Bike Repositioning Optimization: MIP Formulation

Indices and Sets

- $u \in \{1, \dots, U\}$: Trucks (with U being the maximum number of trucks).
- $i, j \in \{1, \dots, N\}$: Stations (with N as the total number of stations).
- $s \in \{0, 1, \dots, S-1\}$: Stop indices along a truck's route (with S being the maximum number of stops per truck).
- $t \in T$: Time periods, where

  $$T = \{\text{Mon}, \text{Wed}\}.$$

  Note: Monday corresponds to the time window $[7 \text{ AM}, 9 \text{ AM})$ and Wednesday to $[4 \text{ PM}, 6 \text{ PM})$.

Parameters

- $C = 40$: Truck capacity.
- $S_{\max} = 5$: Maximum number of stops per truck.
- $F = 50\$$: Fixed cost per truck used.
- $c_d = 3\$$: Transportation cost per kilometer.
- $D_{\max} = 50km$: Maximum distance a truck can travel in a time period.
- $p_s = 50\$$: Surplus penalty (for bikes left over at stations with a surplus).
- $p_d = 50\$$: Deficit penalty (for unmet bike demand at stations with a deficit).
- d_{ij}: Distance (in km) between station i and station j.
- Δ_i: Inventory imbalance at station i, defined as

  $$\Delta_i = \text{Target Bikes}_i - \text{Current Bikes}_i.$$

  Define:

  - $\mathcal{S} = \{i : \Delta_i < 0\}$ (stations with surplus bikes),
  - $\mathcal{D} = \{i : \Delta_i > 0\}$ (stations with bike deficits).

Decision Variables

- $y_{u,t} \in \{0, 1\}$: Equals 1 if truck u is active in time period t, and 0 otherwise.
- $x_{u,i,s,t} \in \{0, 1\}$: Equals 1 if truck u visits station i as its s-th stop in period t, and 0 otherwise.
- $z_{u,i,j,s,t} \in \{0, 1\}$: Equals 1 if truck u travels from station i to station j immediately after stop s in period t, for $i \neq j$.
- $p_{u,i,s,t} \in \mathbb{Z}_{\geq 0}$: Number of bikes picked up by truck u at station i during stop s in period t.
- $d_{u,i,s,t} \in \mathbb{Z}_{\geq 0}$: Number of bikes dropped off by truck u at station i during stop s in period t.
- $\ell_{u,s,t} \in \mathbb{Z}_{\geq 0}$: Load (number of bikes) carried by truck u after stop s in period t, with $\ell_{u,s,t} \leq C$.
- $D_{u,t} \geq 0$: Total distance traveled by truck u in period t.
- For surplus stations ($i \in \mathcal{S}$): $r_i^s \in \mathbb{Z}_{\geq 0}$ represents the number of surplus bikes not moved.
- For deficit stations ($i \in \mathcal{D}$): $r_i^d \in \mathbb{Z}_{\geq 0}$ represents the number of deficit bikes not fulfilled.

Objective Function

Minimize the total cost comprising the truck activation cost, transportation cost, and penalties for surplus and deficit:

$$\min \quad \sum_{t \in T} \sum_{u=1}^{U} \Big(F\, y_{u,t} + c_d\, D_{u,t} \Big) + \sum_{i \in \mathcal{S}} p_s\, r_i^s + \sum_{i \in \mathcal{D}} p_d\, r_i^d.$$

Explanation: This objective function consists of three main components:

- **Fixed truck activation costs:** Each truck that is used in a time period incurs a fixed cost F.
- **Transportation costs:** Trucks incur a variable cost c_d proportional to the distance traveled $D_{u,t}$.

- **Penalty costs:** These are incurred when inventory imbalances at stations are not fully resolved — p_s penalizes uncollected surplus bikes, and p_d penalizes unmet demand (deficits).

The optimization model seeks to minimize the sum of these costs across all trucks and time periods while repositioning bikes efficiently to meet station demands.

Constraints

(1) Truck Activation Constraints:

$$\sum_{i=1}^{N}\sum_{s=0}^{S-1} x_{u,i,s,t} \le S_{\max}\, y_{u,t}, \quad \forall u,\ t \in T. \tag{1}$$

Explanation: This constraint ensures that if truck u is not activated in time period t (i.e., $y_{u,t} = 0$), then it cannot visit any station (all $x_{u,i,s,t}$ must be 0). If truck u is activated (i.e., $y_{u,t} = 1$), then it is allowed to make at most $S_{\max}$ stops.

(2) Load Management Constraints:

$$\ell_{u,0,t} = \sum_{i=1}^{N} \Big(p_{u,i,0,t} - d_{u,i,0,t}\Big), \quad \forall u,\ t \in T, \tag{2}$$

$$\ell_{u,s,t} = \ell_{u,s-1,t} + \sum_{i=1}^{N} \Big(p_{u,i,s,t} - d_{u,i,s,t}\Big), \quad s = 1, \ldots, S-1,\ \forall u,\ t \in T, \tag{3}$$

$$\ell_{u,s,t} \le C, \quad \forall u,\ s,\ t \in T. \tag{4}$$

Explanation:

- The first equation defines the truck's load after the initial stop (stop 0) as the net number of bikes picked up minus the bikes dropped off.
- The second equation updates the load for each subsequent stop by adding the net change (pickups minus dropoffs) from the current stop to the load from the previous stop.
- The third equation ensures that the load carried by the truck never exceeds its capacity C.

(3) Visit-Operation Linkage:

$$p_{u,i,s,t} \leq C\, x_{u,i,s,t}, \quad \forall u,\ i,\ s,\ t \in T, \tag{5}$$

$$d_{u,i,s,t} \leq C\, x_{u,i,s,t}, \quad \forall u,\ i,\ s,\ t \in T, \tag{6}$$

$$\sum_{i=1}^{N} x_{u,i,s,t} \leq y_{u,t}, \quad \forall u,\ s,\ t \in T. \tag{7}$$

Explanation:

- The first two constraints link the operational decisions (picking up or dropping off bikes) to the visit decision. They ensure that if the truck does not visit station i at stop s (i.e., $x_{u,i,s,t} = 0$), then no bikes can be picked up or dropped off there.
- The third constraint ensures that a truck can only perform an operation at a stop if it is active in that time period (i.e., if $y_{u,t} = 0$, then no station visits occur).

(4) Route Continuity Constraints:

$$\sum_{i=1}^{N} x_{u,i,s,t} \geq \sum_{i=1}^{N} x_{u,i,s+1,t}, \quad \forall u,\ s = 0, \ldots, S-2,\ t \in T, \tag{8}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} z_{u,i,j,s,t} = x_{u,i,s,t}, \quad \forall u,\ i,\ s = 0, \ldots, S-2,\ t \in T, \tag{9}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{N} z_{u,i,j,s,t} = x_{u,j,s+1,t}, \quad \forall u,\ j,\ s = 0, \ldots, S-2,\ t \in T. \tag{10}$$

Explanation:

- The first constraint ensures that if a truck makes a stop at position $s+1$, then it must have also made a stop at position s. In other words, the sequence of stops is cumulative.
- The second constraint ties the visit decision at stop s to the route decision: if the truck visits station i at stop s, then it must depart from that station to some other station immediately after.
- The third constraint ensures that if the truck arrives at station j as the $(s+1)$-th stop, it must have come directly from some other station at the previous stop.