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Assortment Optimization under a Single Transition Choice Model

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In this study, we consider a new customer choice model which we call the single transition choice model. In this model, there is a universe of products and customers arrive at each product with a certain probability. If the arrived product is unavailable, then the seller can recommend a subset of available products and the customer will purchase one of the recommended products or choose not to purchase with certain transition probabilities. The distinguishing features of the model are that the seller can control which products to recommend depending on the arrived product, and each customer either purchases a product or leaves the market after one transition. We study the assortment optimization problem under this model. Particularly, we show that it is NP-Hard even if the customer can transition from each product to at most two products. Despite the computational complexity, we provide polynomial time algorithms or approximation algorithms for several special cases, such as when the customer can only transition from each product to at most a given number of products and the size of each recommended set is bounded. Our approximation algorithms are developed by invoking the submodularity arguments, or connecting the problem with maximum constraint satisfaction problem and applying randomized rounding techniques to its semidefinite programming relaxation. We also provide a tight worst-case performance bound for revenue-ordered assortments. In addition, we propose a compact mixed-integer program formulation, which is efficient for moderate size problems. Finally, we conduct numerical experiments to demonstrate the effectiveness of the proposed algorithms.

Key words: assortment optimization; choice model; approximation algorithms; revenue-ordered assortment
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1. Introduction

The fast development of information technology and rapid growth of online sales have presented great opportunities—as well as challenges—for retailers to use data to increase their bottom lines. One of the challenges is to determine which subset of products to make available to customers in order to maximize the expected revenue. Such a problem is often referred to as the assortment optimization problem. According to McKinsey & Company (2019), effective assortment management based on analytical tools can significantly improve the financial performance of retailers. In particular, two important levers in assortment optimization, systematic delisting and strategic listing, when properly employed, can steer customers toward higher-margin products from lower-margin ones, resulting in an up to 0.5 percentage point lift in profit margin or 2–4 percent of revenue growth. To realize the potential of analytics-based assortment optimization, a key issue is to identify an appropriate model to characterize the choice behavior of customers when facing a subset of products. Much recent research has focused on finding such a choice model and solving the associated assortment optimization problem.

One of the customer choice models that has recently gained much attention is the Markov chain choice model proposed by Blanchet et al. (2016). One interpretation of the Markov chain choice model is as follows: Suppose there is a universe of products, among which the seller makes some available to the customers (i.e., chosen in the assortment). Customers arrive at each of the products with a certain exogenous probability. If the product a customer arrived at is unavailable (not in the offered assortment), then the customer will transition to other products (including the no-purchase option) with certain transition probabilities. This process stops when the customer finds an available product to purchase or leaves the

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market.[1] Such a new perspective in modeling customer choice has been validated by empirical data (see, e.g., Blanchet et al. 2016, Şimşek and Topaloglu 2018) and has been extensively studied since then. We will provide a more detailed review of the Markov chain choice model in section 2.

While the above explanation of the Markov chain choice model provides a nice way to interpret the customer choice behavior and the associated decision-making problem, there are two possible improvements for the model under certain scenarios. First, in the above interpretation of the Markov chain choice model, it is assumed that customer's transition between products follows an exogenous process (according to the transition probabilities). However, in practice, after a customer's arrival, it is often the seller's recommendation that will determine the transition of customers. For example, in an online retail environment, customers sometimes search a certain type of product that they have in mind. If they find that the product is unavailable, then they are likely to click other products displayed in the recommendation section associated with this unavailable product. Second, in the Markov chain choice model, it is implicitly assumed that customers could transition arbitrarily many times among unavailable products, whereas in practice, customers often have much less patience and if a customer transitions to an unavailable product, he/she may just leave the market altogether. This is especially likely in the situation when this product is a recommended product—in such a case, it is not hard to imagine that customers may become frustrated and as a result, abandon the entire purchase session.

In this study, motivated by the two points above, we propose a new choice model which we call the single transition choice model, or STCM in short. Particularly, in the STCM, there is also a universe of products and the seller first chooses a subset to include in the assortment (make them available to purchase). Customers initially arrive at each product with a certain probability as in the Markov chain choice model. However, if the product a customer arrived at is not available, under the STCM, the seller can choose a subset of *available* products to recommend to the customer, and the customer will transition to each product in the recommended set (as well as the no-purchase option) based on a set of transition probabilities. Note that under the STCM, a customer will either purchase a product or leave the market after a single transition. A possible scenario that can be captured by the STCM is motivated from online retailing. A customer who shops at an online retailer (e.g., Amazon or Bestbuy) is interested in purchasing a certain product (e.g., a laptop or a microwave). She already has some specific product in mind (e.g., through a friend's recommendation or based on a prior purchase). To start with, she searches for that product. If the product is available (in the offered assortment), then she will purchase that product. Otherwise, if the product is not in the offered assortment, then the seller could recommend some similar products to this customer which will be displayed on the same page as the searched product (see Figure 1 for a screenshot of such a scenario. Note that for those products that are not chosen in the assortment, often there is still a webpage for that product showing that it is unavailable, and the seller may suggest alternative products on the same page). Then the customer will choose among those recommended products or decide to leave without purchasing.

The STCM exhibits the following distinctive features:

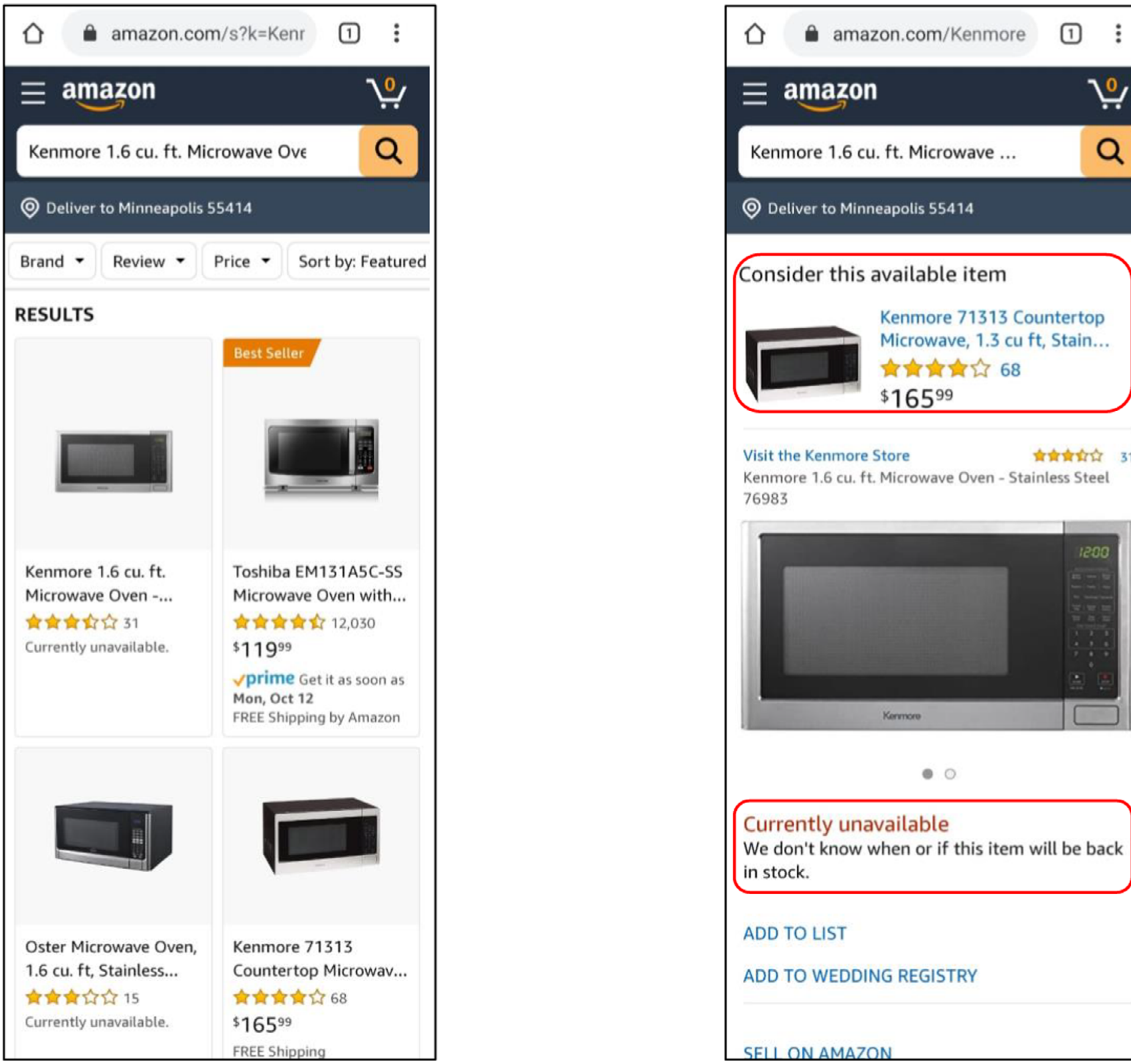
1. Allow more flexible control for the seller in terms of which set of products to recommend after a customer searches an unavailable product, which may give opportunities to achieve higher revenue;
2. By requiring that the seller can only recommend available products, the transition of customers is more realistic, which better reflects the limited patience of customers.

In addition to the STCM introduced above, we also consider a restricted version of the STCM in which the customer can only transition from each product to at most V products (in addition to the no-purchase option), and the size of recommended set of each product is at most C. We refer to this problem as the STCM(V,C) problem. The STCM(V,C) problem allows us to capture or approximate the scenario where a customer is only willing to consider a limited number (V) of close substitutes if his/her favorite product is unavailable, and the retailer can only choose a limited number (C) of products to recommend. Such limited considerations of customers and constrained recommendation capacity of the seller have been widely observed in practice and studied in the literature (we will provide a thorough review of related studies in section 2). Thus, it is of interest to consider this restricted version of the STCM.

Here, we study the properties of the STCM and the associated assortment optimization problem (both the general and the restricted version). In particular, we make the following contributions:

1. We show that the assortment optimization problem under the STCM is NP-Hard in general and does not admit a fully polynomial time approximation scheme (FPTAS). Nevertheless, we establish a tight worst-case performance bound for the revenue-ordered assortment;

Figure 1 A Screenshot from the Website of Amazon (mobile version) [Color figure can be viewed at wileyonlinelibrary.com]



2. When the customer can only transition from each product to one other product or the transition probabilities are homogeneous with respect to the starting points (the initial products the customer arrived at), we show that the assortment optimization problem under the STCM can be solved in polynomial time of the number of products;
3. We study the restricted assortment optimization problem STCM(V,C), in which the customer can only transition from each product to at most V products (in addition to the no-purchase option), and the size of recommended set of each product is limited to at most C. We show that even STCM(2,1) is NP-hard, and does not admit an FPTAS unless P=NP. We then investigate the approximability of various cases of STCM(V,C). Particularly, we have the following results.
   (a) We show that the objective functions of STCM(n,1) and STCM(2,2) are non-monotone submodular functions. Therefore, there exists a (randomized) 1/2-approximation algorithm for STCM(n,1) and STCM(2,2) by applying the result of maximization of submodular functions (Buchbinder et al. 2015). However, we provide a counterexample that the objective function of STCM(3,2) could be

non-submodular. Thus, in general, one may not use the submodular function optimization approach to obtain a 1/2-approximation algorithm for STCM(V,C) when $V \geq 3$ and $C \geq 2$.

(b) To improve upon the above results, we develop a (randomized) 0.588-approximation algorithm for STCM(n,1). The main idea of the algorithm is to transform the problem to a maximum constraint satisfaction problem (MAX CSP), then we apply a randomized rounding technique to its semidefinite programming (SDP) relaxation. The analysis of performance ratio of this approximation algorithm is built upon the results of MAX 2-AND and MAX Not-All-Equal-Satisfiability (NAE SAT) problem, and a nonsymmetric correlation inequality for normal distribution by Szarek and Werner (1999), which is different from the previous literature of the MAX CSP.

(c) Furthermore, we propose a (randomized) 0.733-approximation algorithm for STCM (2,1), which is based on converting the problem into a MAX 3-CSP problem (Zwick 1998) and utilizing SDP relaxation to obtain an approximation.

In Table 1, we summarize the results for various STCM and STCM(V,C) cases and the techniques studied in this work. We note that in the second column of Table 1, "poly" indicates that this case can be solved in polynomial time, and otherwise this case is NP-Hard and the value in the cell refers to the performance guarantee of our proposed approximation algorithm.

We also evaluate the performance of these SDP-based randomized rounding algorithms and other proposed algorithms for STCM(V,C) by numerical experiments.

4. We establish compact mixed integer program (MIP) formulations for both the general and the restricted STCM problems. We show by numerical experiments that the MIP formulation can be solved efficiently for problems of moderate size. We also conduct numerical experiments to demonstrate the benefit for the retailer of being able to recommend different subsets toward customers who arrive at different products.

The remainder of this study is organized as follows: In section 2, we review related literature to our work. In section 3, we formally state the STCM and its associated assortment optimization problem. In section 4, we study the general STCM problem, analyzing its computational complexity, investigating the worst-case performance bound of revenue-ordered assortments, and proposing an MIP formulation for this problem. Then in section 5, we consider the restricted problem STCM(V,C) and propose approximation algorithms for various cases. We present some numerical experiment results in section 6. Finally, we provide some concluding remarks in section 7.

2. Related Work

In this section, we review related literature to our work. At the high level, our work is related to the fast growing research area of choice models and the associated revenue management problems. Particularly, our work is closely related to the recently proposed Markov chain choice model (Blanchet et al. 2016) and the associated assortment optimization problem. In the following, we focus our literature review on these lines of works and refer the readers to Train (2009) and Özer and Phillips (2012) for comprehensive reviews of general research about choice models and revenue management respectively.

The Markov chain choice model is first introduced by Blanchet et al. (2016). One interpretation of the Markov chain choice model is as follows: There is a line of products and each customer arrives at each product with a certain probability. If the product is available, then the customer will purchase it, otherwise the customer will transition to another product or leave with certain transition probabilities. The process terminates until the customer purchases a certain product or chooses to leave. In Blanchet et al. (2016), the authors show that such a model is a good approximation to several well-studied choice models. Moreover, they show that the assortment optimization problem under the Markov chain choice model can be solved in polynomial time by relating it to an optimal stopping problem on a Markov chain. There are several follow-up works that study the Markov chain choice model. For example, Feldman and Topaloglu (2017) study the network revenue management problem under the Markov chain choice model and

Table 1 Summary of Results

Problem	Performance guarantee	Technique
general STCM	$\max\left\{\frac{1}{d}, \frac{1}{1+\log\frac{r_{\max}}{r_{\min}}}\right\}$	revenue-ordered assortment
homogeneous STCM	poly	boils down to MNL
STCM(1,1)	poly	dynamic programming
STCM(2,1)	0.733	MAX CSP and SDP rounding
STCM(n,1)	0.5882	MAX CSP and SDP rounding
STCM(2,2)	0.5	submodularity

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present an algorithm based on linear optimization. Désir et al. (2020a) study the capacity constrained assortment optimization problem. Désir et al. (2015) study the robust assortment optimization under this model. As described in the introduction, the STCM we propose differs from the Markov chain choice model in two dimensions: (i) we add a control for the seller to decide which set of products to recommend when customers arrive at an unavailable product, and (ii) we only allow a single transition of the customers. We propose solution methods for solving the assortment optimization problem under this new model.

Our work is also related to the growing literature that studies assortment optimization problem. Assortment optimization is a central problem in revenue management in which the seller selects a subset of products to offer in order to maximize the expected revenue. The assortment optimization problem was first considered by Talluri and van Ryzin (2004), in which the authors study a single resource revenue management problem. They show that a class of revenue-ordered assortment is optimal when customers choose according to a multinomial logit (MNL) choice model. There are many subsequent works studying the assortment optimization problem under various customer choice models. Some examples include the nested logit model (see, e.g., Davis et al. 2014, Gallego and Topaloglu 2014, Li and Rusmevichientong 2014, Li et al. 2015); the mixture of multinomial logit (MMNL) model (see, e.g., Désir et al. 2020b, Rusmevichientong et al. 2014), the general attraction model (Gallego et al. 2015), the ranking-based choice models (see, e.g., Aouad et al. 2018, Bertsimas and Mišić 2019, Honhon et al. 2012, the Mallows model (see, e.g., Désir et al. 2016), and more recently, more complicated choice models such as the consider-then-choose model (Aouad et al. 2015), the MNL model with endogenous network effects (Wang and Wang 2017), the MNL model with reference price effects (Wang 2018), choice model when customer searches for product information (Sahin and Wang 2017), etc. In another direction, there has also been interest in studying assortment optimization problems under nonparametric choice models (see Farias et al. 2013, Jagabathula 2016, Jagabathula and Rusmevichientong 2017) or in a dynamic environment motivated by various purposes such as when the parameters in the model are unknown and need to be learned (Kallus and Udell 2020, Rusmevichientong et al. 2010), when there are inventory constraints (Aouad et al. 2019, Bernstein et al. 2015, Chen et al. 2016, Golrezaei et al. 2014), or when the assortment is built over time (Davis et al. 2015). In this work, we consider the assortment optimization problem under a new model, the STCM. Similar to other works that study the assortment optimization problem, we study the complexity of the problem under different cases and propose efficient algorithms to solve the problem.

In the study of assortment optimization problem, an important class of assortment is called the revenue-ordered assortment, in which the assortment is selected according to the revenue order of each product. Revenue-ordered assortment has been shown to be optimal in many assortment optimization problems or has a worst-case performance guarantee (see, e.g., Rusmevichientong and Topaloglu 2012, Rusmevichientong et al. 2014, Talluri and van Ryzin 2004, Wang and Wang 2017). Indeed, as shown in Berbeglia and Joret (2020), the optimal revenue-ordered assortment achieves a worst-case ratio of the optimal revenue under a class of “regular discrete choice models.” In this study, we show a tight worst-case performance guarantee for the revenue-ordered assortment for the general STCM problem.

In addition to the general STCM, we consider a restricted version of the STCM in which the customer can only transition from each product to at most V products and the size of recommended set of each product is at most C. In the revenue management literature, such limitation on customer’s attention/preference and the seller’s offering size is quite common. For example, Feldman et al. (2019) study the setting of ranking-based choice model in which the customers consider a limited number of products. They also provide some references to suggest that the customers prefer a limited substitution when the purchasing bias is high or the cost of leaving the system is low, such as in automobile purchase (Hauser et al. 2009, Lapersonne et al. 1995) and travelers’ destination choice (Crompton and Ankomah 1993). In addition, some works in retailing and consumer psychology suggest that sometimes it may not be beneficial for the retailer to offer a boarder recommendation set. Simonson (1999) provides a comprehensive review and extensive empirical studies on how the product assortment influences the buyer preferences. One observation is that the customer may tend to delay their purchase when they consider two or more similarly attractive alternatives. Therefore, a potentially profitable strategy for the retailer is to offer one and only option that is clearly superior to the others for that customer, instead of offering a large assortment. Similar observations have been obtained in Schwartz (2004), Chernev et al. (2010), and Scheibehenne et al. (2010).

Several closely related work to ours include the work by Paul et al. (2016), Kök and Fisher (2007), and Feldman et al. (2019). In Paul et al. (2016), the authors consider a two-step choice model and a two-product nonparametric choice model. Both of those models are based on the Markov chain choice model and only allow customers to make a single transition. In

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particular, the former one only allows homogeneous transition probabilities with respect to the starting point, while the latter one allows heterogeneous transition probabilities. They study the assortment optimization problem under these models, showing that both problems are NP-Hard, while proposing an FPTAS for the former problem and a 1/2-approximation algorithm for the latter problem. Similar models have also been considered in Kök and Fisher (2007) in which the authors mainly consider the estimation problem and propose heuristic algorithms for the assortment optimization problem. More recently, Feldman et al. (2019) consider a more general k-product nonparametric choice model, and propose a 0.874-approximation algorithm based on an SDP relaxation when $k=2$ and a $(2/k)\cdot(1-1/k)^{k-1}$-approximation algorithm for general k based on a linear programming (LP) relaxation. In contrast to those models, in the STCM, we allow the seller to choose which subset of products to recommend after a customer reaches an unavailable product instead of assuming the customer transitions are exogenous. As we will show in our study, such flexibility will lead to different optimization problems and require different algorithms.

One technique used in this work when developing approximation algorithms for the STCM problem is to formulate the problem as a Boolean constraint satisfaction problem (CSP), and then apply randomized rounding methods to its SDP relaxation. After the breakthrough of the 0.878-approximation algorithm for the maximum cut (MAX-CUT) and maximum 2-satisfiability (MAX 2-SAT) problem (Goemans and Williamson 1995), there have been extensive studies on the performance guarantee of SDP-based randomized rounding algorithms for various types of CSP problems. We refer the readers to Makarychev and Makarychev (2017) and references therein for a comprehensive review. Our work is most related to the MAX 2-AND problem (see Lewin et al. 2002) and the MAX SAT problem (see Avidor et al. 2006). In particular, we represent our assortment optimization problems by the CSP constraints combining with these two types of constraints. However, the performance guarantee for such specific type of CSP constraint is only studied when there are at most three variables per constraint (Zwick 1998). For general MAX CSP problem with at most k variables per constraint, the best approximation ratio is $\frac{0.62661k-o(1)}{2^k}$ (Makarychev and Makarychev 2014, the little $o(1)$ tends to 0 as $k\to\infty$), which is poor if we directly apply it to solve our problem. To obtain stronger approximation results, we develop approximation algorithms based on a strengthened SDP relaxation introduced by Andersson and Engebretsen (1998) and Zhang et al. (2004), and analyze the performance guarantee built upon approximation results for the MAX 2-AND problem and MAX NAE SAT problem. In addition, we incorporate a nonsymmetric correlation inequality for normal distribution by Szarek and Werner (1999) in the analysis of performance guarantee for our proposed algorithm. Therefore, although our analysis is built upon previous works, new methods and results are developed for solving our specific problem.

3. Model

In this section, we introduce the STCM and the associated assortment optimization problem. Consider a seller who manages a collection of n products denoted by $\mathcal{N}=\{1,\ldots,n\}$ with 0 being the no-purchase option. We assume each product j has a fixed revenue r_j (one can also view r_j as the profit margin of product j if the objective is to maximize profit). Without loss of generality, we assume that the products are sorted in the non-increasing order based on the revenue, that is, $r_1\geq r_2\geq\cdots\geq r_n$. In the STCM, the seller first selects an assortment $S\subseteq\mathcal{N}$ to offer to the customers (in other words, the seller makes available a subset of products S). Then, with probability $\lambda_j\geq 0$, a customer arrives at product j. (Naturally, we assume that $\sum_{j=1}^{n}\lambda_j=1$.) When a customer arrives at product j, if product j is in the offered assortment S, then the customer purchases product j and the seller gains revenue r_j. Otherwise, the seller can recommend a subset of products within the assortment, that is, a subset $R_j\subseteq S$, and the customer will either select one of the products in R_j or leave the market without purchasing any product. In particular, there is a weight $v_{ji}\geq 0$ for any $j\in\mathcal{N}$ and $i\in\mathcal{N}\cup\{0\}$, satisfying $\sum_{i\in\mathcal{N}\cup\{0\}}v_{ji}=1$ for any j, such that the probability a customer will transition to product $i\in R_j$ or leave the market (transition to product i = 0) given that the customer arrived at product $j\notin S$ is

$$\rho_{ji}=\frac{v_{ji}}{\sum_{i\in R_j}v_{ji}+v_{j0}}.$$

We stipulate that $v_{jj}=0$ for each $j\in\mathcal{N}$, as it is unrealistic that a customer will transition to the same product as he/she knows it is unavailable. The assortment optimization problem under the STCM (we will call it the STCM problem for short in the following) is to find an assortment $S\subseteq\mathcal{N}$, and for each j\not\in$S$, a recommended subset of products $R_j\subseteq S$, such that the expected total revenue is maximized.

Now we derive the formal expression of the STCM problem. Let $\Pr_i(S,\{R_j\}_{j\notin S})$ be the probability that product i is purchased by the customer when the offered assortment is S and the recommended set for j \not\in$S$ is R_j. For short, we let $\mathcal{R}^S=\{R_j\}_{j\notin S}$ and the

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corresponding probability be $\Pr_i(S,\mathscr{R}^S)$. We sometimes omit the superscript S of $\mathscr{R}^S$ and denote the probability by $\Pr_i(S,\mathscr{R})$ when there is no confusion. We have

$$\Pr_i(S,\mathscr{R})=\begin{cases}\lambda_i+\sum_{j:j\notin S,i\in R_j}\lambda_j\frac{v_{ji}}{\sum_{i\in R_j}v_{ji}+v_{j0}}, & \text{if } i\in S,\\ \sum_{j:j\notin S}\lambda_j\frac{v_{j0}}{\sum_{i\in R_j}v_{ji}+v_{j0}}, & \text{if } i=0,\\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

Let $\mathbf{R}(S,\mathscr{R})$ be the expected revenue when the offered assortment is S and the recommended set for j\not\in$S$ is R_j. Then we have

$$\mathbf{R}(S,\mathscr{R})=\sum_{i\in S}r_i\Pr_i(S,\mathscr{R})=\sum_{i\in S}\lambda_i r_i+\sum_{j\notin S}\sum_{i\in R_j}\lambda_j r_i\frac{v_{ji}}{\sum_{i\in R_j}v_{ji}+v_{j0}}$$

and the assortment optimization problem under the STCM can be written as

$$\max_{S\subseteq\mathscr{N},R_j\subseteq S,\forall j\notin S}\mathbf{R}(S,\mathscr{R})=\max_{S\subseteq\mathscr{N},R_j\subseteq S,\forall j\notin S}\sum_{i\in S}\lambda_i r_i+\sum_{j\notin S}\sum_{i\in R_j}\lambda_j r_i\frac{v_{ji}}{\sum_{i\in R_j}v_{ji}+v_{j0}}. \tag{2}$$

We remark that in the case of $v_{j0}=0$ for some $j\in S$, if the seller recommends none of the products (i.e., $R_j=\emptyset$), or only recommends products with zero transition weights ($v_{ji}=0$), then the ratio in (2) would become $0/0$. To avoid confusion in those cases, we stipulate that $0/0=0$ in the rest of our discussion.

By examining (2), if we have chosen an assortment S, then the recommended set for any $j\notin S$ can be found by solving $\max_{R_j\subseteq S}\frac{\sum_{i\in R_j}r_iv_{ji}}{\sum_{i\in R_j}v_{ji}+v_{j0}}$. Note that this is an assortment optimization problem under an attraction model, for which it has been shown that a revenue-ordered assortment is optimal (Talluri and van Ryzin 2004). Therefore, we can efficiently find the optimal recommended set for each j\not\in$S$ if the assortment S has been determined. However, finding an optimal S to offer initially is still a difficult task. As we will show in section 4, the problem is NP-Hard in general, and we will develop algorithms for solving this problem in the following sections.

Before we end this section, we further discuss the position of the STCM among choice models. First, we note that since there is an additional (decision) layer in the STCM which is the recommended sets, the STCM is different from other choice models in the sense that it is no longer a mapping from a vector of utilities (or a vector of arrival probabilities and transition probabilities) to a vector of choice probabilities. In the remaining discussions in this section, we refer to the STCM as the choice model with the optimal recommended sets in the second stage. In this case, the revenue of the products will affect the recommended sets in the second stage, which will further affect the choice probabilities. This makes the STCM very different from other commonly used choice models such as the random utility models (RUM) or the Markov chain choice model, in which the choice probabilities are independent of the revenue of the products. Moreover, even given all parameters fixed (including the revenue of each product), the STCM does not belong to the RUM (therefore any subclass of the RUM) because it does not satisfy the regularity axiom (we will discuss this in more detail in section 4).

However, despite of these differences, in the following, we consider two more restrictive versions of the STCM and discuss connections between them and other choice models.

The first model is to impose a restriction on the STCM that the recommended set R is only a function of the assortment S, that is, all the customers have the same recommended set R, regardless of which product they arrive first. We use STCM$_R$ to denote this model. The assortment optimization problem under the STCM$_R$ can be formulated as follows:

$$\max_{S\subseteq\mathscr{N},R\subseteq S}\mathbf{R}(S,R)=\max_{S\subseteq\mathscr{N},R\subseteq S}\sum_{i\in S}\lambda_i r_i+\sum_{j\notin S}\sum_{i\in R}\lambda_j r_i\frac{v_{ji}}{\sum_{i\in R}v_{ji}+v_{j0}}. \tag{3}$$

The second model is an even more restrictive model than the STCM and the STCM$_R$, where the seller recommends the entire set S to the customers. We call this model STCM$_S$ and the corresponding assortment optimization problem (called the STCM$_S$ problem) is stated as follows:

$$\max_{S\subseteq\mathscr{N}}\mathbf{R}(S)=\max_{S\subseteq\mathscr{N}}\sum_{i\in S}\lambda_i r_i+\sum_{j\notin S}\sum_{i\in S}\lambda_j r_i\frac{v_{ji}}{\sum_{i\in S}v_{ji}+v_{j0}}. \tag{4}$$

We note that the STCM$_S$ problem can be viewed as a generalization of the optimal assortment problem under the mixed multinomial logit (MMNL, see McFadden and Train 2000) model. We have the following result:

Proposition 1. *The assortment optimization problem under the MMNL model can be reduced to* STCM$_S$ *problem in polynomial time.*

Because of this relation, we know that a restrictive version of the STCM problem, the STCM$_S$ problem contains the assortment optimization problem under the MMNL model. We remark that the assortment optimization problem under the MMNL model is NP-Hard and does not have a polynomial time approximation algorithm with performance guarantee better than $O(1/K^{1-\delta})$ for any $\delta>0$ unless NP⊆BPP, where K is the number of mixtures and BPP stands for the complexity class of bounded-error probabilistic polynomial time (Désir et al. 2020b). However, the result in Proposition 1 does not directly imply that an MMNL assortment optimization problem can be reduced to an STCM problem, because imposing the restriction that the seller can only recommend the entire set of products does not necessarily make the problem easier.

4. General STCM Problem

In this section, we consider the general STCM problem. In particular, we study the computational complexity of the general STCM problem, a tight worst-case performance bound for revenue-ordered assortments as well as propose efficient algorithms for solving the STCM problem. We also discuss some connections between the STCM problem and the assortment optimization problem under the classical MNL model and the Markov chain choice model. We start with the following theorem stating that the STCM problem is hard to solve in general.

Theorem 1. *The STCM problem is NP-Hard and does not admit an FPTAS unless P = NP, even if we restrict that the customer can only transition from each product to at most two products.*

The proof of Theorem 1 and all subsequent results are relegated to Appendix A in the online appendix. The proof of Theorem 1 is based on a reduction from the independent set problem (see, e.g., Garey and Johnson 1979). By Theorem 1, it is not possible to obtain polynomial time algorithms or FPTAS for solving the STCM problem in general unless P = NP.

In the revenue management literature, a class of assortment called the revenue-ordered assortment is of special interest. A revenue-ordered assortment consists of products that follow the revenue order, that is, it is of form {1,. . .,k} for some k. It is shown that under the MNL model, there is always an optimal assortment that is a revenue-ordered assortment (Talluri and van Ryzin 2004). Since revenue-ordered assortments have a simple structure, it is often of interest to study whether it is optimal to only consider this class of assortments, and if not, what is the worst-case revenue gap between the best revenue-ordered assortment and the optimal assortment. In the following, we investigate this question for the STCM problem and provide a tight worst-case performance bound for revenue-ordered assortments for the STCM problem.

To start our discussion, we review some results obtained in Berbeglia and Joret (2020). In Berbeglia and Joret (2020), the authors study the worst-case performance guarantee of the revenue-ordered assortments under a class of choice models, which they call *regular discrete choice model*. Particularly, a regular discrete choice model should satisfy the regularity axiom, that is, $\Pr_i(S) \le \Pr_i(S')$, $\forall S' \subseteq S \subseteq \mathcal{N}$ and $i \in S' \cup \{0\}$. It indicates that if one shrinks the choice set, then the probability of choosing every alternative that remains in the choice set will increase. Berbeglia and Joret (2020) establish the worst-case performance bound for the best revenue-ordered assortment for any regular discrete choice model.

In the STCM, we define the choice probability of product i under set S as the choice probability under S with the optimal recommended sets for all $j \notin S$. That is,

$$\Pr_i(S) = \begin{cases} \lambda_i + \sum\limits_{j: j\notin S, i\in R_j^*} \lambda_j \dfrac{v_{ji}}{\sum_{i\in R_j^*} v_{ji} + v_{j0}}, & \text{if } i \in S, \\ \sum\limits_{j: j\notin S} \lambda_j \dfrac{v_{j0}}{\sum_{i\in R_j^*} v_{ji} + v_{j0}}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{where } R_j^* = \arg\max_{R_j \subseteq S} \frac{\sum_{i\in R_j} r_i v_{ji}}{\sum_{i\in R_j} r_i v_{ji} + v_{j0}}. \quad (5)$$

In the following, for the ease of discussion, we assume that R_j^* is unique for every S. (Note that this can be achieved by adding an infinitesimal perturbation to the v_{ji}s and the impact on the revenue can be made arbitrarily small.) The next example shows that the STCM in general does not satisfy the regularity axiom, thus is not a regular discrete choice model.

Example 1. *We consider an example with four products with revenues $r_1 = 4$, $r_2 = 3$, $r_3 = 2$, and $r_4 = 1$. The arrival probabilities are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4$. The transition probabilities from product 1 to other products are $(v_{11}, v_{12}, v_{13}, v_{14}, v_{10}) = (0,0,0,1/2,1/2)$, and the transition probabilities from product 2 to other products are $(v_{21}, v_{22}, v_{23}, v_{24}, v_{20}) = (1/6,0,1/6,1/3,1/3)$. (We do not specify the transition probabilities for other pairs of products, as they are immaterial to the result.) Consider the assortments S = {1,3,4} and $S' = \{3,4\}$, and the choice probability of product 3. It can be calculated that the best recommended set for product 2 under S is {1,3}, and the best recommended set for product 2 under S' is {3,4}. One can calculate that the choice probability $\Pr_3(S) = \lambda_3 + \lambda_2 \frac{v_{23}}{v_{21}+v_{23}+v_{20}} = 0.3125$ is strictly greater than $\Pr_3(S') = \lambda_3 + \lambda_2 \frac{v_{23}}{v_{23}+v_{24}+v_{20}} = 0.3$, which violates the regularity axiom.*

An intuitive explanation for Example 1 is that when one shrinks the assortment from a larger set S to a smaller set S', the recommended sets of some product not in S (product 2) will change and some products may be added to the recommendation set (product 4). In such a case, the choice probability of some product in S (product 3) may not necessarily increase, since the product not in S may transition to the newly added product with a larger transition probability than before (product 2 transitions to product 4 with a higher probability than 1 and 3). In addition, such scenario may lead to the violation of the regularity axiom.

We remark that Example 1 also implies that the STCM is not a RUM since it has been shown (in Berbeglia and Joret 2020, Lemma 4.1) that any RUM must satisfy the regularity axiom. Despite that the STCM in general does not satisfy the definition of regular discrete choice model thus we cannot directly apply the results in Berbeglia and Joret (2020), we are still able to prove that the optimal revenue-ordered assortment achieves a certain fraction of the optimal revenue in the STCM problem using a similar idea as in Berbeglia and Joret (2020). We have the following theorem:

THEOREM 2. *The optimal revenue-ordered assortment gives at least* $\max\left\{\frac{1}{d}, \frac{1}{1+\log\frac{r_{\max}}{r_{\min}}}\right\}$ *fraction of the optimal revenue for the STCM problem, where d is the number of distinctive revenues among the alternatives and* $r_{\max}$ *and* $r_{\min}$ *are the largest and the smallest revenue among all products, respectively. Furthermore, the performance bound is tight in that there exist instances such that the optimal revenue of the best revenue-ordered assortment is* $O(\frac{1}{d})$ *or* $O\left(\frac{1}{1+\log\frac{r_{\max}}{r_{\min}}}\right)$ *fraction of the optimal revenue, respectively.*

Theorem 2 states that although not necessarily optimal, the optimal revenue-ordered assortment still has an expected revenue close to that of the optimal assortment for the STCM problem, especially when d is small or the revenues of the products are close to each other.

Next, we provide a compact MIP formulation for solving the STCM problem. The formulation will be useful to solve problems of moderate size. To obtain the MIP formulation, we define binary variables x_j and y_{ji}, $\forall i,j = 1,\ldots,n$, where $x_j = 1$ indicates that product j is selected in the assortment S, and $y_{ji} = 1$ indicates that product i is in the recommended set R_j of product j. Then the optimization problem (2) can be formulated as the following nonlinear integer optimization problem (6):

$$\max \sum_{j=1}^{n} \lambda_j r_j x_j + \sum_{j=1}^{n} \lambda_j (1-x_j) \frac{\sum_{i=1}^{n} r_i v_{ji} y_{ji}}{\sum_{i=1}^{n} v_{ji} y_{ji} + v_{j0}} \tag{6a}$$

$$\text{s.t. } y_{ji} \le x_i, \quad \forall i,j = 1,\ldots,n, \tag{6b}$$

$$y_{ji} \le 1 - x_j, \quad \forall i,j = 1,\ldots,n, \tag{6c}$$

$$x_i, y_{ji} \in \{0,1\}, \quad \forall i,j = 1,\ldots,n. \tag{6d}$$

In the above formulation, constraints (6b) and (6c) guarantee that a product i can be recommended after product j is visited only if product j is not included in the assortment but product i is. In the following, we assume that $v_{j0} \neq 0$ for all $j \in \mathcal{N}$ (otherwise we can set v_{j0} to be an infinitesimal number, and all the results follow in the same way). Similar to the idea in Davis et al. (2013), we can rewrite (6) as the following MIP (7):

$$\max \sum_{j=1}^{n} \lambda_j r_j x_j + \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} r_i z_{ji} \tag{7a}$$

$$\text{s.t. } z_{ji} \le x_i, \quad \forall i,j = 1,\ldots,n, \tag{7b}$$

$$\sum_{i=1}^{n} z_{ji} + z_{j0} = 1 - x_j, \quad \forall j = 1,\ldots,n, \tag{7c}$$

$$0 \le z_{ji} \le \frac{v_{ji}}{v_{j0}} z_{j0}, \quad \forall i,j = 1,\ldots,n, \tag{7d}$$

$$x_i \in \{0,1\}, \quad \forall i = 1,\ldots,n. \tag{7e}$$

It can be easily seen that (7) is a relaxation for (6) (by noting that one can set $z_{ji} = (1-x_j)\frac{v_{ji} y_{ji}}{\sum_{i=1}^{n} v_{ji} y_{ji} + v_{j0}}$ and $z_{j0} = (1-x_j)\frac{v_{j0}}{\sum_{i=1}^{n} v_{ji} y_{ji} + v_{j0}}$). In the following, we show that the relaxation is exact. We have the following theorem, which uses similar idea as in Davis et al. (2013).

THEOREM 3. *If* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *is an optimal solution to (6), then there exists* $\boldsymbol{z}^*$ *such that* $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ *is an optimal solution to (7). Conversely, if* $(\boldsymbol{x}^*, \boldsymbol{z}^*)$ *is an optimal solution to (7), then there exists* $\boldsymbol{y}^*$ *such that* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *is an optimal solution to (6). Furthermore, the optimal values to (6) and (7) are equal.*

Note that there are only n binary variables in (7). In section 6.1, we will provide some numerical experiments on the MIP approach. As one will see, the MIP formulation is quite efficient for solving the STCM problem of moderate size.

Finally, we remark that there is a special case of the STCM problem that boils down to the assortment optimization problem under the MNL model. We call it the homogeneous case of the STCM problem, in which the transition probabilities of customers are homogeneous with respect to the starting product. More precisely, we consider the case in which there exists $v_i \geq 0$ for $i \in \mathcal{N} \cup \{0\}$ such that $v_{ji} = v_i$ for all $i \in \mathcal{N} \cup \{0\}$ and $j \in \mathcal{N}$. This condition ensures that the transition probability to a product i given a recommended set R is independent of the previous product j. Then this special case of the STCM problem boils down to a special case of the assortment optimization problem under the Markov chain choice model. Moreover, by the result of Blanchet et al. (2016), we can show that this case of assortment optimization under the Markov chain choice model can be further reduced to an assortment optimization problem under the MNL model, in which a revenue-ordered assortment is optimal. We will provide the details in Appendix B of the online appendix.

5. Restricted STCM Problem

In this section, we study a restricted version of the STCM problem. Let STCM(V,C) denote the STCM problem in which the customer can only transition from each product to at most V products (in addition to the no-purchase option), and the size of recommended set for each product not in the assortment is bounded by C. More precisely, in STCM(V,C), for each $j \in \mathcal{N}$, there is at most V products $i \in \mathcal{N}$ such that $v_{ji} > 0$. In addition, for each j\not\in$S$, the number of products in the recommended set R_j is bounded by C. As discussed in section 2, this case captures or approximates the scenario in which a customer is only willing to consider a limited number (V) of substitutes if his/her favorite product is unavailable, and the retailer can only choose a limited number (C) of products to recommend. As shown in Figure 1 and discussed in section 2, such cases with limited consideration of customers and limited recommendation capacity of the retailers are relevant in practice, even for the case with C = 1.

By the definition of the STCM(V,C) problem, it is without loss of generality to assume $V \geq C$ in our following discussion. Also, it is easy to see that STCM(n, n) is the original STCM problem. In the following, we concentrate on the complexity and approximability of various settings of the STCM(V,C) problem.

1. STCM(1, 1): The customer can transition from each product to only one product. We first consider STCM(1,1). In this case, the customer can only transition from one product to one other product (in addition to the no-purchase option). It turns out that this special case can be reduced to the assortment optimization problem under a general tree choice model, which has been studied recently in Paul et al. (2018). For self-completeness, we present a dynamic programming formulation for STCM(1, 1) in Appendix C of the online appendix, and also discuss its connection with the work in Paul et al. (2018).

We also mention that the performance bound $\max\left\{\frac{1}{d}, \frac{1}{1+\log\frac{r_{\max}}{r_{\min}}}\right\}$ of revenue-ordered assortment is still tight even for the STCM(1,1) problem. Since in the tight example given in the proof of Theorem 2, the customer can only transition from each product to one other product in addition to the no-purchase option.

2. STCM(V, 1): Recommended sets contain only one product. Next we study the STCM(V, 1) problem, that is, for each product not offered in the assortment, the retailer can only recommend one product. In this case, if a customer arrives at product j that is not selected in the assortment S, then obviously it is of the best interest to the retailer to recommend him/her the product with the highest expected revenue. In other words, $R_j \subseteq \arg\max_{i \in S} \frac{r_i v_{ji}}{v_{ji}+v_{j0}}$ and the recommended sets $\mathscr{R}$ of all unavailable products can be easily determined given any assortment S. For simplicity of notation, we denote the objective function by $\mathbf{R}(S)$ in the following discussions when all the recommended sets $\mathscr{R}$ are clear, and we write $\tilde{r}_{ji} = \frac{r_i v_{ji}}{v_{ji}+v_{j0}}$. Then, the problem can be formulated as the following optimization problem (8):

$$\max_{S \subseteq \mathcal{N}} \mathbf{R}(S) = \max_{S \subseteq \mathcal{N}} \sum_{j \in S} \lambda_j r_j + \sum_{j \notin S} \lambda_j \max_{i \in S} \tilde{r}_{ji}, \tag{8}$$

We first have the following result about the complexity of the problem.

Proposition 2. *For any $V \geq 2$, $C \geq 1$, the STCM(V,C) problem is NP-Hard and does not admit any FPTAS unless P = NP.*

Because of the hardness of the problem, in the following, we focus on studying approximation algorithms for STCM(n,1). Note that STCM(V,1) is a special case of STCM(n,1), thus the following results also apply to STCM(V,1) for any $V \geq 2$.

First, we can show that the objective function $\mathbf{R}(S)$ is a non-monotone submodular function on set S, and thus problem (8) is an unconstrained submodular maximization problem. Here a set function $f: 2^{\mathcal{N}} \rightarrow \mathbb{R}$ is a submodular function if for any $S, T \subseteq \mathcal{N}$, we have $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$. Furthermore, if f satisfies $f(S) \leq f(T)$ for any $S \subset T$, then we say that it is monotone. It is well-known that there exists a randomized linear time 1/2-approximation algorithm (see, e.g., Buchbinder et al. 2015) for unconstrained non-monotone submodular maximization problem, which is based on a randomized greedy strategy. For completeness, we describe this greedy algorithm in Algorithm 1, and present the performance results in Theorem 4. The performance guarantee bound can be directly obtained from the result in Buchbinder et al. (2015).

Algorithm 1 Randomized Greedy Algorithm for STCM$(n,1)$ (Buchbinder et al. 2015)

1: $S_0^1 := \emptyset$, $S_0^2 := \mathcal{N}$.
2: **for** $i = 1$ to n **do**
3: $\mu_i := \mathbf{R}(S_{i-1}^1 \cup \{i\}) - \mathbf{R}(S_{i-1}^1)$, $\nu_i := \mathbf{R}(S_{i-1}^2 \setminus \{i\}) - \mathbf{R}(S_{i-1}^2)$, where $\mathbf{R}(S)$ is defined in (8).
4: $\tilde{\mu}_i := \max\{\mu_i, 0\}$, $\tilde{\nu}_i := \max\{\nu_i, 0\}$, $p_i := \frac{\tilde{\mu}_i}{\tilde{\mu}_i + \tilde{\nu}_i}$ (set $p_i := 1$ if $\tilde{\mu}_i = \tilde{\nu}_i = 0$).
5: With probability p_i, set $S_i^1 := S_{i-1}^1 \cup \{i\}$, $S_i^2 := S_{i-1}^2$; otherwise, with probability $1 - p_i$, set $S_i^1 := S_{i-1}^1$, $S_i^2 := S_{i-1}^2 \setminus \{i\}$.
6: **end for**
7: **return** $S = S_n^1$ (or equivalently S_n^2).

THEOREM 4. *The objective function* $\mathbf{R}(S)$ *of problem (8) is a non-monotone submodular function. Moreover, Algorithm 1 is a (randomized) 1/2-approximation algorithm for STCM(n,1).*

Below we propose an improved approximation algorithm with performance ratio better than $1/2$ for STCM (n,1). The main idea of the algorithm is to formulate the problem as a MAX CSP, and to apply a randomized rounding technique to its SDP relaxation. The MAX CSP problem is defined as follows: given a set of Boolean (TRUE/FALSE) variables and constraints, where each constraint has a weight, the goal is to find an assignment (TRUE/FALSE) to the variables that maximizes the total weight of satisfied constraints. If each constraint depends on k variables, then the problem is referred to as MAX k-CSP. We note that in the following discussion, a variable has value 1 indicates that it is TRUE in the corresponding CSP instance, and the variable has value 0 indicates that it is FALSE[2]. First, we can formulate (8) as a MAX CSP problem, which can be further written as the following integer quadratic program:

$$\max \sum_{j=1}^{n} \sum_{i=0}^{n-1} r_j^i z_{ji} \tag{9a}$$

$$\text{s.t. } z_{j0} = w_j, \quad \forall j = 1, \ldots, n, \tag{9b}$$

$$\begin{aligned} & z_{ji} \le (1 - w_j)\left(\sum_{k=1}^{i} w_{j(k)}\right), \quad \forall i = 1, \ldots, n-1, j = 1, \ldots, n. \\ & w_j, z_{ji} \in \{0,1\} \quad \forall i = 1, \ldots, n-1, j = 1, \ldots, n. \end{aligned} \tag{9c}$$

TTo relate to our assortment optimization problem, the binary variable $w_j \in \{0,1\}$ represents the selection of product j, where w_j is 1 if product j is selected into the assortment and 0 otherwise. We relate the selection of product j and its transition to other product i by several CSP constraints. Particularly, z_{j0} also represents the selection of product j, which has value $z_{j0} = 1$ if j is selected and 0 otherwise. This is captured by the constraint (9b). The meaning of other z_{ji}s is slightly more complicated. For each $j \in \mathcal{N}$, let $(j(1), \ldots, j(n-1))$ be the permutation of $\mathcal{N} \setminus \{j\}$ such that $\tilde{r}_{jj(1)} \ge \tilde{r}_{jj(2)} \ge \cdots \ge \tilde{r}_{jj(n-1)}$, which can be preliminarily computed within a polynomial time in n. The variables $z_{ji} \in \{0,1\}$ is associated to the transition from product j to product $j(i)$ (not product i), and can be represented by the CSP constraint $z_{ji} = \bar{w}_j \wedge (w_{j(1)} \vee \cdots w_{j(i)})$ where $\bar{w} = 1 - w$, $\wedge$ is the conjunction operation (AND), and $\vee$ is the disjunction operation (OR). In other words, if $j \in S$, then $z_{j0} = 1$ and $z_{ji} = 0$ for all $i = 1, \ldots, n-1$; otherwise if the customer first arrives at an unavailable product j and will transition to an available product $j(i)$, namely, $j, j(1), \ldots, j(i-1)$\not\in$S$ and $j(i) \in S$, then $z_{j1} = \cdots = z_{j,i-1} = 0$ and $z_{ji} = z_{j,i+1} = \cdots = z_{j,n-1} = 1$. This is captured by the constraint (9c). The value r_j^i in the objective function (9a) is given by $r_j^0 = \lambda_j r_j$, $r_j^i = \lambda_j(\tilde{r}_{jj(i)} - \tilde{r}_{jj(i+1)})$ (where $r_j^{n-1} = \lambda_j \tilde{r}_{jj(n-1)}$), for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$. It guarantees that if product j is selected, then the revenue gained from the customer first arrives at j is exactly $\lambda_j r_j$; otherwise if the customer will transition to $j(i)$, then the revenue gained is exactly $\lambda_j \tilde{r}_{jj(i)}$.

We can derive the standard SDP relaxation to the integer quadratic program (9) and apply the randomized rounding method to it. We refer to it as the standard SDP relaxation to the STCM(n,1) problem, and provide the formulation in Appendix D.1 of the online appendix. However, as mentioned in section 2, the performance guarantee of SDP relaxation for the CSP clause with type $\bar{w}_j \wedge (w_{j(1)} \vee \cdots w_{j(i)})$ has only been discussed when there are at most three variables, that is, only known for the case $\bar{w}_j \wedge (w_{j(1)} \vee w_{j(2)})$ (Zwick 1998). For such clause with $i>2$, the performance guarantee of SDP relaxation is not well-studied in the literature. If we directly apply the best known approximation result for the general MAX k-CSP problem, which is $\frac{0.62661k-o(1)}{2^k}$ by Makarychev and Makarychev (2014), then the approximation ratio is poor. Therefore, we consider a more complicated SDP relaxation based on a strengthened SDP relaxation for MAX NAE SAT problem presented in Zhang et al. (2004), which is also available for solving the MAX 2-AND and MAX SAT problems. We remark that although we have not analyzed the theoretical performance bound for the standard SDP relaxation of (9), it is still of interest in practice since it contains fewer variables and constraints. We will use it in the empirical experiments in section 6. As we will see, for the STCM(V, 1) instances with a few products and less dense structure, the SDP rounding algorithm based on (9) can efficiently solve the problem and return solutions with good performance.

Now we derive a strengthened SDP relaxation based on Zhang et al. (2004)'s strengthen SDP relaxation for MAX NAE SAT problem. For convenience, we introduce variable $x_j \in \{-1,1\}$ for product j (in this case, a variable has value 1 indicates that it is TRUE in the corresponding CSP instance, and the variable has value -1 indicates that it is FALSE), which is the same as the binary variable w_j in (9) except that it has value $x_j=-1$ if j is not selected into the assortment. Let $y_{ji}=(x_{j(1)}\vee\cdots\vee x_{j(i)})$, which have value $y_{ji}=1$ if at least one of the product $j(1),\ldots, j(i)$ is selected into the assortment, and -1 otherwise. The description of z_{ji} is similar to that in (9), in which z_{j0} indicates the selection of product j, and z_{ji} indicates the transition from j to at least one product among $j(1),\ldots, j(i)$. Particularly, we show that the STCM(n, 1) problem can be reduced to the following integer quadratic problem:

$$\max \sum_{j=1}^{n}\sum_{i=0}^{n-1} r_j^i z_{ji} \tag{10a}$$

$$\text{s.t. } z_{j0}=\frac{1-x_0x_j}{2}, \quad \forall j=1,\ldots,n, \tag{10b}$$

$$\frac{1-x_0y_{ji}}{2} \le \frac{1}{i}\sum_{0\le l_1<l_2\le i}\frac{1-x_{j(l_1)}x_{j(l_2)}}{2}, \quad \forall i=1,\ldots,n-1, j=1,\ldots,n, \tag{10c}$$

$$\sum_{0\le l_1<l_2\le i} x_{j(l_1)}x_{j(l_2)} \ge -\left\lfloor\frac{i+1}{2}\right\rfloor \quad \forall i=1,\ldots,n-1, j=1,\ldots,n, \tag{10d}$$

$$z_{ji}\le\frac{1+x_0x_j-x_0y_{ji}-x_jy_{ji}}{4} \quad \forall i=1,\ldots,n-1, j=1,\ldots,n, \tag{10e}$$

$$\begin{aligned} & z_{ji}\le\frac{1+x_0x_j}{2} \quad \forall i=1,\ldots,n-1, j=1,\ldots,n,\\ & x_0=-1, x_j, y_{ji}\in\{-1,1\}, z_{j0}, z_{ji}\in\{0,1\} \quad \forall i=1,\ldots,n-1, j=1,\ldots,n, \end{aligned} \tag{10f}$$

where $\lfloor x\rfloor$ denotes the maximum integer that is at most x. In the integer quadratic program (10), the constraints (10c) and (10d) stand for the NAE constraint of $x_{j(0)}, x_{j(1)},\ldots,x_{j(i)}$ introduced by Andersson and Engebretsen (1998) and Zhang et al. (2004). Here a NAE constraint $\text{NAE}(x_{j(0)},x_{j(1)},\ldots,x_{j(i)})$ is satisfied if at least one of the variable is set to be TRUE and at least one of the variable is set to be FALSE, and is unsatisfied otherwise. Since $x_{j(0)}=x_0$ is always set to be FALSE, the NAE constraint $\text{NAE}(x_{j(0)},x_{j(1)},\ldots,x_{j(i)})$ is exactly the SAT constraint $x_{j(1)}\vee\cdots\vee x_{j(i)}$. The constraint (10e) is standard for the AND clause of $\bar{x}_j$ and y_{ji} (see, e.g., Goemans and Williamson 1995, Zhang et al. 2004). Note that (10f) is redundant in this formulation; however it would become useful later when we relax it to an SDP formulation. We summarize the details of the reduction from (8) to the MAX CSP instance and the proof of its equivalence with (10) as Lemma 1.

Lemma 1. *The STCM(n,1) problem (8) can be reduced to a MAX CSP instance, which is equivalent to the integer quadratic program (10).*

Now we derive the SDP relaxation of problem (10). Let $\boldsymbol{u}_j\in\mathbb{R}^{n^2+1}$ be a relaxation of x_j, $\boldsymbol{u}_0=(-1,0,\ldots,0)$, and $\boldsymbol{v}_{ji}\in\mathbb{R}^{n^2+1}$ be a relaxation of y_{ji}. The corresponding SDP relaxation is stated as follows:

$$\max \sum_{j=1}^{n}\sum_{i=0}^{n-1} r_j^i z_{ji} \tag{11a}$$

$$\text{s.t. } z_{j0}=\frac{1-\boldsymbol{u}_0\cdot\boldsymbol{u}_j}{2}, \quad \forall j=1,\ldots,n, \tag{11b}$$

$$\frac{1-\boldsymbol{u}_0\cdot\boldsymbol{v}_{ji}}{2} \le \frac{1}{i}\sum_{0\le l_1<l_2\le i}\frac{1-\boldsymbol{u}_{j(l_1)}\cdot\boldsymbol{u}_{j(l_2)}}{2}, \quad \forall i=1,\ldots,n-1, j=1,\ldots,n, \tag{11c}$$

$$\sum_{0\le l_1<l_2\le i} \boldsymbol{u}_{j(l_1)}\cdot\boldsymbol{u}_{j(l_2)} \ge -\left\lfloor\frac{i+1}{2}\right\rfloor \quad \forall i=1,\ldots,n-1, j=1,\ldots,n \tag{11d}$$

$$z_{ji}\le\frac{1+\boldsymbol{u}_0\cdot\boldsymbol{u}_j-\boldsymbol{u}_0\cdot\boldsymbol{v}_{ji}-\boldsymbol{u}_j\cdot\boldsymbol{v}_{ji}}{4} \quad \forall i=1,\ldots,n-1, j=1,\ldots,n, \tag{11e}$$

$$z_{ji}\le\frac{1+\boldsymbol{u}_0\cdot\boldsymbol{u}_j}{2} \quad \forall i=1,\ldots,n-1, j=1,\ldots,n, \tag{11f}$$

$$\begin{aligned} & \boldsymbol{u}_0=(-1,0,\ldots,0), \boldsymbol{u}_j\cdot\boldsymbol{u}_j=1, \boldsymbol{v}_{ji}\cdot\boldsymbol{v}_{ji}=\\ & 1, \boldsymbol{u}_j, \boldsymbol{v}_{ji}\in\mathbb{R}^{n^2+1} \forall i=1,\ldots,n-1, j=1,\ldots,n,\\ & 0\le z_{ji}\le 1 \quad \forall i=0,\ldots,n-1, j=1,\ldots,n. \end{aligned} \tag{11g}$$

We will refer this SDP relaxation as SDP_{NAE} in the numerical experiments of section 6.2. Given any positive error $\epsilon>0$, the SDP relaxation of (10) can be solved in polynomial time in the input size and $\log(1/\epsilon)$ within an additive error of ϵ (see, e.g., Vandenberghe and Boyd 1996). Let $\boldsymbol{u}_j$, $\boldsymbol{v}_{ji}z_{ji}$ be an optimal solution to (11). Our approximation algorithm uses the randomized hyperplane method and outward rotation, which has been frequently used in designing SDP rounding approximation algorithms (Zhang et al. 2004, Zwick 1998, 1999). The idea is to first generate a random hyperplane $\boldsymbol{h}$ according to the standard normal distribution. Then we round each variable x_j to 1 or −1 according to the inner product of $\boldsymbol{u}_j$ and $\boldsymbol{h}$, with a rotation of a parameter θ. This step is called outward rotation of the randomized rounding, which is often used to improve the performance guarantee of the SDP relaxation (see Zwick 1999 for more details). Then with a certain probability p, we independently set the value of each variable to its negation. It is often called the perturbation step and is also used to improve the performance guarantee of the SDP relaxtion (see, e.g., Andersson and Engebretsen 1998, Zhang et al. 2004). Finally, the algorithm selects the products that has value 1 into the assortment. The above parameters θ and p are selected to optimize the performance guarantee of the algorithm according to the analysis. We present the details of our algorithm in Algorithm 2 and its performance guarantee in Theorem 5. We also remark that since the variables x_j's are not rounded independently but rely on the choice of random hyperplane $\boldsymbol{h}$, and thus we will incorporate nonsymmetric correlation inequality for normal distribution by Szarek and Werner (1999) to analyze the performance guarantee of our algorithm.

Theorem 5. *Algorithm 2 is a (randomized) 0.5882-approximation algorithm for STCM(n,1).*

Theorem 5 shows that for STCM(n,1), one can obtain better approximation result than simply invoking the submodularity function maximization approach. In Mahajan and Ramesh (1999), the authors propose a complicated conditional probability technique, which is applicable to derandomize a variety of SDP-based randomized rounding algorithms. In particular, one can use their method and derandomize Algorithm 2 to a deterministic algorithm in polynomial time within a loss of arbitrarily small accuracy. We refer the details to Mahajan and Ramesh (1999).

Next we show that for a more special case, STCM (2,1), we can obtain even better results via formulation (9). To derive the result, we first formulate STCM(2,1) as an integer quadratic programming problem (12) as follows, in which $i \in \{1,2\}$ for each $j \in \{1,\ldots,n\}$ and the binary variable $w_j \in \{0,1\}$ in (9) is transformed into $x_j \in \{-1,1\}$:

$$\max \sum_{j=1}^{n}\sum_{i=0}^{2} r_j^i z_{ji} \tag{12a}$$

$$\text{s.t. } z_{j0} = \frac{1-x_0x_j}{2}, \quad \forall j=1,\ldots,n, \tag{12b}$$

$$z_{j1} \le \frac{1+x_0x_j-x_0x_{j(1)}-x_jx_{j(1)}}{4}, \quad \forall j=1,\ldots,n, \tag{12c}$$

$$\begin{aligned} &z_{j2} \le \frac{2+2x_0x_j-x_0x_{j(1)}-x_0x_{j(2)}-x_jx_{j(1)}-x_jx_{j(2)}}{4}, \quad \forall j=1,\ldots,n,\\ &x_0=-1, x_j\in\{-1,1\}, z_{ji}\in\{0,1\} \quad \forall i=0,1,2, j=1,\ldots,n, \end{aligned} \tag{12d}$$

where $x_j=1$ indicates that it is set to be TRUE, and −1 otherwise, x_0 is an additional variable that is set to be −1, the constraint (12b) corresponds to the clause x_j, the constraint (12c) corresponds to the clause $\bar{x}_j \wedge x_{j(1)}$, and the constraint (12c) corresponds to the clause $\bar{x}_j \wedge (x_{j(1)} \vee x_{j(2)})$.

Algorithm 2 SDP Rounding Algorithm for STCM(n, 1)

1: Let $\theta := 0.9795$, $p := 0.095$, $S := \emptyset$.

2: Solve the SDP relaxation (11) and let $\boldsymbol{u}_j$, $\boldsymbol{v}_{ji}$, z_{ji}, $i,j=1,\ldots,n$ be the optimal solution.

3: Choose a random vector $\boldsymbol{h}=(h_1,\ldots,h_{n^2+1}) \in \mathbb{R}^{n^2+1}$ and $h_{n^2+2},\ldots,h_{n^2+n+1}$, where h_i are independent standard normal random variables for $i=1,\ldots,n^2+n+1$.

4: (Randomized rounding and outward rotation) For $j=1,\ldots,n$, set $\tilde{x}_j$ to 1 if $\sqrt{1-\theta}\boldsymbol{u}_j \cdot \boldsymbol{h} + \sqrt{\theta}h_{n^2+1+j} \le 0$ and −1 otherwise.

5: (Perturbation) For each $j=1,\ldots,n$, independently let $x_j=-\tilde{x}_j$ with probability p and $x_j=\tilde{x}_j$ with probability $1-p$.

6: Select the product j into S if $x_j=1$, **return** S.

The SDP relaxation of (12) can be derived analogously, and we provide it in the proof of Theorem 6. Note that such formulation has been studied in Zwick (1998) in which an approximation algorithm with approximation ratio 0.733 is proposed. We adopt the algorithm in Zwick (1998) and have the following results.

THEOREM 6. *There is a (randomized) 0.733-approximation algorithm (Algorithm A.1 in Appendix A) for STCM(2,1).*

Similar to Algorithm 2, we can derandomize the algorithm in Theorem 6 and obtain a deterministic approximation algorithm by the method of Mahajan and Ramesh (1999), which can be done in polynomial time and only loses an arbitrarily small accuracy.

3. STCM(V, 2): Recommended sets contain two products. Now we study the case when the number of products in the recommended set is more than 1. In such cases, we mainly focus on the submodularity of the objective function, as it is often directly related to efficient approximation algorithms (see, e.g., Buchbinder et al. 2015). We have the following results.

THEOREM 7. *For STCM(2,2), the objective function $\mathbf{R}(S)$ of problem (8) is a non-monotone submodular function. However, for STCM(V,2) with V>2, the objective function $\mathbf{R}(S)$ of problem (8) could be non-submodular.*

Therefore, we can use the same idea as in Algorithm A.2 to obtain a 1/2-approximation algorithm. The detailed algorithm is given in Algorithm A.2. We have the following corollary.

COROLLARY 1. *There exists a (randomized) 1/2-approximation algorithm (Algorithm A.2) for the STCM(2,2).*

We remark that one can also extend the idea of SDP-based rounding technique for STCM(n, 1), and STCM(2, 1) to the general STCM(V, C) when the recommended sets contain $C \geq 1$ products. For instance, we can formulate the STCM(2, 2) problem as a MAX 3-CSP instance, then apply the best known SDP-based rounding algorithm (Zwick 1999) and obtain a (randomized) 1/2-approximation algorithm as well (See Appendix E in the online Appendix F or more details). However, with larger V and C, the resulting MAX CSP instance requires more variables in the clauses, which does not have approximation algorithm with good performance guarantee in general. We will provide more discussion on the extensions of SDP-based rounding algorithm to STCM(V, C) with $C>1$ in Appendix E of the online appendix.

Finally, to end this section, we extend the MIP formulation shown in section 4 to solve the restricted STCM(V,C) problem. Particularly, we show in Proposition 3 that the STCM(V,C) problem is equivalent to the following MIP formulation (13):

$$\max \sum_{j=1}^{n} \lambda_j r_j x_j + \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} r_i z_{ji} \tag{13a}$$

$$\text{s.t. } z_{ji} \leq x_i, \quad \forall i,j = 1,\ldots,n, \tag{13b}$$

$$\sum_{i=1}^{n} z_{ji} + z_{j0} = 1 - x_j, \quad \forall j = 1,\ldots,n, \tag{13c}$$

$$\sum_{i=1}^{n} \frac{v_{j0}}{v_{ji}} z_{ji} \leq C z_{j0}, \quad \forall j = 1,\ldots,n, \tag{13d}$$

$$0 \leq z_{ji} \leq \frac{v_{ji}}{v_{j0}} z_{j0}, \quad \forall i,j = 1,\ldots,n, \tag{13e}$$

$$x_i \in \{0,1\}, \quad \forall i = 1,\ldots,n. \tag{13f}$$

PROPOSITION 3. *The optimal solution to STCM(V,C) can be obtained by solving the MIP formulation (13).*

Note that similar as the MIP formulation (7), formulation (13a)–(13f) only contains n binary variables. Therefore, the MIP formulation could provide an efficient approach to solve this problem when the size of problem is not too large.

6. Numerical Experiments

In this section, we conduct numerical experiments to test the STCM and the proposed algorithms. The numerical experiments mainly consist of three parts: First, in section 6.1, we demonstrate the efficiency of the MIP formulation for solving the STCM problem and test the performance of the revenue-ordered assortments. In section 6.2, we evaluate the performance of the proposed algorithms for solving the restricted STCM problem. Then in section 6.3, we illustrate the benefit of allowing the seller to choose different recommendation sets for different products. Furthermore, in Appendix F of the online appendix, we provide an estimation method of this model using historical data.

All numerical experiments in this section are conducted on a PC with 16GB Memory and Intel(R) Core (TM) i7-8700U CPU (@3.20GHz) using Julia 1.5.1 (Julia Computing, Inc. https://julialang.org/). All the MIPs are solved using Gurobi 9.0.3 (Gurobi Optimization, Inc. 2020) and all the SDPs are solved using MOSEK 9.2.21 (MOSEK ApS, Denmark. 2020).

6.1. Performance of the MIP and Revenue-Ordered Assortment for General STCM

In this section, we test the performance of the MIP formulation (7) proposed in section 4 as well as the performance of revenue-ordered assortments. We start by describing the setup of the experiments. In our experiments, we consider problem instances with $n \in \{5,10,30,50,100,200,500\}$ products. Such a range should be enough to cover most practical scenarios. We consider the following ways to generate the parameters in the numerical experiments:

1. Revenue: The revenue of the products r_js are generated by first generating n independent random numbers from uniform distributions on [0,1] and then sorting them in decreasing order.[3]
2. Arrival and transition probabilities: For the arrival probabilities, we generate n independent random numbers Λ_j, each from a uniform distribution on [0,1], and set the arrival probabilities to be $\lambda_j = \Lambda_j / \sum_{j=1}^{n} \Lambda_j$ for all j. For the transition probabilities, we consider two methods of generating them. In the first method, for each j, we generate $n+1$ independent random numbers Λ_{ji}, each from a uniform distribution on [0,1]. Then we set $v_{ji} = \Lambda_{ji} / \sum_{i=0}^{n} \Lambda_{ji}$. We use **DEN** (meaning *dense*) to denote this method. In the second method, we make the transition matrix sparse to reflect that customer interested in one product typically is only interested in a few other products. Specifically, for each Λ_{ji}, with probability $1 - 1/\sqrt{n}$, we set it to be zero and with probability $1/\sqrt{n}$, we set it to be a uniformly distributed random number on [0,1] (for Λ_{j0}, we always set it to be a uniformly distributed random number on [0,1], meaning that the customer can always transition to the no-purchase option). Then we set $v_{ji} = \Lambda_{ji} / \sum_{i=0}^{n} \Lambda_{ji}$. We use **SPA** (meaning *sparse*) to denote this method.

In the presentation of the results, we use the notation $(n,\mathbf{A})$ to denote the combination in which there are n products, method $\mathbf{A}$ (one of **DEN** or **SPA**) is used to generate the transition probabilities.

The results are shown in Table 2. In Table 2, each number is calculated by averaging 100 test instances in the corresponding combination of parameters. The first and second columns show the average runtime (in seconds) of the MIP formulation and the average runtime for calculating the best revenue-ordered assortment, respectively. The third and fourth columns show the average and worst-case performance ratio between the best revenue-ordered assortment and the optimal assortment, respectively. From Table 2, we can see that the MIP formulation is very efficient. Particularly, it can solve problems with up to 500 products within a matter of minutes, and for problems that have sparse structure, it is even more efficient. (In comparison, if one enumerates all possible assortments, then even for problems that have 30 products, it will take hours to solve.) For the revenue-ordered assortment, we can see that for the dense cases, it performs quite well, with average performance ratio being more than 99%, and the performance is especially good when the problem size is large. However, for sparse problems, the revenue-ordered assortment performs less well, sometimes giving rise to performance ratios around 70%. Therefore, given its efficiency, we suggest using the MIP formulation for problems of moderate sizes.

6.2. Performance of the Algorithms for the Restricted STCM(V,C) Problem

In this section, we test the performance of algorithms proposed for the restricted STCM(V,1) problem in section 5. We will give some additional numerical experiments for STCM(V,2) in Appendix E of the online appendix.

Particularly, we compare the running time and performance of the MIP formulation, the randomized rounding algorithms of various SDP relaxations, the randomized greedy algorithm and the revenue-ordered assortment. We first conduct numerical test on STCM(2,1), and then on STCM(V,1). In the two tests, the revenue and the arrival probabilities of products are generated in the same way as described in section 6.1. However, the generation method of transition probabilities is slightly different. For each j, we first sample V products from the other $n-1$

Table 2 Running Time and Performance of the Mixed Integer Program (MIP) Formulation and Revenue-Ordered Assortments for STCM

Parameters	MIP	RO		
(n,A)	Time (s)	Time (s)	Ave. Ratio	Min. Ratio
(5,**DEN**)	0.0051	0.0001	99.89%	95.30%
(10,**DEN**)	0.0079	0.0001	99.78%	96.98%
(30,**DEN**)	0.0456	0.0002	99.92%	99.47%
(50,**DEN**)	0.1302	0.0003	99.95%	99.54%
(100,**DEN**)	0.5948	0.0007	99.98%	99.81%
(200,**DEN**)	4.5528	0.0019	99.99%	99.93%
(500,**DEN**)	31.959	0.0101	99.99%	99.99%
(5,**SPA**)	0.0047	0.0001	98.93%	67.99%
(10,**SPA**)	0.0058	0.0002	98.24%	85.53%
(30,**SPA**)	0.0219	0.0004	98.35%	93.63%
(50,**SPA**)	0.0486	0.0008	98.37%	93.99%
(100,**SPA**)	0.1785	0.0022	98.74%	96.77%
(200,**SPA**)	0.8134	0.0019	98.96%	97.98%
(500,**SPA**)	7.7312	0.0987	99.36%	98.84%

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products uniformly without replacement ($V = 2$ for the STCM(2,1) problem), which guarantees that the customer can only transition from product j to at most V products. We note that each product other than j is selected with equal probability $V/(n-1)$. Let $i_1,\ldots,i_V$ be these V products. Then for each j, we generate $V+1$ independent random numbers Λ_{ji}, $i = 0,\ldots,V$, each from a uniform distribution on [0,1]. Then we set $v_{ji_l} = \Lambda_{jl}/\sum_{l=0}^{V}\Lambda_{jl}$, for $l = 0,\ldots,V$ (where $i_0 = 0$) and $v_{ji} = 0$, for $i \notin \{i_0,\ldots,i_V\}$.

In the first test, we generate 100 random test instances of STCM(2,1) with $n \in \{5,10,15,30,50,100\}$ products. For each instance, we solve STCM(2,1) using the MIP formulation (13), randomized rounding based on the standard SDP relaxation (see Theorem 6 and Algorithm A.1), the randomized greedy algorithm (Algorithm 1) and the best revenue-ordered assortment, respectively. In the presentation of the results, we use the notation (n,V) to denote the combination in which there are n products, the customer can only transition from each product to at most V products. The results are presented in Table 3.[4]

We mention that the greedy algorithm and the best revenue-ordered assortment are faster than the other two algorithms, which can return solution within 0.01s when there are $n = 100$ products. In comparison, the MIP formulation and the SDP relaxation can return a solution around 0.1s. Moreover, we can see that for moderate size of STCM(2, 1) problems ($n \le 100$), the SDP rounding algorithm is able to solve problems faster than the MIP formulation in general. We report the performance of the algorithms in Table 3. The first and second columns show the average and worst-case performance ratio between the assortment returned by SDP rounding (Algorithm A.1) and the optimal assortment on the 100 test instances, respectively. The third to sixth columns show the similar numbers for the randomized greedy algorithm, and the best revenue-ordered assortment, respectively. Note that the MIP formulation always returns the optimal assortment. Particularly for each combination of parameter (each row), we highlight the results of the algorithm that returns the best average/worst-case performance ratio. Particularly when the problem size is relatively small ($n<30$), the performance of the SDP rounding algorithm is significantly better than the randomized greedy algorithm and the best revenue-ordered assortment. The SDP rounding algorithm can provide 97%–99% of the optimal total expected revenue on average and 93–96% in the worst case, while the other two algorithms sometimes return solution with performance ratio below 70%. For the problems with larger size ($n>30$), the SDP rounding algorithm can perform well in some of the worst cases, but is generally outperformed by the randomized greedy algorithm and the best revenue-ordered assortment on average.

Table 3 Performance of the Algorithms for STCM(2, 1)

Parameters	SDP (Theorem 6)		GREEDY		RO	
(n,V)	Ave. Ratio	Min. Ratio	Ave. Ratio	Min. Ratio	Ave. Ratio	Min. Ratio
(5,2)	<u>99.09%</u>	<u>93.19%</u>	96.02%	61.02%	98.33%	85.92%
(10,2)	<u>99.13%</u>	<u>95.68%</u>	96.66%	83.54%	97.60%	87.29%
(15,2)	<u>98.93%</u>	<u>96.02%</u>	96.40%	78.23%	97.56%	90.41%
(30,2)	95.80%	<u>92.08%</u>	<u>96.77%</u>	87.58%	96.70%	88.93%
(50,2)	91.14%	86.77%	<u>96.74%</u>	90.82%	96.48%	<u>92.47%</u>
(100,2)	84.40%	81.15%	<u>96.73%</u>	92.72%	96.04%	<u>93.08%</u>

In the second test, we generate 100 instance for STCM(V,1) problem with $n \in \{5,15,30\}$ products and different values of V. For each instance, we solve the problem by randomized rounding algorithms based on various SDP relaxations, as well as other proposed algorithms. Particularly, we denote SDP_{NAE} as the randomized rounding algorithm based on the SDP relaxation (11), that is, the 0.5882-approximation algorithm shown in Algorithm 2 and Theorem 5. We also study the numerical performance of the randomized rounding method based on the standard SDP relaxation of (9) (see also in Appendix D.1) of the online appendix, which can be viewed as an extension of Algorithm A.1 to the case with $V \ge 2$. We denote it as SDP_{std}. Moreover, we consider the randomized rounding method based on a further relaxation of (9), in which we only consider the constraints $i \in \{1,\ldots,P\}$ in (9c) for each j instead of all the constraints $i \in \{1,\ldots,n-1\}$ and P is a value in $\{1,\ldots,V\}$. In other words, we ignore the revenue and constraints of the products $j(P+1),\ldots,j(V)$ that have relatively smaller revenue in the implementations. (We provide the formulation in (D.3) of Appendix D.1.) We refer to the result based on this SDP relaxation as $\text{SDP}_{\text{partial}}$. In our experiments, we set P to be $\lceil V/6 \rceil$. Note that SDP_{std} and $\text{SDP}_{\text{partial}}$ have much fewer constraints and smaller dimensions of decision variables than the SDP_{NAE}, and thus can be solved more efficiently. The results are presented in Table 4.

In Table 4, we show the average and worst-case performance ratio between the assortment returned by the above algorithms and the optimal assortment. Particularly for each combination of parameter (each row), we highlight the results of the algorithm that returns the best average/worst-case performance ratio among the $\text{SDP}_{\text{partial}}$, random greedy algorithm, and the best revenue-ordered assortment. We mention that when the number of products n is large and V is close to n, the SDP rounding based on SDP_{NAE}, SDP_{std} can be slower than solving the MIP formulation. In contrast, the $\text{SDP}_{\text{partial}}$ is faster than the MIP formulation in all the test instances. Meanwhile, we can see from Table 4 the loss of accuracy between $\text{SDP}_{\text{partial}}$ and SDP_{std} is no more than 1% on the

Table 4 Performance of the Algorithms for STCM(V, 1) with $n \in$ {5,15,30} Products

Parameters	SDP_{NAE}		SDP_{std}		$SDP_{partial}$		GREEDY		RO	
(n,V)	Ave. Ratio	Min. Ratio	Ave. Ratio	Min. Ratio	Ave. Ratio	Min. Ratio	Ave. Ratio	Min. Ratio	Ave. Ratio	Min. Ratio
(5,2)	99.74%	96.29%	99.78%	97.77%	99.73%	95.61%	97.24%	73.41%	99.44%	88.71%
(5,3)	99.70%	94.73%	99.79%	95.17%	99.72%	93.05%	97.90%	72.28%	99.61%	92.43%
(5,4)	99.68%	89.70%	99.78%	94.59%	99.87%	95.13%	96.91%	57.17%	99.43%	91.68%
(15,2)	97.39%	92.64%	97.36%	90.18%	97.42%	89.52%	97.03%	89.51%	96.92%	88.68%
(15,6)	98.46%	95.20%	98.68%	95.40%	98.60%	95.14%	96.94%	87.23%	99.11%	91.39%
(15,8)	98.81%	95.70%	98.81%	96.62%	98.74%	96.26%	97.42%	85.79%	99.16%	93.18%
(15,14)	98.87%	95.48%	99.18%	96.33%	98.97%	95.82%	97.89%	85.66%	99.46%	95.22%
(30,2)	96.44%	93.70%	96.44%	93.35%	96.24%	93.35%	97.81%	88.49%	98.10%	92.76%
(30,6)	97.27%	94.06%	97.32%	93.84%	97.29%	93.37%	97.09%	91.58%	98.67%	95.76%
(30,12)	97.47%	94.65%	97.51%	94.18%	97.49%	94.01%	97.31%	91.49%	99.00%	96.49%
(30,18)	97.52%	94.21%	97.58%	94.40%	97.52%	93.94%	97.68%	88.56%	99.21%	96.65%
(30,29)	97.60%	95.40%	97.59%	95.19%	97.54%	94.70%	98.40%	91.60%	99.26%	96.96%

average performance ratio and no more than 2% on the worst performance ratio. Although we have not studied the theoretical worst performance ratio of randomized rounding based on SDP_{std} and $SDP_{partial}$, as we observe from the experimental results, the performance of them is quite well and close to (sometimes better than) SDP_{NAE}, which we show is a (randomized) 0.5882-approximation algorithm for STCM(n,1). Therefore, with an appropriate selection of the value P, that is, the size of the SDP relaxation, the SDP rounding method can be practical and efficient in solving the restricted STCM problem. For the randomized greedy and the revenue-ordered assortment algorithms, they are much faster than the SDP rounding algorithm and the MIP formulation. However, for problems with smaller size (e.g., n = 5 or 15), the performances of these two algorithms are sometimes less well. The randomized greedy algorithm returns a solution with performance ratios below 60% in some case. For large problems, the randomized greedy algorithm performs better in the sparse instances (smaller V) while the revenue-ordered assortment performs better in the dense instances (larger V). This is consistent with our observation in the first test and the test for general STCM in section 6.1, where the revenue-ordered assortment algorithm is efficient when the problem is dense and has large size.

To conclude this section, we mention that when the problem size is much larger, solving the SDP relaxation problem will be time-consuming. Although the theoretical performance bound is generally poor, the revenue-ordered assortment is capable of efficiently solving the very large size and dense problems with a good numerical performance. For problems with relatively small size and sparse structure, the SDP rounding method and randomized greedy algorithm can have potential to improve the efficiency with a slight loss of accuracy in practical use, and can provide decent theoretical performance guarantees as well. In Appendix E, we will provide more results of these algorithms on solving the STCM(V,2) problem.

6.3. The Benefit of Allowing Recommending Different Subsets

In this section, we study the benefit of allowing the seller to customize the recommended sets based on the first arrival product of the customer. We compare the optimal values of three models: the STCM, the $STCM_R$ and the $STCM_S$ which are described in section 3. In particular, in the $STCM_R$, the recommended set R is only a function of the assortment S, regardless of the first arrival product of the customer. In the $STCM_S$, the recommended set is exactly the assortment S for each $j \notin S$.

In our numerical tests, we generate 1000 random test instances with $n \in$ {5, 10, 30, 50, 100, 200} products according to the methods described in section 6.1. For each instance, we solve the optimal assortment under each model, respectively, and calculate the optimal revenue under each model. Note that for the STCM problem, we use the MIP formulation in section 4 to solve for the optimal assortment. For the $STCM_R$ problem, its optimization problem can be formulated as follows:

$$\max \sum_{j=1}^{n} \lambda_j r_j x_j + \sum_{j=1}^{n} \lambda_j (1-x_j) \frac{\sum_{i=1}^{n} r_i v_{ji} y_i}{\sum_{i=1}^{n} v_{ji} y_i + v_{j0}} \tag{14a}$$

$$\text{s.t. } y_j \le x_j, \quad \forall j = 1,\ldots,n, \tag{14b}$$

$$x_j, y_j \in \{0,1\}, \quad \forall j = 1,\ldots,n. \tag{14c}$$

To obtain the optimal assortment S and recommended set R in $STCM_R$, we adapt the linearization technique in Bront et al. (2009) for the fractional

Table 5 Comparison Between the STCM and Other Models

Parameters (n, V)	STCM/STCM$_R$			STCM/STCM$_S$		
	$OPT_{STCM} > OPT_{STCMR}$	Ave. Ratio	Max. Ratio	$OPT_{STCM} > OPT_{STCMS}$	Ave. Ratio	Max. Ratio
(5,**DEN**)	5.7%	1.0009	1.0973	12.8%	1.0026	1.1251
(10,**DEN**)	36.8%	1.0038	1.0830	48.1%	1.0063	1.1196
(30,**DEN**)	90.1%	1.0062	1.0579	92.1%	1.0070	1.0579
(50,**DEN**)	98.5%	1.0061	1.0510	98.9%	1.0066	1.0520
(100,**DEN**)	100%	1.0056	1.0246	100%	1.0059	1.0269
(200,**DEN**)	100%	1.0047	1.0137	100%	1.0047	1.0232
(5,**SPA**)	0.0%	1.0000	1.0000	8.0%	1.0007	1.0173
(10,**SPA**)	4.0%	1.0002	1.0051	12.0%	1.0011	1.0130
(30,**SPA**)	12.8%	1.0007	1.0097	36.0%	1.0021	1.0138
(50,**SPA**)	57.0%	1.0157	2.8253	72.1%	1.0212	3.8244
(100,**SPA**)	92.0%	1.0113	1.8261	92.0%	1.0943	6.0382
(200,**SPA**)	100%	1.1505	2.0221	100%	1.3176	5.5167

programming problem. We introduce $z_j = (1 - x_j)/(\sum_{i=1}^{n} v_{ji} y_i + v_{j0})$ for each $j = 1, \ldots, n$, and reformulate (14) as follow:

$$\max \sum_{j=1}^{n} \lambda_j r_j x_j + \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} r_i v_{ji} z_j y_i \tag{15a}$$

$$\text{s.t.} \sum_{i=1}^{n} v_{ji} z_j y_i + z_j v_{j0} = 1 - x_j, \quad \forall j = 1, \ldots, n, \tag{15b}$$

$$y_j \leq x_j, \quad \forall j = 1, \ldots, n, \tag{15c}$$

$$x_j, y_j \in \{0,1\}, z_j \geq 0 \quad \forall j = 1, \ldots, n. \tag{15d}$$

Then term $z_j y_i$ can be further linearized by replacing $z_j y_i$ with w_{ji} and adding the following linear inequalities: $z_j - w_{ji} \leq M - My_i$, $w_{ji} \leq z_j$, $w_{ji} \leq My_i$ and $w_{ji} \geq 0$, where $M = 1/\min_{i,j=1,\ldots,n}\{v_{ji}\}$. We then obtain a MIP reformulation of (14), and we use it to solve for the optimal assortment to STCM$_R$. For the STCM$_S$, the optimization problem is similar to that for STCM$_R$, except that the y_i in (14b) is replaced by x_i for each $i = 1, \ldots, n$ and the constraints (14b) are removed. We can analogously obtain the optimal assortment to STCM$_S$, by applying the same linearization technique in Bront et al. (2009) and solving the corresponding MIP.

The results are shown in Table 5. In Table 5, the first section shows the comparison between the STCM and the STCM$_R$. Particularly, the first subcolumn shows the percentage of times that the optimal value of the STCM problem (denoted by OPT_{STCM}) is strictly larger than that of the STCM$_R$ problem (denoted by OPT_{STCMR}) among the 1000 test instances. Note that $OPT_{STCM} \geq OPT_{STCMR}$, therefore the remaining percentages in each cell represents the proportion of test instances that $OPT_{STCM} = OPT_{STCMR}$. The second and third subcolumns show the average and maximum ratios between OPT_{STCM} and OPT_{STCMR} among the 1000 test instances, respectively. Similarly, the second section shows the comparison between the STCM and the STCM$_S$. Particularly, the first subcolumn shows the percentage of times that the optimal value of the STCM problem is larger than that of the STCM$_S$ problem (denoted by OPT_{STCMS}) among the 1000 test instances. The second and third subcolumns show the average and maximum ratios between OPT_{STCM} and OPT_{STCMS} among the 1000 test instances, respectively.

As we can see from Table 5, the STCM generates optimal assortments with strictly higher expected revenue frequently, especially when the problem size is large. We can see that for problems that have more than 50 products, the optimal value of the STCM is strictly higher than that for the other two models for nearly all the test instances. Moreover, we can observe that some of the ratios of OPT_{STCM}/OPT_{STCMR} or OPT_{STCM}/OPT_{STCMS} in Table 5 could be large, e.g., exceed 5 in some sparse cases. Indeed, as we show in the following proposition, we can construct examples in which the gap between the optimal revenue under the STCM model and that under STCM$_R$ or STCM$_S$ can be unbounded. This means that in general, there could be significant benefit of allowing the seller to customize different recommended sets based on the arrived product.

Proposition 4. *For any $M \geq 0$, there exists an example such that the ratios OPT_{STCM}/OPT_{STCMS} and OPT_{STCM}/OPT_{STCMS} are larger than M.*

Besides the computational efficiency and the benefit of allowing recommendations based on the initial product, another important issue in practice is how to calibrate the STCM model using data. For this question, we provide an estimation method for the STCM. The detail of this method is relegated to Appendix F of the online appendix.

7. Conclusion

In this study, we considered a STCM for customers. In particular, customers arrive at each product initially, and if the arrived product is unavailable, then the seller can recommend a subset of available products to the customer and the customer will purchase each product (or leave) with certain transition probabilities. We studied the assortment optimization problem under this model. Specifically, we showed that the general case of the assortment optimization problem is NP-Hard and proposed a MIP formulation for this problem. We also showed a tight worst-case performance bound for revenue-ordered assortment. Then we studied various special cases of this problem. Particularly, we showed that for certain cases, there are polynomial-time algorithms for solving the optimal assortment, while for other cases, we proposed approximation algorithms. Finally, we demonstrated via numerical experiments the efficiency of the algorithms proposed and also showed that this model has the potential to increase the seller's revenue.

There are several future directions for research. First, theoretically, it would be interesting to see if there are other practical cases in which the assortment optimization problem under the STCM can be solved in polynomial time or be approximated efficiently. Second, it might be possible to obtain theoretical lower bound for the performance of the assortment solved from the STCM when the model is misspecified. Lastly, it would be very interesting to test this approach on more real data and evaluate its performance.

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Notes

[1]We note that the Markov chain choice model may be explained from different perspectives, not necessarily viewed as the results of customers' transitions. In particular, it can be just viewed as a box of parameters to calibrate the choice probabilities. However, we mainly discuss the perspective of customers' transitions in this study to introduce the idea of the model we propose.

[2]In some later formulations, we may also use −1 to indicate FALSE. We will specify when it occurs.

[3]We also conducted experiments in which the revenues are generated by other methods, such as from exponential distributions. The results are very similar and thus we do not report them for conciseness.

[4]Particularly, in each instance of the randomized algorithms in these two tests, we generate $10n$ random hyperplanes/numbers and calculate the average of the ratios of the returned solutions. Then we calculate the average and minimum of these values among the 100 test instances and report them in Table 3 (denoted as Ave. Ratio and Min. Ratio). We also mention that if we return the best solution among the $10n$ random solutions within each instance, then in most of our test cases, the SDP relaxation and randomized greedy algorithm can return the optimal solutions for all the 100 test instances, that is, the corresponding Ave. Ratio and Min. Ratio are 100%.

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Supporting Information

Additional supporting information may be found online in the Supporting Information section at the end of the article.

Online Appendix for "Assortment Optimization under a Single Transition Choice Model"

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