(5) $\forall x (P(x) \rightarrow \neg Q(x)) \vdash \neg(\exists x (P(x) \land Q(x)))$

1	$\forall x (P(x) \rightarrow \neg Q(x))$	premise
2	$\exists x\, P(x) \land Q(x)$	assume
3	x_0	
4	$P(x_0) \land Q(x_0)$	assume
5	$P(x_0) \rightarrow \neg Q(x_0)$	$\forall x e\ 1$
6	$P(x_0)$	$\land e_1\ 4$
7	$Q(x_0)$	$\land e_2\ 4$
8	$\neg Q(x_0)$	$\rightarrow e\ 5, 6$
9	$\bot$	$\neg e\ 7, 8$
10	$\bot$	$\exists x e\ 2, 3-9$
11	$\neg(\exists x (P(x) \land Q(x)))$	$\neg i\ 2-10$

① $(y=0) \wedge (y=x) \vdash 0=x$

①	$(y=0) \wedge (y=x)$	premise
②	$y=0$	$\wedge e_1$, 1
③	$y=x$	$\wedge e_2$, 1
④	$0=x$	$=e$ 2, 3

③ $(\forall x\, P(x)) \rightarrow (\forall x\, Q(x)) \vdash \forall x((\forall x\, P(x)) \rightarrow Q(x))$

7 $P(a) \vdash \forall(a = x \to P(x))$

1	$P(a)$	premise
2	x_0	
3	$a = x_0$	assume
4	$P(x_0)$	$=e$ 3,1
5	$a = x_0 \to P(x_0)$	$\to i$ 3-4
6	$\forall x((a = x) \to P(x))$	$\forall x i$ 2-5

(6) $\exists x \forall y\, P(x, y) \vdash \forall y \exists x\, P(x, y)$

1.	$\exists x \forall y\, P(x, y)$	premise
2.	y_0	
3.	x_0	
4.	$\forall y\, P(x_0, y)$	assume
5.	$P(x_0, y_0)$	$\forall y$e 4
6.	$\exists x\, P(x, y_0)$	$\exists x$i 5
7.	$\exists x\, P(x, y_0)$	$\exists x$e 1, 3–6
8.	$\forall y \exists x\, P(x, y)$	$\forall$i 2–7

(3) $T(x)$ = x is a taxpayer

$P(x)$ = x is a politician

$Ph(x)$ = x is a philantropist

$$\exists x\, T(x) \rightarrow \forall x(P(x) \rightarrow T(x)),$$

$$\exists x\, Ph(x) \rightarrow \forall x(T(x) \rightarrow Ph(x))$$

$$\vdash \exists x\,(T(x) \wedge Ph(x)) \rightarrow \forall x\,(P(x) \wedge Ph(x))$$

1

(a) All red things are in a box.

$$\forall x\,(R(x) \rightarrow B(x))$$

$R(x) \rightarrow$ x is Red

$B(x) \rightarrow$ x is in a the box

(b) Only red things are in the box.

$$\forall x\,(B(x) \rightarrow R(x))$$

$B(x) \rightarrow$ x is in box

$R(x) \rightarrow$ x is Red

(c) No animal is both a cat and dog.

$$\neg \exists x\,(A(x) \wedge D(x) \wedge C(x))$$

$A(x) \rightarrow$ x is an animal

$D(x) \rightarrow$ x is a Dog

$C(x) \rightarrow$ x is a Cat.

(d) Every prize was won by a boy

$$\forall x \, (P(x) \rightarrow \exists y (B(y) \wedge W(x,y)))$$

$P(x) =$ x is a prize.

$B(y) =$ y is a boy

$W(x,y) =$ x was won by y.

(e) A boy won every prize.

$$\exists y \, (B(y) \wedge \forall x (P(x) \rightarrow W(x,y)))$$

(2) (a) $\forall x \, \forall y \, (x = y)$

All two things are equal

Or

There is only one thing.

(b) $\exists x \, \exists y \, (x \neq y)$

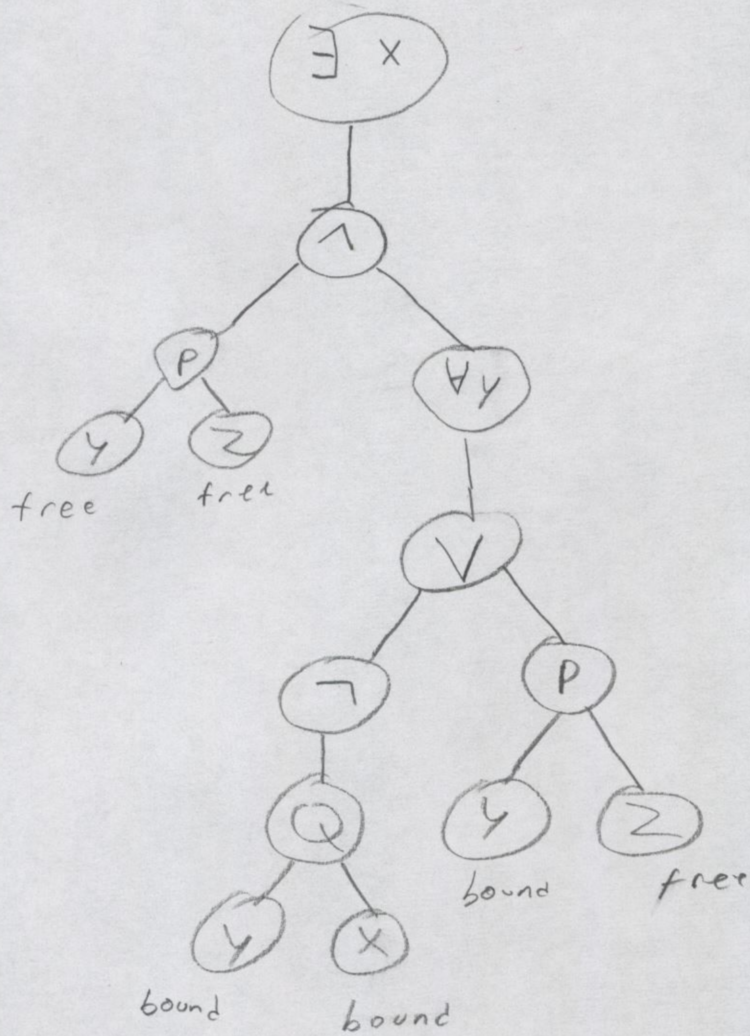
Not all things are equal.

(c) $\exists x \, \exists y \, (x \neq y \wedge \forall z \, (z \neq x \rightarrow z = y)$

Having two items that are not equal means that there is a third element equal to one of these two elements.

2 Parse Trees, Variables and Substitutions

(a) $\Phi = \exists x \, (P(y,z) \wedge (\forall y (\neg Q(y,x) \vee P(y,z))))$



(b) $\Phi = \exists x \, (P(y,z) \wedge (\forall y (\neg Q(y,x) \vee P(y,z))))$

bound bound

free Variables

bound free

3. $w, f(x), g(x,y)$

a. All of these variables are free for x in Φ.

$w, f(x), g(x,y)$

b. w is free for y in Φ.

$p \to \neg q \neq \neg(p \wedge q)$

c. $\Phi[w/x]$

$= \exists x(P(y,z) \wedge (\forall y(\neg Q(y,x) \vee P(y,z))))$

$\Phi[f(w),y]$

$= \exists x(P(f(w),z) \wedge (\forall y(\neg Q(y,x) \vee P(y,z))))$

c

$\Phi[g(w,z)/z]$

$= \exists x(P(y,g(w,z)) \wedge (\forall y(\neg Q(y,x) \vee P(y,g(w,z))))$