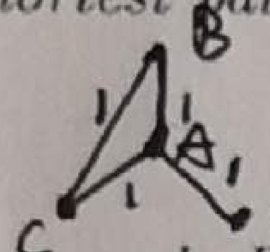
1: (3*2'+4') Dijkstra's Algorithm

Question 1. *Judge whether the following statement is true or false and explain why. Give a counter-example if it is false.*

(a) *Suppose G is strongly connected with integer edge weights, and has shortest paths from some vertex v (i.e. a finite weight shortest path exists from v to all nodes). Then shortest paths can be found from every vertex to every other vertex.* It is false. For example:



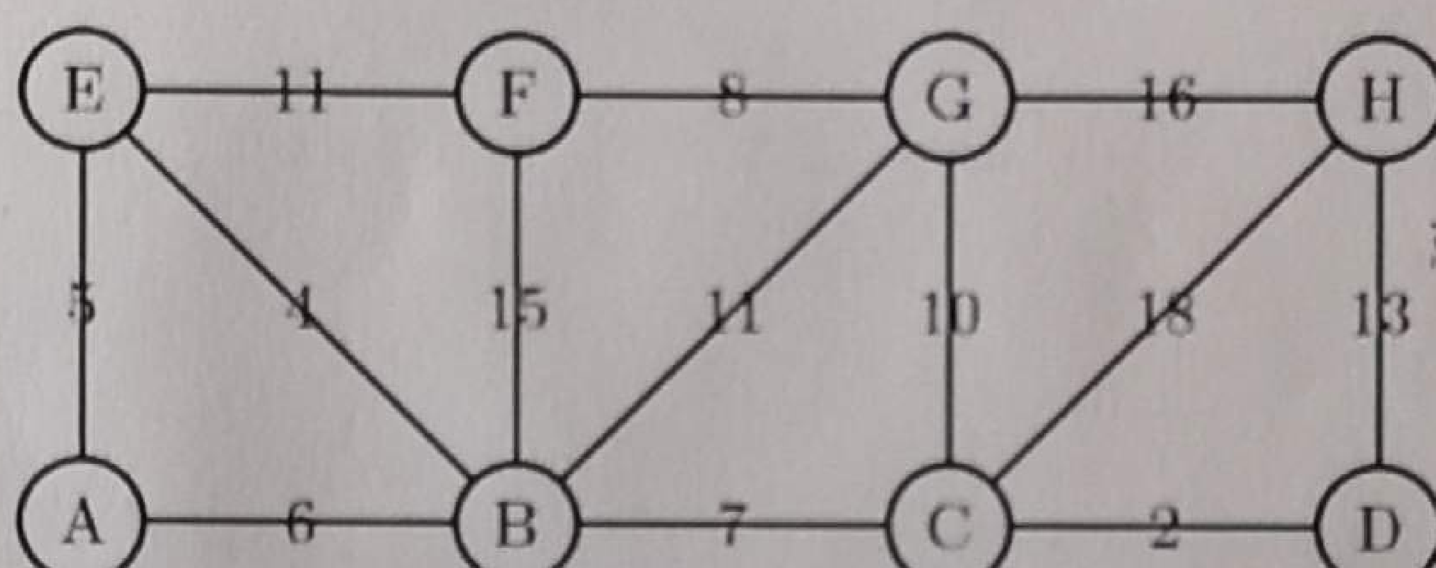
A has shortest paths from some vertex. However, we cannot find a shortest path from B to C containing the shortest paths of A.

(b) *If G is a connected and undirected graph without negative cycles, we can apply Dijkstra's algorithm to find the shortest path.* It is true. Apply Dijkstra's algorithm first, and suppose the shortest path from vertex v_i to vertex v_j is $v_i \cdots v_j$, and the shortest distance is d. Since there are no negative cycles, then we can not find a path containing a negative path from v_i to v_j. (If there exists a negative path from v_i to v_j, then it will form a negative cycle) $\Leftarrow$ Hence Dijkstra's algorithm is applable.

(c) *Suppose G is a DAG. We can find the longest path by negating all edge lengths and then run Dijkstra's algorithm from every source node.*

It is true. Since G is a DAG, there is no cycle, so the path from a source node to v_i is sole, which is the longest path.

Question 2. *Given a weighted graph below, please run Dijkstra's algorithm using vertex A as the source. Write down the vertices in the order which they are marked and the updated distances at each step.*



Solution:

step	vertex
1	A
2	E
3	B
4	C
5	D
6	F
7	G
8	H

step	dist[A]	dist[B]	dist[C]	dist[D]	dist[E]	dist[F]	dist[G]	dist[H]
1	0	6	∞	∞	5	∞	∞	∞
2	0	6	∞	∞	5	16	∞	∞
3	0	6	13	∞	5	16	17	∞
4	0	6	13	15	5	16	17	31
5	0	6	13	15	5	16	17	28
6	0	6	13	15	5	16	17	28
7	0	6	13	15	5	16	17	28
8	0	6	13	15	5	16	17	28

2: (2'+3') Floyd-Warshall Algorithm

B

Question 3. *Let $G = (V, E)$ be a connected, undirected graph with edge weights $w : E \to \mathbb{Z}$. Which of the following statements are True about the Floyd-Warshall algorithm applied to G?*

(A) *Since G is undirected, we cannot apply Floyd-Warshall algorithm.* X

(B) *Since G is undirected, Floyd-Warshall will be asymptotically faster than on directed graphs.*

(C) *Since G is undirected, Floyd-Warshall will be unable to detect negative-weight cycles.* X

(D) *None of the above.*

Question 4. *Consider the following implementation of the Floyd-Warshall algorithm. Assume $w_{ij} = \infty$ where there is no edge between vertex i and vertex j, and assume $w_{ii} = 0$ for every vertex i.*

Algorithm 1 Floyd-Warshall

```
for i = 1 to n do
  for j = 1 to n do
    A[i, j, 0] = w_ij
    P[i, j] = -1
  end for
end for
for k = 1 to n do
  for i = 1 to n do
    for j = 1 to n do
      A[i, j, k] = A[i, j, k - 1]
      if A[i, j, k] > A[i, k, k - 1] + A[k, j, k - 1] then
        A[i, j, k] = A[i, k, k - 1] + A[k, j, k - 1]
        P[i, j] = k
      end if
    end for
  end for
end for
```

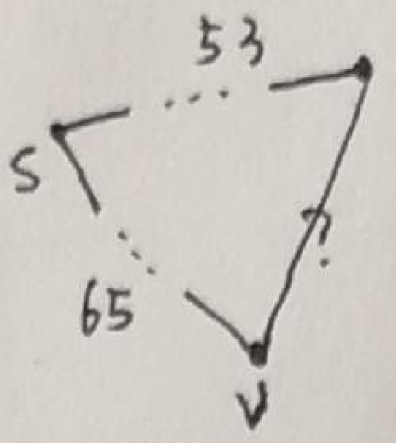
Assume matrix P, the output of the above algorithm is given. Consider the following matrix for graph G with 7 vertices. What is the shortest path from vertex 5 to vertex 7 in graph G?

P	1	2	3	4	5	6	7
1	-1	5	4	-1	4	4	-1
2	5	-1	5	5	-1	5	-1
3	4	5	-1	-1	-1	-1	6
4	-1	5	-1	-1	3	3	1
5	4	-1	-1	3	-1	3	6
6	4	5	-1	3	3	-1	-1
7	-1	-1	6	1	6	-1	-1

$5 \to 3 \to 6 \to 7$

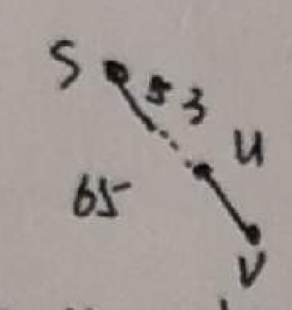
3: (3'+3'+4') Shortest Path

Question 5. *Consider a weighted undirected graph with positive edge weights and let (u, v) be an edge in the graph. It is known that the shortest path from the source vertex s to u has weight 53 and the shortest path from s to v has weight 65. Which is the range of the weight of the edge (u, v)?*



If uv is in the shortest path from s to v

$w(u,v) = 65-53 = 12$



If uv is not in the shortest path from s to v, then

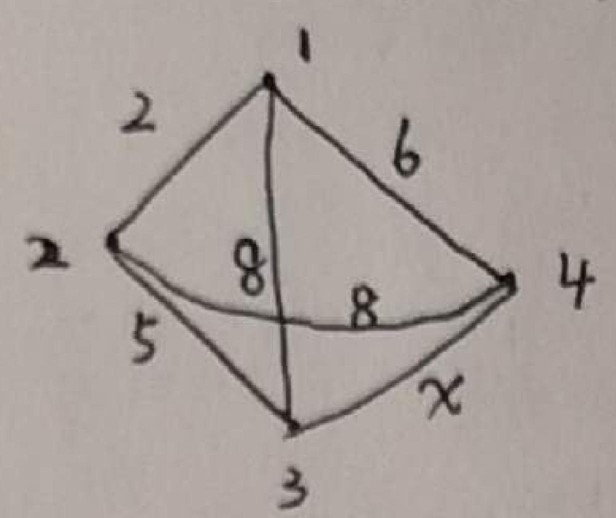
$w(s,u) + w(u,v) > w(s,v) \Rightarrow w(u,v) > 12$

$w(s,v) + w(v,u) > w(s,u)$, $w(u,v) > 0 \Rightarrow w(u,v) > 0$

In all, $w(u,v) \geq 12$

Question 6. *Consider the weighted undirected graph with 4 vertices, where the weight of edge $\{i, j\}$ is given by the entry $W_{i,j}$ in the matrix W*

$$W = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 2 & 8 & 6 \\ 2 & 0 & 5 & 8 \\ 8 & 5 & 0 & x \\ 6 & 8 & x & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$



We want to find the largest possible integer value of x, for which at least one shortest path between some pairs of vertices will definitely contain the edge with weight x. What is this largest possible integer value of such x? Explain your reason briefly. **When breaking tie, the path may be random.**

Denote the weight of shortest path which does not contain x is SP

1→2 : SP=2 $6+x+5 < 2$ $x<-11$
$8+x+8 < 2$ $x<-14$

1→3 : SP=7 $6+x<7$ $x<1$

1→4 : SP=6 $8+x<6$ $x<-2$
$7+x<6$ $x<-1$

2→3 : SP=5 $8+x<5$ $x<-3$

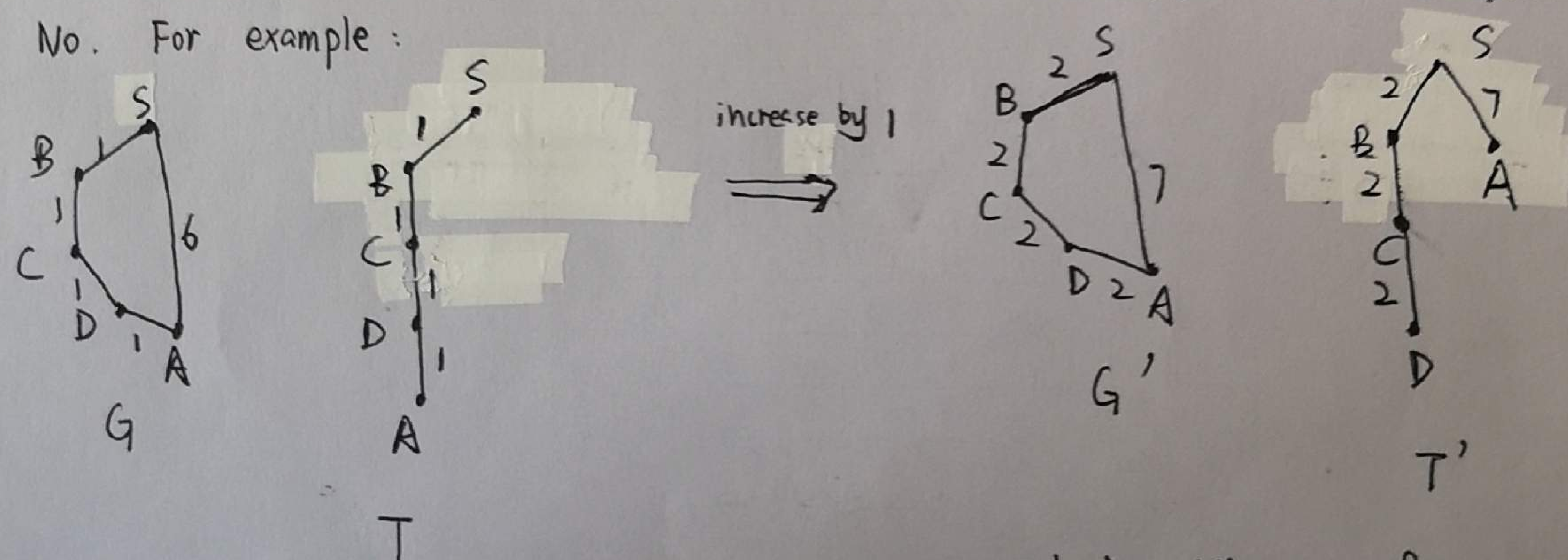
2→4 : SP=8 $5+x<8$ $x<3$

3→4 : SP=13 $x<13$

Hence the largest possible integer value of x is 12.

Question 7. *Suppose $G = (V, E)$ is a weighted graph and T is its shortest-path tree from source s. If we increase all weights in G by the same amount, i.e., $\forall e \in E$, $w'_e = w_e + c$. Is T still the shortest-path tree (from source s) of the new graph? If yes, prove the statement. Otherwise, give a counter example.*

No. For example :



T' is different from T