UNIVERSIDADE EDUARDO MONDLANE
FACULDADE DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA E INFORMÁTICA

Teste 1 de Análise Matemática III

Data: 27 de Abril de 2022 | **V2** | **Duração: 100 Minutos**

1. **[2v]** Especifique e represente graficamente o conjunto dos pontos no plano complexo representados por: $D = \left\{ z \in C : \frac{1}{4} < \operatorname{Re}\left(\frac{1}{z}\right) - \operatorname{Im}\left(\frac{1}{z}\right) < \frac{1}{2} \right\}$.

2. **[3v]** Verifique se a função $f(z) = e^z + z^2$ é analítica em C, usando as condições de Cauchy-Riemann.

3. **[3v]** Dada a função $u(x, y) = shx.seny$. Mostre que a função é harmónica e recupere a função analítica $f(z)$ em que $f(x+iy) = u(x, y) + iv(x, y)$ e $f(0) = i$.

4. **[3v]** Calcule o integral $\int_{-1-i}^{-1+i} \frac{\ln(1+z)}{\sqrt{1+z}} dz$, $\sqrt{-1} = -i$.

5. **[3v]** Aplicando a fórmula integral de Cauchy, calcule $\oint_{\Gamma^+} \frac{zdz}{(z-4)(z-2)^2}$, $\Gamma : |z| = 3$.

6. **[3v]** Aplicando o teorema de resíduos, calcule o integral: $\oint_{\Gamma^+} \frac{senzdz}{z^2(z-2)^2}$, $\Gamma : |z| = 3$

7. **[3v]** Desenvolva a função $f(z) = \frac{z+3}{z^2+1}$, $|z - i| < 2$ em series de Laurent.

Bom Trabalho!
O grupo da disciplina

TESTE 1 - AM III - 2022

① $D = \{ z \in \mathbb{C} : \frac{1}{4} < Re\left(\frac{1}{z}\right) - Im\left(\frac{1}{z}\right) < \frac{1}{2} \}$

I) $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$, $Rez\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$; $Im\left(\frac{1}{z}\right) = \frac{-y}{x^2+y^2}$

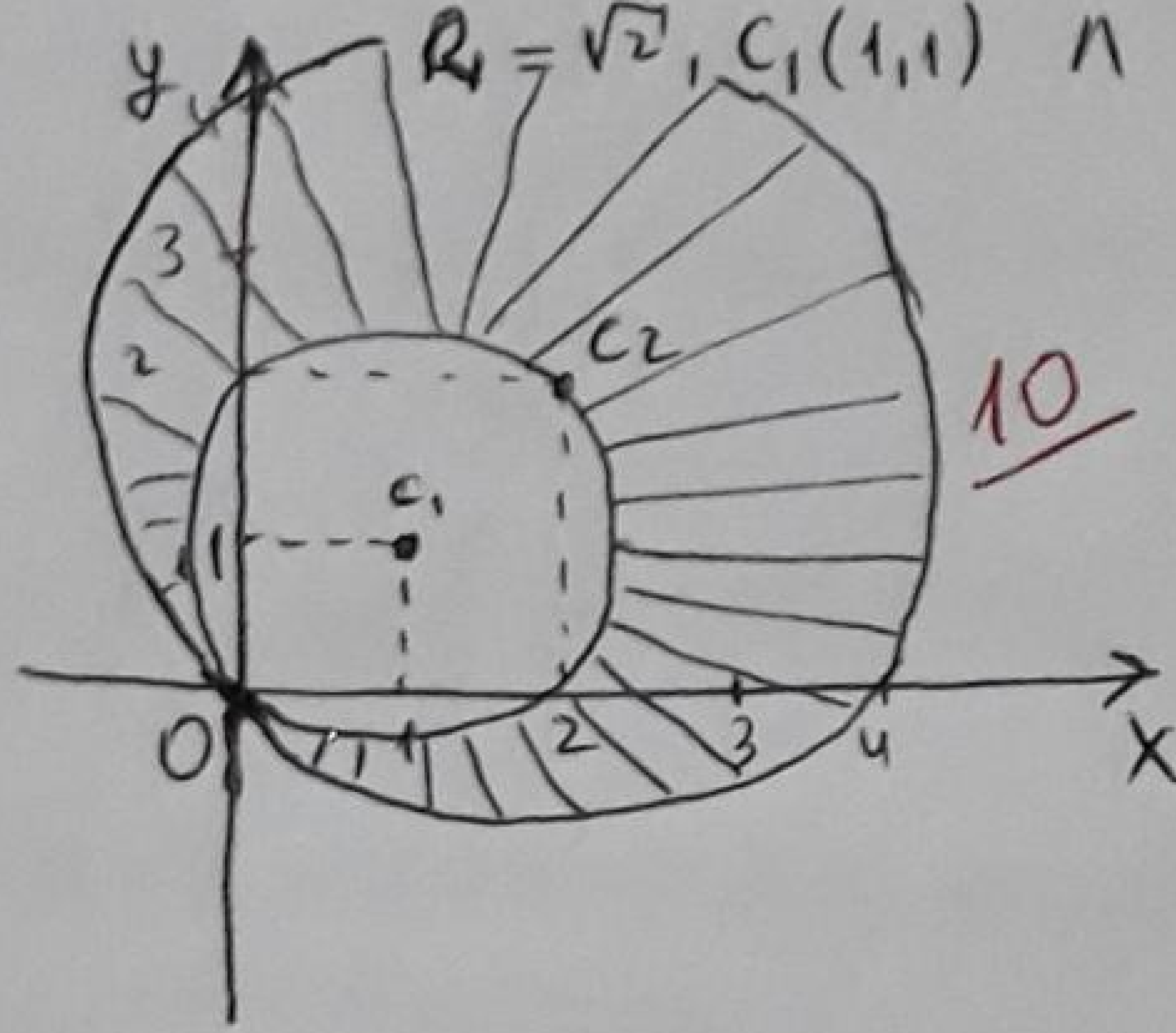
II) $\frac{1}{4} < \frac{x+y}{x^2+y^2} < \frac{1}{2} \Longleftrightarrow x^2+y^2 > 2(x+y) \wedge x^2+y^2 < 4(x+y)$

$\Longleftrightarrow \boxed{(x-1)^2+(y-1)^2 > 2 \wedge (x-2)^2+(y-2)^2 < 8}$ 10

$R_1 = \sqrt{2}, C_1(1,1) \wedge R_2 = \sqrt{8}; C_2(2,2)$

III) Graficamente:

OBS: As circunferências (fronteira) não fazem parte do conjunto D dado



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② $f(z) = e^z + z^2$

$f(z) = f(x+iy) = e^{x+iy} + (x+iy)^2 = e^x \cdot e^{iy} + x^2 + 2xyi - y^2$

$= e^x(\cos y) + x^2 - y^2 + i\left[e^x \sin y + 2xy\right]$

$u = e^x \cos y + x^2 - y^2$ 5 ; $v = e^x \sin y + 2xy$ 5

$u'_x = e^x \cos y + 2x$

$u'_y = -e^x \sin y - 2y$

$v'_x = e^x \sin y + 2y$

$v'_y = e^x \cos y + 2x$

condições de Cauchy - Riemann

$\Rightarrow \begin{cases} u'_x = v'_y \\ u'_y = -v'_y \end{cases}$ 10 $\Rightarrow$ Satisfeito!

R: A função $f(z) = e^z + z^2$ é analítica em $\mathbb{C}$.

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③ $u(x,y) = \sinh x \sin y, \quad f(0) = i$

I) $u(x,y)$ é harmônica, então $u''_{xx} + u''_{yy} = 0$.

$u'_x = \cosh x \sin y, \quad u''_{xx} = \sinh x \sin y$

$u'_y = \sinh x \cos y, \quad u''_{yy} = -\sinh x \sin y \Rightarrow u''_{xx} + u''_{yy} = 0$

$u(x,y) = \sinh x \sin y$ é harmônica! 5

II) $v(x,y) = -\int_0^x u'_y(x,0)\,dx + \int_0^y u'_x(x,y)\,dy + c_1, \quad P_0(0,0) \in D(u)$

$v(x,y) = -\int_0^x \sinh x\,dx + \int_0^y \cosh x \sin y\,dy + c_1$

$v(x,y) = -\cosh x \Big|_0^x \cancel{+} - \cosh x \cos y \Big|_0^y + c_1$

$v(x,y) = -\cancel{\cosh x} + 1 - \cosh x \cos y + \cancel{\cosh x} + c_1$

$\boxed{v(x,y) = -\cosh x \cos y + c}$ 10 $(c = c_1 + 1)$

III) $f(x+iy) = \sinh x \sin y + i[-\cosh x \cos y + c]$

com $f(0) = i \Rightarrow i = 0 + i(-1+c) \Rightarrow \boxed{c = 2}$

$f(x+iy) = \sinh x \sin y - i\cosh x \cos y + 2i$

$f(x+iy) = -i(\cosh x \cos y + i \sinh x \sin y) + 2i$

$\boxed{f(z) = -i\cosh z + 2i}$ 15

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④ $\displaystyle\int_{-1-i}^{-1+i} \frac{\ln(1+z)}{\sqrt{1+z}}\,dz$, se $\sqrt{-1}=i$ (isto é, $\boxed{k=1}$)

$$\int \frac{\ln(1+z)}{\sqrt{1+z}}\,dz \quad \left| \begin{array}{l} \text{seja } t=1+z \\ dt=dz \\ t_1=-i \\ t_2=i \end{array} \right| \quad \int \frac{\ln t}{\sqrt{t}}\,dt \quad \left| \begin{array}{l} u=\ln t,\ du=\frac{dt}{t} \\ dv=\frac{dt}{\sqrt{t}},\ v=2\sqrt{t} \end{array} \right.$$

$$I = 2\sqrt{t}\ln t - \int 2\sqrt{t}\cdot\frac{dt}{t} = 2\sqrt{t}\ln t - 2\int \frac{dt}{\sqrt{t}}$$
$$= 2\sqrt{t}\ln t - 4\sqrt{t}$$ 10

$$I = \int_{-i}^{i} \frac{\ln t}{\sqrt{t}}\,dt = \left(2\sqrt{t}\ln t - 4\sqrt{t}\right)\Big|_{-i}^{i} =$$

$\sqrt{i} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$\sqrt{-i} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$\ln i = \frac{5\pi}{2}i$ 5

$\ln(-i) = \frac{3\pi}{2}i$

$$= \left(2\sqrt{i}\ln i - 4\sqrt{i}\right) - \left(2\sqrt{-i}\ln(-i) - 4\sqrt{-i}\right)$$

$$= 2\sqrt{i}\left(\ln i - 2\right) - 2\sqrt{-i}\left(\ln(-i) - 2\right)$$ 10

$$= -(\sqrt{2})(1+i)\left(\frac{5\pi}{2}i - 2\right) - \sqrt{2}\cdot(-1+i)\left(\frac{3\pi}{2}i - 2\right)$$

$$= \sqrt{2}\left(-\frac{5\pi}{2}i + \cancel{2} + \frac{5\pi}{2} + 2i + \frac{3\pi}{2}i - \cancel{2} + \frac{3\pi}{2} + 2i\right)$$

$$= \sqrt{2}\left(4i - \pi i + 4\pi\right) = 4\sqrt{2}\pi + i\sqrt{2}(4-\pi)$$ 5

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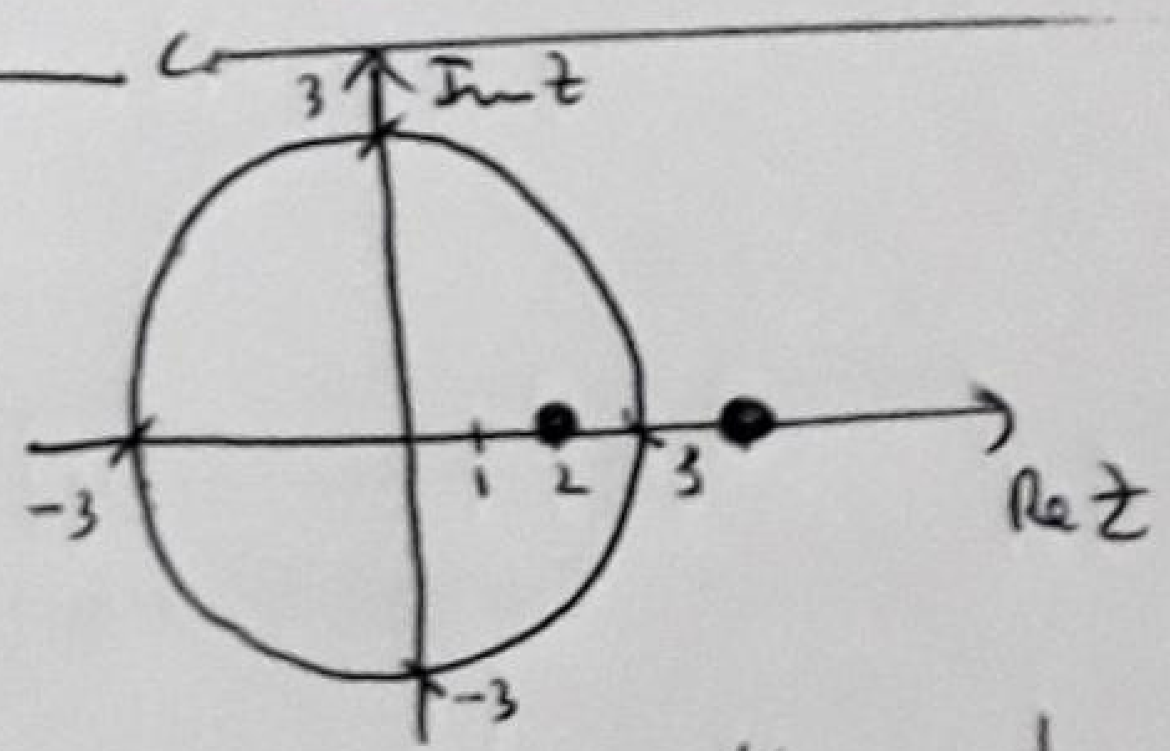
⑤ $\displaystyle\oint_{\Gamma^+} \frac{z\,dz}{(z-4)(z-2)^2}$; $\Gamma: |z|=3$



$z=4 \notin D(\Gamma)$, $z=2 \in D(\Gamma)$

$$I = \oint_{\Gamma^+} \frac{\frac{z}{z-4}}{(z-2)^2}\,dz = 2\pi i\left(\frac{z}{z-4}\right)'_{z=2} = 2\pi i\,\frac{-4}{(+z-4)^2}\Big|_{z=2}$$

$$= 2\pi i(-1)$$

$$= -2\pi i$$

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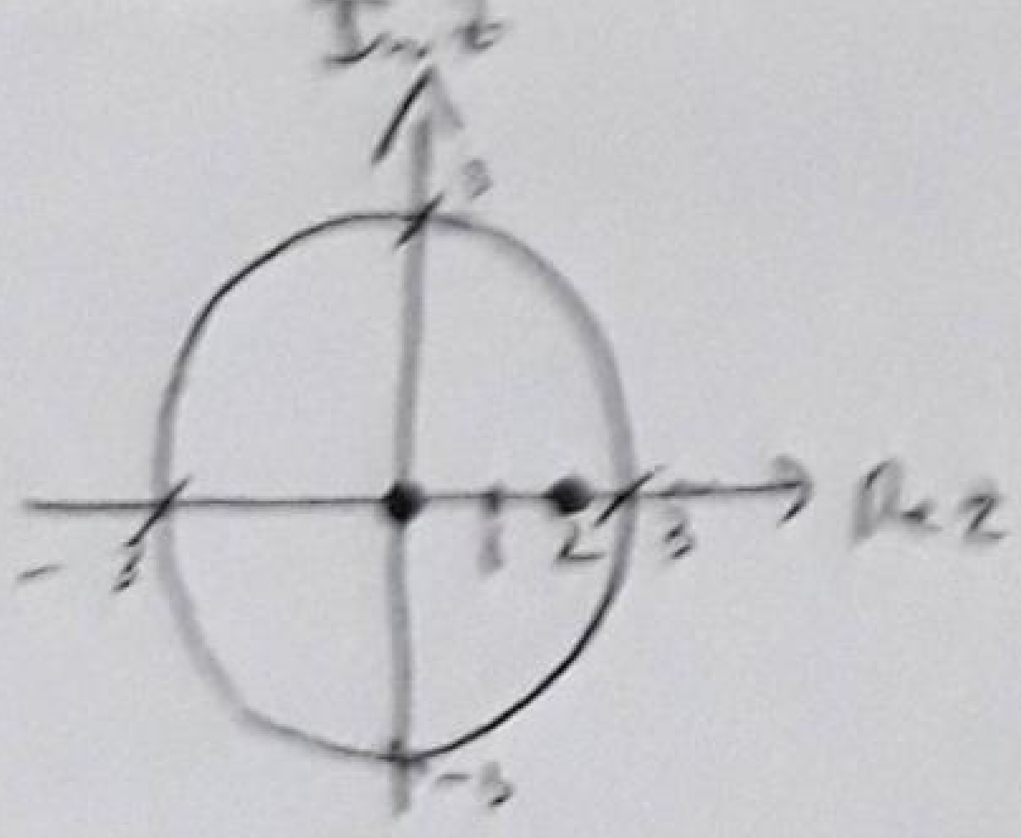
⑥ $\oint_{\Gamma^+} \frac{\sin z\,dz}{z^2(z-2)^2}$; $\Gamma : |z| = 3$



$z=0$ é pólo simples
$z=2$ é pólo duplo

$$I = 2\pi i \cdot \left[\operatorname{Res} f(0) + \operatorname{Res} f(2) \right]$$

$$= 2\pi i \left[\lim_{z\to 0} z \cdot f(z) + \lim_{z\to 2} \left[(z-2)^2 \cdot f(z) \right]' \right] \quad 5 \qquad f(z) = \frac{\sin z}{z^2(z-2)^2}$$

$$= 2\pi i \left[\lim_{z\to 0} \frac{\sin z}{z(z-2)^2} + \lim_{z\to 2} \frac{z^2 \cos z - 2z \sin z}{z^4} \right] \quad 10$$

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$$= 2\pi i \left[\frac{1}{4} + \frac{\cos 2 - \sin 2}{4} \right] = \frac{\pi i}{2}\left(1 + \cos 2 - \sin 2\right) \quad //$$

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⑦ $f(z) = \frac{z+3}{z^2+1}$, $\quad |z-i| < 2 \iff \left|\frac{z-i}{2}\right| < 1 \iff \left|\frac{z-i}{2i}\right| < 1$

$$\frac{z+3}{z^2+1} = \frac{z+3}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}$$

$$z+3 = A(z+i) + B(z-i)$$

$$z = i \Rightarrow i+3 = 2iA \Rightarrow \boxed{A = \frac{i+3}{2i}}$$

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$$z = -i \Rightarrow -i+3 = -2iB \Rightarrow \boxed{B = \frac{i-3}{2i}}$$

$$f(z) = \frac{\frac{i+3}{2i}}{z-i} + \frac{\frac{i-3}{2i}}{z+i} = \frac{i+3}{2i(z-i)} + \frac{i-3}{2i(z-i+2i)} =$$

$$= \frac{i+3}{2i(z-i)} + \frac{i-3}{(2i)^2\left(1 + \frac{z-i}{2i}\right)} = 10$$

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$$= \frac{i+3}{2i(z-i)} + \frac{i-3}{(2i)^2} \cdot \left(1 - \frac{z-i}{2i} + \frac{(z-i)^2}{(2i)^2} - \frac{(z-i)^3}{(2i)^3} + \dots\right)$$

$$= \frac{i+3}{2i(z-i)} + \frac{i-3}{(2i)^2} - \frac{(i-3)(z-i)}{(2i)^3} + \frac{(i-3)(z-i)^2}{(2i)^4} - \frac{(i-3)(z-i)^3}{(2i)^5} + \dots \quad 10$$