these exterior points! It is this distinction that captures the subtleties of the second special case above.

One consequence of this is that a function defined in an isolated point, say $f\colon \{1\} \to \{2\}$, defined by $f(1) = 2$, is continuous at that point $x = 1$ (try to prove it using the above defintion!). This might seem strange, but it turns out to be a good property to have, that moreover agrees with much more abstract definitions of continuity found in the field of topology.

Another interesting takeaway from this definition is that in principle a function is ***discontinuous*** (not continuous) in any point not in its domain. However, when we study a function as a whole, we will consider only points in the domain:

Definition 2.2.3 (Continuity of a function)**.** We say that the function $f\colon X \to Y$ is a ***continuous function*** if and only if the function is continuous for all $x \in X$.

This means that there are functions which we call continuous, despite actually being discontinuous in one or more point; it's just that these points are outside of the domain!

Example 2.2.4. The function $f\colon \mathbb{R} \setminus \{7\} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x - 7}$$

is discontinuous at $x = 7$, but continuous for all other $x \in \mathbb{R}$. But because $x = 7$ doesn't belong to the domain, f is a continuous function anyway! ▲

We started off talking about the graph of a continuous function sort of not having a jump. Clearly, this isn't entirely true, as evidenced by this example. So with what we now know in hand, a better (and also correct) characterisation is that a continuous function has a graph that has no jumps in any interval in its domain.

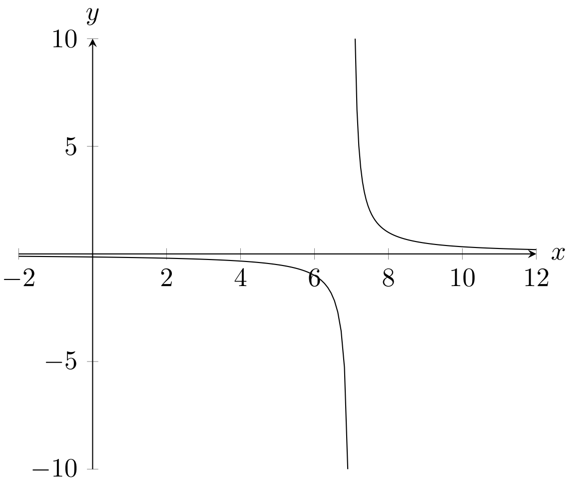


Figure 2.2.1: Plot of $f(x) = \dfrac{1}{x-7}$.

2.3 Continuous Functions

Many functions we are familiar with are continuous.

Examples 2.3.1. All polynomials are continuous, as are all rational functions, and all rational powers $x^{m/n}$, $m, n \in \mathbb{Z}$, $n \neq 0$.

The trigonometric functions (sine, cosine, tangent, etc.) are all continuous.

The absolute value $|x|$ is continuous as well. ▲

We can also combine continuous functions to get new continuous functions:

Theorem 2.3.2 (Combining continuous functions)**.** *If both f and g are functions defined on some neighbourhood containing c and both are continuous at c, then the following functions are continuous at c:*

(i) *$f + g$ and $f - g$;*

(ii) *fg;*

(iii) *kf, where $k \in \mathbb{R}$ is a constant;*

(iv) *$\dfrac{f}{g}$, provided $g(c) \neq 0$;*

(v) *$(f)^{1/n}$, $n \in \mathbb{Z}$, provided $f(c) > 0$ if n is even.*

Proof. We simply use the rules for calculating limits from the last lecture. For instance:

$$\begin{aligned} \lim_{x \to c} (f \pm g)(x) &= \lim_{x \to c} \big(f(x) \pm g(x) \big) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) \\ &= f(c) \pm g(c) = (f \pm g)(c) \end{aligned}$$

and

$$\lim_{x \to c} (fg)(x) = \lim_{x \to c} \big(f(x) g(x) \big) = \big(\lim_{x \to c} f(x) \big) \big(\lim_{x \to c} g(x) \big) = f(c)g(c) = (fg)(c).$$

Note that for endpoints of closed intervals in the domains of the functions, we would of course have to replace the above by appropriate one-sided limits. □

A trickier result to prove, but one that is extremely useful is the following:

Theorem 2.3.3. *If $f(g(x))$ is defined on some neighbourhood of c, and if f is continuous at L, with*
$$\lim_{x\to c} g(x) = L,$$
then
$$\lim_{x\to c} f(g(x)) = f(L) = f\big(\lim_{x\to c} g(x)\big).$$
Furthermore, if g is continuous at c (i.e. $L = g(c)$), then $f \circ g$ is continuous at c as well:
$$\lim_{x\to c} f(g(x)) = f(g(c)).$$

Proof. Let $\varepsilon > 0$. Because we will need to keep track of whether we're in the domain of f or g, we will denote them by X_f and X_g, respectively. Further we use $X_{f\circ g}$ to refer to the domain of the composition. We now establish what we know from the assumptions in the theorem.

Since f is continuous at $L = \lim_{x\to c} g(x)$, there exists some $\delta_1 > 0$ such that for all $y \in X_f$ with
$$\big|y - \underbrace{\lim_{x\to c} g(x)}_{=L}\big| < \delta_1,$$
we have that
$$\Big|f(y) - f\big(\lim_{x\to c} g(x)\big)\Big| < \varepsilon. \tag{2.3.1}$$
Moreover since $\lim_{x\to c} g(x) = L$, there exists some $\delta_2 > 0$ such that for all $x \in X_g$ with $|x - c| < \delta_2$ we have $|g(x) - L| < \delta_1$.

We take a moment to point out what happened in the very last step there: in the definition of the limit of g, we are able to take *any* ε, so in particular we can fix it to a specific value, namely δ_1, and find a corresponding δ_2 which makes the definition work.

If we now take $x \in X_{f\circ g}$ such that $|x - c| < \delta_2$, then $|g(x) - L| < \delta_1$, so by the construction of δ_1, Equation (2.3.1) with $y = g(x)$ becomes
$$|f(g(x)) - f(L)| = \Big|f(g(x)) - f\big(\lim_{x\to c} g(x)\big)\Big| < \varepsilon,$$
so $f \circ g$ is continuous at $x = c$. □

This means that not only are many of the elementary functions we are used to continuous, but almost any conceivable combination of them are as well!

2.4 Removable Discontinuities

Recalling how functions are by definition discontinuous at any point where they are undefined, it might sometimes be possible to therefore define a function at some point where it wasn't previously defined, in such a way that the new function is continuous. We demonstrate this by example:

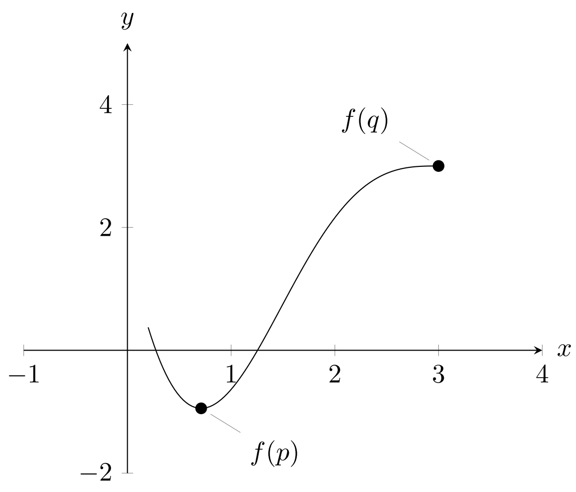


Figure 2.5.1: Example of the Max-min theorem.

Example 2.4.1. Recall the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

from Example 1.3.5 in the previous lecture. It is not continuous over all of $\mathbb{R}$, since it isn't even defined in $x = 1$, but it can be redefined to become continuous everywhere.

Since $\lim_{x \to 1} f(x) = 2$, we *define* a new function F which is $F(x) = f(x)$ for all $x \neq 1$, and $F(1) = 2$, so

$$F(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases} = x + 1,$$

which is continuous everywhere. In this case, $x = 1$ is said to be a ***removable discontinuity***, and $F(x)$ is said to be a ***continuous extension*** of $f(x)$. ▲

2.5 Continuous Functions on Closed, Finite Intervals

Continuous functions defined on closed and finite intervals have a few special properties.

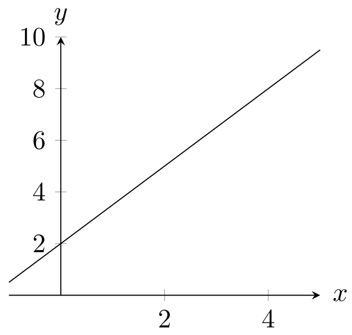
Theorem 2.5.1 (Max-min theorem)**.** *If f is continuous on the closed finite interval $[a, b]$, then there exists $p, q \in [a, b]$ such that for all $x \in [a, b]$ we have*

$$f(p) \leq f(x) \leq f(q).$$

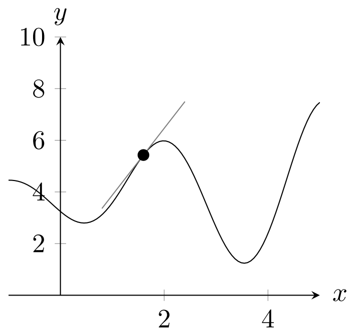
In other words, $f(p)$ is an global minimum of the function, attained on the point $x = p$, and $f(q)$ is an global maximum, at the point $x = q$.

A related result (it turns out) is the following:

Theorem 2.5.2 (Intermediate value theorem)**.** *If f is continuous on the interval $[a, b]$, with $a < b$ being some constants, and if s is a number between $f(a)$ and $f(b)$, then there exists a number $c \in [a, b]$ such that $f(c) = s$.*



(a) Plot of a straight line.



(b) Plot of a less straight line.

Figure 3.1.1: Examples of tangents of functions.

We demonstrate their usefulness by a brief example.

Example 2.5.3. Show that $f(x) = x^3 - 4x + 1$ has a root in the interval $[1, 2]$.

Note first that $f(1) = 1^3-4(1)+1 = 1-4+1 = -2$, and $f(2) = 2^3-4(2)+1 = 8 - 8 + 1 = 1$. Moreover, since $-2 \leq 0 \leq 1$, and since f is continuous (since all polynomials are continuous everywhere), by the intermediate value theorem there must exist some $c \in [1, 2]$ such that $f(c) = 0$.

Note that this tells us that a root of the polynomial exists in this interval, but it doesn't tell us what exactly it is! ▲

Some Notes on Proofs

Both the Max-min theorem and the Intermediate value theorem seem quite obvious, but actually proving them require some deep insights into upper bounds of sets of real numbers and axiomatic details to do with real numbers. It is beyond the scope of this course, but if we have some spare time and if there is interest we might prove them anyway. The curious student may find details in [Ad13, Appendix III].

Lecture 3 The Derivative[4]

3.1 Intuition

Differentiation, ultimately, is the problem of finding the slope[5] of a function at some point.

If the function at hand is a (nonvertical) straight line, this is simple. We read of a coefficient, or we compute rise over run, et cetera.

If, on the other hand, the function isn't a straight line, it can be slightly more complicated. To do it, we simply fall back on what is essentially basic geometry in order to attempt to approximate the tangent line of the function at the point of interest. We demonstrate by drawing on Figure 3.1.1b above, attempting to find the slope at $x = a$.

[4] Date: January 26, 2017.

[5] Or the rate of change, but since we are in only one variable these mean the same things.

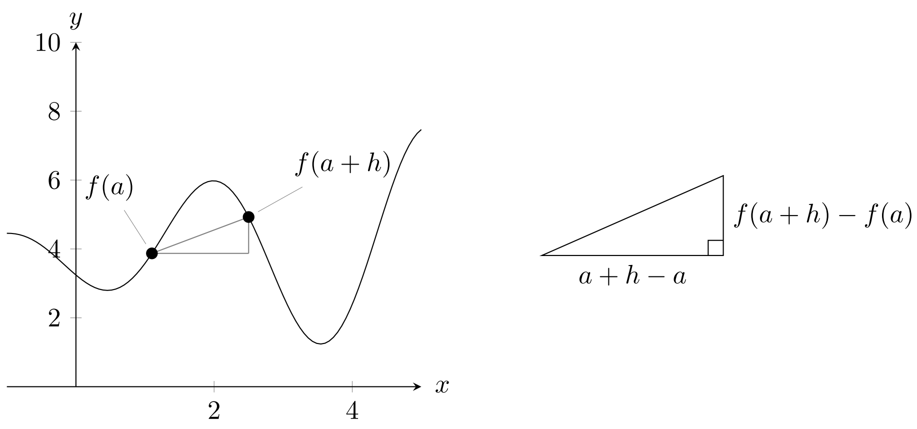


Figure 3.1.2: Constructing a difference quotient.

In other words, we let the point $(a, f(a))$ that we are interested in be one of the corners of a triangle, and pick some real number h to construct another point $(a + h, f(a + h))$ on the curve.

As illustrated below, that creates a right triangle with base $a+h-a = h$ and height $f(a + h) - f(a)$. Hence to find the slope of the hypotenuse, we compute the rise over the run and arrive at

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(a+h) - f(a)}{h}$$

of the line we just constructed. The expression in the right-hand side is called a ***difference quotient***, or sometimes a ***Newton quotient***.

The trick is to now let h go to 0, but never quite reach it. That way we will approach the slope at $x = a$, which is all right, because we know limits!

3.2 Definition and Computaitonal Rules

Having done this we are ready to formalise the definition of the derivative.

Definition 3.2.1 (Derivative, differentiability)**.** The ***derivative*** of a function f is another function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (in the sense that it is finite (and real, in this course)).

If $f'(x)$ exists (at a given x), we say that f is ***differentiable*** at x.

Remark 3.2.2. The domain of f' may be smaller than the domain of f, since f need not be differentiable everywhere. Points where f is not differentiable (and that aren't endpoints of closed intervals in the domain) are called ***singular points***.

Moreover, note that the limit in the definition prevents a function from being differentiable at an isolated point. That is to say, for a function to be differentiable at a point, it must also be defined at some neighbourhood around that point.

Remark 3.2.3. An equivalent difference quotient limit that some people prefer is

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

To see that this accomplishes the same thing, try drawing the corresponding picture.

Example 3.2.4. By definition, $f'(x)$ is the slope of $y = f(x)$ at the point $x = x_0$, where it exists. Therefore the tangent line to $y = f(x)$ at $(x_0, f(x_0))$ is

$$y = f(x_0) + f'(x_0)(x - x_0). \qquad \blacktriangle$$

It should be noted that the derivative comes with many different notations. If our function is $y = f(x)$, the following all mean the same thing:

$$D_x y = y' = \frac{dy}{dx} = \frac{d}{dx} f = f' = D_x f = Df.$$

In some of these, the variable x is implied, whereas in others it is explicit. For us, doing calculus in only one variable, there will probably never be any ambiguity in not specifying the variable, but once one starts doing calculus in more variables, it becomes a very good idea to specify.

Finally, before we go on to prove various interesting properties of the derivative, we make a note about what happens at endpoints of closed intervals in the domains of functions, thereby also clarifying what we meant by the parentheses in Remark 3.2.2 above.

Remark 3.2.5 (Left and right derivatives). Suppose our function f that we wish to differentiate is defined on a closed interval $[a, b]$. We have a problem if we try to compute $f'(a)$ or $f'(b)$.

Consider, for instance, $x = a$. What happens if $h < 0$ in the difference quotient? Well, in $f(a + h)$ we have $x = a + h < a$, and for such a value of x the function f isn't defined.

The solution is to use one-sided limits where appropriate, to define ***left*** and ***right derivatives***:

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \text{and} \qquad f'_-(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}.$$

There is an important connection between differentiability and continuity.

Theorem 3.2.6. *If f is differentiable at x, then it is also continuous at x.*

Proof. Since f is differentiable at x, the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. We then take the difference between $\lim_{h \to 0} f(x + h)$ and $f(x)$:

$$\begin{aligned}
\lim_{h \to 0}(f(x+h)) - f(x) = \lim_{h \to 0}(f(x+h) - f(x)) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot h \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} h = f'(x) \cdot 0 = 0.
\end{aligned}$$

This means that the distance between the limit of f at x and $f(x)$ is 0, whereby they are equal, so by definition f is continuous at x. $\square$

The converse is not true, i.e. there exists continuous functions that aren't differentiable.

Counterexample 3.2.7. Consider the absolute value function $f(x) = |x|$. This function is continuous at the point $x = 0$ ($\delta = \varepsilon$ works fine in the definition of continuity, for instance), but it is not differentiable there, since the the limit defining the derivative doesn't exist there; from the left the limit is -1, whereas from the right the limit is 1. ▲

We now formulate and prove various computational rules of derivatives that we will continue using throughout the course.

Theorem 3.2.8 (Addition, subtraction, and multiplication by constant)**.** *Let f and g be differentiable functions at x, and let k be some constant. Then $f+g$, $f-g$, and kf are all differentiable at x, and*

(i) $(f+g)'(x) = f'(x) + g'(x)$*;*

(ii) $(f-g)'(x) = f'(x) - g'(x)$*; and*

(iii) $(kf)'(x) = kf'(x)$*.*

Proof. We simply rearrange the difference quotients and use our rules for limits. For (i) and (ii), this becomes

$$\begin{aligned}(f \pm g)'(x) &= \lim_{h\to 0} \frac{(f \pm g)(x+h) - (f \pm g)(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) \pm g(x+h) - (f(x) \pm g(x))}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{(g(x+h) - g(x))}{h} = f'(x) \pm g'(x).\end{aligned}$$

For (iii), we get a similarly trivial computation:

$$(kf)'(x) = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} = k \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = kf'(x). \qquad \square$$

What happens when multiplying two functions together is perhaps less intuitive:

Theorem 3.2.9 (Product rule)**.** *Let f and g be functions differentiable at x. Then fg is also differentiable at x, and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Proof. We study the difference quotient and add 0 in a clever way:

$$\begin{aligned}(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \Big(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \Big) \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + \lim_{h\to 0} f(x) \cdot \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}.\end{aligned}$$

The crucial last step is to now recall Theorem 3.2.6, whereby g is continuous, and therefore the above becomes

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x). \qquad \square$$

Dealing with quotients is maybe even less intuitive.

Theorem 3.2.10 (Quotient rule)**.** *If f and g are differentiable functions at x and $g(x) \neq 0$, then the quotient f/g is differentiable at x, and*

$$\Big(\frac{f}{g}\Big)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

To prove it, we first prove an easier result.

Lemma 3.2.11 (Reciprocal rule)**.** *If g is differentiable at x with $g(x) \neq 0$, then $1/g$ is differentiable at x and*

$$\Big(\frac{1}{g}\Big)' = -\frac{g'(x)}{(g(x))^2}.$$

Proof. The trick lies entirely in writing the numerator with common denominator:

$$\begin{aligned}
\frac{d}{dx}\Big(\frac{1}{g(x)}\Big) &= \lim_{h\to 0}\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h\to 0}\frac{\frac{g(x)-g(x+h)}{g(x+h)g(x)}}{h} \\
&= \lim_{h\to 0}\frac{g(x)-g(x+h)}{hg(x+h)g(x)} = -\lim_{h\to 0}\frac{g(x+h)-g(x)}{hg(x+h)g(x)} \\
&= -\lim_{h\to 0}\frac{1}{g(x)g(x+h)}\cdot\frac{g(x+h)-g(x)}{h} \\
&= -\frac{g'(x)}{(g(x))^2}. \qquad \square
\end{aligned}$$

With this at our disposal proving the quotient rule becomes a special case of the product rule.

Proof of the Quotient rule. We write f/g as $f \cdot 1/g$ and use the product rule:

$$\begin{aligned}
\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) &= \frac{d}{dx}\Big(f(x)\cdot\frac{1}{g(x)}\Big) = f'(x)\frac{1}{g(x)} + f(x)\Big(\frac{-g'(x)}{(g(x))^2}\Big) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}. \qquad \square
\end{aligned}$$

In keeping with how we dealt with continuity, after addition, subtraction, multiplication and division comes composition.

Theorem 3.2.12 (Chain rule)**.** *If f is differentiable at $y = g(x)$, and g is differentiable at x, then $f\circ g$ is differentiable at x and $(f\circ g)'(x) = f'(g(x))g'(x)$.*

Proof. We construct a slightly tricky function E, defined as

$$E(k) = \begin{cases} 0, & \text{if } k = 0 \\ \frac{f(y+k)-f(y)}{k} - f'(y), & \text{if } k \neq 0. \end{cases}$$

Notice how

$$\lim_{k\to 0} E(k) = \lim_{k\to 0} \frac{f(y+k)-f(y)}{k} - \lim_{k\to 0} f'(y) = f'(y) - f'(y) = 0 = E(0),$$

meaning that $E(k)$ is continuous at $k = 0$. Moreover

$$E(k) + f'(y) = \frac{f(y+k)-f(y)}{k},$$

which implies that

$$f(y+k) - f(y) = (E(k) + f'(y)) \cdot k.$$

If we now take $y = g(x)$ and $k = g(x+h) - g(x)$, meaning that $y + k = g(x+h)$, and inserting this into the above equation we get

$$f(g(x+h)) - f(g(x)) = \big(f'(g(x)) + E(k)\big)(g(x+h) - g(x)). \tag{3.2.1}$$

By assumption g is differentiable at x, meaning that

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h},$$

and also that g is continuous at x, whereby $\lim\limits_{h\to 0} g(x+h) - g(x) = 0$.

We are now ready to compute the limit of the difference quotient of the composition, using Equation 3.2.1 straight away:

$$\frac{d}{dx} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0} (f'(g(x)) + E(k)) \frac{g(x+h)-g(x)}{h}.$$

Now since E is continuous at $0 = \lim\limits_{h\to 0} g(x+h) - g(x) = \lim\limits_{h\to 0} k$, we have

$$\lim_{h\to 0} E(k) = E\Big(\lim_{h\to 0} k\Big) = E(0) = 0,$$

whereby the above becomes

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x). \qquad \square$$

3.3 Derivatives of Trigonometric Functions

We state without proof that sin and cos are continuous functions everywhere.

Moreover, we claim that $\lim\limits_{x\to 0} \sin(x) = 0$ and $\lim\limits_{x\to 0} \cos(x) = 1$.

Exercise 3.3.1. Try to prove it.

Hint: for the first one, use $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos(\alpha)\sin(\beta)$ *to manipulate* $|\sin(x) - \sin(a)|$ *into a single* sin. *For both this and the* sin *limit, it will help to prove that* $0 \le |\sin(x)| \le |x|$.

In closing, we will prove an important limit, which we will use next time to compute the derivatives of sin and cos.

Lemma 3.3.2. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1.$

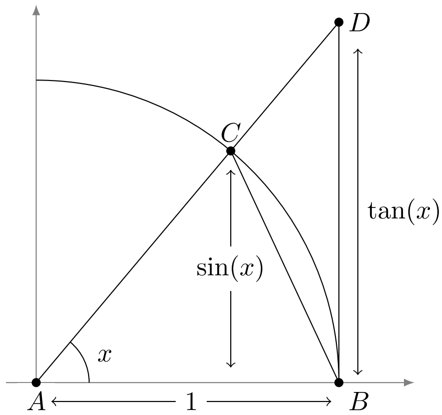


Figure 3.3.1: A geometrical representation of sin and tan on the unit circle.

Proof. This proof relies very much on the geometrical construction of sin and tan found in Figure 3.3.1. Let $0 < x < \pi/2$.

We make three claims:

(i) The area of the triangle $\triangle ABC$ is $\frac{1}{2}\sin(x)$ (since $\sin(x)$ is its height).

(ii) The area of the sector ABC is $\frac{1}{2}x$.

(iii) The area of the triangle $\triangle ABD$ is $\frac{1}{2}\tan(x)$.

Moreover, it is clear that these areas are in ascending order as listed, i.e.

$$\frac{1}{2}\sin(x) \leq \frac{1}{2}x \leq \frac{1}{2}\tan(x).$$

If we divide by $\frac{1}{2}\sin(x)$, which is positive since $0 < x < \pi/2$, we then get

$$1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)}.$$

Finally we take reciprocals, thereby reversing the inequalities,

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1.$$

Note how 1, $\sin(x)/x$, and $\cos(x)$ are all even functions, so this chain of inequalities is true also for $-\pi/2 < x < 0$ by symmetry. This is good, because it allows us to consider the two-sided limit we are interested in, despite the geometric picture only discussing positive x.

Finally we remark that since $\lim\limits_{x\to 0}\cos(x) = 1$, the Squeeze theorem tells us that

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1,$$

as desired. $\square$

Lecture 4 The Mean-Value Theorem[6]

4.1 Trigonometric Derivatives, continued

Recall from last time how
$$\lim_{x\to 0}\frac{\sin(x)}{x}=1.$$
Using this we can prove

Proposition 4.1.1. $\dfrac{d}{dx}\sin(x)=\cos(x)$.

Proof. We need the identity $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$. Then

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}\\
&= \Big(\lim_{h\to 0}\sin(x)\Big)\cdot\Big(\lim_{h\to 0}\frac{\cos(h)-1)}{h}\Big)+\Big(\lim_{h\to 0}\cos(x)\Big)\cdot\Big(\lim_{h\to 0}\frac{\sin(h)}{h}\Big)\\
&= \sin(x)\cdot 0+\cos(x)\cdot 1=\cos(x).
\end{aligned}$$

The only mystery in the above computation is the $\cos(h)$ limit in the penultimate line. We shed light on it by using $\cos(h)=1-2(\sin(h/2))^2$, like so

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}=-\lim_{h\to 0}\frac{2(\sin(h/2))^2}{h},$$

which if we let $x=h/2$ (which also approaches 0 as h does) becomes

$$-\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\sin(x)=-1\cdot 0=0.$$ □

Since we have a chain rule, it is immediate to find the derivative of cos once we know the derivative of sin.

Corollary 4.1.2. $\dfrac{d}{dx}\cos(x)=-\sin(x)$.

Proof. We know that $\cos(x)=\sin(\pi/2-x)$, and $\sin(x)=\cos(\pi/2-x)$, whence by the chain rule

$$\begin{aligned}
\frac{d}{dx}\cos(x)=\frac{d}{dx}\sin\Big(\frac{\pi}{2}-x\Big)&=\cos\Big(\frac{\pi}{2}-x\Big)\cdot\frac{d}{dx}\Big(\frac{\pi}{2}-x\Big)\\
&=-\cos\Big(\frac{\pi}{2}-x\Big)=-\sin(x).
\end{aligned}$$ □

Exercise 4.1.3. Use the computational rules to find the derivatives of tan, sec, cot, and csc, in the event that you like these functions.

[6]Date: January 30, 2017.

Since so far the discussion about derivatives has been quite heavy on theory, we close it off with a number of computational problems, so as to hopefully get some practice with how the definition of the derivative works. In the future we will often end up computing derivatives directly using results such as these, instead of falling back on the definition in terms of the limit of a difference quotient.

Exercises 4.1.4. Show the following:

(a) $\dfrac{d}{dx}c = 0$, where c is constant.

(b) $\dfrac{d}{dx}kx = k$, where k is constant.

(c) $\dfrac{d}{dx}\dfrac{1}{x} = -\dfrac{1}{x^2}$.

(d) $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$.

(e) $\dfrac{d}{dx}x^n = nx^{n-1}$, for all $n \in \mathbb{Z}$, perhaps using induction.

Note how, for instance, (a), (b), and (e) combined, together with our computational rules, tell us how to differentiate every possible polynomial.

The majority of this lecture will be about the Mean-value theorem and some consequences thereof, but first we will wrap up the discussion of the derivative itself somewhat.

4.2 Higher Order Derivatives

Since, by definition, the derivative of a function $y = f(x)$ is again a function, it can happen that the derivative itself is differentiable (at some point(s)).

We call this derivative of a derivative the ***second derivative*** of the original function. Like the first derivative, there are many notations used for this:

$$y'' = f'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{d}{dx}f = \frac{d^2}{dx^2}f = D_x^2 y = D_x^2 f.$$

We can continue this to achieve a general nth order derivative,

$$y^{(n)} = f^{(n)} = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n}f = D_x^n y = D_x^n f,$$

defined inductively as the derivative of the $(n-1)$st order derivative.

Remark 4.2.1. We use $y^{(n)}$ or $f^{(n)}$, with the parentheses, in order to distinguish higher order derivatives from exponents or compositions,

$$y^n = \underbrace{y \cdot y \cdot \ldots \cdot y}_{n \text{ lots of } y}, \qquad \text{or} \qquad f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ lots of } f}.$$

For convenience the function itself is usually taken to be its own zeroth derivative, $f = f^{(0)}$.

Example 4.2.2. The classical example is the reason calculus was invented: take $x = x(t)$ to be the position (of something) at the time t. Then its velocity is $v = \frac{dx}{dt} = x'$, and its acceleration is $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = x''$. This exemplifies why we often want to think of the derivative as the rate of change of something. ▲

These higher order derivatives are often very important in physics, since it turns out that much in the real world can be modeled using so-called differential equations, which we will deal more with toward the end of the course. For now, we'll discuss it briefly in an example.

Example 4.2.3. Show that $y(t) = A\cos(kt) + B\sin(kt)$, for A, B, and k constants, is a solution to the ***second-order differential equation***

$$\frac{d^2y}{dt^2}(t) + k^2y(t) = 0.$$

To do this we compute the second derivative and test it in the equation (when we deal with differential equations more thoroughly in the future we will learn how to find these solutions on our own):

$$\frac{d^2y}{dt^2}(t) = \frac{d}{dt}(-Ak\sin(kt) + Bk\cos(kt)) = -Ak^2\cos(kt) - Bk^2\sin(kt).$$

We then insert it into the differential equation,

$$\frac{d^2y}{dt^2}(t) + k^2y(t) = -k^2(A\cos(kt) + B\sin(kt)) + k^2(A\cos(kt) + B\sin(kt)) = 0$$

for all t. ▲

4.3 The Mean-Value Theorem

Example 4.3.1. Suppose we leave on a trip at 3 o'clock, traveling 100 kilometres away, and arriving 2 hours later, at 5 o'clock. Then of course our average speed was

$$\frac{100\,\text{km}}{2\,\text{h}} = 50\,\text{km/h}.$$

We might not have traveled at this speed constantly (indeed it's unlikely we did, since we presumably started at rest and accelerated), but we *must* have traveled at 50 km/h at least once!

If not, our speed was either always above or always below 50 km/h, in which case we would have arrived either before or after 5 o'clock. ▲

In the language of calculus, this is called the Mean-value theorem, and the crucial detail that makes it true is, of course, that our speed whilst traveling can't 'jump'. I.e., we can't go from 25 km/h to 75 km/h without ever passing 50 km/h. Once again in the language of calculus, this is what we call continuity, and indeed since we have acceleration of some kind everywhere, even if constant, our speed (or velocity) must also have a derivative.

Theorem 4.3.2 (Mean-value theorem)**.** *Let f be a function continuous on the closed and finite interval $[a, b]$ that is differentiable on $]a, b[$. Then there exists a point $c \in]a, b[$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

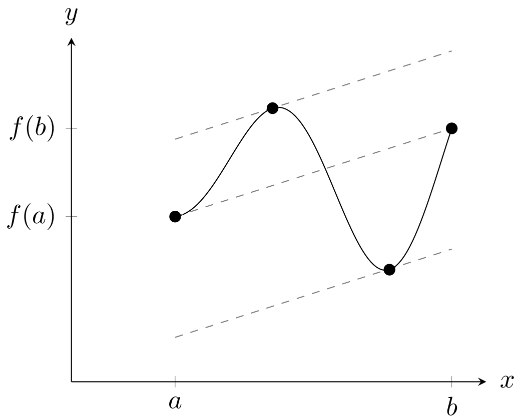
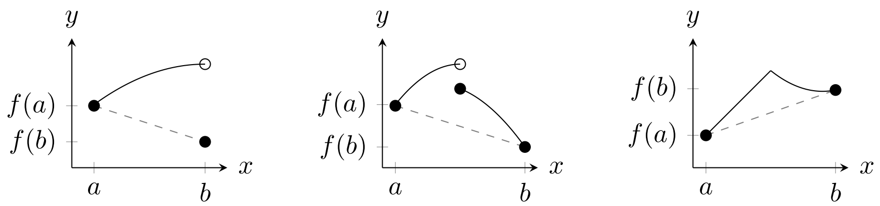


Figure 4.3.1: Illustration of the Mean-value theorem.



(a) Discontinuous at endpoint

(b) Discontinuous at interior point

(c) Not differentiable at interior point

Figure 4.3.2: Counterexamples of the Mean-value theorem where essential conditions are violated.

That is to say, there must exist some point(s) in the interior of the interval with the same slope as the line segment connecting the two endpoints.

An example of this theorem is demonstrated in Figure 4.3.1, which also demonstrates that the c in the theorem needn't be unique. We will prove this in a bit, but for now we will establish some more theory and also motivate why both the continuity and the differentiability is required in Figure 4.3.2.

This demonstrates that if we lack continuity anywhere in the closed interval, then we can construct a counter example, and likewise if we lack differentiability anywhere in the open interval, we may do the same.

We now prove an intermediary result we need in order to prove the Mean-value theorem.

Lemma 4.3.3. *If f is defined on an open interval $]a,b[$ and achieves a maximum (or minimum) at $c \in\,]a,b[$, then if $f'(c)$ exists, it must be* 0.

Such points where the derivative is 0 are called ***critical points***.

Proof. If f has a maximum at $c \in\,]a,b[$, then $f(x) - f(c) \leq 0$ for all $x \in\,]a,b[$. We consider two cases:

(i) If $c < x < b$, then

$$\frac{f(x) - f(c)}{x - c} \leq 0,$$

since $c < x$, which, if we recall the alternative difference quotient,[7] implies the sign of the right derivative at c,

$$f'_+(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \leq 0.$$

(ii) Similarly if $a < x < c$, then we have

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

which similarly implies

$$f'_-(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0.$$

By assumption $f'(c)$ exists, so it is equal to the left and right derivative at that point, so

$$0 \leq f'_-(c) = f'(c) = f'_+(c) \leq 0,$$

whereby $f'(c) = 0$.

For the case of the minimum, the proof is almost identical, except we change the direction of certain inequalities. □

We now show a special case of the Mean-value theorem, with which we later prove the full version.

Theorem 4.3.4 (Rolle's theorem)**.** *Suppose that f is a continuous function on the closed and finite interval $[a, b]$ and that it is differentiable on $]a, b[$. If $f(a) = f(b)$, then there exists a point $c \in\ ]a, b[$ such that $f'(c) = 0$.*

Proof. There are two cases to consider. If $f(x) = f(a)$ for all $x \in [a, b]$, then f is constant and $f'(c) = 0$ for all $c \in\ ]a, b[$.

Suppose that there exists some $x \in\ ]a, b[$ such that $f(x) \neq f(a)$, so that either $f(x) > f(a)$ or $f(x) < f(a)$.

Consider the $f(x) > f(a)$ case. Since f is continuous on $[a, b]$, by the Max-min theorem f must have a maximum at some point $c \in [a, b]$. Thus

$$f(c) \geq f(x) > f(a) = f(b),$$

so c cannot be equal to a or b, whereby $c \in\ ]a, b[$, where by assumption f is differentiable. Thus $f'(c)$ exists, and so by the previous lemma $f'(c) = 0$.

For the $f(x) < f(a)$ case, we instead use the Max-min theorem to ensure the existence of a minimum, and proceed accordingly. □

With this we are finally ready to prove the Mean-value theorem.

Proof of Mean-value theorem. Recall that f is continuous on $[a, b]$ and differentiable on $]a, b[$. We define a new function g by

$$g(x) = f(x) - \Big(f(a) + \frac{f(b) - f(a)}{b - a}\,(x - a)\Big).$$

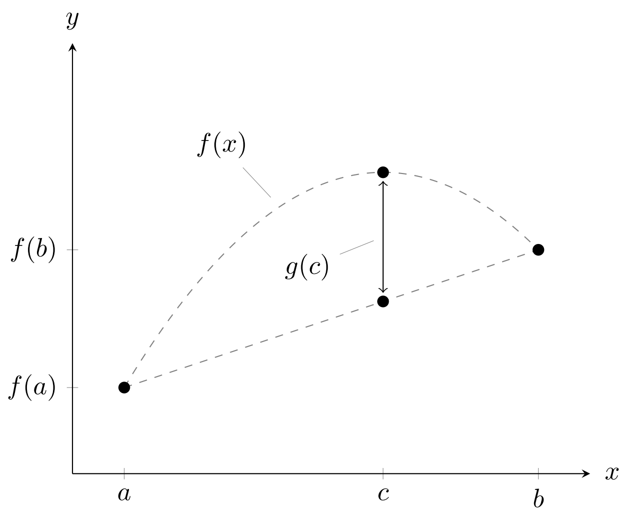


Figure 4.3.3: The geometrical construction of g in the proof of the Mean-value theorem.

The expression in the parentheses is the line segment joining $(a, f(a))$ and $(b, f(b))$, whereby $g(x)$ is the vertical distance between this line segment and f itself at each point x, as illustrated in Figure 4.3.3.

Since f is continuous on $[a, b]$ and differentiable on $]a, b[$, g is as well. Moreover $g(a) = g(b) = 0$ (feel free to verify), whence by Rolle's theorem there exists a $c \in\ ]a, b[$ such that $g'(c) = 0$.

By differentiating g we get

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so at $x = c$ we have

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \iff \qquad f'(c) = \frac{f(b) - f(a)}{b - a}. \qquad \square$$

Corollary 4.3.5 (Generalised Mean-value theorem)**.** *If f and g are both continuous on $[a, b]$ and differentiable on $]a, b[$), and if $g'(x) \neq 0$ for all $x \in\ ]a, b[$, then there exists a number $c \in\ ]a, b[$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Note that $g(b) \neq g(a)$ since otherwise there would exist an $x \in\ ]a, b[$ such that $g'(x) = 0$ by the Rolle's theorem. Thus the division in the left-hand side is well-defined.

We apply the Mean-value theorem to

$$h(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - g(a)).$$

[7]It is of course possible to compute the same with the $h \to 0$ type difference quotient, it's just marginally more cumbersome.

Since $h(a) = h(b) = 0$ (again, feel free to verify) and h has the appropriate continuity and differentiability, there exists a $c \in]a, b[$ such that $h'(c) = 0$. Therefore

$$h'(c) = (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c) = 0$$

which by rearranging becomes

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \qquad \square$$

We end today's lecture with two nice consequences of the Mean-value theorem.

Definition 4.3.6 (Increasing and decreasing functions)**.** Suppose f is defined on some interval I and that x_1 and x_2 are points on I.

(*i*) If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is ***increasing*** on I.

(*ii*) If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is ***decreasing*** on I.

(*iii*) If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, then f is ***nondecreasing*** on I.

(*iv*) If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, then f is ***nonincreasing*** on I.

Remark 4.3.7. Note that a nonincreasing or nondecreasing function might be constant.

In the event that we have differentiability, we can easily detect the above features of a function.

Theorem 4.3.8. *Let J be an open interval and let I be an interval consisting of J and possibly one or both of J's endpoints. Suppose f is continuous on I and differentiable on J.*

(*i*) *If $f'(x) > 0$ for all $x \in J$, then f is increasing on I.*

(*ii*) *If $f'(x) < 0$ for all $x \in J$, then f is decreasing on I.*

(*iii*) *If $f'(x) \geq 0$ for all $x \in J$, then f is nondecreasing on I.*

(*iv*) *If $f'(x) \leq 0$ for all $x \in J$, then f is nonincreasing on I.*

Proof. Let $x_1, x_2 \in I$ with $x_2 > x_1$. By the Mean-value theorem there exists a $c \in (x_1, x_2) \subset J$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

implying that $f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$. Since $x_2 - x_1 > 0$, $f(x_2) - f(x_1)$ has the same sign as $f'(c)$ and may be 0 if $f'(c) = 0$. From this all the four cases follow according to the definition. $\square$

If a function is constant on an interval, then its derivative is 0 on this interval('s interior), which was one of the exercises left at the end of the last lecture. The converse is also true.

Theorem 4.3.9. *If f is continuous on an interval I and $f'(x) = 0$ at every interior point of I, then $f(x) = k$ is constant.*

Proof. Pick any point $x_0 \in I$ and let $k = f(x_0)$. If x is any other point of I, then by the Mean-value theorem there exists a c between x_0 and x such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(c).$$

Then c must belong to I and cannot be an endpoint. Moreover $f'(c) = 0$ for all c in the interior of I, so $f(x) - f(x_0) = f(x) - k = 0$ for all $x \in I$, which is equivalent with $f(x) = k$ for all $x \in I$. □

Lecture 5 The Natural Logarithm[8]

5.1 Implicit Differentiation

So far we have only computed derivatives of things on the form $y = f(x)$, but not everything is this nice.

Consider for example $F(x, y) = x^2 + y^2 - 1 = 0$, the equation of the unit circle. This certainly has some kind of slope everywhere (though vertical at $(1, 0)$ and $(-1, 0)$). How do we find these slopes?

In this particular case we can of course solve for y, getting $y = \pm\sqrt{1 - x^2}$, depending on which half plane we are in, and this we can differentiate using the chain rule:

$$y' = \pm\frac{1}{2}(1 - x^2)^{-1/2} \cdot -2x = \mp\frac{x}{\sqrt{1 - x^2}}.$$

We say that there are ***implicit functions***[9] of x, in the sense that we can somehow extract functions in one variable x describing the relation in two variables x and y. We can't always find these explicitly as above, however! But we can still differentiate, as it turns out.

Example 5.1.1. Differentiate implicitly $x^2 + y^2 - 1 = 0$.

The idea is to differentiate both sides of the equation with respect to our variable x, meanwhile treating $y = y(x)$ as a function of x. This means that whenever we run into a y, we must use the chain rule:

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2 - 1) = \frac{d}{dx}(0) \quad &\iff \quad \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(1) = 0 \\ \iff \quad 2x + 2y\frac{dy}{dx} = 0 \quad &\iff \quad \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}, \end{aligned}$$

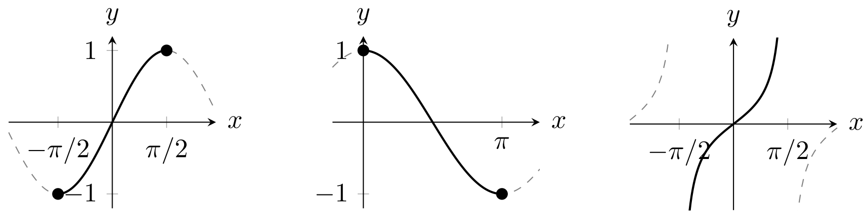
which if we substitute $\pm\sqrt{1 - x^2}$ for y becomes

$$\frac{dy}{dx} = -\frac{x}{\pm\sqrt{1 - x^2}} = \mp\frac{x}{\sqrt{1 - x^2}},$$

just as above. ▲

[8]Date: February 2, 2017.

[9]This is actually a remarkably important idea, leading up to the Implicit function theorem in Multivariable calculus, but you'll learn more about this in a future course.



(a) Sine (b) Cosine (c) Tangent

Figure 5.2.1: Restricted domains of trigonometric functions.

5.2 Derivatives of Inverse Functions

Recall from previous experience that a bijective (i.e. injective (one-to-one) and surjective (onto)) function $f\colon X \to Y$ has an ***inverse function*** $f^{-1}\colon Y \to X$ such that

$$y = f(x) \qquad \iff \qquad x = f^{-1}(y)$$

and

$$f(f^{-1}(y)) = y, \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

Using one of these so-called cancellation identities together with implicit differentiation we can find a general formula for the derivative of inverse functions:

$$\frac{d}{dx}\big(f(f^{-1}(x))\big) = \frac{d}{dx}(x) \qquad \iff \qquad f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1}(x)) = 1$$

which if we solve for $\frac{d}{dx}(f^{-1}(x))$ becomes

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}.$$

To see why this is useful, let us attempt to differentiate (some of) the inverse trigonometric functions.

Of course trigonometric functions are periodic and therefore not one-to-one, so in order to invert them we restrict their domains, as demonastrated in Figure 5.2.1. We therefore have

$$\begin{aligned}
\arcsin(\sin(x)) &= x, && -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}; \\
\sin(\arcsin(x)) &= x, && -1 \leq x \leq 1; \\
\arccos(\cos(x)) &= x, && 0 \leq x \leq \pi; \\
\cos(\arccos(x)) &= x, && -1 \leq x \leq 1; \\
\arctan(\tan(x)) &= x, && -\frac{\pi}{2} < x < \frac{\pi}{2}; \\
\tan(\arctan(x)) &= x, && -\infty < x < \infty.
\end{aligned}$$

We can now find the derivatives of the inverse functions, since if $y = \arcsin(x)$, we have $x = \sin(y)$, whence

$$\frac{d}{dx}(x) = \frac{d}{dx}\sin(y) \quad \iff 1 = \cos(y)\,\frac{dy}{dx}.$$

Since $-\pi/2 \leq y \leq \pi/2$, we have $\cos(y) \geq 0$, whereby the trigonometric identity $(\cos(y))^2 + (\sin(y))^2 = 1$ is equivalent with

$$\cos(y) = \sqrt{1 - (\sin(y))^2} = \sqrt{1 - x^2},$$

whence

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - x^2}}.$$

Note that arcsin isn't differentiable at the endpoints $x = -1$ and $x = 1$ since we get division by 0 there, and the slope approaches infinity.

Similarly for arccos we get

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1 - x^2}}$$

and for arctan we get

$$\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}.$$

Exercise 5.2.1. Prove the above two claims about arccos and arctan. Note that for the arctan function we will require the derivative of $\tan(x)$, which is quite useful and turns out to be $(\sec(x))^2 = 1 + (\tan(x))^2$.

5.3 The Natural Logarithm

We will now *define* a function $\ln(x)$, called the ***natural logarithm*** of x. This definition might look quite strange, but we will show that it has the properties we expect of logarithms.

Definition 5.3.1 (Natural logarithm)**.** Take $x > 0$ and let $A(x)$ be defined as the area of the plane bounded by the curve $y = 1/t$, the t axis, and the vertical lines $t = 1$ and $t = x$ (see Figure 5.3.1).

The function ln is defined as

$$\ln(x) = \begin{cases} A(x), & \text{if } x \geq 1 \\ -A(x), & \text{if } 0 < x < 1. \end{cases}$$

From the definition it follows that $\ln(1) = 0$, $\ln(x) > 0$ if $x > 1$, and $\ln(x) < 0$ if $0 < x < 1$. Finally since the area must increase as we move x_0 rightward on the figure, the function is increasing, and so injective (one-to-one).

This function has a plethora interesting properties, maybe chief amongst which is

Theorem 5.3.2. *If $x > 0$, then*

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

Proof. If $x > 0$ and $h > 0$, then $\ln(x+h) - \ln(x)$ is the area bounded by $y = 1/t$, $y = 0$, $t = x$, and $t = x + h$, as illustrated in Figure 5.3.2a.

Clearly this area is greater than $h \cdot \frac{1}{x+h}$ and smaller than $h \cdot \frac{1}{x}$, as illustrated in Figure 5.3.2b, meaning that

$$\frac{h}{x+h} < \ln(x+h) - \ln(x) < \frac{h}{x}. \tag{5.3.1}$$

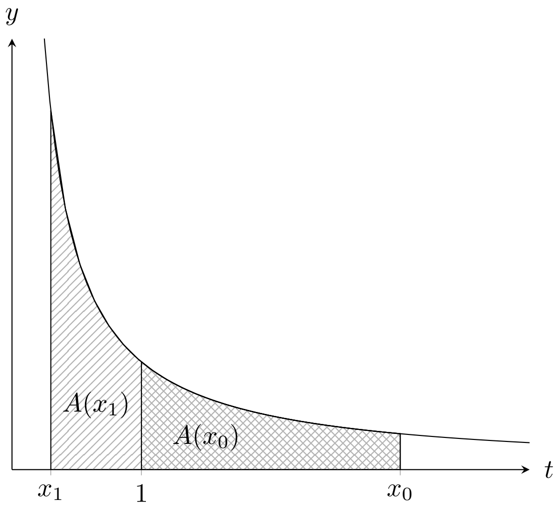


Figure 5.3.1: The area under $y = 1/t$.

If we divide this through by h (which is positive by assumption) we get

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x},$$

which by the Squeeze theorem (for one-sided limits) gives

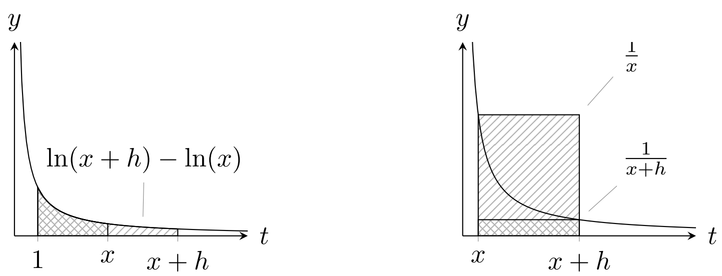
$$\lim_{h\to 0^+} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x}$$

since

$$\lim_{h\to 0^+} \frac{1}{x+h} = \lim_{h\to 0^+} \frac{1}{x} = \frac{1}{x}.$$

Taking $h < 0$ instead (and we also need to guarantee that $x + h > 0$, so take h such that $0 < x + h < x$) we get the following from Equation (5.3.1):

$$\frac{1}{x} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x+h},$$



(a) The area $\ln(x + h) - \ln(x)$. (b) The area $\ln(x + h) - \ln(x)$.

Figure 5.3.2: The area under $y = 1/t$.

which by the same argument yields

$$\lim_{h\to 0^-} \frac{\ln(x+h)-\ln(x)}{h} = \frac{1}{x},$$

so in all

$$\frac{d}{dx}\ln(x) = \lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h} = \frac{1}{x}. \qquad \square$$

Note, by the way, that $1/x > 0$ for all $x > 0$, which is therefore another way to see that ln is increasing.

Other interesting properties of the ln function are those we associate with logarithms:

Theorem 5.3.3. *If x and y are positive, then*

(i) $\ln(xy) = \ln(x) + \ln(y)$*;*

(ii) $\ln(x/y) = \ln(x) - \ln(y)$*, with the special case* $\ln(1/x) = -\ln(x)$*;*

(iii) $\ln(x^r) = r\ln(x)$*, for now assuming* $r \in \mathbb{Q}$.[10]

Proof. (i) Let $y > 0$ be constant. Then by the chain rule

$$\frac{d}{dx}(\ln(xy) - \ln(x)) = \frac{y}{xy} - \frac{1}{x} = 0$$

for all $x > 0$. Since the derivative is 0 for all $x > 0$,

$$\ln(xy) - \ln(x) = C \tag{5.3.2}$$

must be constant by Theorem 4.3.9 last lecture. Take $x = 1$, then $\ln(y) - 0 = C$, so $C = \ln(y)$, whence Equation (5.3.2) gives

$$\ln(xy) = \ln(x) + \ln(y).$$

(ii) Consider

$$\frac{d}{dx}\ln\left(\frac{1}{x}\right) = \frac{1}{1/x}\cdot -\frac{1}{x^2} = -\frac{1}{x}.$$

So mimicking (i), look at $\ln(1/x) + \ln(x)$ since

$$\frac{d}{dx}\Big(\ln\big(\frac{1}{x}\big) + \ln(x)\Big) = -\frac{1}{x} + \frac{1}{x} = 0,$$

so $\ln(1/x) + \ln(x) = C$ is constant, and in particular $C = 0$ (to see this, take $x = 1$). Therefore

$$\ln\left(\frac{1}{x}\right) = -\ln(x).$$

To finish the proof, take this together with (i).

(iii) In the same way, study

$$\frac{d}{dx}\ln(x^r) = \frac{1}{x^r}\cdot rx^{r-1} = r\cdot x^{r-1-r} = \frac{r}{x},$$

[10] This is because we have not yet showed that arbitrary exponentials are continuous. If they are, we could define $\ln(x^\alpha)$ for $\alpha \in \mathbb{R}$ to be $\lim_{r\in\mathbb{Q},\, r\to\alpha} \ln(x^r)$, but we can't take the limit inside the exponential until we know it is continuous.

at least for $r \in \mathbb{Q}$ so far, whereby

$$\frac{d}{dx}\big(\ln(x^r) - r\ln(x)\big) = 0,$$

so $\ln(x^r) - r\ln(x) = C$ for every x. Plugging in $x = 1$ gives $C = 0$, whence

$$\ln(x^r) = r\ln(x). \qquad \square$$

5.4 The Exponential Function

Since $\ln\colon \mathbb{R}_{>0} \to \mathbb{R}$ is injective and clearly[11] surjective, it is bijective and therefore has an inverse function. We call this inverse function the ***exponential function***, $\exp\colon \mathbb{R} \to \mathbb{R}_{>0}$,

$$y = \exp(x) \qquad \Longleftrightarrow \qquad x = \ln(y),$$

for $y > 0$. We already know a bit about this function. Since $\ln(1) = 0$, we also have $\exp(0) = 1$. Since it's an inverse, we have

$$\ln(\exp(x)) = x$$

for all $x \in \mathbb{R}$ and

$$\exp(\ln(x)) = x$$

for all $x > 0$.

Due to the last theorem we can also quickly deduce a few more properties of this function:

Theorem 5.4.1. *If x and y are any real numbers, then*

(i) $(\exp(x))^r = \exp(rx)$*;*

(ii) $\exp(x + y) = \exp(x)\exp(y)$*;*

(iii) $\exp(x - y) = \exp(x)/\exp(y)$*, with the special case* $\exp(-x) = 1/\exp(x)$*.*

Proof. We use that it is the inverse of ln alongside the previous theorem.

(i) If $y = (\exp(x))^r$, then $\ln(y) = \ln(\exp(x)^r) = r\ln(\exp(x)) = rx$, whereby if we take exponents of both sides gives us $y = \exp(rx)$.

(ii) Take $z = \exp(x + y)$, which is equivalent with

$$\ln(z) = x + y = \ln(\exp(x)) + \ln(\exp(y)) = \ln(\exp(x)\exp(y)).$$

Now take exponents of both sides to get $z = \exp(x)\exp(y)$.

(iii) Same idea: $y = \exp(-x)$ is equivalent with

$$\ln(y) = -x = -\ln(\exp(x)) = \ln\Big(\frac{1}{\exp(x)}\Big).$$

Again the more general result follows from this together with *(ii)*. $\square$

[11] Hopefully. Think about it!

Lecture 6 The Number e and L'Hôpital's Rules[12]

6.1 The Number e

We now *define* the number $e = \exp(1)$, in other words $\ln(e) = 1$, making e the *unique* number that makes the area under $1/t$ precisely 1.

Now comes a *crucial* point: $\exp(x)$ is defined for *all* $x \in \mathbb{R}$ (since it is the inverse of ln), so we can extend

$$\exp(r) = \Big(\exp(1 \cdot r) = \exp(1)^r\Big) = e^r$$

to be defined not only for $r \in \mathbb{Q}$, but for all r! Note that this is not an equality we prove, but a definition we take. We therefore *define*

$$e^x = \exp(x)$$

for all $x \in \mathbb{R}$.

This is a very special function, for it is the only (nontrivial) function which is its own derivative everywhere.

Theorem 6.1.1. $\dfrac{d}{dx} e^x = e^x$.

Proof. By defintion $y = e^x$ is equivalent with $x = \ln(y)$. If we differentiate this implicitly we get

$$1 = \frac{1}{y} \cdot \frac{dy}{dx},$$

which yields

$$y = \frac{dy}{dx} = e^x. \qquad \square$$

The fact that e^x is now defined for all x helps us to define arbitrary exponentials a^x, for $a > 0$. For rational r we have

$$\ln(a^r) = r \ln(a),$$

which is equivalent with

$$a^r = e^{r \ln(a)}.$$

Notice how the right-hand side is defined for all r, since e^x is, so we take this as the definition of a^r for all $r \in \mathbb{R}$, with no contradiction arising if $r \in \mathbb{Q}$.

Definition 6.1.2. Let $a > 0$ and $x \in \mathbb{R}$. Then

$$a^x = e^{x \ln(a)}$$

by definition.

Using the chain rule we can now compute

$$\frac{d}{dx}\, a^x = \frac{d}{dx}\, e^{x \ln(a)} = e^{x \ln(a)} \cdot \ln(a) = a^x \ln(a).$$

Moreover we are finally able to prove the general power rule for derivatives:

[12] Date: February 6, 2017.

Proposition 6.1.3. *Provided $x > 0$ and $\alpha \in \mathbb{R}$, we have*

$$\frac{d}{dx}\, x^{\alpha} = \alpha x^{\alpha-1}.$$

Proof. Straight forward computation by writing $x^{\alpha} = e^{\alpha \ln(x)}$:

$$\frac{d}{dx}\, x^{\alpha} = \frac{d}{dx}\, e^{\alpha \ln(x)} = e^{\alpha \ln(x)} \cdot \frac{\alpha}{x} = x^{\alpha} \cdot \alpha \cdot x^{-1} = \alpha x^{\alpha-1}. \qquad \square$$

We can go further. Our exponential e^x is continuous everywhere (since it has a derivative everywhere), whereby we can finally prove for all $\alpha \in \mathbb{R}$ that

$$\lim_{x\to c} \big(f(x)\big)^{\alpha} = \Big(\lim_{x\to c} f(x)\Big)^{\alpha},$$

provided $\lim\limits_{x\to c} f(x) > 0$. Namely since now by definition $(f(x))^{\alpha} = e^{\alpha \ln(f(x))}$, so

$$\lim_{x\to c} \big(f(x)\big)^{\alpha} = e^{\lim_{x\to c}(\alpha \ln(f(x)))} = e^{\alpha \ln\left(\lim_{x\to c} f(x)\right)} = \Big(\lim_{x\to c} f(x)\Big)^{\alpha}.$$

Example 6.1.4. Compute $f'(x)$ when $f(x) = x^x$.

Neither $(a^x)' = a^x \ln(a)$ nor $(x^a)' = ax^{a-1}$ apply (why?).

On the other hand,

$$\frac{d}{dx}\, x^x = \frac{d}{dx}\, e^{x \ln(x)} = e^{x\ln(x)} \cdot \Big(x \cdot \frac{1}{x} + \ln(x)\Big) = x^x(1 + \ln(x)). \qquad \blacktriangle$$

6.2 How Fast is Exponential and Logarithmic Growth?

Both exponential and logarithmic growth approach infinity, but we ask ourselves how quickly. In fact, e^x grows faster than any positive power, and $\ln(x)$ increases more slowly than any positive power. To show this and to compute some useful limits, we first show that $\ln(x) \le x - 1$ for all $x > 0$:

Let $g(x) = \ln(x) - (x-1)$ for $x > 0$. We have $g(1) = 0$, and $g'(x) = 1/x - 1$, which is positive for $0 < x < 1$ and negative for $x > 1$.

Therefore $g(x)$ is increasing on $]0, 1[$ and decreasing on $]1, \infty[$, whence $g(x) \le g(1) = 0$ for all $x > 0$, meaning that $\ln(x) - (x-1) \le 0$, which by adding $x - 1$ to both sides yields $\ln(x) \le x - 1$.

This method of studying the derivative of the difference of two functions is a very effective method to show that one function is greater than or less than another function.

Theorem 6.2.1. *If $a > 0$, then*

(i) $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{x^a} = 0,$

(ii) $\displaystyle\lim_{x\to 0^+} x^a \ln(x) = 0,$

(iii) $\displaystyle\lim_{x\to\infty} \frac{x^a}{e^x} = 0,$

(iv) $\displaystyle\lim_{x\to -\infty} |x|^a e^x = 0.$

Proof. (*i*) Let $x > 1$, $a > 0$, and $s = a/2$. Since $\ln(x^s) = s\ln(x)$, our above result $\ln(x) \le x - 1$ gives us $\ln(x^s) = s\ln(x) \le x^s - 1 < x^s$, which dividing by s produces

$$0 < \ln(x) < \frac{x^s}{s}.$$

By dividing again by $x^a = x^{2s}$ we arrive at

$$0 < \frac{\ln(x)}{x^a} < \frac{x^s}{sx^{2s}} = \frac{1}{sx^s}.$$

Now

$$\lim_{x\to\infty} \frac{1}{s}\frac{1}{x^s} = 0$$

since $s > 0$, so by the Squeeze theorem

$$\lim_{x\to\infty} \frac{\ln(x)}{x^a} = 0$$

as well.

(*ii*) We use (*i*) and substitute $x = 1/t$. Then as $x \to 0^+$ we have $t \to \infty$, whereby

$$\lim_{x\to 0^+} x^a \ln(x) = \lim_{t\to\infty} \frac{\ln(1/t)}{t^a} = -\lim_{t\to\infty} \frac{\ln(t)}{t^a} = -0 = 0.$$

(*iii*) Again in (*i*), substitute $x = \ln(t)$, so that $t \to \infty$ corresponds to $x \to \infty$,

$$\lim_{x\to\infty} \frac{x^a}{e^x} = \lim_{t\to\infty} \frac{(\ln(t))^a}{t} = \lim_{t\to\infty} \left(\frac{\ln(t)}{t^{1/a}}\right)^a = 0^a = 0.$$

(*iv*) Finally in (*iii*) we substitute $x = -t$:

$$\lim_{x\to-\infty} |x|^a e^x = \lim_{t\to\infty} |-t|^a e^{-t} = \lim_{t\to\infty} \frac{t^a}{e^t} = 0. \qquad \square$$

Another interesting limit, and one which is sometimes taken to be the definition of e, is

Theorem 6.2.2. *For every real x,*

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Proof. For $x = 0$ it is trivially true.

For $x \ne 0$, take $y = x/n$, with x being fixed. Clearly y tends to 0 as n tends to infinity, whereby

$$\begin{aligned}\lim_{n\to\infty} \ln\left(\left(1 + \frac{x}{n}\right)^n\right) &= \lim_{n\to\infty} n\ln\left(1 + \frac{x}{n}\right) = \lim_{n\to\infty} \frac{\frac{x}{n} n \ln\left(1 + \frac{x}{n}\right)}{x/n} \\ &= x\lim_{y\to 0} \frac{\ln(1+y)}{y} = x\lim_{y\to 0} \frac{\ln(1+y) - \ln(1)}{y},\end{aligned}$$

which is of course $xf'(1)$ where $f(x) = \ln(x)$, so the limit is $x \cdot 1/1 = x$. Since $\ln(x)$ is continuous, we can take limits in and out of it, so the above becomes

$$\lim_{n\to\infty} \ln\left(\left(1 + \frac{x}{n}\right)^n\right) = \ln\left(\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n\right) = x.$$

Taking exponentials we arrive at

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n. \qquad \square$$

6.3 Intermediate Forms and L'Hôpital's Rules

Recall how some time ago we computed

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1,$$

despite $\sin(x) \to 0$ and $x \to 0$ as $x \to 0$. We call $\sin(x)/x$ an ***intermediate form*** of the type "0/0" or [0/0], depending on the textbook.

The reason we call this an intermediate form is that a limit approaching 0/0 can be equal to anything, or indeed not exist at all.

Examples 6.3.1. Consider the following limits:

$$\lim_{x\to 0} \frac{kx}{x} = k, \quad \lim_{x\to 0} \frac{x}{x^3} = \infty, \quad \text{and} \quad \lim_{x\to 0} \frac{x^3}{x^2} = 0. \qquad \blacktriangle$$

There are other types of intermediate forms:

$$\text{“}\infty/\infty\text{”}, \quad \text{“}0\cdot\infty\text{”}, \quad \text{“}\infty-\infty\text{”}, \quad \text{“}0^0\text{”}, \quad \text{“}\infty^0\text{”}, \quad \text{and} \quad \text{“}1^\infty\text{”}.$$

We saw a few of these, in particular the first two, in the logarithm and exponential limits.

Encountering "0/0" is perhaps the most common, and they can often be resolved by simplifying or using the Squeeze theorem.

Another method for computing limits of the form "0/0" or "∞/∞" are the two l'Hôpital's rules. Most other forms can be made into these two forms by some algebraic manipulation of change of variable.

Theorem 6.3.2 (L'Hôpital's first rule)**.** *Suppose f and g are differentiable on $]a, b[$, and*

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0.$$

If $g'(x) \neq 0$ for all $x \in \,]a, b[$, and the limit

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$$

exists, then $g(x) \neq 0$ for all $x \in \,]a, b[$ and

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

Remark 6.3.3. The same holds for left limits, two-sided limits, and limits at $\pm\infty$, assuming f and g are differentiable on the relevant punctured neighbourhoods.

Proof. This proof is based on defining continuous extensions of f and g, namely $F(x) = f(x)$ and $G(x) = g(x)$ for $a < x < b$ and $F(a) = G(a) = 0$. Then F and G are continuous on $[a, x]$ for all $a < x < b$. By the Mean-value theorem,

$$g(x) = G(x) - G(a) = g'(c)(x - a)$$

for some $c \in \,]a, x[$, so since $g'(c) \neq 0$ by assumption, we must have $g(x) \neq 0$ for all $a < x < b$, since the right-hand side is never 0.

By the Generalised mean-value theorem, for each $x \in]a,b[$, there exists a $c \in]a,x[$ (depending on x) such that

$$\frac{f(x)}{g(x)} = \frac{F(x)}{G(x)} = \frac{F(x)-F(a)}{G(x)-G(a)} = \frac{f'(c)}{g'(c)}.$$

Now since $a < c < x$, c will approach a from above as x does. Therefore

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{c\to a^+}\frac{f'(c)}{g'(c)} = L. \qquad \square$$

The proof for the left-hand limits is almost identical, and the two-sided result is just the combination of the two.

For limits at $\pm\infty$, we take $y = 1/x$ and study the functions $F(y) = f(1/y)$ and $G(y) = g(1/y)$. Then

$$F'(y) = -\frac{f'(1/y)}{y^2} \qquad \text{and} \qquad G'(y) = -\frac{g'(1/y)}{y^2},$$

so

$$\frac{F'(y)}{G'(y)} = \frac{f'(1/y)}{g'(1/y)},$$

whereby

$$\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = \lim_{y\to 0^\pm}\frac{F(y)}{G(y)} = \lim_{y\to 0^\pm}\frac{F'(y)}{G'(y)} = \lim_{x\to\pm\infty}\frac{f'(x)}{g'(x)}.$$

The second rule of Hôpital concerns the "∞/∞" case:

Theorem 6.3.4 (L'Hôpital's second rule)**.** *L'Hôpital's rule also holds if we have* $\lim|f(x)| = \lim|g(x)| = \infty$.

Proof. We'll consider limits as $x \to a^-$. The other cases follow in the same was as described above.

Let $\varepsilon > 0$. We want to show that

$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon$$

if $a - x < \delta$ for some $\delta > 0$. Since

$$\frac{f'(x)}{g'(x)} \to L$$

and $|g(x)| \to \infty$ as $x \to a^-$, we can pick an $x_0 < a$ in such a way that $|f'(x)/g'(x) - L| < \varepsilon/2$ and $g(x) \neq 0$ for all $x \in]x_0, a[$. The Generalised mean-value theorem gives

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(c)}{g'(c)}$$

for some $x_0 < c < x$, meaning that

$$\left|\frac{f(x)-f(x_0)}{g(x)-g(x_0)} - L\right| < \frac{\varepsilon}{2} \tag{6.3.1}$$

for $x_0 < x < a$ since $x_0 < c < a$.

By some algebraic manipulation we have

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x)}}{1 - \frac{g(x_0)}{g(x)}},$$

which if solved for $f(x)/g(x)$ becomes

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} - \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \cdot \frac{g(x_0)}{g(x)} + \frac{f(x_0)}{g(x)}.$$

As x tends to a from below, $|g(x)| \to \infty$ by assumption, so $f(x_0)/g(x)$ and $g(x_0)/g(x)$ tend to 0, so the last two terms tend to 0 since the difference quotient is bounded by Equation (6.3.1).

Thus for x sufficiently close to a from below,

$$\Big|\frac{f(x)}{g(x)} - \frac{f(x) - f(x_0)}{g(x) - g(x_0)}\Big| < \frac{\varepsilon}{2}.$$

Adding this and Equation (6.3.1) together yields

$$\Big|\frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L\Big| + \Big|\frac{f(x)}{g(x)} - \frac{f(x) - f(x_0)}{g(x) - g(x_0)}\Big| < \varepsilon,$$

so by using the Triangle inequality backwards we have

$$\Big|\frac{f(x)}{g(x)} - L\Big| < \varepsilon. \qquad \square$$

Lecture 7 Sketching Functions[13]

7.1 Extreme Values

As we saw when discussing the Mean-value theorem, the sign of the derivative tells us whether the function is increasing or decreasing. We will now explore how to use this information to find maximum and minimum values, collectively known as ***extreme values***.

Recall how a function has an ***absolute*** or ***global maximum value*** $f(x_0)$ at $x = x_0$ if $f(x) \leq f(x_0)$ for all x in the domain of f.

(Similarly it has an ***absolute*** or ***global minimum value*** if $f(x) \geq f(x_0)$.)

Note that a function can have at most one global maximum value, and at most one global minimum value, but they can attain these values for more than one x-value.

Example 7.1.1. The function $f(x) = \sin(x)$ attains its global maximum value 1 at $x = \pi/2 + 2\pi n$ and its global minimum value -1 at $x = -\pi/2 + 2\pi n$, for $n \in \mathbb{Z}$. ▲

Functions don't always have global extreme values.

Example 7.1.2. The function plotted in Figure 7.1.1 has no global maximum value, since it goes off to infinity close to the dashed line, however it does have a ***local maximum value*** in the right part. ▲

Definition 7.1.3 (Local extreme values)**.** A function f has a ***local maximum value*** $f(x_0)$ at x_0 in its domain if there exists a number $h > 0$ such that $f(x) \leq f(x_0)$ whenever x is in the domain of f and $|x - x_0| < h$.

Similarly for ***local minimum value***, with $f(x) \geq f(x_0)$.

A function f can have local extreme values only at points of three special types:

(i) ***critical points*** of f, i.e. points of the domain where $f'(x) = 0$;

(ii) ***singular points*** of f, i.e. points of the domain where $f'(x)$ is undefined; and

(iii) ***endpoints*** of the domain of f, i.e. points of the domain that do not belong to any open interval contained in the domain of f.

The first item about critical points follows from Lemma 4.3.3 related to the Mean-value theorem, and the others are fairly obvious.

A point being a local extreme value implies (i), (ii), or (iii), but the other way around need not be true; critical points, singular points and endpoints need not be extreme points.

Counterexample 7.1.4. Consider the function $f(x) = x^3$. We have $f'(x) = 3x^2$, whence $f'(0) = 0$, so by definition $x = 0$ is a critical point, yet $f(0) = 0$ is no local extreme value. ▲

We can decide whether the three types of points are local minimum or maximum values by studying the derivative close to the points.

Theorem 7.1.5 (First derivative test)**.** *Let $f : X \to Y$ be a function differentiable on the relevant intervals below. The theorem has two parts:*

[13] Date: February 9, 2017.

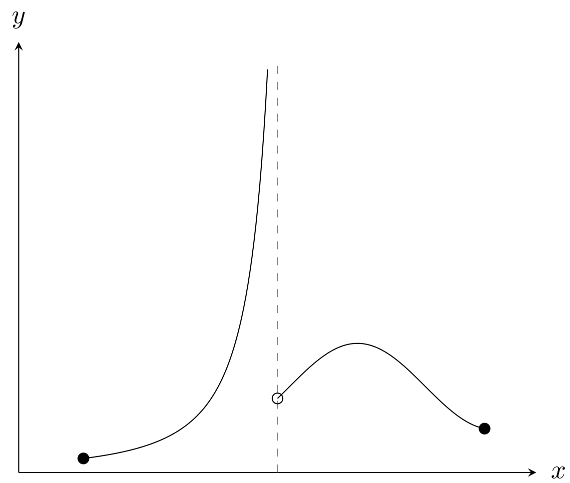


Figure 7.1.1: A function with no global maximum.

Testing critical and singular points

Suppose f is continuous at x_0 and that x_0 isn't an endpoint of X.

(*i*) *If there exists an open interval $]a,b[\ \ni x_0$ such that $f'(x) > 0$ on $]a,x_0[$ and $f'(x) < 0$ on $]x_0,b[$, then f has a local maximum value at x_0.*

(*ii*) *Similarly f has a local minimum value at x_0 if $f'(x) < 0$ on $]a,x_0[$ and $f'(x) > 0$ on $]x_0,b[$.*

Testing endpoints of X

Suppose a is a left endpoint of X and that f is right continuous at a.

(*iii*) *If $f'(x) > 0$ on some interval $]a,b[$, then f has a local minimum at a.*

(*iv*) *If $f'(x) < 0$ on some $]a,b[$, then f has a local maximum at a.*

Suppose b is a right endpoint of X and that f is left continuous at b.

(*v*) *If $f'(x) > 0$ on some $]a,b[$, then f has a local maximum at b.*

(*vi*) *If $f'(x) < 0$ on some $]a,b[$, then f has a local minimum at b.*

Exercise 7.1.6. To prove this, recall Definition 4.3.6.

Note that if $f'(x)$ has the same sign on both sides of a critical or singular point, then f has neither maximum nor minimum at that point (like the x^3 example).

Example 7.1.7. Consider
$$f(x) = \frac{1}{3}x^3 - x$$
defined on $[-3,3]$. Potential extreme values are at the endpoints $x = -3$ and $x = 3$, and critical points (it has no singular points).

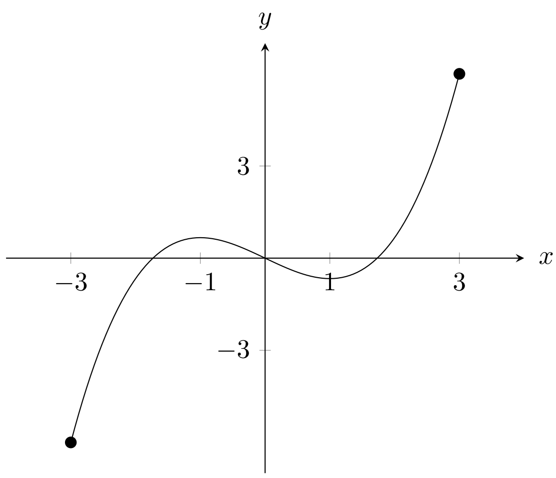
We have
$$f'(x) = \frac{1}{3} \cdot 3 \cdot x^2 - 1 = x^2 - 1 = (x-1)(x+1),$$
so its critical points are $x = 1$ and $x = -1$.

We analyse the sign of the derivative around these points:

x	-3	$-3 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	min	$\nearrow$	max	$\searrow$	min	$\nearrow$	max

We (naturally) see the same behaviour by sketching the function:

▲



7.2 Concavity

We can also use the second derivative to find out useful things about a function.

In particular, it tells us if the slope (i.e. derivative) is an increasing or decreasing function. We have fancy names for these behaviours.

Definition 7.2.1 (Concavity)**.** We say that f is ***concave up*** on an open interval I if it is differentiable there and the derivative f' is an increasing function on I. We say that f is ***concave down*** on I if f' exists and is decreasing on I.

Example 7.2.2. The function $f(x) = x^2$ is concave up on the entirety of $\mathbb{R}$. ▲

Example 7.2.3. The third degree polynomial from two examples ago is concave down on $x > 0$ and concave up on $x < 0$. ▲

This exemplifies how functions sometimes change concavity on either side of a point:

Definition 7.2.4 (Inflection point)**.** We say that f has an ***inflection point*** at x_0 if f has a (possibly vertical) tangent line there and the concavity is opposite on opposing sides of x_0.

We've already seen one example, namely $x = 0$ for $f(x) = x^3/3 - x$.

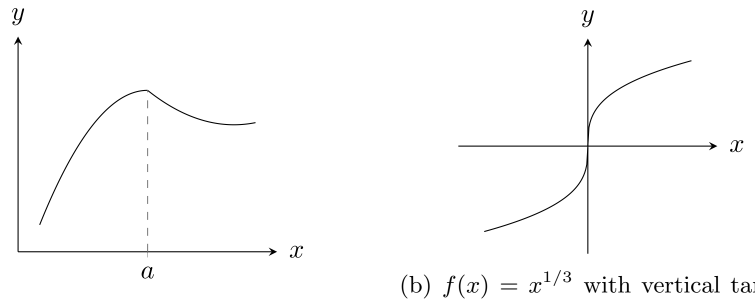
We demonstrate two more interesting examples that exemplify the requirement about the tangent line.

Examples 7.2.5. In Figure 7.2.1a we see a function which changes concavity around $x = a$, however it is by definition *not* an inflection point since there is no tangent line at this point.

Moreover in Figure 7.2.1b we demonstrate an inflection point at a point where the function at hand, $f(x) = x^{1/3}$, is not differentiable, but it has a tangent line. In this sense the condition on the existence of a tangent line is stronger than the existence of the derivative. ▲

Since concavity depends on the derivative being increasing or decreasing, we can test it using the derivative of the derivative.

Theorem 7.2.6. *Let f be a twice differentiable function on some interval I.*



(a) No tangent line at $x = a$

(b) $f(x) = x^{1/3}$ with vertical tangent line at $x = 0$

Figure 7.2.1: Counterexamples of the Mean-value theorem where essential conditions are violated.

(*i*) *If* $f''(x) > 0$ *on the interval, then* f *is concave up on* I*.*

(*ii*) *If* $f''(x) < 0$ *on the interval, then* f *is concave down on* I*.*

(*iii*) *If* f *has an inflection point at* x_0 *and* $f''(x_0)$ *exists, then* $f''(x_0) = 0$*.*

Proof. As suggested above, (i) and (ii) are clear by Theorem 4.3.8, wherein the sign of the derivative implies that a function is increasing or decreasing.

For (iii), if f has an inflection point at x_0 and $f''(x_0)$ exists, then f must be differentiable on an open interval containing x_0. Since f' is increasing on one side of x_0 and decreasing on the other, it must have a local extreme value at x_0, and therefore $f''(x_0) = 0$. □

Combining this with what we know about critical points we get a very useful result.

Theorem 7.2.7 (Second derivative test)**.** *Let* f *be a twice differentiable function.*

(*i*) *If* $f'(x_0) = 0$ *and* $f''(x_0) < 0$*, then* f *has a local maximum value at* x_0*.*

(*ii*) *If* $f'(x_0) = 0$ *and* $f''(x_0) > 0$*, then* f *has a local minimum value at* x_0*.*

Proof. Suppose $f'(x_0) = 0$ and $f''(x_0) < 0$. Since

$$\lim_{h\to 0} \frac{f'(x_0 + h)}{h} = \lim_{h\to 0} \frac{f'(x_0 + h) - f'(x_0)}{h} = f''(x_0) < 0,$$

it follows that $f'(x_0 + h) > 0$ for all sufficiently small negative h. Thus by the first derivative test, f must have a local maximum value at x_0.

The case of the local minimum is similar. □

Remark 7.2.8. If $f'(x_0) = f''(x_0) = 0$, no conclusion can be drawn from this information alone.

Example 7.2.9. Consider $f(x) = x^4$. Here $f'(0) = f''(0) = 0$, and $f(0) = 0$ is a local minimum value, whereas for $g(x) = x^3$ we also have $g'(0) = g''(0) = 0$, yet $x = 0$ is an inflection point. ▲

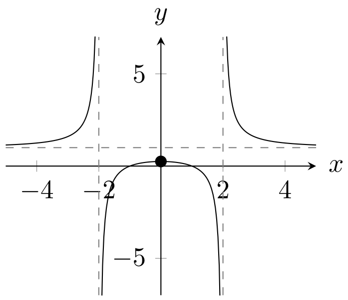


Figure 7.3.1: Plot of $f(x) = \dfrac{x^2-1}{x^2-4}$.

7.3 Sketching Functions

Combining all of this, and maybe some other things, we can quite accurately sketch graphs of functions.

Example 7.3.1. Sketch

$$f(x) = \frac{x^2-1}{x^2-4} = \frac{(x-1)(x+1)}{(x-2)(x+2)}.$$

We take stock of what we know.

The domain is all $x \neq \pm 2$. There are vertical asymptotes at $x = \pm 2$. Because

$$\lim_{x\to\pm\infty} \frac{x^2-1}{x^2-4} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} \cdot \frac{1-1/x^2}{1-4/x^2} = 1,$$

$y = 1$ is a horizontal asymptote.

We also observe that f is even, since $f(-x) = f(x)$.

It crosses the x-axis at $x = \pm 1$, and it intersects the y-axis at $f(0) = 1/4$.

Finally we compute the first and second derivatives:

$$f'(x) = \frac{2x \cdot (x^2-4) - 2x \cdot (x^2-1)}{(x^2-4)^2} = \frac{-6x}{(x^2-4)^2},$$

and

$$f''(x) = \frac{-6 \cdot (x^2-4)^2 + 6x \cdot 2 \cdot (x^2-4) \cdot 2x}{(x^2-4)^4} = \frac{18x^2+24}{(x^2-4)^3}.$$

Using this we sketch the function in Figure 7.3.1. Note how we know $x = 0$ is a local maximum point since $f'(0) = 0$ and $f''(0) < 0$. ▲

Let's now tackle a rather more difficult function to sketch.

Example 7.3.2. Sketch the graph of $y = xe^{-x^2/2}$.

We compute the first two derivatives:

$$y' = xe^{-x^2/2} \cdot (-x) + e^{-x^2/2} = (1-x^2)e^{-x^2/2},$$

and

$$\begin{aligned} y'' &= (1-x^2)e^{-x^2/2} \cdot (-x) + e^{-x^2/2} \cdot (-2x) \\ &= -x(1-x^2+2)e^{-x^2/2} = x(x^2-3)e^{-x^2/2}. \end{aligned}$$

Thus from y itself we know that the domain is all of $\mathbb{R}$. We have a horizontal asymptote at $y = 0$ since

$$\lim_{x\to\pm\infty} xe^{-x^2/2} = \lim_{t\to\infty} \sqrt{2t}e^{-t} = 0,$$

wherein we did the change of variable $t = x^2/2$.

Moreover y is odd, since $y(-x) = -y(x)$, so it is symmetric about the origin. It also goes through the origin since $y(0) = 0$.

From y' we know that there are critical points at $x = \pm 1$.

Finally from y'' we observe that $y''(0) = y''(\sqrt{3}) = y''(-\sqrt{3}) = 0$.

Let us study the signs of the derivatives around these points:

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	$-$		$-$	0	$+$		$+$	0	$-$		$-$
y''	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
y	$\searrow$		$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$		$\searrow$
Concavity	$\frown$	Inflection	$\smile$		$\smile$	Inflection	$\frown$		$\frown$	Inflection	$\smile$

With this at our disposal we are able to sketch the function in Figure. ▲

Lecture 8 Overdue Proofs and Antiderivatives[14]

8.1 Proofs of Max-min and Intermediate Value Theorem

A long time ago (all the way back in Lecture 2) we stated the Max-min theorem and the Intermediate value theorem, but we did not prove them. We will now, though note that strictly speaking this is not a part of this course.

We will need to remember the definitions of limit, continuity at a point, and what we meant by a function being continuous on an interval.

As suggested back then, part of the mystery behind the proofs has to do with upper bounds of sets of real numbers.

Definition 8.1.1. A number u is said to be an ***upper bound*** for a nonempty set S if $x \le u$ for all $x \in S$.

The number u^* is called the ***least upper bound*** or ***supremum*** of S if u^* is an upper bound for S and $u^* \le u$ for every upper bound u of S. It is denoted $u^* = \sup(S)$.

[14]Date: February 13, 2017.

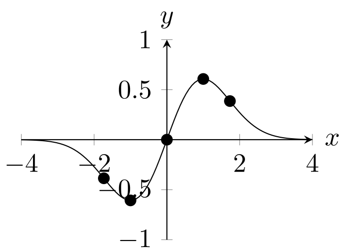


Figure 7.3.2: Plot of $y = xe^{-x^2/2}$.

Similarly ℓ is a ***lower bound*** of S if $\ell \leq x$ for all $x \in S$ and $\ell^* = \inf(S)$ is the ***greatest lower bound*** or ***infimum*** of S is $\ell \leq \ell^*$ for all lower bounds ℓ of S.

Example 8.1.2. Let $S_1 = [2, 3]$. Any $u \geq 3$ is an upper bound for S_1, with $\sup(S_1) = 3$.

On the other hand, $S_2 =]2, \infty[$ has no upper bound, but many lower bounds ($\ell \leq 2$), and $\inf(S_2) = 2$. ▲

An incredibly important property of the real numbers is called ***completeness***. Roughly speaking this means that there are no gaps on the real number line (unlike e.g. $\mathbb{Q}$). In the language of of sup and inf:

1. A nonempty set of real numbers that has an upper bound must have a least upper bound that is a real number.

2. A nonempty set of real numbers that has a lower bound must have a greatest lower bound that is a real number.

It is the last part that is the crucial difference between, for instance, $\mathbb{Q}$ and $\mathbb{R}$. Consider for example the set of all rational numbers whose square is less than or equal to 2, that is $\{\, x \in \mathbb{Q} \mid x^2 \leq 2 \,\}$. This set certainly has an upper bound (take any $u \geq \sqrt{2}$, but it has no *least* upper bound *within* $\mathbb{Q}$ itself, since $\sqrt{2}$ isn't a rational number!

This property, called the ***supremum property***, of the real numbers is (to us in this course) an ***axiom***. That is to say, it cannot be proven, instead it is something we *decide* to be true about the reals. (If we have a way of actually construction, of building the real numbers, then it becomes a theorem that one proves.)

This allows us to prove several interesting results.

Theorem 8.1.3. *If $x_1, x_2, \dots, x_n, x_{n+1}, \dots$ is an increasing sequence that is bounded above;*

$$x_{n+1} \geq x_n \qquad \text{and} \qquad x_n \leq K$$

for some K and all $n = 1, 2, 3, \dots$, then

$$\lim_{n\to\infty} x_n = L$$

exists.

The same theorem is true for decreasing sequences bounded below.

Proof. The set $S = \{\, x_n \,|\, n = 1, 2, 3, \dots \}$ has an upper bound K by assumption, therefore by the Supremum property there exists a least upper bound $L = \sup(S)$.

For every $\varepsilon > 0$, there exists some N such that $x_N > L - \varepsilon$, because otherwise $L - \varepsilon$ would be an upper bound for S that is smaller than the least upper bound L.

If $n \geq N$, then $L - \varepsilon < x_N \leq x_n \leq L$, whereby $|x_n - L| < \varepsilon$, so

$$\lim_{n\to\infty} x_n = L. \qquad \square$$

We will now prove some theorems we already established for functions, but this time for sequences.

Theorem 8.1.4. *If $a \leq x_n \leq b$ for all $n = 1, 2, 3, \ldots$, and $\lim_{n\to\infty} x_L = L$, then $a \leq L \leq b$.*

Proof. Suppose $L > b$ and let $\varepsilon = L - b$. Since

$$\lim_{n\to\infty} x_n = L$$

there exists an n such that $|x_n - L| < \varepsilon$. Therefore $x_n > L - \varepsilon = L - (L - b) = b$, so $x_n > b$, a contradiction. Therefore $L \leq b$.

Similarly for $L \geq a$. □

Theorem 8.1.5. *If f is continuous on $[a, b]$, if $a \leq x_n \leq b$ for all $n = 1, 2, 3, \ldots$, and if $\lim_{n\to\infty} x_n = L$, then*

$$\lim_{n\to\infty} f(x_n) = f(L).$$

Proof. Similar to the limits of compositions of continuous functions. □

These are the tools we require to prove the main result behind the two theorems we are after.

Theorem 8.1.6. *If f is continuous on $[a, b]$, then f is bounded there (i.e there exists some K such that $|f(x)| \leq K$ for all $x \in [a, b]$).*

Proof. We show boundedness from above.

For each positive integer n, let

$$S_n = \{\, x \in [a, b] \mid f(x) > n \,\}.$$

Note that this is the set of x values that makes this happen.

We would like to show that S_n is empty for some n. It would follow that $f(x) \leq n$ for this n, so n would be an upper bound.

Suppose S_n is nonempty for all n. For each $n = 1, 2, 3, \ldots$, since S_n is bounded below (a is a lower bound by construction), by the Supremum axiom there exists a greatest lower bound, call it x_n.

Since $f(x) > n$ at some point in $[a, b]$, and continuous at that point, $f(x) > n$ on some interval contained in $[a, b]$. Thus $x_n < b$ and $f(x_n) \geq n$ (otherwise by continuity $f(x) < n$ for some distance to the right of x_n and x_n could not be $\inf(S_n)$).

For each n, $S_{n+1} \subseteq S_n$ implies that $x_{n+1} \geq x_n$, so $x_1, x_2, \ldots, x_n, \ldots$ is an increasing sequence bounded above by b, meaning that it converges to, say,

$$\lim_{n\to\infty} x_n = L.$$

By Theorem 8.1.4 we have $a \leq L \leq b$, and f is continuous at L so

$$\lim_{n\to\infty} f(x_n) = f(L)$$

exists by Theorem 8.1.5. But since $f(x_n) \geq n$ for every n,

$$\lim_{n\to\infty} f(x_n)$$

cannot exist, since it grows arbitrarily large, so we have a contradiction. Therefore our assumption of S_n always being nonempty is wrong, ergo there exists some n such that S_n is empty, whereby f is bounded above on $[a, b]$.

For boundedness below do almost the same thing. □

We are now able to prove the Max-min theorem and the Intermediate value theorem.

Theorem 8.1.7 (Max-min theorem)**.** *If f is continuous on $[a, b]$, then there exists some $u, v \in [a, b]$ such that $f(v) \leq f(x) \leq f(u)$ for all $x \in [a, b]$.*

Proof. By the previous theorem, $S = \{\, f(x) \,|\, x \in [a, b] \,\}$ has an upper bound and by the Supremum axiom it also has a least upper bound, call it $M = \sup(S)$. Suppose there is no $u \in [a, b]$ such that $f(u) = M$. Then $1/(M - f(x))$ is continuous on $[a, b]$ so again by the last theorem there exists some K such that $1/(M - f(x)) \leq K$ for all $x \in [a, b]$.

By rearranging, this implies that

$$f(x) \leq M - \frac{1}{K},$$

which contradicts M being the *least* upper bound for S, therefore there must exist some $u \in [a, b]$ such that $f(x) \leq f(u) = M$.

The existence of v is similar. □

Theorem 8.1.8 (Intermediate value theorem)**.** *If f is continuous on $[a, b]$ and s is a number between $f(a)$ and $f(b)$, then there exists some $c \in [a, b]$ such that $f(c) = s$.*

Proof. Assume $f(a) < s < f(b)$ (the other case, where $f(b) < s < f(a)$ is treated in almost the same way) and let $S = \{\, x \in [a, b] \mid f(x) \leq s \,\}$.

The set S is nonempty (at the very least it contains a) and bounded above (b is an upper bound), so by the Supremum axiom there exists a least upper bound $\sup(S) = c$.

Now suppose $f(c) > s$. Then $c \neq a$ and by continuity $f(x) > s$ on some interval $]c - \delta, c]$, with $\delta > 0$. But this says that $c - \delta < c$ is an upper bound for S, so c wasn't the *least* upper bound, which is a contradiction, meaning that instead we must have $f(c) \leq s$.

We then perform the same argument to achieve $f(c) \geq s$, meaning that $f(c) = s$. □

8.2 Antiderivatives

So far we have studied the problem of finding the derivative f' given a function f. The reverse problem, finding f given a derivative f', is also interesting and important!

Definition 8.2.1 (Antiderivative)**.** An ***antiderivative*** of a function f on an interval I is another function F satisfying

$$F'(x) = f(x)$$

for all x in I.

Example 8.2.2. The function $F(x) = x$ is an antiderivative to $f(x) = 1$ on any interval since $F'(x) = 1 = f(x)$ everywhere.

Similarly, $G(x) = -1/x$ is an antiderivative to $g(x) = 1/x^2$ and $H(x) = e^{2x} + 17$ is an one for $h(x) = 2e^{2x}$. ▲

The last example is meant to exemplify that antiderivatives aren't unique; since all constants have derivative 0, we can always add one and get the same derivative.

More importantly, *all* antiderivatives of f on an interval I can be obtained by adding constants to *any particular* antiderivative.

The proof of this is easy; if F and G are two antiderivatives to f on I, then

$$\frac{d}{dx}(F(x) - G(x)) = f(x) - f(x) = 0,$$

whereby $F(x) - G(x) = C$, with C constant.

8.3 Indefinite Integrals

The *general* antiderivative of a function f on an interval I is $F(x) + C$, for all constants C, where F is any particular antiderivative of f on I.

Definition 8.3.1 (Indefinite integral)**.** The ***indefinite integral*** of f on an interval I is

$$\int f(x)\,dx = F(x) + C,$$

for all $C \in \mathbb{R}$, provided $F'(x) = f(x)$ for all $x \in I$.

We already know how to compute many indefinite integrals.

Examples 8.3.2. We have, amongst others,

$$\int dx = \int 1\,dx = x + C \quad \text{and} \quad \int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C,$$

if $\alpha \neq -1$, with

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln(x) + C,$$

if $x > 0$.

We have the special result

$$\int e^x\,dx = e^x + C.$$

We also know about some trigonometric results, e.g.

$$\int \sin(x)\,dx = -\cos(x) + C \quad \text{and} \quad \int (\sec(x))^2\,dx = \tan(x) + C.$$

We can even compute some pretty complicated indefinite integrals if we just take some care. Consider, for instance,

$$\int \frac{\ln(x)}{x}\,dx.$$

If we see this as a product of $1/x$ and $\ln(x)$, we might, if we're trying to guess what this is the derivative of, come to think of the product rule, since

$$\frac{d}{dx}\ln(x)\ln(x) = \frac{2\ln(x)}{x},$$

so the above indefinite integral must be $(\ln(x))^2/2 + C$. ▲

Since the graphs of an indefinite integral for the various C are just vertically displaced, just one of them will pass through a given point.

Thus if we know just *one* point of the antiderivative some problem is looking for, we can find the right one.

Example 8.3.3. Find the function f the derivative of which is $f'(x) = 3x^3 + 2x - 3$ for all real x and for which $f(1) = 0$.

We compute the indefinite integral

$$f(x) = \int 3x^3 + 2x - 3\,dx = \frac{3}{4}x^4 + x^2 - 3x + C,$$

so

$$f(1) = \frac{3}{4} + 1 - 3 + C = \frac{3}{4} - 2 + C = -\frac{5}{4} + C = 0,$$

whereby $C = 5/4$, so $f(x) = 3/4x^4 + x^2 - 3x + 5/4$. ▲

This kind of problem is called an ***initial-value problem***, and rewriting the formulation we get

$$\frac{dy}{dx} = 3x^3 + 2x - 3,$$

and we see that this is a differential equation.

Example 8.3.4. Solve the inital-value problem

$$\begin{cases} y' = \frac{3+2x^2}{x^2} \\ y(-2) = 1. \end{cases}$$

As before we compute the indefinite integral and then insert the known information:

$$y = \int \frac{3}{x^2} + 2\,dx = -\frac{3}{x} + 2x + C,$$

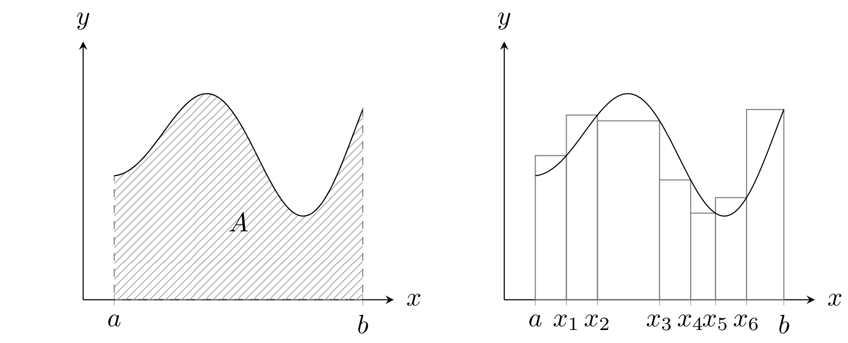
so

$$1 = y(-2) = \frac{3}{2} - 4 + C,$$

whereby $C = 7/2$.

This $y = -3/x + 2x + 7/2$ is a solution to the initial-value problem on $]-\infty, 0[$ (because this is the largest interval which contains the initial point $x = -2$ but not $x = 0$, where y (and y') is undefined). ▲

This, confusingly, is not really where the notion of integration comes from. Instead integration has to do with areas, but just how the indefinite integral has much to do with the derivative, so do these areas!



(a) The precise area under some curve (b) An approximation of the same area

Figure 9.1.1: The area under a curve, both exact and as estimated by a sum.

Lecture 9 The Integral[15]

9.1 Areas Under Curves

We will soon have to deal a lot with sums, usually of very many (maybe infinitely many) terms, and for this we will use sigma notation, which hopefully the student is familiar with.

Suppose we have some function f, and we want to find the area between $x = a$, $x = b$, the x-axis, and $y = f(x)$. Now, if this region is a polygon, it's easy, but it isn't necessarily. See, for instance, Figure 9.1.1a.

It *can* however be estimated using polygons. We do this by ***partitioning*** (dividing) the interval $[a, b]$ into n parts (not necessarily equal in length),

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

Thus the smaller sections are $[x_{i-1}, x_i]$ for all $i = 1, 2, 3, \ldots, n$. We denote by $\Delta x_i = x_i - x_{i-1}$ the length of these subintervals.

The point of this is that we can now approximate the area A we are interested in by taking the sum of the areas of the rectangles we form as in Figure 9.1.1b, namely

$$S_n = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n = \sum_{i=1}^{n} f(x_i)\Delta x_i.$$

Of course S_n is not *quite* equal to A, since it sometimes overestimates a bit and sometimes underestimates a bit, but it seems reasonable that if we make n greater, so that we're splitting the interval up into finer parts, and at the same time make sure the Δx_i are getting smaller, then we'll approach A, i.e.

$$A = \lim_{\substack{n\to\infty \\ \max\{\,\Delta x_i\,\}\to 0}} S_n.$$

Example 9.1.1. Find the area of the region bounded by $y = x^2$, $y = 0$, $x = 0$, and $x = b$, where $b > 0$.

[15]Date: February 16, 2017.

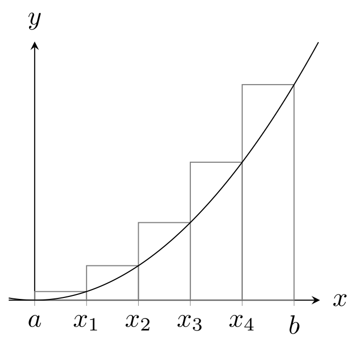


Figure 9.1.2: Estimating the area under $y = x^2$ using rectangles.

We divide the interval $[0, b]$ into n equal parts, as illustrated in Figure 9.1.2, each with width $\Delta x_i = b/n$. The height of the rectangle at $x = x_i$ is then $(ib/n)^2$, whereby[16]

$$S_n = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \frac{b}{n} = \frac{b^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6}.$$

Taking the limit, we find that

$$\lim_{n\to\infty} b^3 \frac{(n+1)(2n+1)}{6n^2} = b^3 \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} = \frac{b^3}{3} = A. \quad \blacktriangle$$

We mentioned earlier that Δx_i needn't all be equal. Indeed, trying to use the same partition in the next problem we'd run into trouble.

Exercise 9.1.2. Let $0 < a < b$, and let $k \neq -1$ be real. Show that the area bounded by $y = x^k$, $y = 0$, $x = a$, and $x = b$ is

$$A = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

Hints: Let $t = (b/a)^{1/n}$, and use the partition $x_0 = a$, $x_1 = at$, $x_2 = at^2$, …, $x_n = at^n = b$.

Also recall what we know about geometric sums, i.e. that

$$\sum_{i=1}^{n} r^i = \frac{r^{n+1} - 1}{r - 1},$$

or maybe more useful,

$$\sum_{i=1}^{n} r^{i-1} = \frac{r^n - 1}{r - 1},$$

for $r \neq 1$.

You'll also at some point change a limit to infinity to a limit at 0^+, and use L'Hôpital's rule.

If you like, try to use the same partition as before as well, to see why this is problematic.

[16] Feel free to verify on your own, perhaps by induction, that the last step is correct.

9.2 Definite Integrals and Riemann Sums

In the following discussion we will assume that f is continuous on the interval $[a, b]$.

Let us go back to partitions, say

$$P = \{ x_0, x_1, x_2, \dots, x_{n-1}, x_n \},$$

such that

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

Now since f is continuous on $[a, b]$, it must also be continuous on the subintervals $[x_{i-1}, x_i]$ of P. Since it is continuous on a closed interval, there must exist $\ell_i, u_i \in [x_{i-1}, x_i]$ such that

$$f(\ell_i) \leq f(x) \leq f(u_i)$$

for all $x_i \leq x \leq x_i$, by the Max-min theorem.

If we perform the same sum machinery as before with these special ℓ_i and u_i, we get areas as small and as large as possible.

Definition 9.2.1 (Riemann sum)**.** The ***lower (Riemann) sum***, $L(f, P)$, and the ***upper (Riemann) sum***, $U(f, P)$, for the function f and the partition P, are

$$L(f, P) = \sum_{i=1}^{n} f(\ell_i)\Delta x_i, \qquad \text{and} \qquad U(f, P) = \sum_{i=1}^{n} f(u_i)\Delta x_i.$$

Remark 9.2.2. If f is negative, we have negative areas in our sum.

Since $L(f, P)$ is summed with the minimum for f in $[x_{i-1}, x_i]$ and $U(f, P)$ its maximum in the same interval, it is clear that

$$L(f, P) \leq U(f, P)$$

for all partitions P. (The only case where this might not be clear is when negative values are involved. . . Think about it! Draw different partitions!)

Definition 9.2.3 (Definite integral)**.** Let f be a function, not necessarily continuous. Suppose there is exactly one number I such that for *every* partition P of $[a, b]$ we have

$$L(f, P) \leq I \leq U(f, P),$$

then we say that f is (Riemann) ***integrable*** on $[a, b]$, and we call I the ***definite integral*** of f on $[a, b]$.

We use the notation

$$I = \int_a^b f(x)\, dx.$$

Here a and b are called ***limits of integration***, f is called the ***integrand***, dx is called the ***differential***, and x is called the ***variable of integration***.

Comparing this with

$$S_n = \sum_{i=1}^{n} f(x_i)\Delta x_i,$$

we can think of the definite integral as the sum of areas of infinitely many rectangles of height $f(x)$ and infinitesimally small widths dx.

We will learn next time how to efficiently calculate these sorts of quantities, but for now we close with an important theorem, the proof of which, like the Max-min theorem and the Intermediate value theorem strictly speaking doesn't belong to the course:

Theorem 9.2.4. *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.*

So far, when studying $\int_a^b f(x)\,dx$, we have required $a < b$. It turns out to be useful to drop this and allow $a = b$ and $b < a$ as well. The extension is pretty straight forward; we still have $a = x_0, x_1, \ldots, x_n = b$, but for $a = b$ they're all the same, so $\Delta x_i = 0$ (making the integral 0), and for $b < a$ we have $\Delta x_i < 0$, so the area switches sign!

We have the following properties:

Theorem 9.2.5. *Let f and g be integrable on an interval containing a, $-a$, b, and c. Then*

(i) $\displaystyle\int_a^a f(x)\,dx = 0$;

(ii) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$;

(iii) *with A and B constants,*

$$\int_a^b Af(x) + Bg(x)\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx;$$

(iv) $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$;

(v) *if $a \le b$, and $f(x) \le g(x)$ for all $a \le x \le b$,*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx;$$

(vi) *(Triangle inequality), if $a \le b$,*

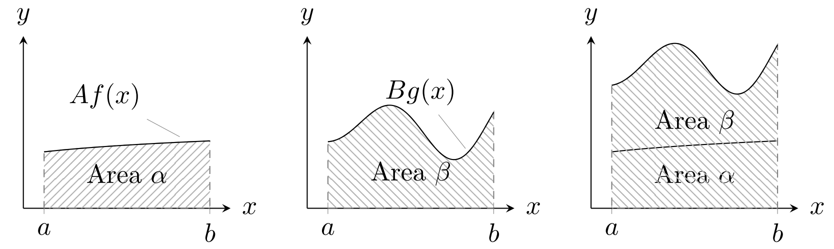
$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx;$$

(vii) *if f is odd (i.e. $f(x) = -f(-x)$),*

$$\int_{-a}^a f(x)\,dx = 0;$$

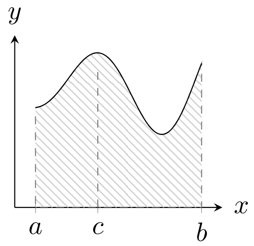
(viii) *if f is even (i.e. $f(x) = f(-x)$),*

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$



(Almost literal) sketch of proof. (i) and (ii) are motivated by the previous discussion.

For (iii), consider the following sketch:
For (iv), at least when $a \leq c \leq b$, consider this picture:



If is isn't true that $a \leq c \leq b$, consider rearranging the same picture, and now some of the integrals will be negative because of the order of the limits of integration.

(v) is probably obvious. If not, consider the Riemann sums; since f is bounded by g, so must the heights in the sums.

(vi) is just (v), since $-|f(x)| \leq f(x) \leq |f(x)|$. The only remaining mystery is to prove that $|f|$ must be integrable on $[a, b]$ when f is (which is in fact true).

For (vii) and ($viii$), draw the appropriately symmetric pictures! $\square$

Using these properties and remembering that fundamentally they represent areas we can sometimes reduce definite integral problems to very, very simple computations.

Examples 9.2.6. Calculate

$$\int_{-2}^{2} 2 + 5x\,dx, \qquad \int_{0}^{3} 2 + x\,dx, \qquad \text{and} \qquad \int_{-3}^{3} \sqrt{9 - x^2}\,dx.$$

We use the linearity of the integral, meaning that

$$\int_{-2}^{2} 2 + 5x\,dx = \int_{-2}^{2} 2\,dx + \int_{-2}^{2} 5x\,dx.$$

The first integral in the right-hand side is just the area of a rectangle of height 2 and width 4, so it is 8. The second integral is the integral of an odd function over a symmetric interval, whereby it is 0, so

$$\int_{-2}^{2} 2 + 5x\,dx = 8 + 0 = 8.$$

For the second one, sketch the graph and we notice that the area we're interested in is just a trapezoid, consisting of one rectangle of height 2 and width 3, on top of which is a triangle of height 3 and base 3, so

$$\int_0^3 2 + x\,dx = 3 \cdot 2 + \frac{1}{2} \cdot 3 \cdot 3 = \frac{21}{2}.$$

For the third and final one we recognise that the integrand $\sqrt{9 - x^2}$ is the expression for the upper half of a circle centred on the origin with radius 3. Therefore

$$\int_{-3}^3 \sqrt{9 - x^2}\,dx = \frac{1}{2} \cdot \pi \cdot 3^2 = \frac{9\pi}{2}. \qquad \blacktriangle$$

Lecture 10 The Fundamental Theorem of Calculus[17]

10.1 Mean-Value Theorem for Integrals

Just like the derivative, definite integrals have a mean-value property.

Theorem 10.1.1 (Mean-value theorem for integrals)**.** *If f is continuous on $[a, b]$, then there exists a point $c \in [a, b]$ such that*

$$\int_a^b f(x)\,dx = (b - a)f(c).$$

Proof. Since f is continuous on $[a, b]$, and $[a, b]$ is a closed and finite interval, we know by the Max-min theorem that there exist $\ell, u \in [a, b]$ such that

$$m = f(\ell) \leq f(x) \leq f(u) = M$$

for all $a \leq x \leq b$.

Now consider the very simple partition P_1 of $[a, b]$ with $a = x_0 < x_1 = b$. Then

$$m(b - a) = L(f, P_1) \leq \int_a^b f(x)\,dx \leq U(f, P_1) = M(b - a).$$

Thus

$$f(\ell) = m \leq \frac{1}{b - a} \int_a^b f(x)\,dx \leq M = f(u).$$

By the Intermediate value theorem, $f(x)$ attains every value between $f(\ell)$ and $f(u)$, so there exists some c between ℓ and u such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x)\,dx \quad \Longleftrightarrow \quad \int_a^b f(x)\,dx = (b - a)f(c). \qquad \square$$

This value

$$\bar{f} = \frac{1}{b - a} \int_a^b f(x)\,dx$$

is often called the ***average value*** or ***mean value*** of f on $[a, b]$. This is because the area under $f(c)$ and above $f(x)$ is the same as the area above $f(c)$ and below $f(x)$. We will show this in the near future.

[17]Date: February 20, 2017.

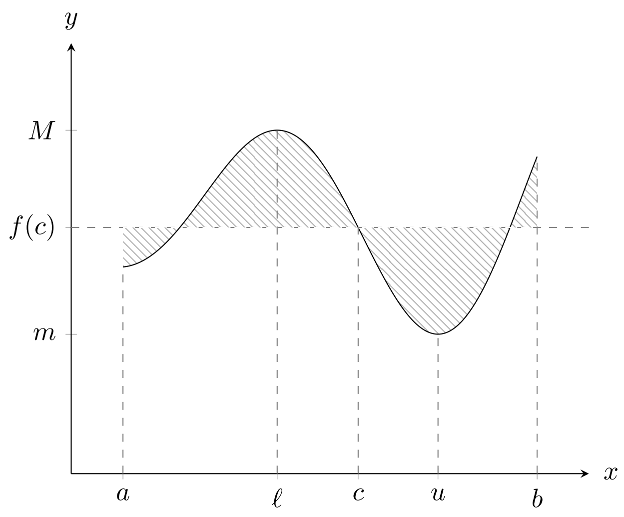


Figure 10.1.1: Visualisation of the Mean-value theorem for integrals.

10.2 Piecewise Continuous Functions

Sometimes we will want to integrate functions that aren't continuous. An important class of such functions are ***piecewise continuous*** functions.

See for instance Figure 10.2.1. Clearly this function isn't continuous, but equally clearly there is an area underneath it.

Definition 10.2.1 (Piecewise continuous function)**.** Let $c_0 < c_1 < c_2 < \ldots < c_n$ be a *finite* set of points. A function f defined on $[c_0, c_n]$, except possibly at some of the c_i, for $0 \le i \le n$, is called ***piecewise continuous*** on the interval if for each $1 \le i \le n$ there exists a function F_i continuous on the closed interval $[c_{i-1}, c_i]$ such that $f(x) = F_i(x)$ on $]c_{i-1}, c_i[$.

In this case we *define* the definite integral of f from c_0 to c_n to be

$$\int_{c_0}^{c_n} f(x)\,dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} F_i(x)\,dx.$$

10.3 The Fundamental Theorem of Calculus

There is an important relation between definite integrals and indefinite integrals. Recall how we defined $\ln(x)$ as a particular area under $1/t$, and as a consequence we found that $\frac{d}{dx}\ln(x) = 1/x$.

This is no coincidence, and is a special case of the following theorem:

Theorem 10.3.1 (Fundamental Theorem of Calculus)**.** *Suppose that f is continuous on the interval I containing the point a.*

Part I

Let F be the function defined by

$$F(x) = \int_a^x f(t)\,dt.$$

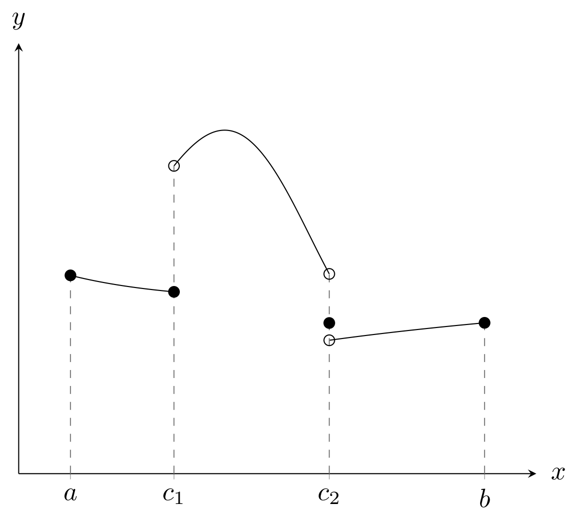


Figure 10.2.1: An example of a piecewise continuous function.

Then F is differentiable on I and $F'(x) = f(x)$ there.
Therefore F is an antiderivative of f on I, i.e.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Part II

If G is any *antiderivative of f on I, so that $G'(x) = f(x)$ on I, then for any b in I we have*

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Proof. The proof of Part I boils down to the definition of the derivative:

$$\begin{aligned} F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} &= \lim_{h\to 0}\frac{1}{h}\Big(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\Big) \\ = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt, & \end{aligned}$$

which by the Mean-value theorem for integrals is equal to

$$\lim_{h\to 0}\frac{1}{h}\cdot h\cdot f(c) = \lim_{h\to 0} f(c)$$

for some $c \in [x, x+h]$, with c depending on h. Now as $h \to 0$, clearly c being squeezed between x and $x+h$ forces c to approach x, so this is in turn equal to

$$\lim_{c\to x} f(c) = f(x),$$

since f is continuous.

For Part II, if $G'(x) = f(x)$, then $F(x) = G(x) + C$ (since they're both antiderivatives they must differ by a constant). Therefore

$$\int_a^x f(t)\,dt = F(x) = G(x) + C.$$

By taking $x = a$ we get

$$\int_a^a f(t)\,dt = 0 = F(a) = G(a) + C,$$

so $C = -G(a)$.

Now instead take $x = b$, and we get

$$\int_a^b f(t)\,dt = F(b) = G(b) + C = G(b) - G(a). \qquad \square$$

Both of these are very useful. The first part tells us how to differentiate a definite integral with respect of its upper limit, and the second part tells us how to evaluate a definite integral if you can find *any* antiderivative of the integrand.

Since we will be doing this second part quite a lot, we'll introduce a new piece of notation that makes life a bit easier: we'll write

$$F(x)\Big|_a^b = F(b) - F(a),$$

so that, for instance,

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b.$$

Example 10.3.2. Evaluate

$$\int_0^a x^2\,dx = \frac{1}{3}x^3\Big|_0^a = \frac{1}{3}a^3 - \frac{1}{3}0^3 = \frac{a^3}{3}. \qquad \blacktriangle$$

Example 10.3.3. Compute

$$\begin{aligned}\int_{-1}^2 x^2 - 3x + 2\,dx &= \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\Big|_{-1}^2 \\ &= \frac{1}{3}\cdot 8 - \frac{3}{2}\cdot 4 + 4 - \left(\frac{1}{3}\cdot -1 - \frac{3}{2}\cdot 1 - 2\right) = \frac{9}{2}. \qquad \blacktriangle\end{aligned}$$

Example 10.3.4. Find the area bounded by the x-axis and the curve $y = 3x - x^2$.

First we need to find where $y = 3x - x^2$ intersects the x-axis: $0 = 3x - x^2 = x(3 - x)$, so $x = 0$ and $x = 3$. Therefore the area is

$$A = \int_0^3 3x - x^2\,dx = \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^3 = \frac{27}{2} - \frac{27}{3} - (0 - 0) = \frac{27}{6} = \frac{9}{2}. \qquad \blacktriangle$$

Example 10.3.5. Find the average value of $f(x) = e^{-x} + \cos(x)$ on the interval $[-\pi/2, 0]$.

We recall from the Mean-value theorem for integrals that what we want to compute it

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

so

$$\begin{aligned}\bar{f} &= \frac{1}{0-(-\pi/2)}\int_{-\pi/2}^{0} e^{-x}+\cos(x)\,dx = \frac{2}{\pi}(-e^{-x}+\sin(x))\Big|_{-\pi/2}^{0}\\ &= \frac{2}{\pi}(-1+0+e^{\pi/2}-(-1)) = \frac{2}{\pi}e^{\pi/2}.\end{aligned}$$

▲

We discuss the following example as a cautionary note.

Counterexample 10.3.6 (Improper integral)**.** We know that $\frac{d}{dx}\ln|x| = 1/x$ if $x \neq 0$. Even so it is ***incorrect*** to say that

$$\int_{-1}^{1}\frac{dx}{x} = \ln|x|\Big|_{-1}^{1} = 0-0 = 0,$$

even though $1/x$ is odd. This is because $1/x$ is undefined at $x = 0$, so $1/x$ is not integrable on $[-1, 0]$ or $[0, 1]$.

All this to say that it might sometimes be important to keep track of where one's integrand is defined.

There are ways to deal with some of these so-called ***improper integrals***, although not this one in particular. ▲

Finally we'll work through some examples that use the Fundamental theorem in interesting ways.

Example 10.3.7. Differentiate

$$F(x) = \int_x^3 e^{-t^2}\,dt.$$

We note that the Fundamental theorem requires the unknown x to be the upper limit of integration, so we first switch the order by negating the integrand:

$$F(x) = -\int_3^x e^{-t^2}\,dt,$$

so by the Fundamental theorem

$$F'(x) = -e^{-x^2}.$$

▲

In the following example we'll combine the Fundamental theorem with the chain rule, that is to say

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x).$$

Example 10.3.8. Differentiate

$$G(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt.$$

First we note that

$$G(x) = \int_0^{x^3} e^{-t^2}\,dt + \int_{x^2}^0 e^{-t^2}\,dt = \int_0^{x^3} e^{-t^2}\,dt - \int_0^{x^2} e^{-t^2}\,dt,$$

so

$$G'(x) = e^{-(x^3)^2} \cdot (3x^2) - e^{-(x^2)^2} \cdot (2x) = 3x^2 e^{-x^6} - 2xe^{-x^4}.$$

▲

Lecture 11 Method of Substitution[18]

11.1 Method of Substitution

So far we have calculated all integrals pretty much by inspection, since we know quite a few derivatives, e.g.

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C,\ r \neq -1, \qquad \int \frac{1}{x}\,dx = \ln|x| + C,\ x \neq 0,$$

$$\int (\sec(x))^2\,dx = \tan(x) + C, \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C,\ a > 0,$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C, \qquad \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C,$$

$$\int b^{ax}\,dx = \frac{1}{a\ln(b)}b^{ax} + C, \qquad \text{et cetera\ldots}$$

However this is not always enough, sometimes we need other techniques. One of the most important ones is the ***method of substitution***, which is really just applying the integral to the chain rule. We know that

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x),$$

whereby

$$\int f'(g(x) \cdot g'(x)\,dx = f(g(x)) + C.$$

In the language of differentials, it looks like this:

Let $u = g(x)$. Then $\frac{du}{dx} = g'(x)$, which, if we treat du and dx as differentials, becomes $du = g'(x)\,dx$, which means that

$$\int f'(g(x)) \cdot \underbrace{g'(x)\,dx}_{=du} = \int f'(u)\,du = f(u) + C = f(g(x)) + C.$$

(We have not spoken about differentials, and we probably aren't going to, but we may consider the above ideas as a way to remember what to do.)

Example 11.1.1. Compute the integral of $x/(x^2+1)$.

We let $u = x^2 + 1$, whereby $du = 2x\,dx$, so $x\,dx = 1/2\,du$. Therefore

$$\begin{aligned}\int \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln(x^2+1) + C = \ln(\sqrt{x^2+1}) + C.\end{aligned}$$

▲

[18]Date: February 23, 2017.

Example 11.1.2. Find the indefinite integral of $\sin(3\ln(x))/x$.

We let $u = 3\ln(x)$, meaning that $du = 3/x\,dx$. Therefore

$$\int \frac{\sin(3\ln(x))}{x}\,dx = \frac{1}{3}\int \sin(u)\,du = -\frac{1}{3}\cos(u) + C$$
$$= -\frac{1}{3}\cos(3\ln(x)) + C. \qquad \blacktriangle$$

Example 11.1.3. Compute the integral of $e^x\sqrt{1+e^x}$.

First let $u = 1 + e^x$, whereby $du = e^x\,dx$, so that

$$\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,dy = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C. \qquad \blacktriangle$$

In all of these examples the substitutions required are fairly obvious. Regrettably, life isn't always that good:

Example 11.1.4. Integrate $1/(x^2 + 4x + 5)$.

A very clever thing to do, which we'll formalise and make systematic in the future, is to complete the square of the denominator. In other words

$$x^2 + 4x + 5 = (x+2)^2 + 1,$$

so

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{dx}{(x+2)^2+1}.$$

If we now let $u = x + 2$ we get $du = dx$, which means that

$$\int \frac{dx}{(x+2)^2+1} = \int \frac{du}{u^2+1} = \arctan u + C = \arctan x + 2 + C. \qquad \blacktriangle$$

Example 11.1.5. Find the indefinite integral of $1/\sqrt{e^{2x}-1}$.

We first break e^x out of the square root, whereby

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{e^x\sqrt{1-e^{-2x}}} = \int \frac{e^{-x}}{\sqrt{1-(e^{-x})^2}}\,dx.$$

By now taking $u = e^{-x}$ we have $du = -e^{-x}\,dx$, which gives us

$$\int \frac{e^{-x}}{\sqrt{1-(e^{-x})^2}}\,dx = -\int \frac{du}{\sqrt{1-u^2}} = -\arcsin(u) + C = -\arcsin(e^{-x}) + C. \qquad \blacktriangle$$

With this said substitution doesn't always work. We will not be able to substitute our way through

$$\int x^2(1+x^8)^{1/3}\,dx,$$

for instance (though we will learn a method that works for this one).

We can of course use substitution in definite integrals as well, but we need to be careful with the limits of integration.

Theorem 11.1.6. *Suppose g is differentiable on $[a, b]$ and that $g(a) = A$ and $g(b) = B$, and suppose that f is continuous on the range of g. Then*

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_A^B f(u)\,du.$$

Proof. Let F be an antiderivative of f, then

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x),$$

which means that

$$\begin{aligned}\int_a^b f(g(x)) \cdot g'(x)\,dx &= F(g(x))\Big|_a^b = F(g(b)) - F(g(a)) \\ &= F(B) - F(A) = F(u)\Big|_A^B = \int_A^B f(u)\,du. \qquad \square\end{aligned}$$

The point being that we cannot keep the x-values when we are dealing with u-values. Unless we switch back before evaluating!

Example 11.1.7. Evaluate

$$I = \int_0^8 \frac{\cos(\sqrt{x+1})}{\sqrt{x+1}}\,dx.$$

Let $u = \sqrt{x+1}$, yielding $du = dx/(2\sqrt{x+1})$. If $x = 0$, then $u = 1$, and if $x = 8$, then $u = 3$, so

$$I = 2\int_1^3 \cos(u)\,du = 2\sin(u)\Big|_1^3 = 2\sin(3) - 2\sin(1),$$

or alternatively

$$I = 2\int_{x=0}^{x=8} \cos(u)\,du = 2\sin(u)\Big|_{x=0}^{x=8} = 2\sin(\sqrt{x+1})\Big|_0^8 = 2\sin(3) - 2\sin(1). \quad \blacktriangle$$

Example 11.1.8. Evaluate

$$I = \int_0^\pi (2 + \sin(x/2))^2 \cos(x/2)\,dx.$$

We let $u = 2 + \sin(x/2)$, meaning that $du = \cos(x/2)/2\,dx$, and if $x = 0$, $u = 0$, and if $x = \pi$, then $u = 3$. Thus

$$I = \int_2^3 = u^2\,du = \frac{2}{3}u^3\Big|_2^3 = \frac{2}{3}(27 - 8) = \frac{38}{3}. \quad \blacktriangle$$

Remark 11.1.9. Note that f being continuous on the range of g is important. Consider

$$\int_{-1}^1 x\csc(x^2)\,dx,$$

with $u = x^2$ and $du = 2x\,dx$ it would seem as though this becomes

$$\frac{1}{2}\int_1^1 \csc(u)\,du = 0,$$

but this is ***incorrect***! Indeed csc is discontinuous at $x = 0$, and we are attempting to compute an infinite area. This is another example of an improper integral.

In general, if the integrand is a quotient where the numerator is (close to) the derivative of the denominator, substitution will lead us beautifully toward a logarithm.

Example 11.1.10. To integrate $\tan(x)$, let $u = \cos(x)$ and $du = -\sin(x)\,dx$, whereby

$$\begin{aligned}\int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{du}{u} = -\ln|u| + C\\ &= -\ln|\cos(x)| + C = \ln\left|\frac{1}{\cos(x)}\right| + C = \ln|\sec(x)| + C.\end{aligned}$$ ▲

We close with another example where the substitution isn't all that obvious.

Example 11.1.11. Integrate $\sqrt{1-x^2}$ between $x = 0$ and $x = 1$. We let $u = \arcsin(x)$, whereby $x = \sin(u)$, and implicit differentiation $dx = \cos(u)\,du$. Thus

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,dx &= \int_0^{\pi/2} \underbrace{\sqrt{1-(\sin(u))^2}}_{=\cos(u)} \cos(u)\,du = \int_0^{\pi/2} (\cos(u))^2\,du\\ &= \int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos(2u)\,du = \frac{u}{2}\Big|_0^{\pi/2} + \frac{1}{4}\sin(2u)\Big|_0^{\pi/2}\\ &= \frac{\pi}{4} + \frac{1}{4}(\sin(\pi) - \sin(0)) = \frac{\pi}{4}.\end{aligned}$$

Note that of course we are calculating the area of the unit disc contained in the first quadrant, which by simple geometry is indeed $\pi/4$. ▲

11.2 Areas Between Curves

Recall first of all that $\int_a^b f(x)\,dx$ measures the area between $y = f(x)$, $y = 0$, $x = a$, and $x = b$, except it treats parts under the x-axis as negative areas.

If, for some reason, we don't want this, it is easily avoided: simply take the absolute value of the integrand!

Consider Figure 11.2.1, where

$$\int_a^b f(x)\,dx = -A_1 + A_2 - A_3,$$

but

$$\int_a^b |f(x)|\,dx = A_1 + A_2 + A_3.$$

In other words we find the parts of the function that are below the axis and compute these areas separately, then add them to the areas under the curve that are positive.

Example 11.2.1. Calculate the positive area bounded by $y = \cos(x)$, $y = 0$, $x = 0$, and $x = 3\pi/2$. We have

$$\begin{aligned}A &= \int_0^{3\pi/2} |\cos(x)|\,dx = \int_0^{\pi/2} \cos(x)\,dx + \int_{\pi/2}^{3\pi/2} -\cos(x)\,dx\\ &= \sin(x)\Big|_0^{\pi/2} - \sin(x)\Big|_{\pi/2}^{3\pi/2} = (1-0) - (-1-1) = 3.\end{aligned}$$ ▲

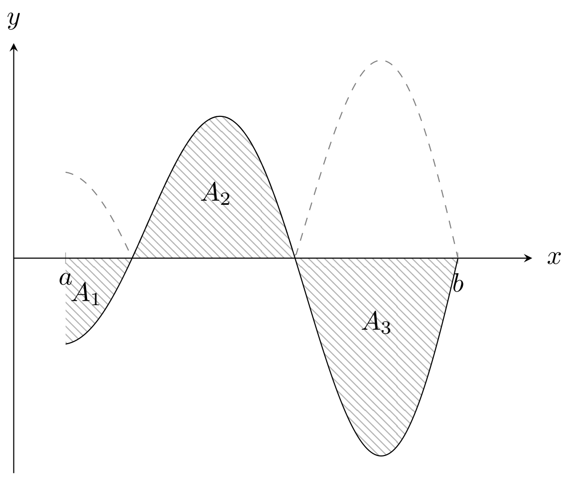


Figure 11.2.1: Making negative areas positive, with A_1, A_2, and A_3 representing the positive areas, with the dashed line being the absolute value of the function.

Suppose we want to compute the area bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. Assume $a < b$ and $f(x) \leq g(x)$ for all $x \in [a, b]$.

Then the area we are looking for is

$$A = \int_a^b g(x) - f(x)\, dx.$$

Example 11.2.2. Find the area bounded by $y = x^2 - 2x$ and $y = 4 - x^2$.

We first find the points of intersection between the two curves:

$$x^2 - 2x = 4 - x^2 \quad \Longleftrightarrow \quad 2x^2 - 2x - 4 = 0 \quad \Longleftrightarrow \quad 2(x-2)(x+1) = 0,$$

so $x = 2$ and $x = -1$. Since $4 - x^2 \geq x^2 - 2x$ for $-1 \leq x \leq 2$, we have

$$\begin{aligned} A &= \int_{-1}^{2} (4 - x^2) - (x^2 - 2x)\, dx = \int_{-1}^{2} 4 - 2x^2 + 2x\, dx \\ &= \Big(4x - \frac{2}{3}x^3 + x^2\Big)\Big|_{-1}^{2} = 4 \cdot 2 - \frac{2}{3} \cdot 8 + 5 - \Big(-4 + \frac{2}{3} + 1\Big) = 9. \end{aligned}$$

Note that if we got the conclusion $4 - x^2 \geq x^2 - 2x$ wrong, we'd have gotten -9 as the area instead. ▲

If we don't have $g(x) \geq f(x)$, but we're still interested in the positive area, we naturally do the same thing as before:

$$A = \int_a^b |f(x) - g(x)|\, dx,$$

where we'd again have to split the integral up into the positive parts and the negative parts.

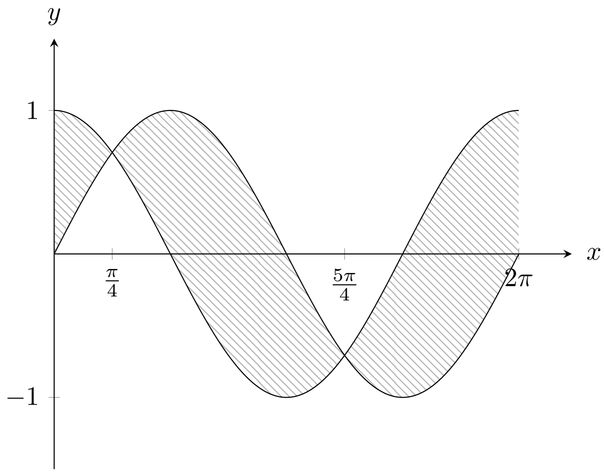


Figure 11.2.2: The area between $\sin(x)$ and $\cos(x)$ for x between 0 and 2π.

Example 11.2.3. Find the area lying between $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = 2\pi$, as seen in Figure 11.2.1.

$$\begin{aligned}
A &= \int_0^{\pi/4} \cos(x) - \sin(x)\, dx + \int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)\, dx + \int_{5\pi/4}^{2\pi} \cos(x) - \sin(x)\, dx \\
&= (\sin(x) - \cos(x))\Big|_0^{\pi/4} - (\cos(x) + \sin(x))\Big|_{\pi/4}^{5\pi/4} + (\sin(x) - \cos(x))\Big|_{5\pi/4}^{2\pi} \\
&= (\sqrt{2} - 1) + (\sqrt{2} + \sqrt{2}) + (1 + \sqrt{2}) = 4\sqrt{2}.
\end{aligned}$$

▲

Lecture 12 Integration by Parts[19]

[19] Date: February 27, 2017.

References

[Ad13] R. A. Adams. *Calculus: A Complete Course*. Prentice-Hall, 1136 pages, 8th revised edition, 2013.

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