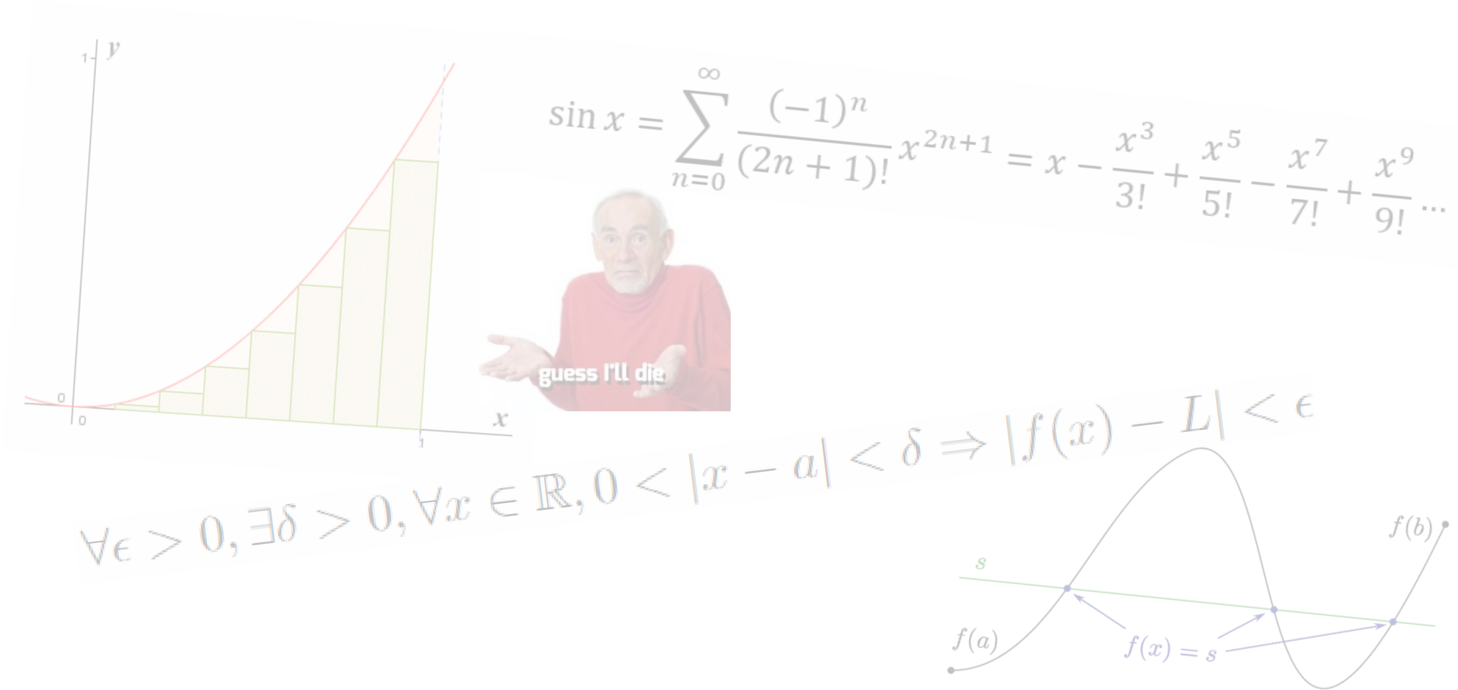
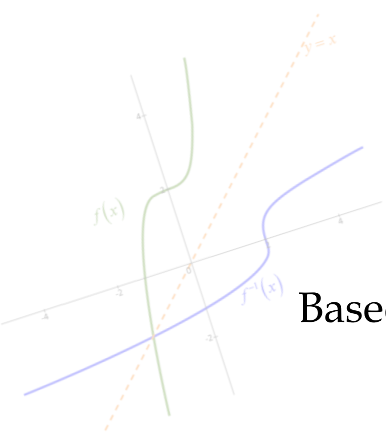
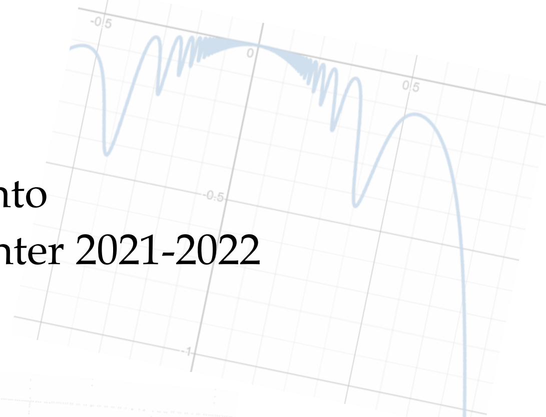


MAT137: Calculus with Proofs

A (Hopefully) Concise Guide





By Richard Yin
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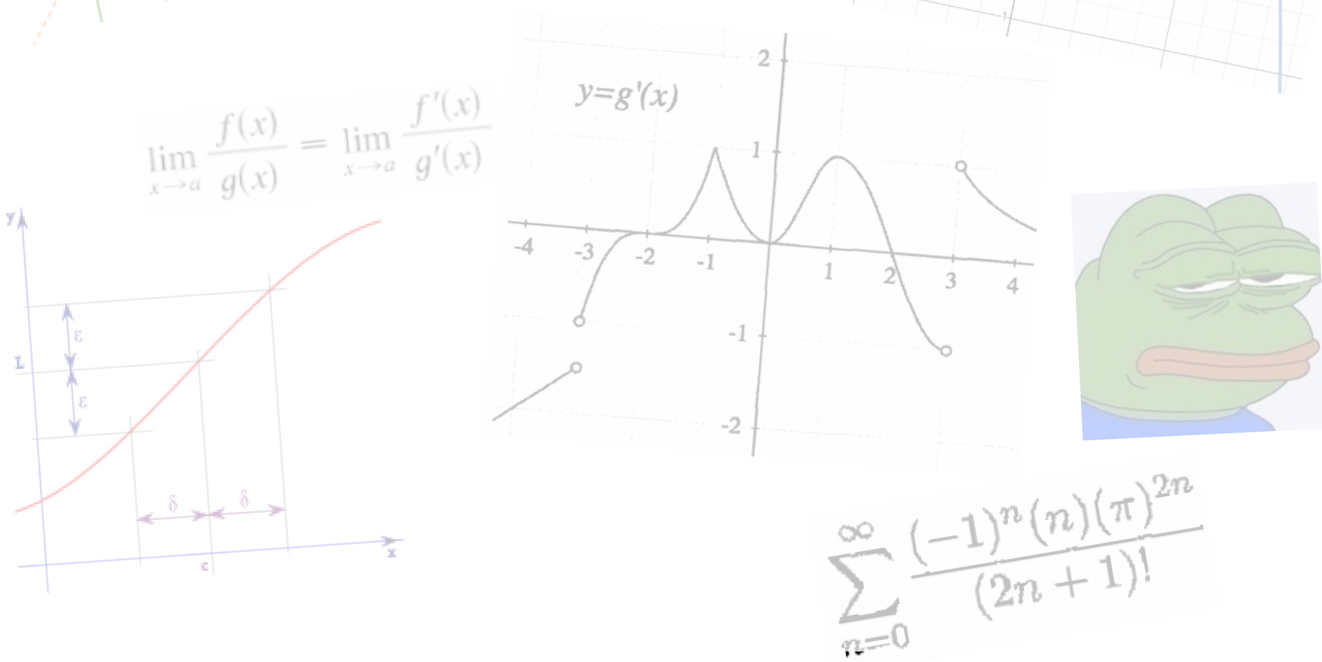
Based on content from Fall-Winter 2021-2022



Preface

These notes are UTSG-centered and based off notes and remarks (which I may not remember correctly) by my MAT137 professor, Xiao Jie, who made a course of suffering more bearable. The purpose of this textbook is to **be as dry and succinct as possible** as to not strain your eyes with paragraphs like most textbooks.

The accompanying document, *Questions in MAT137*, has practice questions (some from past exams) with partial solutions. It is recommended you check them out.

Choosing a Calculus Course

All first-year calculus courses are full-credit (Y) or two half-credit (F/S) courses. Consider this guide.

Course	Recommended For	Description
MAT133	Commerce	Commerce-focused, easy and shallow. Crams calculus AND linear algebra (1.0 + 0.5 credits) into a 1.0 credit course. Not a valid prerequisite for all second-year math courses.
MAT135/ MAT136	Science Social Sciences	Split into derivatives (MAT135) and integrals (MAT136). The standard calculus courses. Mostly applied math, no theory. Computation, modelling, and application-focused.
MAT137	Computer Science Physics Statistics Actuarial Science Economics	A combination of theory and computation, balancing the extremes of MAT135/136 and MAT157. Introduces mathematical rigour, definitions, and proofs, whose initial learning curve is very steep.
MAT157	Math Specialists	Mostly theory, the most difficult one. Delves into complex proofs & abstract mathematics. "Analysis" is essentially advanced calculus.

There's also MAT134 (a UTM thing, similar to MAT135/136). Here're some course-specific content:

12. Consider the predator-prey system $\frac{dR}{dt} = 15R - 3RW$, $\frac{dW}{dt} = -8W + 2RW$. When the system is in equilibrium with $W \neq 0$, $R \neq 0$, then $RW =$

4. Show that

$$\ln(x) \geq x - 1 - \frac{(x-1)^2}{2}$$

for $x \geq 1$ using the Racetrack Principle. [8 marks]

13. (2 points) Let $P(\alpha)$ be the cumulative distribution function related to $p(\alpha)$. We say that a user u is *inclined towards science* if $\alpha(u)$ is negative. Which of the following represent the probability that a user is inclined towards science?

MAT135/136 computations are hard. Unique concepts include, probability models, and differential equations.

But…you can just formally study differential equations in MAT244 and probability in STA237/257.

12. Consider the sequence given by $a_0 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$ for all $n \geq 0$.

(a) *[8 points]* Prove the sequence converges.

(b) Write a formal proof for the following theorem, directly from the definition of limit. Do not use any of the limit laws.

Theorem
Let f and g be functions with domain $\mathbb{R}$. Let $h = f + g$. Let $L, M \in \mathbb{R}$.

- IF $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$
- THEN $\lim_{x\to\infty} h(x) = L + M$

MAT137's computations generally *aren't as hard* as 135/136. Exams have a **proofs section** worth roughly 25%.

There're less word questions and application-based questions.

3. (i) *[5 marks]* Suppose $A_1, A_2, A_3, \ldots$ are sets and $A_i \subseteq (-1, 1), \forall i$. In particular, each A_i is bounded above by 1 and below by -1, and so each A_i has a supremum: let $\alpha_i = \sup A_i$, $i = 1, 2, 3, \ldots$.

Prove the following statement or provide a counterexample, including an explanation of why it is a counterexample: let $A = \cup_i A_i$. Let $\alpha = \sup A$. Then

$$\begin{aligned} \alpha &= \sup_i \alpha_i, \\ \text{i.e.,} \quad \sup(\cup_i A_i) &= \sup_i(\sup(A_i)). \end{aligned}$$

MAT157 is frightening. Proofs make up ~33% of the exam, and are **very abstract** – even the computations.

Computations often combine many concepts (limit, integral, derivative).

10. *[10 marks]* Prove the following result. Suppose $r < a < t$, and $f(x)$ is defined on (r,t). Then $f(x)$ is continuous at $x = a$ if and only if for every sequence $\{a_n\}$ of elements of (r,t) that converges to a, we have that the sequence $\{f(a_n)\}$ converges to $f(a)$.

1. (ii) *[5 marks]* Define a function $g(x)$ on $(0,1)$ as follows:

$$g(x) = \begin{cases} \frac{a}{b^2}, & \text{if } x = \frac{a^2}{b^2} \in \mathbb{Q} \text{ in lowest terms, with } a > 0, b > 0, \\ 0, & \text{if } x \neq r^2, \text{ for any } r \in \mathbb{Q}. \end{cases}$$

Evaluate $\lim_{x \to 0^+} g(x)$.

MAT157 unlocks the same courses as 137, plus a few unique extras, like:

- MAT257 (Analysis II),
- MAT267 (Advanced Ordinary Differential Equations)
- MAT327 (Topology)

Check out these 157 notes: #1, #2

Anything with MAT135/136 as a prerequisite accepts 137 too. Anything that accepts 137 also accepts 157.

You can take MAT138 (Introduction to Proofs) alongside these courses to get familiar with proofs, but I don't recommend it – they're so many other cool things you can learn in university!

If you didn't take MAT157 but want to take courses only accepting MAT157 as prerequisite, usually MAT246 (Concepts in Abstract Mathematics) or PHL245 (Modern Symbolic Logic) can unlock some.

Linear Algebra

MAT223 (Linear Algebra I) is easier than MAT137 in my opinion. Its computations are dry and mechanical; the hard part is terminology and visualizing the problems.

MAT224 (Linear Algebra II) gets theoretical and is more comparable to MAT137.

MAT240 (Algebra I) and MAT247 (Algebra II) are the MAT157/257 equivalents of linear algebra, and even require taking 157/257 as corequisites.

Linear algebra is very important in statistics, modelling, AI and machine learning, economics, quantum mechanics, theoretical chemistry, etc. So if you're going into those fields, at least consider MAT223!

Advice

1. This textbook probably teaches in a <u>different order</u> than your professor.
2. MAT137 is hardest in the first 2-3 months: the learning curve of grasping how limit proofs work.
3. For test/exam review, don't review concepts; <u>do practice questions</u>. You learn math via example.
   - For proofs, it builds up an intuition of how to approach problems.
   - For computations, it helps you easily remember formulas.
4. If there're questions that stump you, use Desmos and Symbolab if it's computation. Otherwise, use office hours and tutorials. Or, if you're in UTSG, use the Math Learning Centers (MLC).
5. In my opinion, <u>the best predictor of university success is work ethic</u> – studying the way you find most effective, and not finishing work the day before it's due.

About this Guide

My credentials are: I'm a UTSG comp-sci student who got 95% on MAT137. In highschool, I took IB SL math.

Here's a refresher on large operators. I will only use summation, but higher-level courses might use more.

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n \qquad \prod_{i=1}^{n} i = 1 \times 2 \times \ldots \times n \qquad \bigcup_{i=1}^{n} S_i = S_1 \cup S_2 \cup \ldots \cup S_n \qquad \bigcap_{i=1}^{n} S_i = S_1 \cap S_2 \cap \ldots \cap S_n$$

All example questions are <u>valid exam-style questions</u> unless otherwise stated. All content in this guide, except for some example questions, was written by me (though the solutions are original).

When I use "", I'm probably paraphrasing my math prof. Credit goes where it's due. I'm not monetizing this or anything so don't sue me aaahh

Table of Contents

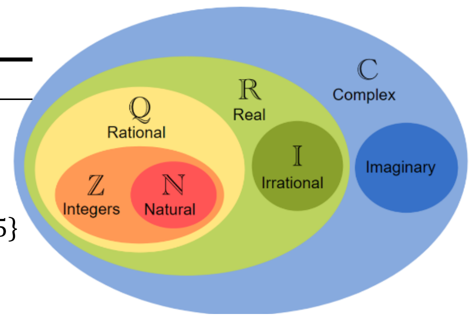
1. Sets

Sets are unordered collections of elements; in math, these elements are numbers. Sets are notated $\{a, b, c\}$.

Notation	Meaning
$\emptyset$	Empty set {}
$x \in A$	x is an element of/in set A
A^c	**Complement** – The set of everything not in set A. If $x \in A$, then $x \notin A^c$ and vice versa
$A \cup B$	**Union** – The set containing everything in A or B *(inclusive "or")*
$A \cap B$	**Intersection** – The set containing everything in A and B
$\lvert A \rvert$	**Cardinality** – The size/length/number of elements in set A
$A \subseteq B$	**Containment** – Set A is a **subset** of set B (everything in A is in B). $\subseteq$ and $\subset$ mean the same.
$A = B$	**Equality** – $A \subseteq B$ and $B \subseteq A$ (ie. A and B are identical)

The above is essentially unused in MAT137; it's more important for Linear Algebra (MAT223). However, the number sets below will be useful references:

Set	Meaning	Example Subsets
$\mathbb{C}$	Complex number	$\{2i + 1, e^i\}$
$\mathbb{R}$	Real number*	$\{1, 2.08, -\pi, e\}$
$\mathbb{I}$	Irrational number	$\{\pi, e, -\sqrt{2}\}$
$\mathbb{Q}$	Rational number *(writable as fraction)*	$\{-1/3, 2/9, -102.22\ldots, 4.5\}$
$\mathbb{Z}$	Integer	$\{-3, -2 - 1, 0, 1, 2, 3\}$
$\mathbb{N}$	Natural number**	$\{1, 2, 3, 4, 5\}$



*$\mathbb{R}^1$ *(the default) refers to all numbers in 1D space.* $\mathbb{R}^n$ *means all numbers in n-D space. MAT223 uses this concept*
***May/may not include 0 in math. Ask your math prof; I'll say it does. Includes 0 in computer science*

Certain notations can describe more specific sets of numbers:

Interval Notation	Set-Builder Notation	English Translation
(a, b)	$\{x \in \mathbb{R} \mid a < x < b\}$	Set of real numbers between a (exclusive) and b (exclusive)
$(a, b]$	$\{x \in \mathbb{R} \mid a < x \leq b\}$	Set of real numbers between a (exclusive) and b (inclusive)
$[a, \infty)$	$\{x \in \mathbb{R} \mid x > a\}$	Set of real numbers greater or equal to a

Set-builder notation, $\{x: y\}$ or $\{x|y\}$, reads as "The set of all x where y." y can be wordy and include conjunctions (ie. *and*, *or*). This notation is flexible and multipurpose, but more common in MAT223.

Interval notation is simpler and used in MAT137. Intervals like (a, b) are **open**, [a, b] are **closed**, and [a, b) are **half-open.** Intervals like (a, ∞) are **unbounded** and aren't very "proper", mathematically speaking, due to the ∞. Note that there must be "(" or ")" next to ∞.

The **maximum/minimum** is the largest/smallest item in a set.

- $[a, b]$ has a **maximum** (b) and **minimum** (a).
- $(a, b]$ has a **maximum** (b), but **no minimum**.

An **upper/lower bound** of a set is greater than or equal to/less than or equal to every set item.

- The **supremum** (sup) is the smallest possible upper bound.
  - $[a, b]$ has a **supremum** (b) and **infimum** (a).
- The **infimum** (inf) is the largest possible lower bound.
  - $(a, b]$ has a **supremum** (b) and **infimum** (a).

The *max/min* of a set is <u>in the set</u>. This isn't necessarily true for *sup/inf*. You'll find formal definitions soon.

2. Proofs

2.1. Logical Symbols

Connective	Name	Meaning
$\neg P$, $\sim P$, not P*	Negation	P is not true
$P \wedge Q$, P and Q	Conjunction	P is true and Q is true
$P \vee Q$, P or Q	Disjunction	P is true or (inclusive) Q is true
$P \Rightarrow Q$	Implication/Conditional	If P is true, Q is true.
$P \Leftrightarrow Q$	Biconditional	If and only if** P is true, Q is true. (ie. $P \Rightarrow Q$ and $Q \Rightarrow P$)

**You see $\neg P$ in CSC110/CSC165 and $\sim P$ in PHL245. My MAT137 prof didn't use any symbols for not/and/or.*
***"If and only if" is often abbreviated to "iff".*

Quantifier	Name	Meaning
$\forall x \in S, P$	Universal Quantifier	For every/all/any x in set S, P is true
$\exists x \in S, P$	Existential Quantifier	For some/there exists x in set S, (such that) P is true

Below is a **truth table** containing all truth/false combinations and all connectives' truth values.

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T*	F
F	F	F	F	T*	T

*This is called **vacuous truth**, where conditional $P \Rightarrow Q$ is true since premise P is false. You can't prove $P \Rightarrow Q$ with an example where P is false.

2.2. Proof Structure

Think of proofs like using a **toolbox** of known things (in math, it's stuff like $x + 1 > x$ or $1 > -1$), to show something new. The following tables show how to start proofs in the face of different statement types.

Implication	Conjunction	Disjunction	Biconditional
eg. Prove $P \Rightarrow Q$	eg. Prove $P \wedge Q$	eg. Prove $P \vee Q$	eg. Prove $P \Leftrightarrow Q$
Assume P *Find a way to show Q*	*Separately show P* *Separately show Q*	*Split into two cases:* (1) Assume P is true Thus P ∨ Q is true (2) Assume P is false *Find a way to show Q*	Show $P \Rightarrow Q$ Assume P *Find a way to show Q* Show $Q \Rightarrow P$ Assume Q *Find a way to show P*
Or, prove an equivalent form – the **contrapositive:** $\neg Q \Rightarrow \neg P$		You can also use cases when Q is true/false.	Often seen in theorems and definitions.

Universal Instantiation	Existential Instantiation
eg. Prove $\forall x \in S, P$	eg. Prove $\exists x \in S, P$
Let $x \in S$ *Find a way to show P*	Pick $x =$ *something in S* *Find a way to show P*
We're picking a random/arbitrary/any x in S, making no assumptions about what x is, only assumptions about S. If the proof still works despite the lack of specifics, we know the proof will work for all $x \in S$.	Since we've claimed P is true for some x, we have to back it up with a specific $x \in S$. Since we've chosen something specific, we can make assumptions based on what x is.

Note that $\begin{array}{c}(\forall x, \forall y, P) = (\forall y, \forall x, P) \\ (\exists x, \exists y, P) = (\exists y, \exists x, P)\end{array}$, but $(\forall x, \exists y, P) \neq (\exists y, \forall x, P)$. How come? Compare "For all x, there's a negative y = -x" to the impossible "There's a specific number y negative to all x. For all x, y = -x".

Set Containment	Set Equality
eg. Prove $A \subseteq B$	eg. Prove $A = B$
Let $x \in A$	*Find a way to show* $A \subseteq B$
Find a way to show $x \in B$	*Find a way to show* $B \subseteq A$

Here are some rules of inference (from PHL245. Learning derivations gives strong proofs foundations). These are examples of what you can conclude without proof based on given assumptions.

Logical Techniques	Name
$P.$ $\quad P \Rightarrow Q.$ $\quad \therefore Q.$	Modus ponens (mp)
$\sim Q.$ $\quad P \Rightarrow Q.$ $\quad \therefore \sim P.$	Modus tollens (mt) – *based on proving the contrapositive*
$\sim P.$ $\quad P \vee Q.$ $\quad \therefore Q.$	Modus tollendo ponens (mtp)
$P \wedge Q.$ $\quad \therefore P.$ $\quad \therefore Q.$	Simplification (s)
$P.$ $\quad \therefore P \vee Q.$	Addition (add)

Some examples of "theorems" you can use without proof are:

- $k = \max\{x, y\} \Rightarrow k \geq x$ and $k \geq y$ – *similar logic for minimums*
- $k = \text{ceil}(x) \Rightarrow k \geq x$ and $k \in \mathbb{Z}$ – *similar logic for floor*
- $(x > y \Rightarrow x \geq y)$ and $(x = y \Rightarrow x \geq y)$ – *similar logic for* $<$
- $|x + y| \leq |x| + |y|$ – *the Triangle Inequality*
- $\big||x| - |y|\big| \leq |x - y|$ – *the Reverse Triangle Inequality*

2.3. Negations and Proof by Contradiction

If a statement is wrong, you show it by proving the statement's negation true. When negating a statement, replace the symbols as shown to the right. Note swapping pairs $(\forall, \exists)$ and $(\wedge, \vee)$.

You must also invert equalities $(=, \neq)$ and inequalities $(>, \leq), (\geq, <)$.

	=	
$\sim(\forall x, P)$		$\exists x, \sim P$
$\sim(\exists x, P)$		$(\forall x, \sim P)$
$\sim(P \wedge Q)$		$\sim P \vee \sim Q$
$\sim(P \vee Q)$		$\sim P \wedge \sim Q$
$P \Rightarrow Q$		$P \wedge \sim Q$
$P \Leftrightarrow Q$		$P \Leftrightarrow \sim Q$

Contradiction
eg. Prove P
Proof by Contradiction, suppose $\sim P$ *Use* $\sim P$ *to arrive at a contradiction somewhere* Thus, P

Proofs by contradiction require you assume a statement's negation. They're generally hard, as often, you're wandering aimlessly for a contradiction. We'll only see these proofs for certain question types in MAT137, but you'll see more if you take PHL245.

2.4. Proof by Induction

Induction works for integers/natural numbers. While not common in MAT137, it's valuable in CSC111/CSC148/CSC165 due to ties to recursion and growth rates (Big-O).

Say you want to prove $\forall n \in \mathbb{N}, 2n \leq 2^n$. You notice $\{0,2,4,6,8,10 \ldots\}$ vs. $\{1,2,4,8,16,32, \ldots\}$ Clearly, $2n$ never catches up to 2^n. But just because it *seems* right doesn't mean it's a proper mathematical proof! Do it like so:

Induction
eg. Prove $\forall n \in \mathbb{N}, P(n)$
Proof by Induction, let $n \in \mathbb{N}$ <u>Base Case:</u> $P(1)$ *Find a way to show* $P(1)$ <u>Inductive Step:</u> $P(n) \Rightarrow P(n+1)$ Assume $P(n)$ *Find a way to show* $P(n+1)$

$P(x)$ is a statement with the variable x, like $P(x)$: $(x+1)^2 \geq 0$.

$P(1)$ refers to the statement when $x = 1$, like $P(1)$: $(1+1)^2 \geq 0$.

Statements with variables are used studied in **predicate logic**, which is more important for computer science than MAT137.

Remember to add "Let …", since the statement starts with $\forall n \in \mathbb{N}$.

The logic behind this is: since $P(1)$ is true, $P(2)$ is true as $P(n) \Rightarrow P(n+1)$ for $n = 1$. Since $P(2)$ is true, $P(3)$ is true, and so on.

3. Limits

3.1. Theory

3.1.1. Definition ($\epsilon - \delta$)

Any value of x has a corresponding distance to a, $|x - a|$.
Any value of $f(x)$ has a corresponding distance to L, $|f(x) - L|$, which we'll call ϵ.

What does $\lim_{x \to a} f(x) = L$ mean?
The closer x gets to a (while not touching it), the closer $f(x)$ gets to L.
As the distance between x and a becomes infinitesimally small (but not 0),
the distance between $f(x)$ and L becomes infinitesimally small (and maybe 0).

If you manually picked a $f(x_1)$ that's close to L (where $x_1 \neq a$), I could always pick a closer $f(x_2)$ to L than what you just picked (where $x_2 \neq a$).

eg. $f(x) = x^2$, and thus $\lim_{x \to 2} f(x) = L = 4$

- If you pick $f(x_1) = 3.5$, I pick $x_2 = 1.9 \neq 2$, so $f(1.9) = 3.61$, which is closer to $L = 4$.
- If you pick $f(x_1) = 3.9$, I pick $x_2 = 1.99 \neq 2$, so $f(1.99) = 3.9601$, which is closer to $L = 4$.
- If you pick $f(x_1) = 3.99$, I pick $x_2 = 1.999 \neq 2$, so $f(1.999) = 3.996001$, which is closer to $L = 4$.

Let's gradually turn the above into a logical statement.

1) No matter what $f(x_1)$ you pick, I can always pick a $x_2 \neq a$ so that $f(x_2)$ is closer to L than $f(x_1)$.
2) For any distance between $f(x_1)$ and L you pick, there is always a $x_2 \neq a$ such that the distance between $f(x_2)$ and L is smaller than the distance between $f(x_1)$ and L.
3) For any $|f(x_1) - L|$, there exists $x \neq a$ such that $|f(x) - L|$ is smaller than $|f(x_1) - L|$.
4) $\forall \epsilon > 0, \exists x \neq a, |f(x) - L| < \epsilon$

Why is $\epsilon > 0$? Because we defined ϵ as a distance (an absolute value); it cannot be negative. It cannot be 0 either, because we've defined the limit as when $|f(x) - L| = e \to 0$ but isn't 0. So $e > 0$.

But this isn't the full definition of limit. Instead of saying there exists $x \neq a$, where $|f(x) - L| < \epsilon$, it's better to say there's a **range** of x values that give closer $f(x)$ values to L than what you choose.

eg. $f(x) = x^2$, and thus $\lim_{x \to 2} f(x) = L = 4$

- If you pick $f(x_1) = 4.5$, I can pick any $x_2 \neq 2$ where $1.870 \ldots < x_2 < 2.121 \ldots$ for a closer $f(x_2)$.
- If you pick $f(x_1) = 4.1$, I can pick any $x_2 \neq 2$ where $1.974 \ldots < x_2 < 2.024 \ldots$ for a closer $f(x_2)$.
- If you pick $f(x_1) = 4.01$, I can pick any $x_2 \neq 2$ where $1.997 \ldots < x_2 < 2.002 \ldots$ for a closer $f(x_2)$.

Let's gradually turn the above into a logical statement.

1) No matter what $f(x_1)$ you pick, I can always pick $x_2 \neq a$ in the range $a - \delta < x_2 < a + \delta$ (where $\delta \in \mathbb{R}$) that'll give me a $f(x_2)$ closer to L than $f(x_1)$.
2) For any distance between $f(x_1)$ and L you pick, there's always a $\delta \in \mathbb{R}$ where any $x_2 \neq a$ in the range $a - \delta < x_2 < a + \delta$ will make $|f(x_2) - L|$ smaller than $|f(x_1) - L|$.
3) For any ϵ, there exists $\delta \in \mathbb{R}$ such that if $a - \delta < x < a + \delta$, $|f(x) - L|$ is smaller than ϵ.
4) $\boxed{\forall \epsilon > 0, \exists \delta > 0,\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon}$

How did I get $0 < |x - a| < \delta$?

$a - \delta < x < a + \delta$	Since $x \neq a$, $|x - a| \neq 0$
$-\delta < x - a < \delta$	$\therefore 0 < |x - a| < \delta$
$|x - a| < \delta$	To make the inequality work, we set $\delta > 0$. If you're confused, see 3.1.2.

3.1.2. Notation

Consider the definition of limit of $f(x)$:	$\forall \epsilon > 0, \exists \delta > 0, 0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \epsilon$
Note that it uses x but never introduces it. The statement should really be:	$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, 0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \epsilon$
You could also write it like:	$\forall x \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, 0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \epsilon$

Don't use the 2nd form; it doesn't work in negations. Takeaway: limit should work for *all numbers* (ie. $\forall x \in \mathbb{R}$)

We could actually further expand this, but at the cost of getting even bulkier.	$\forall \epsilon \in \mathbb{R}, \exists \delta \in \mathbb{R}, \forall x \in \mathbb{R},$ $(\epsilon > 0 \text{ and } \delta > 0) \Rightarrow (0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \epsilon)$

So, beware of these notation shortcuts in MAT137 and future math courses:

- If there's an un-introduced x or $f(x)$, assume it's $\forall x \in \mathbb{R}$ and $\forall f(x)$
- If there's a variable like $\exists \alpha > 0$ with specified no number set (eg. $\mathbb{R}, \mathbb{N}, \mathbb{Z}$), assume it's in $\mathbb{R}$

By saying $\lim_{x \to a} f(x) = L$, we implicitly mean $\lim_{x \to a} f(x)$ exists.
L is a placeholder for $\lim_{x \to a} f(x)$, which is clunky, but you may need to write that, so get familiar with:

$$\forall \epsilon > 0, \exists \delta > 0, 0 < |x - a| < \delta \Rightarrow \left| f(x) - \lim_{x \to a} f(x) \right| < \epsilon$$

3.1.3. Absolute Value

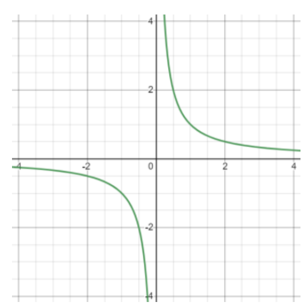
Limit proofs heavily involve absolute value, so make sure you're familiar with how they work.

$\lvert x \rvert = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$	$\lvert xy \rvert = \lvert x \rvert \lvert y \rvert$ *("Multiplicative")* $\lvert x \rvert = 0 \Leftrightarrow x = 0$ *("Non-Degenerate")*	$\lvert x \rvert < \alpha \Leftrightarrow -\alpha < x < \alpha$ $\lvert x \rvert > \alpha \Leftrightarrow \{x \in \mathbb{R} : x > \alpha \text{ or } x < -\alpha\}$

3.1.4. Types of Limits

One-Sided Limits

Consider $f(x) = \frac{1}{x}$.



$\lim_{x \to 0} f(x)$ does not exist (DNE).	$\lim_{x \to 0^+} f(x) = \infty$ (the limit from the right/+ side)	$\lim_{x \to 0^-} f(x) = -\infty$ (the limit from the left/- side)

$$\boxed{\lim_{x \to a} f(x) \text{ exists} \Leftrightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x)}$$

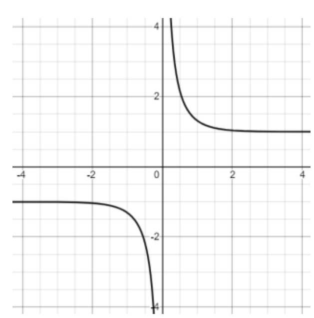
$$\boxed{\lim_{x \to a^+} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, 0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon}$$

$$\boxed{\lim_{x \to a^-} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, 0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon}$$

$\lim_{x \to 0^+} f(x) = \infty$ is *notational shorthand*. As a mathematical statement, it's meaningless as we can't compute with ∞. Also, alternate notation: $\lim_{x \to 0^+} f(x) = \lim_{x \downarrow 0} f(x) = \lim_{x \searrow 0} f(x)$ and $\lim_{x \to 0^-} f(x) = \lim_{x \uparrow 0} f(x) = \lim_{x \nearrow 0} f(x)$.
Lastly, my prof saw one-sided limits as a lower-level math thing, but I don't remember why.

Approaching Infinity

Consider $f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ shown on the right.



$\lim_{x \to \infty} f(x) = 1$ (the limit going to the right)
$\lim_{x \to -\infty} f(x) = -1$ (the limit going to the left)

In both cases, our $\epsilon - \delta$ limit definition breaks down & stops making sense:

$\forall \epsilon > 0, \exists \delta > 0, 0 < |x - \infty| < \delta \Rightarrow |f(x) - L| < \epsilon$

So, instead of a value of δ defining a range of x, we use a lower bound N:

$$\boxed{\forall \epsilon > 0, \exists N \in \mathbb{R}, x > N \Rightarrow |f(x) - L| < \epsilon}$$

The intuition is that I can always "choose" a bigger x with an $f(x)$ closer to L than the $f(x)$ you chose.

3.1.5. Proving a Limit

eg. Prove $\lim\limits_{x\to 1}\frac{(x-1)(x+1)^2}{x-1}=4$	
$\forall\epsilon>0, \exists\delta>0, 0<\lvert x-1\rvert<\delta \Rightarrow \left\lvert\frac{(x-1)(x+1)^2}{x-1}-4\right\rvert<\epsilon$ *(Let $x\in\mathbb{R}$, let $f(x)=\frac{(x-1)(x+1)^2}{x-1}$)* Let $\epsilon>0$ Pick $\delta=\ldots$ Assume $0<\lvert x-1\rvert<\delta$ Show $\left\lvert\frac{(x-1)(x+1)^2}{x-1}-4\right\rvert<\epsilon$ …	*Expand the definition of limit.* *Set-up the proof by introducing necessary variables. DON'T pick a δ value initially! You don't know what it has to be yet, so leave it blank for now.* *You can add "Let $x\in\mathbb{R}$" under "Pick $\delta=\ldots$". It's optional. Note you can't set $\delta=x$, as you haven't introduced x yet when you introduced δ. It's picky.*
<u>Rough Work</u> *The goal is to get to $\left\lvert\frac{(x-1)(x+1)^2}{x-1}-4\right\rvert<\epsilon$.* *Our assumption, $0<\lvert x-1\rvert<\delta$, is key to helping us.* *Turn $\left\lvert\frac{(x-1)(x+1)^2}{x-1}-4\right\rvert$ into something involving $\lvert x-1\rvert$.* $\left\lvert\frac{(x-1)(x+1)^2}{x-1}-4\right\rvert=\lvert(x+1)^2-4\rvert$ $=\lvert x^2+2x-3\rvert$ $=\lvert(x+3)(x-1)\rvert$ $=\lvert x+3\rvert\lvert x-1\rvert$ *Since $0<\lvert x-1\rvert<\delta$,* $\lvert x-1\rvert<\delta$ $\lvert x+3\rvert\lvert x-1\rvert<\lvert x+3\rvert\delta$ *If $\lvert x+3\rvert\delta<\epsilon$, then we proved our objective. Let's try simplifying $\lvert x+3\rvert$ first, using our assumption.* $\lvert x-1\rvert<\delta$ $-\delta<x-1<\delta$ $-\delta+4<x+3<\delta+4$ *Uh oh. We can't smoothly turn $x+3$ into $\lvert x+3\rvert$ here.* *What if we picked a δ such that $\delta=1$?* $\lvert x-1\rvert<\delta=1$ $\lvert x-1\rvert<1$ $-1<x-1<1$ $3<x+3<5$ $\lvert x+3\rvert<5$ *Then $\lvert x+3\rvert\delta<5\delta$. But wait, $\delta=1$, so $5\delta=5$, but ϵ is arbitrary and positive, so $5\le\epsilon$ is not always true?* *Yes. So, let's also pick δ such that $\delta=\frac{\epsilon}{5}$. Then we'd get* $\lvert x+3\rvert\delta<5\delta=5\cdot\frac{\epsilon}{5}=\epsilon$ *How can we pick $\delta=1$ and $\delta=\frac{\epsilon}{5}$? Remember a theorem from 2.2, $k=\max\{x,y\}\Rightarrow k\ge x$ and $k\ge y$. If we pick $\delta=\min\{1,\frac{\epsilon}{5}\}$ we get $\delta\le 1$ and $\delta\le\frac{\epsilon}{5}$, which works.*	*In the rough work, we work backwards from the solution (since we can't do that in a normal proof)* *Why? So we can use the assumption. You'll see how.* *Remember in a limit, $x\neq a$. Here, $a=1$, so we're not actually dividing by 0 when we do this step.* *As shown by 3.1.3's refresher on absolute value* *Since $\lvert x+3\rvert>0$, multiplying it on both sides doesn't flip the inequality.* *Some might pick $\delta=\frac{\epsilon}{2\lvert x+3\rvert}$, resulting in* $\lvert x+3\rvert\delta=\lvert x+3\rvert\frac{\epsilon}{2\lvert x+3\rvert}=\frac{\epsilon}{2}<\epsilon$ *We <u>cannot</u> do this. Why? Recall the limit definition: … $\exists\delta>0, \forall x\in\mathbb{R}$, … Since we introduced x after δ, we cannot base δ on x.* *Remember we can pick any $\delta>0$. Choosing $\delta=1$ is arbitrary; the proof works with other choices. But <u>$\delta=1$ is simple and almost always works</u>, so I'd recommend it as the default. We'll see when it doesn't work.* *I picked $\frac{\epsilon}{5}$ to turn 5δ into the elegant, readable ϵ. Any smaller number works, but isn't as pretty.* *There're many concepts here that might make you think "How was I supposed to think of that?" The answer is…experience. But don't worry – all other "prove a limit" questions will have a similar format.*

Let $\epsilon > 0$ Pick $\delta = \min\{1, \frac{\epsilon}{5}\}$, then $\delta \le 1$ and $\delta \le \frac{\epsilon}{5}$. Assume $0 < \|x-1\| < \delta$ Show $\left\|\frac{(x-1)(x+1)^2}{x-1} - 4\right\| < \epsilon$ $\left\|\frac{(x-1)(x+1)^2}{x-1} - 4\right\| = \|(x+1)^2 - 4\| = \|x^2+2x-3\| = \|x+3\|\|x-1\|$ Since $0 < \|x-1\| < \delta$, $\|x+3\|\|x-1\| < \|x+3\|\delta$ Since $\delta \le 1$, $0 < \|x-1\| < \delta \le 1$ $\|x-1\| < 1$ $-1 < x-1 < 1$ $3 < x+3 < 5$ $\|x+3\| < 5$ $\|x+3\|\delta < 5\delta$ Since $\delta \le \frac{\epsilon}{5}$, $5\delta \le 5 \cdot \frac{\epsilon}{5} = \epsilon$ ■	*Return back to our formal proof.* *Note that $\{1, \frac{\epsilon}{5}\}$ is just one solution pair. If you don't pick $\delta \le 1$, you can still solve the question, but you might have to change $\frac{\epsilon}{5}$ to a different $\frac{\epsilon}{k}$.* *Essentially just copy your rough work in.* *To summarize, our chain of logic in this proof is:* $\left\|\frac{(x-1)(x+1)^2}{x-1} - 4\right\| = \|x+3\|\|x-1\|$ $\|x+3\|\|x-1\| < \|x+3\|\delta$ *as* $\|x-1\| < \delta$ $\|x+3\|\delta < 5\delta$ *as* $\|x+3\| < 5$ $5\delta \le \epsilon$ *as* $\delta \le \epsilon/5$ *Odds are, most limit proofs will follow this logic.*

I never proved one-sided limits in MAT137, so I doubt it'll be a thing on the exam.

3.1.6. Proving a Limit ($x \to \infty$)

eg. Prove $\lim_{x\to\infty}\left(1 + \frac{1}{x+1} + \frac{1}{x-1}\right) = 1$	
$\forall \epsilon > 0, \exists N \in \mathbb{R}, x > N \Rightarrow \left\|\left(1 + \frac{1}{x+1} + \frac{1}{x-1}\right) - 1\right\| < \epsilon$ $\forall \epsilon > 0, \exists N \in \mathbb{R}, x > N \Rightarrow \left\|\frac{1}{x+1} + \frac{1}{x-1}\right\| < \epsilon$ *(Let $x \in \mathbb{R}$, let $f(x) = 1 + \frac{1}{x+1} + \frac{1}{x-1}$)* Let $\epsilon > 0$ Pick $N = \dots$ Assume $x > N$ Show $\left\|\frac{1}{x+1} + \frac{1}{x-1}\right\| < \epsilon$ …	*Expand the definition of limit. I choose to re-write it, simplified here, for clarity's sake, but it's not strictly necessary; you can "derive" it in the last steps of your proof anyways.* *As usual, leave the variable you're picking blank and do some rough work first.*
<u>Rough Work</u> *We need* $\left\|\frac{1}{x+1} + \frac{1}{x-1}\right\| < \epsilon$. *If* $\left\|\frac{1}{x+1}\right\| < \frac{\epsilon}{2}$ *and* $\left\|\frac{1}{x-1}\right\| < \frac{\epsilon}{2}$, *then we can achieve that.* $\left\|\frac{1}{x+1}\right\| < \frac{\epsilon}{2}$ $-\frac{\epsilon}{2} < \frac{1}{x+1} < \frac{\epsilon}{2}$ $-\epsilon(x+1) < 2 < \epsilon(x+1)$ $-\epsilon(x+1) < 2$; $x+1 < -2/\epsilon$; $x < -2/\epsilon - 1$ $2 < \epsilon(x+1)$; $2/\epsilon < x+1$; $2/\epsilon - 1 < x$	*Splitting the inequality into two equations*

<table>
<tr>
<td>There are two solutions: $x < -{}^{2}/_{\epsilon} - 1$ or $x > {}^{2}/_{\epsilon} - 1$.
Repeat this above calculation with $\left|\frac{1}{x-1}\right|$ instead to get $x < -{}^{2}/_{\epsilon} + 1$ and $x > {}^{2}/_{\epsilon} + 1$.

Remember that x > 0 and we choose N where x > N, so we have to choose $x > {}^{2}/_{\epsilon} + 1$ or $x > {}^{2}/_{\epsilon} - 1$. The former has a stricter bound, so we'll set N based on that.</td>
<td>Why choose the stricter bound? Because if $x > {}^{2}/_{\epsilon} + 1$, then by extension, $x > {}^{2}/_{\epsilon} - 1$. We get more tools to work with this way.</td>
</tr>
<tr>
<td>Let $\epsilon > 0$
Pick $N = \frac{2}{\epsilon} + 1$
Assume $x > N$
Show $\left|\frac{1}{x+1} + \frac{1}{x-1}\right| < \epsilon$
Since $x > N = \frac{2}{\epsilon} + 1$,
$$x > \frac{2}{\epsilon} + 1$$ $$\epsilon x > 2 + \epsilon$$ $$2 < \epsilon(x-1)$$ $$\frac{1}{x-1} < \frac{\epsilon}{2}$$ $$\left|\frac{1}{x-1}\right| < \frac{\epsilon}{2}$$ Since $x > \frac{2}{\epsilon} + 1$ and $\frac{2}{\epsilon} + 1 > \frac{2}{\epsilon} - 1$,
$$x > {}^{2}/_{\epsilon} - 1$$ $$\epsilon x > 2 - \epsilon$$ $$2 < \epsilon(x+1)$$ $$\frac{1}{x+1} < \frac{\epsilon}{2}$$ $$\left|\frac{1}{x+1}\right| < \frac{\epsilon}{2}$$ Thus
$$\left|\frac{1}{x+1}\right| + \left|\frac{1}{x-1}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$ From the triangle inequality,
$$|\alpha + \beta| \leq |\alpha| + |\beta|$$ $$\left|\frac{1}{x+1} + \frac{1}{x-1}\right| \leq \left|\frac{1}{x+1}\right| + \left|\frac{1}{x-1}\right| < \epsilon$$ ■</td>
<td>Make sure to check that since $\epsilon > 0$, ${}^{2}/_{\epsilon} > 0$ and thus $x = {}^{2}/_{\epsilon} + 2$ satisfies $x > 0$. This is another reason why out of the 4 possible solutions, we chose ${}^{2}/_{\epsilon} + 1$ – it's the only one guaranteed to be positive!

We can divide by $x - 1$ as we know $x > \frac{2}{\epsilon} + 1$, thus $x - 1 > \frac{2}{\epsilon} > 0$. By the same reason, applying absolute value changes nothing since $\frac{1}{x-1} > 0$.

Recall the theorems section in 2.2. The triangle inequality is often used in limit proofs because of the absolute values involved.

This proof type is less common than 3.1.5.</td>
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To prove $\lim_{x \to a} f(x) = \infty$, do $\boxed{\forall \epsilon > 0, \exists \delta > 0, 0 < |x - a| < \delta \Rightarrow f(x) > \epsilon}$

To prove $\lim_{x \to \infty} f(x) = \infty$, do $\boxed{\forall \epsilon > 0, \exists N \in \mathbb{R}, x > N \Rightarrow f(x) > \epsilon}$

I've never seen these proof types in my experience of MAT137, so don't worry about it too much.

3.1.7. Disproving a Limit

<table>
<tr>
<td>eg. Prove $\lim_{x \to 1} \frac{1}{|2x|} \neq 1$</td>
<td></td>
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<tr>
<td>$\exists \epsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}, 0 < |x - 1| < \delta$ and $|f(x) - 1| \geq \epsilon$
Pick $\epsilon = \ldots$
Let $\delta > 0$
Consider $x = \ldots$
Show $0 < |x - 1| < \delta$
Show $|f(x) - 1| \geq \epsilon$
…</td>
<td>Negate the definition and prove it true.

I use "Consider", not "Pick" since it's a counterexample, but I think both work. I'm fuzzy on the notation myself; just do what your math professor does.</td>
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<u>*Rough Work*</u> *To satisfy* $0 < \|x-1\| < \delta$, *we can set something like* $x = 1 + \frac{\delta}{2}$. *To satisfy* $\|f(x) - 1\| \geq \epsilon$, *let's derive it from* x *and* $f(x)$. *Since* $x = 1 + \frac{\delta}{2} > 0$, $f(x) = \frac{1}{\|2x\|} = \frac{1}{2x} = \frac{1}{2\left(1+\frac{\delta}{2}\right)} = \frac{1}{2+\delta}$ $0 < \frac{1}{2+\delta} < \frac{1}{2}$ $-1 < \frac{1}{2+\delta} - 1 < -\frac{1}{2}$ $-1 < f(x) - 1 < -\frac{1}{2}$ $\frac{1}{2} < \|f(x) - 1\| < 1$ *So we need to set* $\epsilon < \frac{1}{2}$ *if we want* $\|f(x) - 1\| > \frac{1}{2} > \epsilon$.	*We can base* x *on* δ *because we introduced* δ *first.* *Since we picked a specific* x, *we can calculate a specific* $f(x)$. *As* $\delta \to 0$, $\frac{1}{2+\delta} \to \frac{1}{2+0} = \frac{1}{2}$ *As* $\delta \to \infty$, $\frac{1}{2+\delta} \to \frac{1}{2+\infty} = 0$ *You need to arrive at this realization to come up with the inequality* $0 < \frac{1}{2+\delta} < \frac{1}{2}$
Pick $\epsilon = \frac{1}{4}$ Let $\delta > 0$ Consider $x = 1 + \frac{\delta}{2}$ Show $0 < \|x - 1\| < \delta$ $\|x-1\| = \left\|\left(1+\frac{\delta}{2}\right) - 1\right\| = \left\|\frac{\delta}{2}\right\| = \frac{\delta}{2}$ Since $\delta > 0$, $0 < \frac{\delta}{2} < \delta$ Show $\|f(x) - 1\| \geq \epsilon$ Since $x > 0$, $f(x) = \frac{1}{\|2x\|} = \frac{1}{2x} = \frac{1}{2\left(1+\frac{\delta}{2}\right)} = \frac{1}{2+\delta}$ $0 < \frac{1}{2+\delta} < \frac{1}{2}$ $-1 < \frac{1}{2+\delta} - 1 < -\frac{1}{2}$ $-1 < f(x) - 1 < -\frac{1}{2}$ $\frac{1}{2} < \|f(x) - 1\| < 1$ $\|f(x) - 1\| > \frac{1}{2} > \frac{1}{4} = \epsilon$ ■	*Any value of* ϵ *where* $0 < \epsilon < \frac{1}{2}$ *works.*

3.2. Computation

3.2.1. General Methods

When computing $\lim_{x \to a} f(x)$, <u>simplify $f(x)$ first</u>, then <u>substitute $x = a$</u>.

eg. Evaluate $\lim_{x \to -3} \frac{x^2+3x}{x^2-x-12}$.
$\lim_{x \to -3} \frac{x^2+3x}{x^2-x-12} = \lim_{x \to -3} \frac{x(x+3)}{(x+3)(x-4)} = \lim_{x \to -3} \frac{x}{x-4} = \frac{(-3)}{(-3)-4} = \boxed{\frac{3}{7}}$

- You can **cancel out** the $x + 3$ because when $x \to -3$, $x \neq -3$.

eg. Evaluate $\lim_{x \to \infty} \frac{24x^6+1}{2x^6+30x^5+1}$.
$\lim_{x \to \infty} \frac{24x^6+1}{2x^6+30x^5+1} = \lim_{x \to \infty} \frac{(24x^6+1)x^{-6}}{(2x^6+30x^5+1)x^{-6}} = \lim_{x \to \infty} \frac{24+x^{-6}}{2+30x^{-1}+x^{-6}} = \frac{24+\frac{1}{\infty^6}}{2+\frac{30}{\infty}+\frac{1}{\infty^6}} = \frac{24+0}{2+0+0} = \boxed{12}$

- *Polynomials*, limit approaches ∞ → divide by the **polynomial with the highest exponent**
- Note that lower powers don't matter: $\lim_{x \to \infty} \frac{24x^6+1}{2x^6+30x^5+1} = \lim_{x \to \infty} \frac{24x^6}{2x^6}$.

eg. Evaluate $\lim_{x\to\infty}\frac{3^x-5^x}{3^{2x}+2^{4x}}$.
$\lim_{x\to\infty}\frac{3^x-5^x}{3^{2x}+2^{4x}}=\lim_{x\to\infty}\frac{3^x-5^x}{9^x+16^x}=\lim_{x\to\infty}\frac{\frac{3^x-5^x}{16^x}}{\frac{9^x+16^x}{16^x}}=\lim_{x\to\infty}\frac{\left(\frac{3}{16}\right)^x-\left(\frac{5}{16}\right)^x}{\left(\frac{9}{16}\right)^x+1}=\frac{\left(\frac{3}{16}\right)^\infty-\left(\frac{5}{16}\right)^\infty}{\left(\frac{9}{16}\right)^\infty+1}=\frac{0-0}{0+1}=\boxed{0}$

- *Exponents*, limit approaches ∞ → divide by the **exponent with the highest base**.

eg. Evaluate $\lim_{x\to-\infty}\frac{3^x-5^x}{3^{2x}+2^{4x}}$.
$\lim_{x\to-\infty}\frac{3^x-5^x}{3^{2x}+2^{4x}}=\lim_{x\to-\infty}\frac{3^x-5^x}{9^x+16^x}=\lim_{x\to-\infty}\frac{1-\left(\frac{5}{3}\right)^x}{3^x+\left(\frac{16}{3}\right)^x}=\frac{1-\left(\frac{5}{3}\right)^{-\infty}}{3^{-\infty}+\left(\frac{16}{3}\right)^{-\infty}}=\frac{1-\left(\frac{3}{5}\right)^\infty}{\left(\frac{1}{3}\right)^\infty+\left(\frac{3}{16}\right)^\infty}=\frac{1-0}{0+0}=\boxed{\infty}$

- *Exponents*, limit approaches $-\infty$ → divide by the **exponent with the lowest base**.

eg. Evaluate $\lim_{x\to-6}\frac{2x+12}{\lvert x+6\rvert}$.
$\lim_{x\to-6}\frac{2x+12}{\lvert x+6\rvert}=\lim_{x\to-6}\frac{2(x+6)}{\lvert x+6\rvert}$
$\lim_{x\to-6^+}\frac{2(x+6)}{\lvert x+6\rvert}=\lim_{x\to-6^+}\frac{2(x+6)}{x+6}=\lim_{x\to-6^-}2=2$ | $\lim_{x\to-6^-}\frac{2(x+6)}{\lvert x+6\rvert}=\lim_{x\to-6^-}\frac{2(x+6)}{-(x+6)}=\lim_{x\to-6^-}-2=-2$
$\therefore\boxed{\lim_{x\to-6}\frac{2(x+6)}{\lvert x+6\rvert}\ DNE}$

- *Absolute values* → split into two **one-sided limits**.

eg. Evaluate $\lim_{x\to0}\frac{1}{x}$.		
$\lim_{x\to0^+}\frac{1}{x}=\frac{1}{+0}=\infty$	$\lim_{x\to0^-}\frac{1}{x}=\frac{1}{-0}=-\infty$	$\therefore\boxed{\lim_{x\to0}\frac{1}{x}\ DNE}$

- *Special case*, $f(x)$ approaches $\pm\infty$ → split into two **one-sided limits**.

eg. Evaluate $\lim_{x\to1}\frac{\sqrt{x+1}-1}{x}$.
$\lim_{x\to0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to0}\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\lim_{x\to0}\frac{\left(\sqrt{x+1}\right)^2-1^2}{x\left(\sqrt{x+1}+1\right)}=\lim_{x\to0}\frac{x+1-1}{x\left(\sqrt{x+1}+1\right)}=\lim_{x\to0}\frac{x}{x\left(\sqrt{x+1}+1\right)}=\lim_{x\to0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{\sqrt{0+1}+1}=\boxed{\frac{1}{2}}$

- *Square roots* → use the relation $a^2-b^2=(a+b)(a-b)$, **multiplying by conjugate**, to simplify

eg. Evaluate $e^{\lim_{x\to1^+}(\ln x)}$.
$e^{\lim_{x\to1^+}(\ln x)}\to\lim_{x\to1^+}(\ln x)=-\infty\to e^{-\infty}=\boxed{0}$

- $f(x)^{\lim g(x)}$ → compute the upper limit first, then the exponent.
- *Note:* In math, it's convention that $\log x=\log_e x$. In science/engineering, $\ln x=\log_e x$ and usually $\log x=\log_{10} x$. In computer science, sometimes $\log x=\log_2 x$. I use "$\ln x$" because I like it.

eg. Let $f(x)=\begin{cases}x & x\in\mathbb{Z}\\ -x & x\notin\mathbb{Z}\end{cases}$. Evaluate $\lim_{x\to1}f(x)$.		
$\lim_{x\to1^+}f(x)=-1$	$\lim_{x\to1^-}f(x)=-1$	$\therefore\boxed{\lim_{x\to1}f(x)=-1}$

- *Piecewise functions* → split into two one-sided limits based on the piecewise function.

In higher-level analysis, **gamma functions** are used for limit/derivative of factorials (eg. $x!$). Also, it's bad practice to write $\frac{24+\frac{1}{\infty^6}}{2+\frac{30}{\infty}+\frac{1}{\infty^6}}$ or $\frac{1-\left(\frac{3}{5}\right)^\infty}{\left(\frac{1}{3}\right)^\infty+\left(\frac{3}{16}\right)^\infty}$ (∞ is mathematically improper). I only write it to be explicitly clear.

3.2.2. Limit Techniques

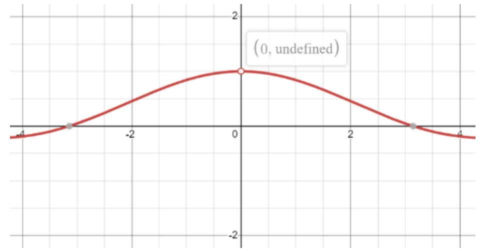
Small Angle Approximation

$\forall a, x \in \mathbb{R}$,

$$\boxed{\lim_{x\to 0}\frac{\sin ax}{ax} = \lim_{x\to 0}\frac{\tan ax}{ax} = \lim_{x\to 0}\frac{\arcsin ax}{ax} = \lim_{x\to 0}\frac{\arctan ax}{ax} = 1}$$

$$\boxed{\lim_{x\to\infty}\sin\frac{1}{x} = \lim_{x\to\infty}\tan\frac{1}{x} = \lim_{x\to\infty}\arcsin\frac{1}{x} = \lim_{x\to\infty}\arctan\frac{1}{x} = \lim_{x\to\infty}\frac{1}{x} = 0}$$

$$\boxed{\lim_{x\to 0}\frac{\cos ax - 1}{ax} = 0}$$



$\frac{\sin x}{x}$ is on the right. You will learn to prove some of these "dodgy" theorems in the Taylor Series section.

eg. Evaluate $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$.

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\left(\frac{\sin 2x}{\sin 3x}\cdot\frac{2x}{2x}\cdot\frac{3x}{3x}\right) = \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\cdot\frac{3x}{\sin 3x}\cdot\frac{2x}{3x}\right) = 1\cdot 1\cdot\frac{2}{3} = \boxed{\frac{2}{3}}$$

- *Sine* → divide by **whatever's inside the sine**, use small angle approximation

eg. Evaluate $\lim_{x\to 3}\frac{\tan(x-3)}{2x-6}$.

$$\lim_{x\to 3}\frac{\tan(x-3)}{2x-6} = \lim_{x\to 3}\left(\frac{\sin(x-3)}{\cos(x-3)}\cdot\frac{1}{2(x-3)}\right) = \lim_{x\to 3}\left(\frac{\sin(x-3)}{x-3}\cdot\frac{1}{2\cos(x-3)}\right) = \lim_{x\to 3}\frac{1}{2\cos(x-3)} = \frac{1}{2\cos 0} = \boxed{\frac{1}{2}}$$

- *Tangent* → **write in terms of sine and cosine**, and use small angle approximation
- Note that $\lim_{x\to 3}\frac{\sin(x-3)}{x-3} = 1$, but $\lim_{x\to 0}\frac{\sin(x-3)}{x-3} \neq 1$. Both numerator and denominator must approach 0

Squeeze Theorem

- $g(x) \le f(x) \le h(x)$ at point a, and
- $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$

$\Rightarrow \quad \lim_{x\to a} f(x) = L$

Most likely, you'll be using squeeze theorem for $0 \le f(x) \le h(x)$, where you must show $\lim_{x\to a} h(x) = 0$.

eg. $\lim_{x\to 0} x\sin\frac{1}{x}$

Since $-1 \le \sin\frac{1}{x} \le 1$,

$$-x \le x\sin\frac{1}{x} \le x$$

Since $\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$, by the squeeze theorem, $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

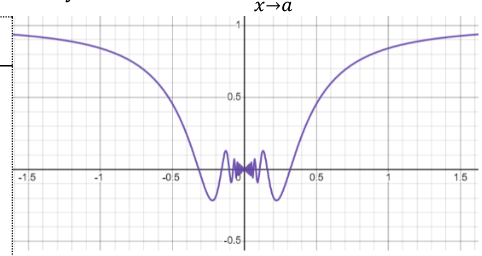
Another way (may be easier, as you only calculate a limit on one side):

Since $-1 \le \sin\frac{1}{x} \le 1$, $\left|\sin\frac{1}{x}\right| \le 1$

$$\left|x\sin\frac{1}{x}\right| \le |x|$$

Since $\lim_{x\to 0} x = 0$, $\lim_{x\to 0}|x| = 0$. *(Absolute Value properties, see 3.1.3)*

By the squeeze theorem, $\lim_{x\to 0}\left|x\sin\frac{1}{x}\right| = 0$, and thus $\boxed{\lim_{x\to 0} x\sin\frac{1}{x} = 0}$.



Squeeze Theorem Tools:

$x^2 \ge 0$

$|x| \ge 0$

$-1 \le \sin x \le 1$

$-1 \le \cos x \le 1$

$-\pi/2 \le \arcsin x \le \pi/2$

$0 \le \arccos x \le \pi$

$-\pi/2 \le \arctan x \le \pi/2$

You'll formally learn these in 6.1.2.

4. Derivatives

4.1. Theory

4.1.1. Definition as Limit

For all $f(x)$, the derivative at any x is:

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

Remember this formula! Sections 4.1.2 and 4.1.3 explain this formula's applications, but aren't important.

$f'(x)$ can be further differentiated to $f''(x)$ and $f'''(x)$ and so forth.
The n-th order derivative, $f^{(n)}(x)$, is the number of times $f(x)$ is differentiated.
$\frac{dy}{dx}$ and $\frac{d}{dx}(y)$ is Leibniz notation, equivalent to y' (where you're differentiating x). $\frac{d^2y}{dx^2}$ is y''.

4.1.2. Definition as Slope

Consider any two points $(a, f(a))$ and $(a+h, f(a+h))$, which are on the graph of $f(x)$. Take their slope:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

And what happens as h becomes infinitesimally small? You get the same formula as 4.1.1.

4.1.3. Definition in Kinematics

Derivative can be understood as "rate of change" in physics.

Distance/Position (d): The net distance an object travels.
Displacement (s): The net distance an object travels, relative to the origin.
Velocity (v): The *rate of change* of an object's displacement/position/position. $s'(t) = v(t)$
Acceleration (a): The *rate of change* of an object's velocity. $v'(t) = a(t)$

Let $s(t)$ be an object's position at time t. The object's average velocity between points of time t and $t+h$ is

$$v_{avg} = \frac{\Delta d}{\Delta t} = \frac{s(t+h)-s(t)}{(t+h)-t} = \frac{s(t+h)-s(t)}{h}$$

And what happens as h becomes infinitesimally small? You get the same formula as 4.1.1.

4.1.4. Derivative Properties

General Theorems	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$(\alpha)' = 0$	$\sin x$	$\cos x$	$\ln x$	$\frac{1}{x}$
$(\alpha f)' = \alpha f'$	$\cos x$	$-\sin x$	x^n	nx^{n-1}
$(f+g)' = f' + g'$	$\tan x$	$\sec^2 x = \frac{1}{\cos^2 x}$	e^x	e^x
$\left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2}$				
$(fg)' = f'g + g'f$	$\frac{1}{\sin x} = \csc x$	$-\csc x \cot x = -\frac{\cos x}{\sin^2 x}$	$(\sin x)^{-1} = \arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$(fgh)' = f'gh + fg'h + fgh'$	$\frac{1}{\cos x} = \sec x$	$\sec x \tan x = \frac{\sin x}{\cos^2 x}$	$(\cos x)^{-1} = \arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
Chain Rule: $\big(f(g)\big)' = f'(g)\cdot g'$	$\frac{1}{\tan x} = \cot x$	$-\csc^2 x = -\frac{1}{\sin^2 x}$	$(\tan x)^{-1} = \arctan x$	$\frac{1}{1+x^2}$

For "Prove …" question with a derivative, plug $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and compute a solution. There is no actual "proving" like with limits. I never did derivative proofs in class, nor were there these proofs on the exam. When computing derivatives, don't use the 4.1.1 formula as it's clunky and awkward. Use the above table.

eg. Prove $(x^2)' = 2x$. *(From MAT137, April exam 2018)*

Let $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} & &= \lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} & &= \lim_{h\to 0}\frac{2hx+h^2}{h} \\ & & &= \lim_{h\to 0}(2x+h) = 2x \end{aligned}$$

This is probably the extent of the derivative proofs you might get on the MAT137 exam.

eg. Prove $(fg)' = f'g + g'f$.

Let $p(x) = f(x)g(x)$.

$$\begin{aligned} p'(x) &= \lim_{h\to 0}\frac{p(x+h)-p(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]}{h}+\lim_{h\to 0}\frac{g(x)[f(x+h)-f(x)]}{h} \\ &= \lim_{h\to 0}f(x+h)\left[\frac{g(x+h)+g(x)}{h}\right]+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= f(x)g'(x)+g(x)f'(x) \end{aligned}$$

I don't recommend practicing derivative proofs with theorems. Just know that this is how to approach it.

This step's tricky to figure out.

4.2. Computation

4.2.1. General Methods

Use the table from 4.1.4 for most, if not all computations. Here are some examples.

eg. Evaluate $\frac{d}{dx}e^{x^e}$.

$$\begin{aligned} \frac{d}{dx}e^{(x^e)} &= e^{x^e}\cdot(x^e)' \\ &= e^{x^e}\cdot ex^{e-1} \\ &= \boxed{e^{x^e+1}x^{e-1}} \end{aligned}$$

Apply $(e^x)' = e^x$
Apply $\big(f(g)\big)' = f'(g)\cdot g'$
Apply $(x^n)' = nx^{n-1}$

eg. Evaluate $\frac{d}{dx}[x^3\arctan(3x+1)]$.

$$\begin{aligned} \frac{d}{dx}[x^3\arctan(3x+1)] &= (x^3)'\arctan(3x+1)+(\arctan(3x+1))'\cdot x^3 \\ &= 3x^2\arctan(3x+1)+\frac{1}{(3x+1)^2+1}\cdot(3x+1)'x^3 \\ &= \boxed{3x^2\arctan(3x+1)+\frac{3x^3}{(3x+1)^2+1}} \end{aligned}$$

Apply $(fg)' = f'g + g'f$
Apply $(x^n)' = nx^{n-1}$
Apply $(\arctan x)' = \frac{1}{x^2+1}$
Apply $\big(f(g)\big)' = f'(g)\cdot g'$

eg. Let $f(x) = \begin{cases}\sin(\cos x) & x\le 1 \\ \ln(\ln x) & x>1\end{cases}$. Find $f'(x)$.

$$f'(x) = \begin{cases}\cos(\cos x)\cdot(\cos x)' = -\sin x\cos(\cos x) & x<1 \\ \frac{1}{\ln x}\cdot(\ln x)' = \frac{1}{x\ln x} & x>1\end{cases}$$

Apply $\big(f(g)\big)' = f'(g)\cdot g'$
Apply $(\sin x)' = \cos x$
Apply $(\cos x)' = -\sin x$
Apply $(\ln x)' = \frac{1}{x}$

$f'(x)$ DNE at $x=1$ as $\lim_{x\to 1^+}f'(x) = \frac{1}{(1)\ln(1)} = \infty \neq \lim_{x\to 1^-}f'(x) = -\sin(1)\cos(\cos 1)\approx -0.72$ *(one-sided limits)*

4.2.2. L'Hôpital's Rule

This is a special case for computing more complex limits:

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

When $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$, apply this rule.

When $\lim_{x\to a}\frac{f(x)}{g(x)}=\pm\frac{\infty}{\infty}$, apply this rule.

When $\lim_{x\to a}[f(x)\times g(x)]=0\times\pm\infty$,

- Re-write $\lim_{x\to a}[f(x)\times g(x)]=\lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}$ if $g(x)$ is simpler than $f(x)$, then apply this rule.
- Re-write $\lim_{x\to a}[f(x)\times g(x)]=\lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}$ if $f(x)$ is simpler than $g(x)$, then apply this rule.

eg. Find $\lim_{x\to 0^+}x\ln(\tan 2x)$.

$$\lim_{x\to 0^+}x\ln(\tan 2x)=0\times-\infty$$

Apply L'Hôpital's Rule,

$\lim_{x\to 0^+}x\ln(\tan 2x)=\lim_{x\to 0^+}\left(\frac{\ln(\tan 2x)}{x^{-1}}\right)$

$=\lim_{x\to 0^+}\left(\frac{\frac{1}{\tan 2x}\cdot(\tan 2x)'}{-x^{-2}}\right)$

$=\lim_{x\to 0^+}\left(-x^2\cdot\frac{1}{\tan 2x}\cdot\left(\frac{1}{\cos^2 2x}\cdot(2x)'\right)\right)$

$=\lim_{x\to 0^+}\left(-2x^2\cdot\frac{\cos 2x}{\sin 2x}\cdot\frac{1}{\cos^2 2x}\right)$

$=\lim_{x\to 0^+}\left(\frac{-2x^2}{\sin 2x\cos 2x}\right)$

$=\lim_{x\to 0^+}\left(\frac{-2x^2}{\sin 2x}\right)=\frac{0}{0}$

You have $\lim_{x\to 0^+}f(x)g(x)$, where $f(x)=x$ and $g(x)=\ln(\tan 2x)$. Since $f(x)$ is simpler, set it to the denominator. This is for simplicity in calculations.

As $x\to 0$, $\cos 2x\to\cos 2(0)=\cos 0\to 1$

Apply L'Hôpital's Rule (again),

$$\lim_{x\to 0^+}\left(\frac{-2x^2}{\sin 2x}\right)$$
$$=\lim_{x\to 0^+}\left(\frac{-4x}{\cos 2x\cdot(2x)'}\right)$$
$$=\lim_{x\to 0^+}\left(\frac{-2x}{\cos 2x}\right)=-\frac{0}{1}=\boxed{0}$$

Some questions will make you have to apply L'Hôpital's rule <u>many</u> times.

L'Hôpital's Rule has some preconditions, which *usually* aren't a problem:

- $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, called **indeterminate form**.
- $\lim_{x\to a}f'(x)$ and $\lim_{x\to a}g'(x)$ exist, and the denominators aren't 0
- $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists or is equal to $\pm\infty$

Its complete proof is way too long-winded for MAT137. Many profs look down on this rule as it's "overused", like a "magic wand" to mindlessly plug numbers without understanding important concepts.

4.2.3. Logarithmic Differentiation

For a <u>product/quotient of functions</u> or <u>exponent functions</u>, like

$$y(x)=\frac{f(x)^2g(x)}{h(x)}\text{ or }y(x)=f(x)^{g(x)},$$

take ln of both sides and differentiate.

Refresher on Logarithm Operations	
$\log(ab)=\log a+\log b$	$\log 1=0$
$\log\left(\frac{a}{b}\right)=\log a-\log b$	$\log_a a=1$
	$a^{\log_a b}=b$
$\log a^b=b\log a$	$e^{\ln x}=x$

eg. Let $f(x)=\frac{\sqrt{x^2+3}(2x-5)^2}{(x^2+2)^2}$. Find $f'(x)$.

You could try doing quotient rule or chain rule instead, but it won't be fun.

$$f(x)=\frac{\sqrt{x^2+3}(2x-5)^2}{(x^2+2)^2}$$

$$\ln f(x)=\ln\left(\frac{(x^2+3)^{\frac{1}{2}}(2x-5)^2}{(x^2+2)^2}\right)$$

$$=\ln(x^2+3)^{\frac{1}{2}}+\ln(2x-5)^2-\ln(x^2+2)^2$$

$$=\frac{1}{2}\ln(x^2+3)+2\ln(2x-5)-2\ln(x^2+2)$$

$$\frac{f'(x)}{f(x)}=\frac{1}{2}\cdot\frac{(x^2+3)'}{x^2+3}+2\cdot\frac{(2x-5)'}{2x-5}-2\cdot\frac{(x^2+2)'}{x^2+2}$$

Using chain rule, $(\ln f(x))'=\frac{f'(x)}{f(x)}$

$$f'(x)=f(x)\left(\frac{x}{x^2+3}+\frac{4}{2x-5}-\frac{4x}{x^2+2}\right)$$

$$=\frac{\sqrt{x^2+3}(2x-5)^2}{(x^2+2)^2}\left(\frac{x}{x^2+3}+\frac{4}{2x-5}-\frac{4x}{x^2+2}\right)$$

I'm not expanding this because it's difficult and not enlightening.

These questions are cruel because it's just mindless and time-crunching. I doubt they'll be on the exam, but just know that the method exists.

4.2.4. Implicit Differentiation

When differentiating an equation where <u>both variables cannot be isolated</u>, treat each variable as a function when taking derivatives.

eg. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of $x^2+4xy+y^3+5=0$ at $(x,y)=(2,-1)$.

$$x^2+4xy+y^3+5=0$$

$$2x+4(xy'+x'y)+3y^2\cdot y'=0$$

$$2x+4xy'+4y+3y^2\cdot y'=0$$

You can simplify x', but not y', so keep y'.

$$y'(4x+3y^2)+2x+4y=0$$

$$y'=-2\cdot\frac{x+2y}{4x+3y^2}$$

It's better to write $\frac{dy}{dx}$*, not y', as the question writes it that way. I use y' to save space.*

$$y''=-2\cdot\frac{(x+2y)'(4x+3y^2)-(4x+3y^2)'(x+2y)}{(4x+3y^2)^2}$$

$$=-2\cdot\frac{(1+2y')(4x+3y^2)-(4+6y\cdot y')(x+4y)}{(4x+3y^2)^2}$$

$$=-2\cdot\frac{\left(1+2\left(-2\cdot\frac{x+2y}{4x+3y^2}\right)\right)(4x+3y^2)-\left(4+6y\cdot\left(-2\cdot\frac{x+2y}{4x+3y^2}\right)\right)(x+4y)}{(4x+3y^2)^2}$$

Replace y' with what you found it to be.

$$=-2\cdot\frac{9y^4+48xy^2+12x^2y-64xy+16x^2}{(4x+3y^2)^3}$$

Implicit differentiation can get nasty past the first-order derivative.

Substitute $(2,-1)$ into both equations:

$$y'=-2\cdot\frac{2+2(-1)}{4(2)+3-(1)^2}=\boxed{0}$$

$$y''=-2\cdot\frac{9(-1)^4+48(2)(-1)^2+12(2)^2(-1)-64(2)(-1)+16(2)^2}{(4(2)+3(-1)^2)^3}=\boxed{-\frac{498}{1331}}$$

You will never find 3+ variables (MAT137 is one-dimensional calculus). I believe implicit differentiation is also not common in higher-level math. Logarithmic differentiation is technically implicit differentiation.

4.3. Application

4.3.1. Optimization

There's a variable that needs to be minimized/maximized, y.
There's a variable that can changed, x.
Calculate $\frac{dy}{dx} = 0$, and solve for x.

eg. Find the farthest point(s) from (0,1) on the ellipse $\frac{x^2}{4} + y^2 = 1$.

(From MAT157, April exam 2018)

The distance between (x, y) and $(0,1)$ is

$$D = \sqrt{x^2 + (y-1)^2}$$

From Pythagorean theorem

We know (x, y) must be on the ellipse $\frac{x^2}{4} + y^2 = 1$.

$$x = \pm\sqrt{4 - 4y^2}$$

Isolate x (you can choose to instead isolate y, but plugging it into the distance equation will be more annoying)

Substitute this into the distance equation:

$$\begin{aligned} D &= \sqrt{\left(\pm\sqrt{4-4y^2}\right)^2 + (y-1)^2} \\ &= \sqrt{4 - 4y^2 + y^2 - 2y + 1} \\ &= \sqrt{-3y^2 - 2y + 5} \\ &= \sqrt{-(y-1)\left(y + {}^5/_3\right)} \\ &= (1-y)^{\frac{1}{2}}\left(y + {}^5/_3\right)^{\frac{1}{2}} \end{aligned}$$

Always simplify until your equation has <u>only 1 variable left</u> in it.

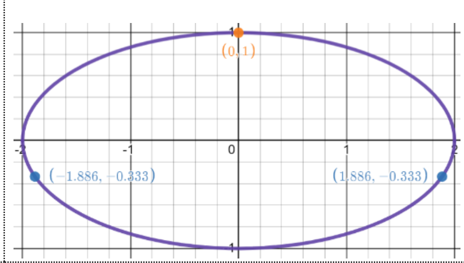
$$\begin{aligned} \frac{dD}{dy} &= -\frac{1}{2}(1-y)^{-\frac{1}{2}}\left(y + \frac{5}{3}\right)^{\frac{1}{2}} + \frac{1}{2}\left(y + \frac{5}{3}\right)^{-\frac{1}{2}}(1-y)^{\frac{1}{2}} \\ &= -\frac{\sqrt{y + {}^5/_3}}{2\sqrt{1-y}} + \frac{\sqrt{1-y}}{2\sqrt{y + {}^5/_3}} \\ &= -\frac{y + {}^5/_3}{2\sqrt{1-y}\sqrt{y + {}^5/_3}} + \frac{1-y}{2\sqrt{1-y}\sqrt{y + {}^5/_3}} \\ &= \frac{-2\left(y + {}^1/_3\right)}{\sqrt{1-y}\sqrt{y + {}^5/_3}} = 0 \end{aligned}$$

The variable to maximize is D
The variable that can change is y.
Do $\frac{dD}{dy} = 0$ and solve for y.

$$y = \boxed{-\frac{1}{3}}$$

$$\begin{aligned} x &= \pm\sqrt{4 - 4\left(-{}^1/_3\right)^2} \\ &= \pm 2\sqrt{1 - {}^1/_9} \\ &= \pm 2\sqrt{{}^8/_9} \\ &= \boxed{\pm\frac{4\sqrt{2}}{3}} \end{aligned}$$

Plug y in the distance formula to find x.



$\therefore \left(\frac{4\sqrt{2}}{3}, -\frac{1}{3}\right)$ and $\left(-\frac{4\sqrt{2}}{3}, -\frac{1}{3}\right)$ are the farthest points from (0,1).

eg. You want to sell a square open box of volume 1,000 cm³ for $1,250: the bottom is a square, the four sides are identical rectangles, and the top is empty. The material for the bottom costs $2/cm² and the material for the four sides costs $3/cm². Is this operation profitable?	
We have a square prism of volume 1000. $$V = bwh = w^2h = 1000$$ $$h = 1000/w^2$$	*Since base = width (it's a square)*
We know the total surface area is: $$A_s = bw + 2bh + 2wh = w^2 + 4wh$$	*The top is empty. It's bw, not 2bw.*
The total cost is based on surface area, and is thus: $$C_{total} = w^2 \cdot C_{bottom} + 4wh \cdot C_{sides}$$ $$= w^2 \cdot 2 + 4w\left(\frac{1000}{w^2}\right) \cdot 3$$ $$= 2w^2 + 12000w^{-1}$$	*For these questions, you will often have to figure out the formulas yourself, which can be hard.*
$$\frac{dC_{total}}{dw} = 4w - 12000w^{-2}$$ $$= 4w^{-2}(w^3 - 3000) = 0$$ $$w = \sqrt[3]{3000}$$	*The variable to minimize is C* *The variable that can change is w* *Do* $\frac{dC}{dw} = 0$ *and solve for y.*
$$C_{total} = 2\left(\sqrt[3]{3000}\right)^2 + 12000\left(\sqrt[3]{3000}\right)^{-1}$$ $$= \boxed{1248.0502 \ldots}$$ Since the cost of a box (≈ $1,248) is exceeded by the profit ($1,250), this operation is barely profitable.	*This is unsolvable without a calculator. Good news: the exam doesn't allow calculators, so you'll get something easy to compute.*

4.3.2. Related Rates

Write an equation relating all variables.
Perform implicit differentiation against variables <u>that change with time (*dt*)</u>, and plug the right values in.

eg. The volume of a cone is increasing at 30 m³/min. Its diameter and height are always equal. How fast is the cone's height increasing when the height is 10 m?	
$$\frac{dV}{dt} = 30, \qquad 2r = h, \qquad h = 10, \qquad \frac{dh}{dt} = ?$$ We know the volume of a cone is	*It's a good idea to write all information given in the problem and what to solve. Kind of like physics questions.*
$$V = \frac{1}{3}\pi r^2 h$$ $$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$ $$= \frac{1}{12}\pi h^3$$	*You can replace h with 2r instead, but remember, the question asks for* $\frac{dh}{dt}$*, not* $\frac{dr}{dt}$
$$\frac{dV}{dt} = \frac{1}{12}\pi\left(3h^2 \cdot \frac{dh}{dt}\right)$$ $$30 = \frac{1}{4}\pi\left((10)^2 \cdot \frac{dh}{dt}\right)$$	*Substitute the initial values*
$$\frac{dh}{dt} = \frac{30 \cdot 4}{\pi \cdot 100} = \boxed{\frac{6}{5\pi}}$$ The cone's height is increasing at a rate of $\frac{6}{5\pi}$ m/min.	*Isolate* $\frac{dh}{dt}$

Related rates is closely tied to implicit differentiation, both concepts originating from physics.

5. Continuity

5.1. Theory

5.1.1. Definition

$$f(x) \text{ continuous at } a \Leftrightarrow \lim_{x\to a} f(x) = f(a)$$
$$f(x) \text{ continuous on } [a,b] \Leftrightarrow \forall \alpha \in [a,b], \lim_{x\to \alpha} f(x) = f(\alpha)$$
$$f(x) \text{ continuous} \Leftrightarrow \forall a \in \mathbb{R}, \lim_{x\to a} f(x) = f(a)$$

All standard functions (ie. $x, x^2, \sqrt{x}, e^x, \sin x, \ln x$) are continuous in their domains.

Adding/subtracting/multiplying/dividing functions retains their continuity (except division by 0)

You might see "*Let $f(x)$ be a function defined at least around on an interval centered around a, except maybe at a.*" It means $f(x)$ is continuous around a, but not necessarily at a – the bare minimum for $\lim_{x\to a} f(x)$ to exist.

5.1.2. Types of Discontinuities

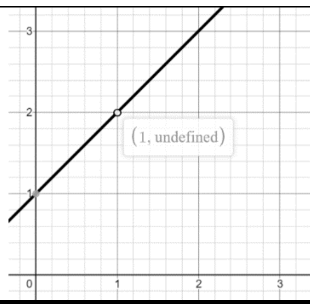
<u>Removable Discontinuity</u>

- $\lim_{x\to a} f(x)$ exists
- $f(a) \neq \lim_{x\to a} f(x)$ or $f(a)$ DNE

eg. $\lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = 2$

$\frac{(x-1)(x+1)}{x-1}$ DNE at $x = 1$

"Removable" as you can redefine $f(x)$ as a piecewise function & erase the discontinuity



<u>Jump Discontinuity</u>

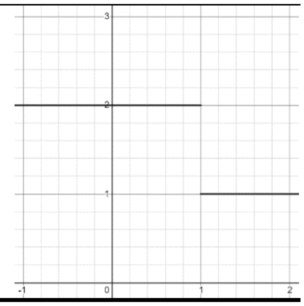
- $\lim_{x\to a^-} f(x) \neq \lim_{x\to a} f(x)$, so $\lim_{x\to a} f(x)$ DNE

eg. $f(x) = \begin{cases} 2 & x \leq 1 \\ 1 & x > 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = 2$

$\lim_{x\to 1^+} f(x) = 1$

$\therefore \lim_{x\to 1} f(x)\ DNE$



<u>Infinite Discontinuity</u>

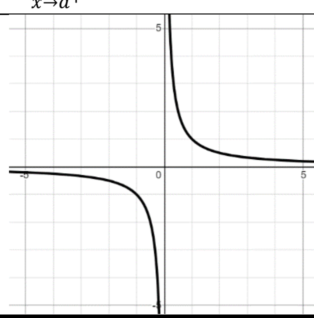
- $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

eg. $\lim_{x\to 0} \frac{1}{x}$

$\lim_{x\to 0^+} \frac{1}{x} = \infty$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$

$\therefore \lim_{x\to 0} \frac{1}{x}\ DNE$

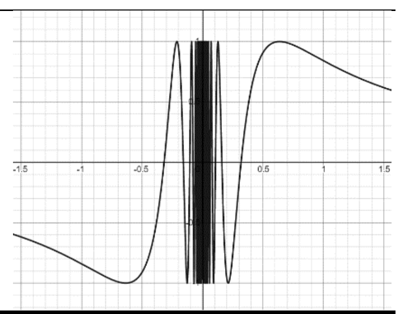


<u>Essential Discontinuity</u>

Anything that doesn't fit into the other types.

eg. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Is described as "oscillating wildly" as it approaches 0



5.2. Limit Laws

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

- $\lim_{x\to a}[kf(x)] = kL$
- $\lim_{x\to a} f(kx) = L$
- $\lim_{x\to a}[f(x) + g(x)] = L + M$ *(additive)*
- $\lim_{x\to a}[f(x) \cdot g(x)] = L \cdot M$ *(multiplicative)*
- $\lim_{x\to a}[f(x) - g(x)] = L - M$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ (if $M \neq 0$)
- $f(x)$ is continuous $\Rightarrow \lim_{x\to a} f(g(x)) = f\left[\lim_{x\to a} g(x)\right]$ (*function composition*)

You've been implicitly using these in your computations; now we've justified them rigorously. Also, the proofs for all these theorems, except multiplying/dividing, are easy enough to be learned in MAT137.

eg. Prove $\lim_{x \to a}[kf(x)] = kL$

Assume $\lim_{x \to a} f(x) = L$, which means
$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \forall x_1 \in \mathbb{R}, 0 < |x_1 - a| < \delta_1 \Rightarrow |f(x_1) - L| < \epsilon_1$
Show $\lim_{x \to a}[kf(x)] = kL$, which means
$\forall \epsilon_2 > 0, \exists \delta_2 > 0, \forall x_2 \in \mathbb{R}, 0 < |x_2 - a| < \delta_2 \Rightarrow |kf(x_2) - kL| < \epsilon_2$
  Let $\epsilon_2 > 0$
  Pick $\delta_2 = \ldots$
  Let $x_2 \in \mathbb{R}$
  Assume $0 < |x_2 - a| < \delta_2$
  Show $|kf(x_2) - kL| < \epsilon_2$

Remember the premise of this theorem.

If your proof involves assuming __another limit already exists__, you'll need to add the usually unnecessary $\forall x \in \mathbb{R}$ part. __Add numerals__ to disambiguate symbols.

__Rough Work:__
Note that $|kf(x_2) - kL| < \epsilon_2$ can be simplified to $|f(x_2) - L| < \frac{\epsilon_2}{k}$
You already know $0 < |x_2 - a| < \delta_2$
You already know $0 < |x_1 - a| < \delta_1 \Rightarrow |f(x_1) - L| < \epsilon_1$
You want to show $|f(x_2) - L| < \frac{\epsilon_2}{k}$
So it's just a matter of picking variables to match and modus ponens.

$$e_1 = \frac{e_2}{k}, x_1 = x_2, \delta_1 = \delta_2$$

But there're some things to watch out for. Consider your assumption
$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \forall x_1 \in \mathbb{R}, 0 < |x_1 - a| < \delta_1 \Rightarrow |f(x_1) - L| < \epsilon_1$,

1) *For ϵ_1, since it's $\forall \epsilon_1$, you can pick anything and what follows will be true. So "Pick $e_1 = \frac{e_2}{k}$" is fine.*
2) *For δ_1, you __may not__ do "Pick $\delta_1 = \ldots$", as you know $\exists \delta_1 > 0$, but not what δ_1 is. Thus, you must do "Pick … = δ_1". Since we need $\delta_1 = \delta_2$ and we can do "Pick $\delta_2 = \ldots$", do $\delta_2 = \delta_1$ instead.*
3) *For x_1, the same principle as ϵ_1 applies. $x_1 = x_2$ is fine.*

Divide both sides by k

Modus ponens was in 2.2. Here's a refresher of it:

$$P. \qquad P \Rightarrow Q. \qquad \therefore Q.$$

You write "Pick $\epsilon_1 = \ldots$", not "Let $\epsilon_1 = \ldots$" despite it being $\forall \epsilon_1$, because this is an already-known assumption.

You're __instantiating__ $\forall \epsilon_1$; that is, since the assumption is true for all ϵ_1, it is thus true for this specific value of ϵ_1."

Assume $\lim_{x \to a} f(x) = L$, which means
$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \forall x_1 \in \mathbb{R}, 0 < |x_1 - a| < \delta_1 \Rightarrow |f(x_1) - L| < \epsilon_1$
Show $\lim_{x \to a}[kf(x)] = kL$, which means
$\forall \epsilon_2 > 0, \exists \delta_2 > 0, \forall x_2 \in \mathbb{R}, 0 < |x_2 - a| < \delta_2 \Rightarrow |kf(x_2) - kL| < \epsilon_2$
  Let $\epsilon_2 > 0$
  Pick $\delta_2 = \delta_1$
  Let $x_2 \in \mathbb{R}$
  Assume $0 < |x_2 - a| < \delta_2$
  Show $|kf(x_2) - kL| < \epsilon_2$
    Pick $\epsilon_1 = \frac{\epsilon_2}{k}$, $x_1 = x_2$, then $0 < |x_2 - a| < \delta_2 \Rightarrow |f(x_2) - L| < \frac{\epsilon_2}{k}$
    Since $0 < |x_2 - a| < \delta_2$, $|f(x_2) - L| < \frac{\epsilon_2}{k}$

$$k|f(x_2) - L| < \epsilon_2$$
$$|kf(x_2) - kL| < \epsilon_2$$

■

eg. Prove $\lim_{x \to a}[f(x) + g(x)] = L + M$

Assume $\lim_{x \to a} f(x) = L$, which means

$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \forall x_1 \in \mathbb{R}, 0 < |x_1 - a| < \delta_1 \Rightarrow |f(x_1) - L| < \epsilon_1$

Assume $\lim_{x \to a} g(x) = M$, which means

$\forall \epsilon_2 > 0, \exists \delta_2 > 0, \forall x_2 \in \mathbb{R}, 0 < |x_2 - a| < \delta_2 \Rightarrow |g(x_2) - M| < \epsilon_2$

Show $\lim_{x \to a}[f(x) + g(x)] = L + M$, which means

$\forall \epsilon_3 > 0, \exists \delta_3 > 0, \forall x_3 \in \mathbb{R}, 0 < |x_3 - a| < \delta_3 \Rightarrow |f(x_3) + g(x_3) - L - M| < \epsilon_3$

Let $\epsilon_3 > 0$

Pick $\delta_3 = \ldots$

Let $x_3 \in \mathbb{R}$

Assume $0 < |x_3 - a| < \delta_2$

Show $|f(x_3) + g(x_3) - L - M| < \epsilon_3$

Remember there are two premises for this theorem.

Theorems have complicated set-ups but much less rough work.

<u>*Rough Work:*</u>

Note that $|f(x_3) + g(x_3) - L - M| = |(f(x_3) - L) + (g(x_3) - M)|$

You already know $0 < |x_3 - a| < \delta_3$

You already know $0 < |x_1 - a| < \delta_1 \Rightarrow |f(x_1) - L| < \epsilon_1$

You already know $0 < |x_2 - a| < \delta_2 \Rightarrow |g(x_2) - M| < \epsilon_2$

You want to show $|(f(x_3) - L) + (g(x_3) - M)| < \epsilon_3$

If you add $|f(x_1) - L| < \epsilon_1$ *and* $|g(x_2) - M| < \epsilon_2$, *you get*

$$|f(x_1) - L| + |g(x_2) - M| < \epsilon_1 + \epsilon_2$$

So we need to choose ϵ_1 *and* ϵ_2 *where* $\epsilon_1 + \epsilon_2 = \epsilon_3$

Similar logic from the previous proof applies. $x_1 = x_2 = x_3$ *and* $\delta_1 = \delta_2 = \delta_3$.

But wait. We need to pick $\delta_3 = \delta_1$ *AND pick* $\delta_3 = \delta_2$. *We can't pick* $\delta_1 = \ldots$ *or* $\delta_2 = \ldots$, *so what do we do? The answer is* $\delta_3 = \min\{\delta_1, \delta_2\}$.

But wait!

$$|f(x_1) - L| + |g(x_2) - M|$$

is different from what we show,

$$|(f(x_3) - L) + (g(x_3) - M)|$$

Luckily, we already have a tool for this: <u>*triangle inequality*</u>*!*

Assume $\lim_{x \to a} f(x) = L$, which means

$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \forall x_1 \in \mathbb{R}, 0 < |x_1 - a| < \delta_1 \Rightarrow |f(x_1) - L| < \epsilon_1$

Assume $\lim_{x \to a} g(x) = M$, which means

$\forall \epsilon_2 > 0, \exists \delta_2 > 0, \forall x_2 \in \mathbb{R}, 0 < |x_2 - a| < \delta_2 \Rightarrow |g(x_2) - M| < \epsilon_2$

Show $\lim_{x \to a}[f(x) + g(x)] = L + M$, which means

$\forall \epsilon_3 > 0, \exists \delta_3 > 0, \forall x_3 \in \mathbb{R}, 0 < |x_3 - a| < \delta_3 \Rightarrow |f(x_3) + g(x_3) - L - M| < \epsilon_3$

Let $\epsilon_3 > 0$

Pick $\delta_3 = \min\{\delta_1, \delta_2\}$, then $\delta_3 \leq \delta_1$ and $\delta_3 \leq \delta_2$

Let $x_3 \in \mathbb{R}$

Assume $0 < |x_3 - a| < \delta_3$

Show $|f(x_3) + g(x_3) - L - M| < \epsilon_3$

Pick $\epsilon_1 = \frac{\epsilon_3}{2}$, $x_1 = x_3$, then $0 < |x_3 - a| < \delta_1 \Rightarrow |f(x_3) - L| < \frac{\epsilon_3}{2}$

Pick $\epsilon_2 = \frac{\epsilon_3}{2}$, $x_2 = x_3$, then $0 < |x_3 - a| < \delta_2 \Rightarrow |g(x_3) - M| < \frac{\epsilon_3}{2}$

Since $0 < |x_3 - a| < \delta_3 \leq \delta_1$, $|f(x_3) - L| < \frac{\epsilon_3}{2}$

Since $0 < |x_3 - a| < \delta_3 \leq \delta_2$, $|g(x_3) - M| < \frac{\epsilon_3}{2}$

$|f(x_3) - L| + |g(x_3) - M| < \frac{\epsilon_3}{2} + \frac{\epsilon_3}{2} = \epsilon_3$

From the triangle inequality,

$$|\alpha + \beta| \leq |\alpha| + |\beta|$$

$$|f(x_3) - L + g(x_3) - M| \leq |f(x_3) - L| + |g(x_3) - M| < \epsilon_3$$

$\blacksquare$

Hopefully with this, you now have a solid grasp of when to use min/max and the triangle inequality.

eg. Prove $f(x)$ is continuous $\Rightarrow \lim_{x\to a} f(g(x)) = f\left[\lim_{x\to a} g(x)\right]$

Assume $\lim_{x\to a} g(x)$ exists, meaning

From the premise. I'm not using placeholder M.

$$\boxed{\forall\epsilon_1 > 0, \exists\delta_1 > 0, \forall x_1 \in \mathbb{R}, 0 < |x_1 - a| < \delta_1 \Rightarrow \left|g(x_1) - \lim_{x\to a} g(x_1)\right| < \epsilon_1}$$

Assume $f(x)$ is continuous

Since $f(x)$ is continuous, it is continuous at $\lim_{x\to a} g(x)$.

Since $f(x)$ is continuous at $\lim_{x\to a} g(x)$, $\lim_{x\to \lim_{x\to a} g(x)} f(x) = f\left[\lim_{x\to a} g(x)\right]$

Just try to follow this proof for now. I'll explain the logic behind all my choices below.

$$\boxed{\forall\epsilon_2 > 0, \exists\delta_2 > 0, \forall x_2 \in \mathbb{R}, 0 < \left|x_2 - \lim_{x\to a} g(x_2)\right| < \delta_2 \Rightarrow \left|f(x_2) - f\left[\lim_{x\to a} g(x_2)\right]\right| < \epsilon_2}$$

Show $\lim_{x\to a} f(g(x)) = f\left[\lim_{x\to a} g(x)\right]$, meaning

$$\boxed{\forall\epsilon > 0, \exists\delta > 0, \forall x \in \mathbb{R}, 0 < |x - a| < \delta \Rightarrow \left|f\big(g(x)\big) - f\left[\lim_{x\to a} g(x)\right]\right| < \epsilon}$$

I boxed all of the big statements for clarity.

Let $\epsilon > 0$

Pick $\delta = \delta_1$

Let $x \in \mathbb{R}$

Assume $0 < |x - a| < \delta$

Show $\left|f\big(g(x)\big) - f\left[\lim_{x\to a} g(x)\right]\right| < \epsilon$

Pick $\epsilon_1 = \delta_2, x_1 = x$, then $0 < |x - a| < \delta \Rightarrow \left|g(x) - \lim_{x\to a} g(x)\right| < \delta_2$

Since $0 < |x - a| < \delta$, $\left|g(x) - \lim_{x\to a} g(x)\right| < \delta_2$ — *Modus ponens*

<u>*Case 1.*</u> $g(x) \neq \lim_{x\to a} g(x)$

Then $\left|g(x) - \lim_{x\to a} g(x)\right| \neq 0$, and $0 < \left|g(x) - \lim_{x\to a} g(x)\right| < \delta_2$

Pick $\epsilon_2 = \epsilon, x_2 = g(x)$, then $0 < \left|g(x) - \lim_{x\to a} g(x)\right| < \delta_2 \Rightarrow \left|f\big(g(x)\big) - f\left[\lim_{x\to a} g(x)\right]\right| < \epsilon$

Since $0 < \left|g(x) - \lim_{x\to a} g(x)\right| < \delta_2$, $\left|f\big(g(x)\big) - f\left[\lim_{x\to a} g(x)\right]\right| < \epsilon$ — *Modus ponens*

<u>*Case 2.*</u> $g(x) = \lim_{x\to a} g(x)$

$$g(x) = \lim_{x\to a} g(x)$$
$$f\big(g(x)\big) = f\left(\lim_{x\to a} g(x)\right)$$
$$f\big(g(x)\big) - f\left(\lim_{x\to a} g(x)\right) = 0$$
$$\left|f\big(g(x)\big) - f\left(\lim_{x\to a} g(x)\right)\right| = 0 < \epsilon$$

■

This theorem is the "Continuity Law for Composition". **Composition** is when you take a function of another function, like $f(g(h(x)))$, also notated $f \circ g \circ h$. This proof is arguably the hardest one of MAT137.

So, what just happened? How are we supposed to come to a conclusion like this on our own?

*Let's start with **what we know**:*

- $0 < |x - a| < \delta$
- $0 < |x_1 - a| < \delta_1 \Rightarrow \left|g(x_1) - \lim_{x\to a} g(x_1)\right| < \epsilon_1$
- $f(x)$ *is continuous*

*And what we **need to show**:*

- $\left|f\big(g(x)\big) - f\left[\lim_{x\to a} g(x)\right]\right| < \epsilon$

With our first two assumptions, we can get $\left|g(x) - \lim_{x\to a} g(x)\right| < \epsilon$.

But our conclusion's structure is different: $\left|g(x) - \lim_{x\to a} g(x)\right|$ *vs.* $\left|f\big(g(x)\big) - f\left[\lim_{x\to a} g(x)\right]\right|$.

Remember $f(x)$ is continuous, $\forall \alpha \in \mathbb{R}, \lim_{x \to \alpha} f(x) = f(\alpha)$. Let's expand that:

$$\forall \alpha \in \mathbb{R}, \forall \epsilon_2 > 0, \exists \delta_2 > 0, \forall x_2 \in \mathbb{R}, 0 < |x_2 - \alpha| < \delta_2 \Rightarrow |f(x_2) - f(\alpha)| < \epsilon_2$$

Pick $\alpha = \lim_{x \to a} g(x)$, then $\forall \epsilon_2 > 0, \exists \delta_2 > 0, \forall x_2 \in \mathbb{R}, 0 < \left|x_2 - \lim_{x \to a} g(x)\right| < \delta_2 \Rightarrow \left|f(x_2) - f\left[\lim_{x \to a} g(x)\right]\right| < \epsilon_2$

Compare $\left|f(x_2) - f\left[\lim_{x \to a} g(x)\right]\right|$ and the conclusion, $\left|f(g(x)) - f\left[\lim_{x \to a} g(x)\right]\right|$. This statement is the missing link!

Our chain of logic for this proof is thus summarized below:

$0 < |x - a| < \delta$

$0 < |x_1 - a| < \delta_1 \Rightarrow \left|g(x_1) - \lim_{x \to a} g(x_1)\right| < \epsilon_1$ $\rightarrow$ $\left|g(x) - \lim_{x \to a} g(x)\right| < \delta_2$

$f(x)$ continuous at $\lim_{x \to a} g(x)$ $\rightarrow$ $\lim_{x \to \lim_{x \to a} g(x)} f(x) = f\left[\lim_{x \to a} g(x)\right]$ $\rightarrow$ $0 < \left|x_2 - \lim_{x \to a} g(x)\right| < \delta_2 \Rightarrow \left|f(x_2) - f\left[\lim_{x \to a} g(x)\right]\right| < \epsilon_2$

$\left|g(x) - \lim_{x \to a} g(x)\right| < \delta_2$

$0 < \left|g(x) - \lim_{x \to a} g(x)\right| < \delta_2 \Rightarrow \left|f(g(x)) - f\left[\lim_{x \to a} g(x)\right]\right| < \epsilon_2$ $\rightarrow$ $\left|f(g(x)) - f\left[\lim_{x \to a} g(x)\right]\right| < \epsilon_2$

Wait a minute, we only have $\left|g(x) - \lim_{x \to a} g(x)\right| < \delta_2$. We need $0 < \left|g(x) - \lim_{x \to a} g(x)\right| < \delta_2$ for the conditional!

That's why we split into two cases:

- *$g(x) \neq \lim_{x \to a} g(x)$, thus $0 < \left|x_2 - \lim_{x \to a} g(x)\right|$ and we can finish the proof as shown above.*
- *$g(x) = \lim_{x \to a} g(x)$, where we have to solve the proof in the other way, as shown in the proof.*

Note that this detail, while mathematically rigorous, is <u>technical</u> and omittable (according to my MAT137 prof)

eg. Explain why the following methodology is incorrect *(Note: Not an exam question)*.

$\lim_{x \to 0}\left[x \sin\left(\frac{1}{x}\right)\right] = \lim_{x \to 0} x \cdot \lim_{x \to 0} \sin\left(\frac{1}{x}\right) = 0 \cdot \lim_{x \to 0} \sin\left(\frac{1}{x}\right) = 0$

The first step uses the limit law $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to 0} f(x) \cdot \lim_{x \to 0} g(x)$.

This law presupposes both $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ exists. However, $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

5.3. Differentiability and Continuity

$f(x)$ is differentiable at $a \Rightarrow f(x)$ is continuous at a $\left(\text{which means } \lim_{x \to a} f(x) = f(a)\right)$

$f(x)$ is differentiable at $a \Leftrightarrow f'(x)$ is continuous at a $\left(\text{which means } \lim_{x \to a} f'(x) = f'(a)\right)$

$f(x)$ is increasing $\Leftrightarrow f'(x) > 0 \Leftrightarrow \forall x_1, x_2 \in [a, b], x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$f(x)$ is decreasing $\Leftrightarrow f'(x) < 0 \Leftrightarrow \forall x_1, x_2 \in [a, b], x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

$f(x)$ is monotonic $\Leftrightarrow f(x)$ is increasing or $f(x)$ is decreasing

eg. Give a function that is continuous but not differentiable at $x = 0$.

Take $\boxed{f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}}$. $\lim_{x \to 0^+}|x| = \lim_{x \to 0^-}|x| = 0 = f(0)$

Or you prove, $\epsilon - \delta$, style, that $\lim_{x \to 0}|x| = 0$. Moving onto derivatives,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

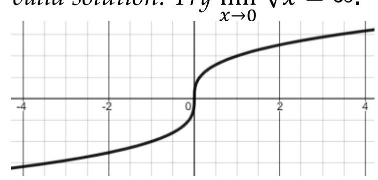
$\lim_{h \to 0^+} \frac{|x+h| - |x|}{h} = \lim_{h \to 0^+} \frac{(x+h) - (x)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$

$\lim_{h \to 0^-} \frac{|x+h| - |x|}{h} = \lim_{h \to 0^-} \frac{-(x+h) + (x)}{h} = \lim_{h \to 0^+} \frac{-h}{h} = -1$

The one-sided limits aren't equal, thus $f'(x)$ DNE at 0 and isn't continuous at 0. Thus $f(x)$ isn't differentiable at 0.

We combined one-sided limits and the definition of continuity here.

Note that $\boxed{f(x) = \sqrt[3]{x}}$ is also a valid solution. Try $\lim_{x \to 0} \sqrt[3]{x} = \infty$.

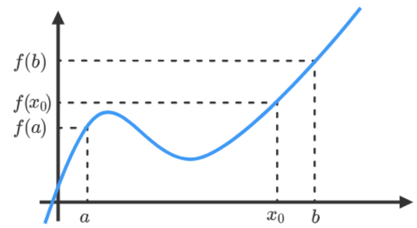


5.4. Important Theorems

These theorems seem obvious intuitively, but their formal proofs are actually too hard for MAT137.

5.4.1. Intermediate Value Theorem (IVT)

$\boxed{f(x) \text{ is continuous at } [a,b] \Rightarrow \forall k \in [f(a), f(b)], \exists c \in [a,b], f(c) = k}$

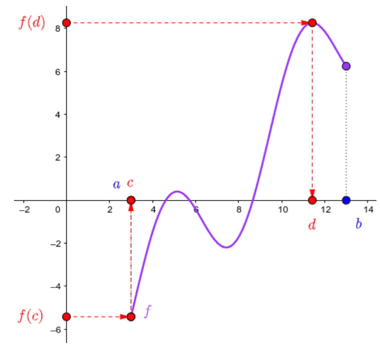


If $f(x)$ is continuous from a to b, there's a $x = k$ where $f(a) \leq f(k) \leq f(b)$

"If $f(a) > 0$ and $f(b) < 0$, $f(x)$ must pass 0 somewhere from a to b."

5.4.2. Extreme Value Theorem (EVT)

$\boxed{f(x) \text{ is continuous at } [a,b] \Rightarrow \exists \min_{x \in [f(a), f(b)]} f(x) \text{ and } \exists \max_{x \in [f(a), f(b)]} f(x)}$



If $f(x)$ is continuous on <u>closed interval</u> $[a,b]$, there's a minimum/maximum $f(x)$ in that interval.

If necessary, you can expand the definition of maximum as such:

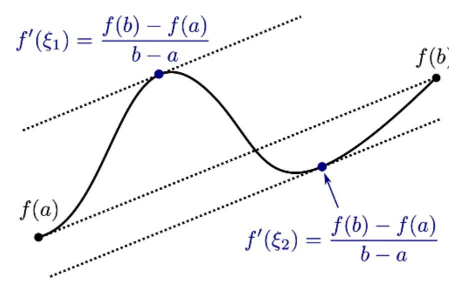
$$\exists c \in [a,b], \forall k \in [a,b], f(c) \geq f(k)$$

5.4.3. Mean Value Theorem (MVT)

$$\boxed{f(x) \text{ is continuous at } [a,b] \text{ and } f(x) \text{ is differentiable at } (a,b) \Rightarrow \exists c \in (a,b), f'(c) = \frac{f(b) - f(a)}{b - a}}$$

If $f(x)$ is continuous between a and b, there's a $f(x)$ with the same slope as the average slope between $f(a)$ and $f(b)$.

"If you accelerate from 80 km/h to 100 km/h, you reach 90 km/h sometime in between."



My prof called this the Most Valuable Theorem (MVT). I think it's because it relates derivatives to continuity. If you're wondering why the boundaries $[a,b]$, (a,b) differ…I don't know.

5.4.4. Proofs with Important Theorems

<u>These theorems tell us $\exists c$, but not how to find c</u>, so proofs with them are often more abstract. It's a good idea to remember these theorems, because proofs with them are not uncommon on exams.

eg. Show there are at least 3 solutions to $\left(\frac{x}{\pi}\right)^3 - 2\sin x + \frac{1}{2} = 0$.

Let $f(x) = \left(\frac{x}{\pi}\right)^3 - 2\sin x + \frac{1}{2}$

$f(-2\pi) = -8 - 2(0) + \frac{1}{2} = -\frac{15}{2} < 0$	$f\left(\frac{\pi}{2}\right) = \frac{1}{8} - 2(1) + \frac{1}{2} = -\frac{11}{8} < 0$
$f(0) = \frac{1}{2} > 0$	$f(\pi) = 1 - 2(0) + \frac{1}{2} = \frac{3}{2} > 0$

Since $f(x)$ is continuous, by the Intermediate Value Theorem (IVT),

$$\exists c_1 \in [-2\pi, 0], f(c_1) = 0$$

$$\exists c_2 \in \left[0, \frac{\pi}{2}\right], f(c_2) = 0$$

$$\exists c_3 \in \left[\frac{\pi}{2}, \pi\right], f(c_3) = 0$$

■

Unfortunately, you have to just guess and check to make sure you have the right values.

f(x) is continuous since it is composed of standard functions, and adding standard functions retains continuity.

eg. Let $f(x)$ be differentiable. If $f'(x)$ has 1 zero on $[a, b]$, how many zeroes does $f(x)$ have at most on $[a, b]$?	
ANSWER: $f(x)$ has at most 2 zeroes. Proof by contradiction, suppose $f(x)$ has at least 3 zeroes on $[a, b]$: $\exists x_1, x_2, x_3 \in [a, b]$ where $f(x_1) = f(x_2) = f(x_3) = 0$. Since $f(x)$ is differentiable, it is continuous. By the Mean Value Theorem, $\exists c_1 \in (x_1, x_2), f'(c_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0-0}{x_2 - x_1} = 0$ $\exists c_2 \in (x_2, x_3), f'(c_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{0-0}{x_2 - x_1} = 0$ Thus $f'(c_1) = f'(c_2) = 0$ and $f'(x)$ has 2 zeros on $[a, b]$. This contradicts the premise that $f'(x)$ has 1 zero on $[a, b]$. Therefore, $f(x)$ has at most 2 zeroes on $[a, b]$. ■	*MVT is a rare instance where you must use <u>proof by contradiction</u>.* *In this case, the opposite of "at most x" is "at least x + 1". You can learn about symbolizing English sentences in PHL245.* *Note that* $a \le x_1 \le c_1 \le x_2 \le c_2 \le x_3 \le b$

This specific usage of MVT is called **Rolle's Theorem:**

$\boxed{f(x) \text{ is continuous at } [a, b] \text{ and } f(x) \text{ is differentiable at } (a, b) \text{ and } f(a) = f(b) \Rightarrow \exists c \in (a, b), f'(c) = 0}$

It's derived from the MVT (ie. a **corollary**). I guess it's included because it's a common usage of MVT.

eg. Let $a < b$. Suppose $f(x)$ is differentiable on (a, b) and $\forall x \in (a, b), f'(x) > 0$. Show $f(x)$ is increasing on (a, b).	*(From MAT137, August exam 2019)*

Let $a < b$
Assume $f(x)$ is differentiable on (a, b)
Assume $\forall x \in (a, b), f'(x) > 0$
Show $f(x)$ is increasing on (a, b), meaning
$\forall x_1, x_2 \in (a, b), x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
  Let $x_1, x_2 \in (a, b)$
  Assume $x_1 < x_2$
  Show $f(x_1) < f(x_2)$
    Proof by contradiction, suppose $f(x_1) \ge f(x_2)$
    Since $f(x)$ is differentiable on (x_1, x_2), by the MVT,
    $\exists c \in (x_1, x_2), f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$
    Since $x_1 < x_2$, $x_2 - x_1 > 0$
    Since $f(x_1) \ge f(x_2)$, $f(x_2) - f(x_1) \le 0$
    Then $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le 0$
    This contradicts assumption $\forall x \in (a, b), f'(x) > 0$
    $\therefore f(x_1) < f(x_2)$
■
	MVT, so proof by <u>contradiction</u>!
	
	Technically, the MVT requires $f(x)$ being continuous on $[x_1, x_2]$, but it's a technical detail. To be rigorous:
	
	Pick α, β such that $(\alpha, \beta) \subseteq (x_1 x_2)$ and $f(\alpha) \ge f(\beta)$. Then you can get $f(x)$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β) to apply MVT.

6. Inverse Functions

6.1. Theory

6.1.1. Definition

Let $f: A \to B$ be a function.
The **domain** is set A (set of all inputs of f).
The **codomain** is set B.
The **image** is the set of all outputs of f. $\text{image}(f) \subseteq \text{codomain}(f)$

*Don't use the word **range** as it's ambiguous: depending on the mathematician, it can mean codomain or image.*

$\boxed{f \text{ is onto/surjective} \Leftrightarrow \text{image}(f) = \text{codomain}(f)}$ → "*$f(x)$ can reach any value in the codomain.*"

$\boxed{f \text{ is one-to-one/injective} \Leftrightarrow (\forall a, b \in \mathbb{R}, f(a) = f(b) \Rightarrow a = b)}$ → "*Each y has exactly one corresponding x.*"

$\boxed{f \text{ is invertible/bijective} \Leftrightarrow f \text{ is onto and } f \text{ is one-to-one}}$

"f is invertible" means there exists an inverse function $f^{-1}: B \to A$ where $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$.

The **Inverse Function Theorem** finds the inverse's derivative. Its generalized proof is very difficult.

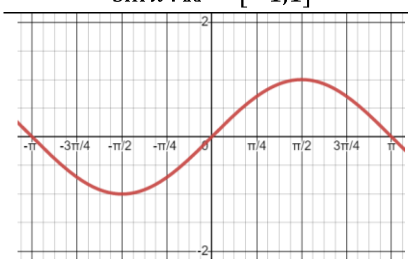
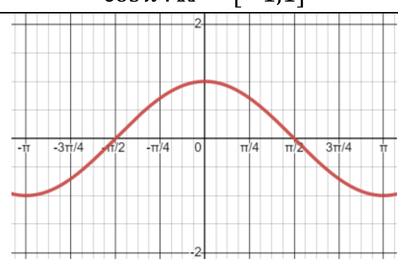
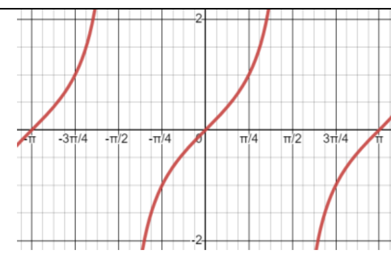
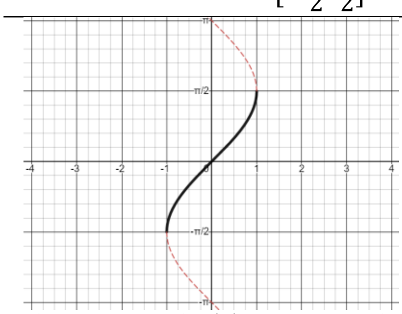
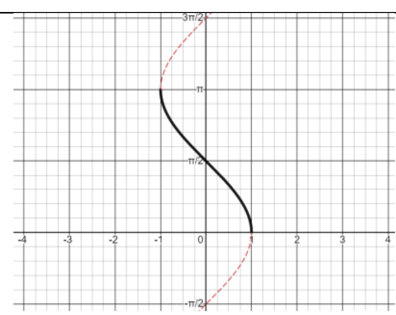
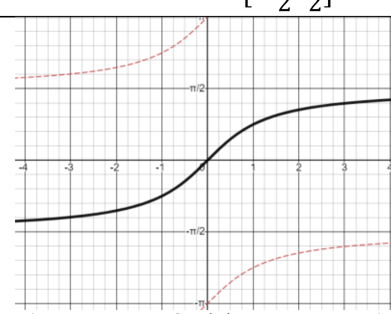
$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

$f(x)$	x^2	x^3	2^x	e^x
$f^{-1}(x)$	$\pm\sqrt{x}$	$\sqrt[3]{x}$	$\log_2 x$	$\ln x$

Important: $\ln x$ is defined as the inverse of e^x.
Visually, f^{-1} looks like f reflected across $y = x$.

Consider $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$.
f is not onto, as $\text{image}(f) = [0, \infty) \neq \mathbb{R} = \text{codomain}(f)$. Restrict **codomain** $f: \mathbb{R} \to [0, \infty)$ to make f **onto**.
f is not one-to-one, as $f(-2) = f(2) = 4$. Restrict the **domain** $f: [0, \infty) \to [0, \infty)$ to make f **one-to-one**.

6.1.2. Inverse Trigonometric Functions

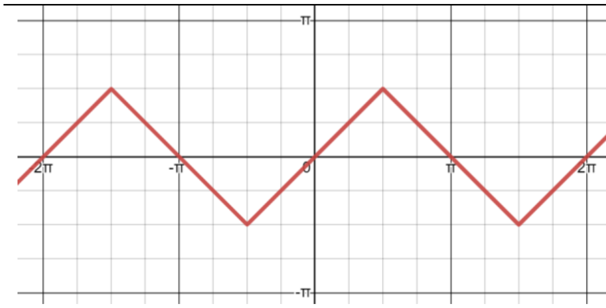
Function		
$\sin x : \mathbb{R} \to [-1,1]$	$\cos x : \mathbb{R} \to [-1,1]$	$\tan x : \mathbb{R} \to \mathbb{R}$
		
Inverse Function		
$\arcsin x : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\arccos x : [-1,1] \to [0, \pi]$	$\arctan x : \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
		
The inverse of $f(x) = \sin x$ with the domain cut to $\left\{-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$ to make it one-to-one and onto	The inverse of $f(x) = \cos x$ with the domain cut to $\{0 \le x \le \pi\}$ to make it one-to-one and onto	The inverse of $f(x) = \tan x$ with the domain cut to $\left\{-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$ to make it one-to-one and onto

Why the domains are cut like that is completely arbitrary – the math community just decided it was convenient this way. Note that $\operatorname{arccsc} x$, $\operatorname{arcsec} x$, and $\operatorname{arccot} x$ exist, but it's uncommon (not in MAT137)

Trigonometric functions on their inverses have interesting graphs. You'll learn how to calculate these:

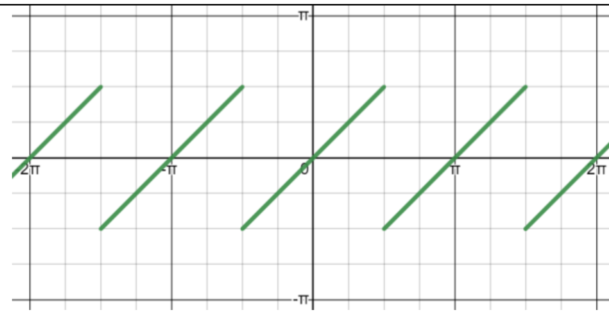
Triangle Wave

$f(x) = \arcsin(\sin x)$

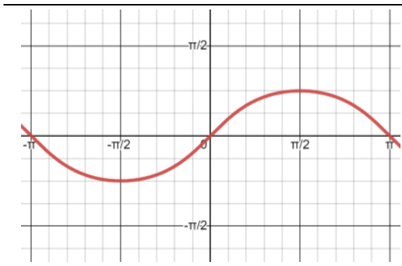


Sawtooth Wave

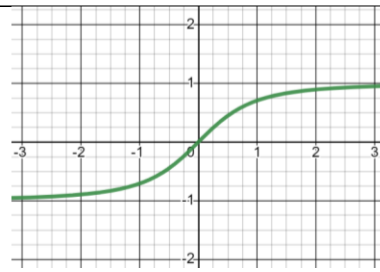
$f(x) = \arctan(\tan x)$



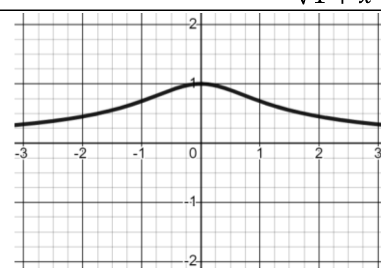
$f(x) = \arctan(\sin x) \left(\neq \frac{\pi}{4}\sin x\right)$



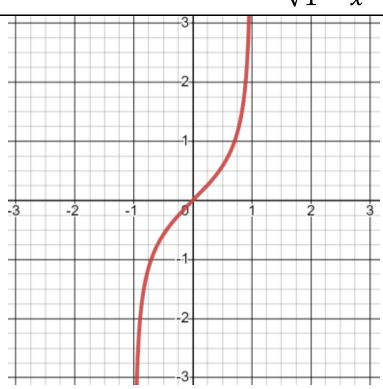
$f(x) = \sin(\arctan x) = \dfrac{x}{\sqrt{1+x^2}}$



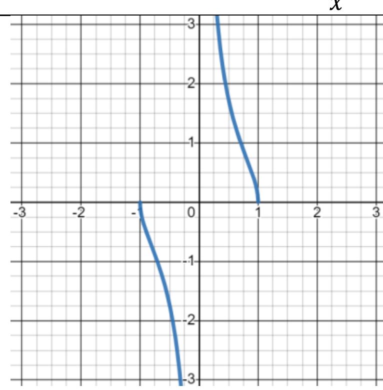
$f(x) = \cos(\arctan x) = \dfrac{1}{\sqrt{1+x^2}}$



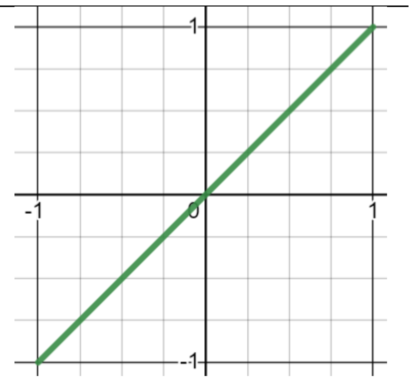
$f(x) = \tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$



$f(x) = \tan(\arccos x) = \dfrac{\sqrt{1-x^2}}{x}$

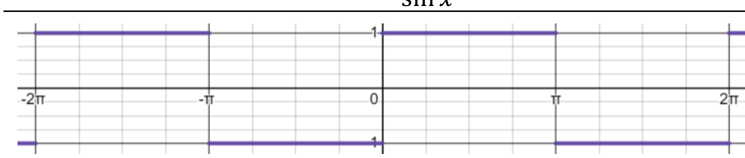


$f(x) = \sin(\arcsin x)$



Square Wave

$f(x) = \dfrac{|\sin x|}{\sin x}$



Waves aren't in MAT137, but I think there're in MAT237's Fourier Series. Waves are used in electrical & computer engineering for electronics & signal processing, as well as in creating electronic music through additive synthesis.

6.2. Computation

<u>Finding $f^{-1}(x)$</u>

Write the equation in terms of y and x.
Write the equation again, replacing y's with x's and vice versa.
Isolate y, and rewrite y as $f^{-1}(x)$.

eg. Find the inverse of $f(x) = 2 - \sqrt{3x}$.

$$y = 2 - \sqrt{3x} \quad (x \geq 0)$$
$$x = 2 - \sqrt{3y} \quad (y \geq 0)$$
$$\sqrt{3y} = 2 - x$$
$$3y = (x-2)^2$$
$$y = \frac{1}{3}(x-2)^2 \quad (y \geq 0)$$

Write the domain in terms of x.

$$y \geq 0$$
$${}^1\!/_3\,(x-2)^2 \geq 0$$
$$x \geq 2$$
$$\boxed{\therefore f^{-1}(x) = \frac{1}{3}(x-2)^2 \ (x \geq 2)}$$

Remember to <u>swap the x and y inside domains</u> as well!

eg. Find the inverse of $f(x) = x^2 + x$.

$$y = x^2 + x$$
$$x = y^2 + y$$
$$\left(y^2 + y + \tfrac{1}{4}\right) - \tfrac{1}{4} = x$$
$$\left(y + \tfrac{1}{2}\right)^2 = x + \tfrac{1}{4}$$
$$y + \tfrac{1}{2} = \pm\sqrt{x + \tfrac{1}{4}}$$
$$\boxed{f^{-1}(x) = \pm\sqrt{x + \tfrac{1}{4}} - \tfrac{1}{2}}$$

There're <u>2 solutions</u> as $f(x)$ isn't onto

The technique you use is called ***completing the square:***

$$ax^2 + bx + c$$
$$= a\left(x + {}^b\!/_{2a}\right)^2 - {}^{b^2}\!/_{4a} + c$$

eg. Find the inverse of $f(x) = \frac{e^x - e^{-x}}{2}$.

$$y = \tfrac{1}{2}(e^x - e^{-x})$$
$$x = \tfrac{1}{2}(e^y - e^{-y})$$
$$2x = e^y - e^{-y}$$
$$2xe^y = e^{2y} - 1$$

Replace e^y with u for clarity.

$$2xu = u^2 - 1$$
$$u^2 - 2xu - 1 = 0$$

Complete the square (hard!)

$$2xu + 1 = u^2$$
$$(x^2u^2 + 2xu + 1) - x^2u^2 = u^2$$
$$(xu+1)^2 = x^2u^2 + u^2$$
$$(xu+1)^2 = u^2(x^2+1)$$

See how this can be <u>cleanly square-rooted</u>.

$$xu + 1 = \pm u\sqrt{x^2+1}$$
$$x + \tfrac{1}{u} = \pm\sqrt{x^2+1}$$
$$\tfrac{1}{u} = -x \pm \sqrt{x^2+1}$$
$$u = \frac{1}{-x \pm \sqrt{x^2+1}}$$
$$e^y = \left(-x \pm \sqrt{x^2+1}\right)^{-1}$$
$$y = \ln\left(-x \pm \sqrt{x^2+1}\right)^{-1}$$
$$= -\ln\left(-x \pm \sqrt{x^2+1}\right)$$

$-x - \sqrt{x^2+1} < 0, \ln$ *of it DNE*

$$\boxed{\therefore y = -\ln\left(-x + \sqrt{x^2+1}\right)}$$

eg. Evaluate $(x^{\sin x})'$ with logarithmic differentiation and with $f(x) = e^{\ln f(x)}$.

Method 1.

$$y = x^{\sin x}$$
$$\ln y = \ln(x^{\sin x})$$
$$= \sin x \ln x$$
$$\frac{y'}{y} = (\sin x)' \ln x + \sin x\,(\ln x)'$$
$$y' = x^{\sin x}\left(\cos x \ln x + {}^{\sin x}\!/_x\right)$$
$$= \boxed{x^{\sin x}\left(\cos x \ln x + {}^{\sin x}\!/_x\right)}$$

Method 2.

$$y = x^{\sin x}$$
$$= e^{\ln(x^{\sin x})}$$
$$= e^{\sin x \ln x}$$
$$y' = e^{\sin x \ln x} \cdot (\sin x \ln x)'$$
$$= x^{\sin x}\left((\sin x)' \ln x + \sin x\,(\ln x)'\right)$$
$$= \boxed{x^{\sin x}\left(\cos x \ln x + {}^{\sin x}\!/_x\right)}$$

Choose the method you're most comfortable with.

The relationship $x = e^{\ln x}$ is ***<u>essential</u>*** *to solving limits and derivatives of this form $f(x)^{g(x)}$.*

eg. Prove $(\log_a x)' = \frac{1}{x \ln a}$ (*Hint*: First show $\log_a x = \ln x / \ln a$)

From the log rules table in 4.2.3,

$$a^{\log_a x} = x$$
$$\ln\left(a^{\log_a x}\right) = \ln x$$
$$\log_a x \ln a = \ln x$$
$$\log_a x = \frac{\ln x}{\ln a}$$
$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$$
$$= \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x)$$
$$= \boxed{\frac{1}{x \ln a}}$$

In all of MAT137, I never saw log with a base that wasn't e. Nor was this in any exams. So this formula isn't important.

eg. Let $g(x) = x^3 + x + 2$ be one-to-one. Find $g^{-1}(4)$ and $(g^{-1})'(4)$.

(From MAT137, April exam 2019)

Finding $g^{-1}(x)$ the way I showed you won't work.

$$x = y^3 + y + 2$$

Good luck trying to isolate y; that is, you can't. Instead, realise this:

To calculate $g(4)$, you substitute $x = 4$ into $y = x^3 + x + 2$.
Thus, to calculate $g'(4)$, you substitute $y = 4$ into $y = x^3 + x + 2$.

$$x^3 + x + 2 = 4$$

When $x = 1$, $\qquad (1)^3 + (1) + 2 = 4$

Thus $\boxed{g^{-1}(4) = 1}$. *We know there're no other solutions as $g(x)$ is one-to-one. From the inverse function theorem,*

$$g^{-1}(x) = \frac{1}{g'(g^{-1}(x))}$$

$$g^{-1}(4) = \frac{1}{g'(g^{-1}(4))} = \frac{1}{g'(1)}$$

Differentiate $g(x)$ to get $g'(x) = 3x^2 + 1$, then

$$g^{-1}(4) = \frac{1}{g'(1)} = \frac{1}{3(1)^2+1} = \boxed{\frac{1}{4}}$$

This question is very gimmicky.

You have to plug in values of x until you guess a working one. Usually, the question is nice and gives you a simple number.

Trigonometric Inverses

Fit x in the right domain using the following rules:

Refresher on Trigonometric Identities, Part 1		
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\sin(\theta \pm \pi) = -\sin\theta$	$\cos(\theta \pm \pi) = -\cos\theta$	$\tan(\theta \pm \pi) = \tan\theta$
$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$	$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$	

eg. Evaluate $\arcsin\left(\sin\frac{2\pi}{3}\right)$.

$\frac{2\pi}{3}$ *is not inside* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *so we change it so it is.*

$$\sin\frac{2\pi}{3} = -\sin\left(\frac{2\pi}{3} - \pi\right) = \sin\left(\pi - \frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$$

$$\therefore \arcsin\left(\sin\frac{2\pi}{3}\right) = \arcsin\left(\sin\frac{\pi}{3}\right) = \boxed{\frac{\pi}{3}}$$

Remember the restricted domain for sine and arcsine:

$$\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$

$$\arcsin x : [1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

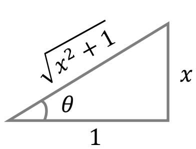
eg. Evaluate $\arcsin(\cos 2)$.

$$\begin{aligned}\arcsin(\cos 2) &= \arcsin\left(\sin\left(2 + {}^{\pi}/_{2}\right)\right)\\ &= \arcsin\left(-\sin\left(2 - {}^{\pi}/_{2}\right)\right)\\ &= \arcsin\left(\sin\left({}^{\pi}/_{2} - 2\right)\right)\\ &= \boxed{{}^{\pi}/_{2} - 2}\end{aligned}$$

Use $\sin\left(\theta + {}^{\pi}/_{2}\right) = \cos\theta$
Use $\sin(\theta \pm \pi) = -\sin\theta$
Use $\sin(-\theta) = -\sin\theta$
Since ${}^{\pi}/_{2} \approx 1.6, {}^{\pi}/_{2} - 2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

eg. Evaluate $\sin(\arctan x)$.

$\sin(\arctan x) = \sin\theta$, where $\theta = \arctan x$



$$\begin{aligned}\tan\theta &= \tan(\arctan x)\\ &= x\end{aligned}$$

This means $\tan\theta = \frac{opposite}{adjacent} = \frac{x}{1}$. *We can calculate* $\sin\theta = \frac{opposite}{hypotneuse} = \boxed{\frac{x}{\sqrt{x^2+1}}}$ *from this triangle ratio.*

Remember tan and arctan's domain:

$$\tan x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$$

$$\arcsin x : \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Thus $\tan(\arctan x) : \mathbb{R} \to \mathbb{R}$

Use Pythagorean Theorem to calculate the hypotenuse.

7. Graphing

7.1. Simplifying Rational Fractions

Polynomial: A function of format $p(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + z$

Degree: A polynomial's highest power (eg. for the example $p(x)$ in the line above, $\deg(p) = n$)

Rational Function: A function of format $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Concepts from 7.1 will always be used in tandem with other concepts on the exam, never on their own.

7.1.1. Long Division

Do this for $\frac{p(x)}{q(x)}$ when the **degree** of $p(x) \geq$ the **degree** of $q(x)$.

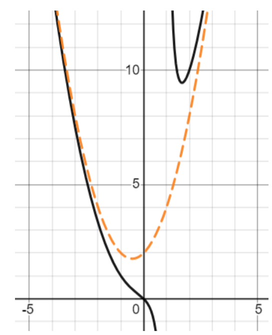
eg. Simplify $\frac{x^3+x}{x-1}$ with long division.

$$\begin{array}{r}
x^2 + x + 2 \\
x - 1 \mid x^3 + 0x^2 + x \\
-(x^3 - x^2) \\
0 + x^2 + x \\
-(x^2 - x) \\
2x \\
-(2x - 2) \\
2
\end{array}$$

$\therefore \frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$

Not much to say other than…just do long division, but instead of base 10, it's base x.

Note: The non-fraction part (orange) is the slant asymptote (see 7.4) if it's a line.



7.1.2. Partial Fractions

Do this for $\frac{p(x)}{q(x)}$ when the **degree** of $p(x) <$ the **degree** of $q(x)$.

eg. Simplify $\frac{x^3+x^2-x+1}{x(x-1)(x^2+1)}$ with partial fractions.

$$\frac{x^3+x^2-x+1}{x(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

$$x(x-1)(x^2+1) \cdot \left(\frac{x^3+x^2-x+1}{x(x-1)(x^2+1)}\right) = x(x-1)(x^2+1) \cdot \left(\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}\right)$$

$$\begin{aligned} x^3 + x^2 - x + 1 &= A(x-1)(x^2+1) + Bx(x^2+1) + (Cx+D)(x)(x-1) \\ &= (A+B+C)x^3 + (-A-C+D)x^2 + (A+B-D)x - A \end{aligned}$$

Note the opposing x^3, x^2, x^1, x^0 on both sides. Set up a system of equations to solve them:

$$\begin{cases} 1 = A + B + C \\ 1 = -A - C + D \\ -1 = A + B - D \\ 1 = -A \end{cases}$$

$$A = -1, B = C = D = 1$$

$$\therefore \frac{x^3+x^2-x+1}{x(x-1)(x^2+1)} = -\frac{1}{x} + \frac{1}{x-1} + \frac{x+1}{x^2+1}$$

How did I know how to choose the numerator?

- Since $\deg(x^2+1) = 2$, we add the form $\alpha x + \beta$ in the numerator.
- Since $\deg(x-1) = \deg(x) = 1$, we add the form α in the numerator.

eg. Expand $\frac{3x^2+x+1}{(x-1)^2(x^2+1)^3}$ into partial fractions.

$$\frac{3x^2+x+1}{(x-1)^2(x^2+1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} + \frac{Gx+H}{(x^2+1)^3}$$

A, B, C, D, E, F, H are all constants that one would need to solve for.

The proof for why partial fractions work is too difficult for MAT137, so just have faith in it.

7.2. Basic Properties

Domain: All possible inputs.

- eg. The domain of $f(x) = \frac{\sqrt{x}}{x-1}$ is $\{x \in \mathbb{R}: x \geq 0 \text{ and } x \neq 1\}$, also writable as $[0, \infty)$ and $x \neq 1$

Image: All possible outputs.

Zeroes: Number of "solutions" or "instances when $f(x) = 0$" or "x-intercepts"
Y-Intercept: The value of $f(x)$ when $x = 0$
X-Intercept(s): The value(s) of x when $f(x) = 0$

Odd: A function where $-f(x) = f(-x)$
Even: A function where $f(-x) = f(x)$

Discontinuities: Areas where $f(x)$ DNE. Review 5.1.2, types of continuities, for how to calculate them.
Tangent Line: A tangent line for $f(x)$ at $x = \alpha$ is a line touching $(\alpha, f(\alpha))$ with slope $f'(\alpha)$.

eg. Let $f(x) = \frac{x^2-1}{1+\cos(\pi x)} \cdot \frac{|x-2|}{x-2}$ with domain [0,4]. Identify all points of discontinuity, and their type.

(From MAT157, April exam 2018)

$f(x)$ is discontinuous when the denominator of $\frac{x^2-1}{1+\cos(\pi x)}$ is 0.

$$1 + \cos(\pi x) = 0$$
$$\cos(\pi x) = -1$$
$$x = -5, -3, -1, 1, 3, 5, \ldots$$
$$= \{2n + 1 : n \in \mathbb{Z}\}$$

We can ignore $\frac{|x-2|}{x-2}$ here because $x \neq 2$, so $\frac{|x-2|}{x-2}$ will just be -1 or 1.

This is an **infinite discontinuity** (dividing by 0 gets infinity).
But the numerator, $x^2 - 1$, is 0 at $x = \pm 1$

$$\lim_{x\to 1} \frac{x^2-1}{1+\cos(\pi x)} = \lim_{x\to 1} \frac{2x}{-\pi \sin(\pi x)} = \frac{2}{0} = \infty$$
$$\lim_{x\to -1} \frac{x^2-1}{1+\cos(\pi x)} = \lim_{x\to -1} \frac{2x}{-\pi \sin(\pi x)} = \frac{2}{0} = \infty$$

We have to make sure the numerator being 0 doesn't change the infinite discontinuity.

Apply L'Hôpital's Rule and realize the limit still diverges.

So the discontinuity is still infinite. Moving onto $\frac{|x-2|}{x-2}$, this is discontinuous when the denominator is 0, or $x = 2$.

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1$$
$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1$$

We can ignore $\frac{x^2-1}{1+\cos(\pi x)}$ here because when $x = 2$, $\frac{x^2-1}{1+\cos(\pi x)}$ is neither 0 nor ∞.

This is a **jump discontinuity** (ie. $f(x)$ "jumps" from – to +).

eg. Find the tangent line to $x + x^2 \cos y + y + 1 = 0$ at $x = 0$.

Substitute $x = 0$: $\quad 0 + (0)^2 \cos y + y + 1 = 0$

$$y = -1$$

Thus the tangent line passes $(0, -1)$. Do implicit differentiation:

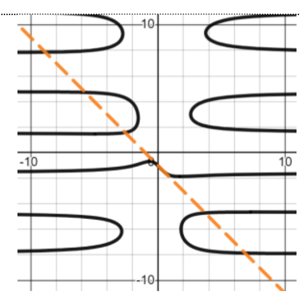
$$1 + (x^2)' \cos y + x(\cos y)' + y' = 0$$
$$1 + 2x \cos y + x(-\sin y) \cdot y' + y' = 0$$
$$1 + 2x \cos y + y'(1 - x \sin y) = 0$$

Substitute $x = 0$: $\quad 1 + 2(0) \cos y + y'(1 - (0) \sin y) = 0$

$$y' = -1$$

Thus $y = -x + b$ is the tangent line. Substitute $(0, -1)$ into it:

$-1 = -(0) + b \quad \Rightarrow \quad b = -1 \quad \Rightarrow \quad \boxed{\therefore y = -x - 1}$



I got something like this in my MAT137 exam. Realize these questions only work in <u>extremely specific conditions</u>: that is, y is isolated if you set x = 0.

7.3. Derivatives and Graphs

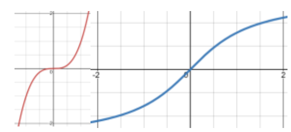
Increasing	*Non-decreasing*	*Decreasing*	*Non-increasing*	*Constant function*	*One-to-one*
$f'(x) > 0$	$f'(x) \geq 0$	$f'(x) < 0$	$f'(x) \leq 0$	$f'(x) = 0$	$f'(x) \neq 0$

Extrema			
Global Extrema		**Local Extrema**	
Global Maximum	**Global Minimum**	**Local Maximum**	**Local Minimum**
Maximum of $f(x)$	Minimum of $f(x)$	Maximum of $f(x)$ relative to surrounding points	Minimum of $f(x)$ relative to surrounding points

c is a local maximum $\Leftrightarrow \exists l \in \mathbb{R}, \forall x \in (c-l, c+l), f(x) \leq f(c)$
c is a local minimum $\Leftrightarrow \exists l \in \mathbb{R}, \forall x \in (c-l, c+l), f(x) \geq f(c)$

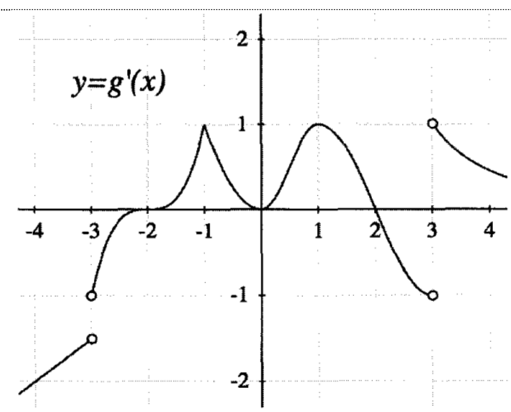
Inflection Point		**Critical Point**	
When $f''(x)$ switches between concave up/down		When $f'(\alpha) = 0$ or $f'(\alpha)\ DNE$	
Concave Up	**Concave Down**	**Local Maximum**	**Local Minimum**
$f(x)$ follows a "∪" shape $f'(x)$ is increasing	$f(x)$ follows a "∩" shape $f'(x)$ is decreasing	$f'(x) > 0$ to its left, $f'(x) < 0$ to its right	$f'(x) < 0$ to its left, $f'(x) > 0$ to its right
$f''(x) > 0$	$f''(x) < 0$	$f'(\alpha) = 0$ and $f''(\alpha) < 0$	$f'(\alpha) = 0$ and $f''(\alpha) > 0$

Visually, notice that x^3 and $\arctan x$ have one inflection point at $x = 0$.
x^3 is concave down → concave up
$\arctan x$ is concave up → concave down



eg. Let $f(x) = 4\pi x + \cos(4\pi x)$. Find its maximum and minimum on $[0,1]$.	*(From MAT157, April exam 2017)*
$f'(x) = 4\pi - \sin(4\pi x) \cdot 4\pi = 4\pi(1 - \sin(4\pi x))$	
Since $\sin(x) \leq 1, \sin(4\pi x) \leq 1$ $-\sin(4\pi x) \geq -1$ $1 - \sin(4\pi x) \geq 0$ $f'(x) = 4\pi(1 - \sin(4\pi x)) \geq 0$	*Doing $-1 < \sin x < 1$ is a common tool you should learn to use.*
$f'(x)$ is non-decreasing, therefore: Its maximum is the right-most $x = 1$, where $\boxed{f(1) = 4\pi + 1}$. Its minimum is the left-most $x = 0$, where $\boxed{f(0) = 1}$.	*This is a "common sense" thing. If $f(x)$ is always growing (or 0), its max is the right-most point in the domain, and vice versa.*

eg. Let g be continuous. Consider the graph of $g'(x)$ on the right. Describe $g(x)$ at the points $x = -3, -2, -1, 0, 2, 3$.



$g'(-3)$ DNE, so it's a **critical point**, but remains negative (no local extrema)

$g'(-2) = 0$, so it's a **critical point**; its sign changes $-$ to $+$, so g's a **local minimum**. $g'(-2)$ is increasing around $x = -2$ (not inflection point)

$g''(-1)$ changes $+$ to $-$, so it's an **inflection point**. $g'(x)$ remains positive (not critical point)

$g'(0) = 0$, so it's a **critical point**, but remains positive (no local extrema). $g''(x)$ changes sign $-$ to $+$, so it's an **inflection point**.

$g'(2) = 0$, so it's a **critical point;** its sign changes $+$ to $-$, so g's a **local maximum**. $g'(2)$ is decreasing around $x = 2$ (not inflection point)

$g'(3)$ DNE, so it's a **critical point**; its sign changes $-$ to $+$, so g's a **local minimum**. $g'(3)$ is decreasing around $x = 3$ (not inflection point)

(From MAT137, April exam 2019)

By "remains negative/positive", I mean $g'(x)$ is negative/positive to the left & right of $g'(x)$

By "changes $-$ to $+$", I mean left of $g'(x) < 0$, and right of $g'(x) > 0$

eg. Find all inflection/critical points and graph $f(x) = 6x^4 - 3x^2 + 2$

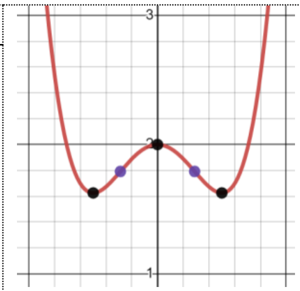
$f'(x) = 24x^3 - 6x = 6x(4x^2 - 1) = 6x(2x+1)(2x-1)$

$f''(x) = 72x^2 - 6 = 6(12x^2 - 1) = 6(2\sqrt{3}x + 1)(2\sqrt{3}x - 1)$

x	...	$-1/2$	...	$-1/2\sqrt{3}$	...	0	...	$1/2\sqrt{3}$	...	$1/2$	...
$f(x)$	↳	13/8	⤴	43/24	↱	2	↴	43/24	↳	13/8	⤴
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$

↳ and ⤴ follow concave up (∪), while ↱ and ↴ follow concave down (∩)

You might've done this in high school. Graph using the table as a guideline.



Critical points are black

Inflection points are blue

7.4. Asymptotes

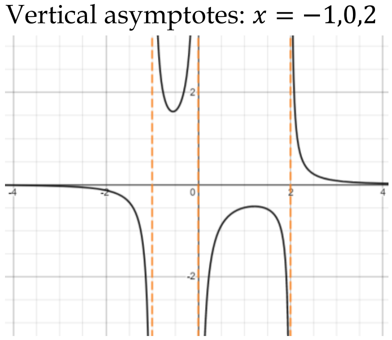
Asymptotes are like tangent lines for $f(x)$ as $x/f(x)$ approaches infinity. $f(x)$ may touch its asymptote(s).

<u>Vertical Asymptotes</u>

Vertical lines. When $f(x) \to \pm\infty$. Find when $f(x)$ is undefined.

eg. $\frac{1}{x(x+1)(x-2)}$

Vertical asymptotes: $x = -1,0,2$

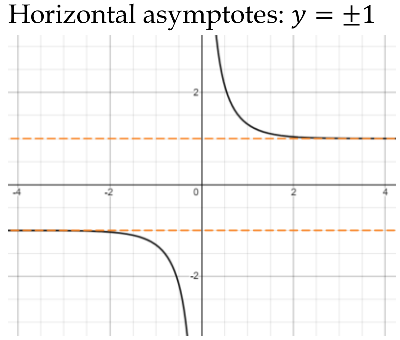


<u>Horizontal Asymptotes</u>

Horizontal lines. When $x \to \pm\infty$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

eg. $\frac{e^x + e^{-x}}{e^x - e^{-x}}$

Horizontal asymptotes: $y = \pm 1$

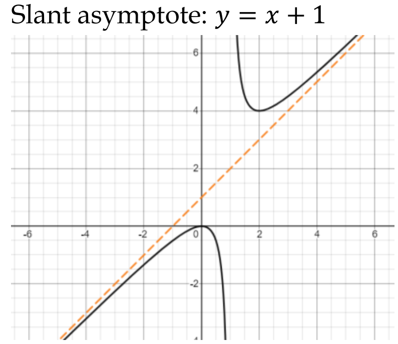


<u>Slant/Oblique Asymptotes</u>

Slanted lines. Special case. Find with fraction long division.

eg. $\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$

Slant asymptote: $y = x + 1$



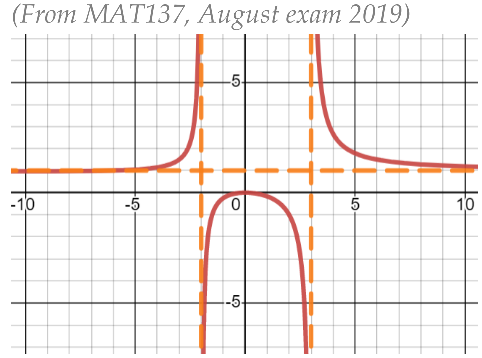
eg. Give an example of a $f(x)$ with vertical tangent lines at $x = 3$ and $x = -2$, plus a horizontal asymptote at $y = 1$.

A vertical tangent line at $x = 3$ means $f'(3) = \pm\infty$. An easy way to do that is with $f(x) = \frac{\cdots}{(x-3)(x+2)}$.

We need a horizontal asymptote at $y = 1$. If we do $f(x) = \frac{x^2}{(x-3)(x+2)}$, then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2 - 5x + 6} = \lim_{x\to\infty} \frac{1}{1 - \frac{5}{x} + \frac{6}{x^2}} = 1$, and we get a horizontal asymptote.

$$\boxed{\therefore f(x) = \frac{x^2}{(x-3)(x+2)}}$$

(From MAT137, August exam 2019)



These questions should be very easy.

8. Integrals

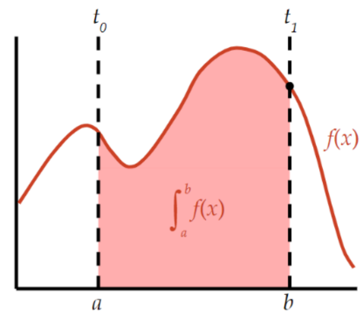
8.1. Theory

8.1.1. Definition as Riemann Sum

What does $\int_a^b f(x)\,dx$ mean?

The integral of $f(x)$ at $[a,b]$ is the **area** between $f(x)$ and the x-axis at $[a,b]$.

- Note that if $f(x) < 0$, the area is negative.
- **True area** refers to all area, negative or positive, while **net area** = positive area – negative area.



One way to calculate area is to *approximate* it by drawing rectangles under the function, and summing the area of those rectangles. Any sums calculated this way are called **Riemann sums**.

The **partition** is how we pick these rectangles. Specifically, it's the values of $t_0, t_1, \ldots$ shown below.

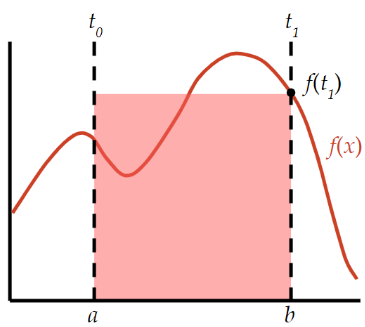
Let's use 1 rectangle, its height defined as the **right-most** endpoint. Its area is

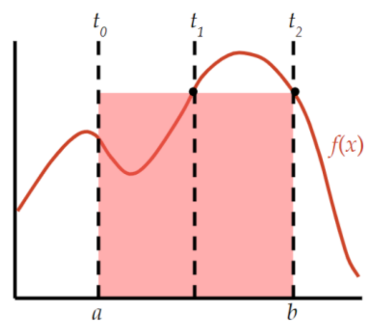
$$A = bh = (t_1 - t_0)f(t_1)$$

which is sucky and inaccurate.

Let's use a partition of 2 evenly-spaced rectangles. The total area is

$$\begin{aligned} A &= b_1h_1 + b_2h_2 \\ &= (t_1 - t_0)\,f(t_1) + (t_2 - t_1)f(t_2) \end{aligned}$$

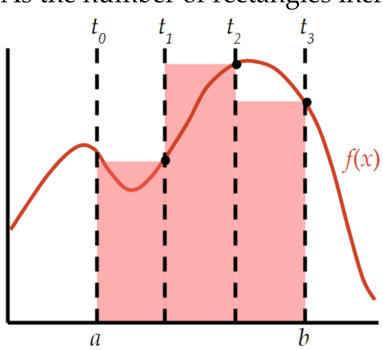


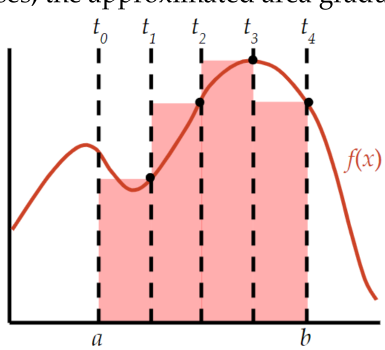


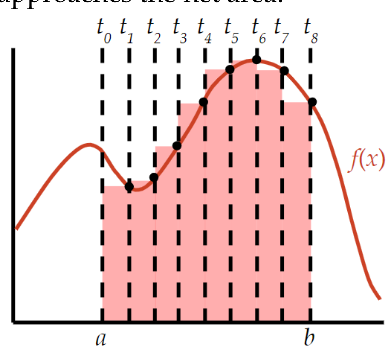
Hopefully, you see the pattern. Given a partition of evenly-spaced points, $P: t_0 \to t_1 \to \cdots \to t_n$, our area is equal to the sum of $(t_i - t_{i-1})\,f(t_i)$ for all i between 1 and n:

$$A = \sum_{i=1}^{n} b_i h_i = \sum_{i=1}^{n} (t_i - t_{i-1}) f(t_i)$$

As the number of rectangles increases, the approximated area gradually approaches the net area.







Since all rectangles in the partition are evenly-spaced, we can simplify this formula to not use t. I'm not going to explain this derivation because it's neither hard nor important.

$$\boxed{A = \sum_{i=1}^{n} b_i h_i = \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i\,\frac{b-a}{n}\right)}$$

Then we add a limit to the formula: $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i\,\frac{b-a}{n}\right)$ to calculate area of infinite rectangles.

We chose the **right-most endpoint** as each rectangle's height as it gives the mathematically simplest formula.

Calculating integrals via Riemann sums is unfortunately very difficult. In practice, we usually calculate integrals using the Fundamental Theorem of Calculus (8.1.6).

8.1.2. Integrability

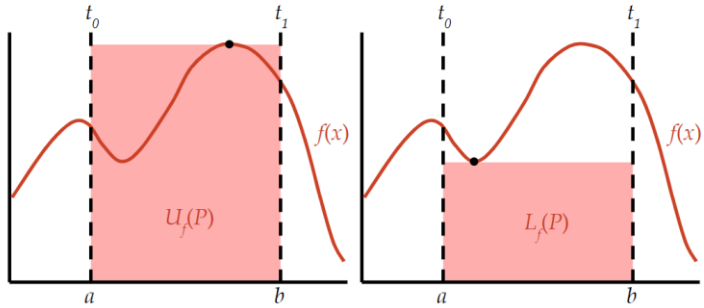
While splitting up area into rectangles isn't good for calculating, it's good for proving integrability.

Let's use a 1-rectangle partition, P.

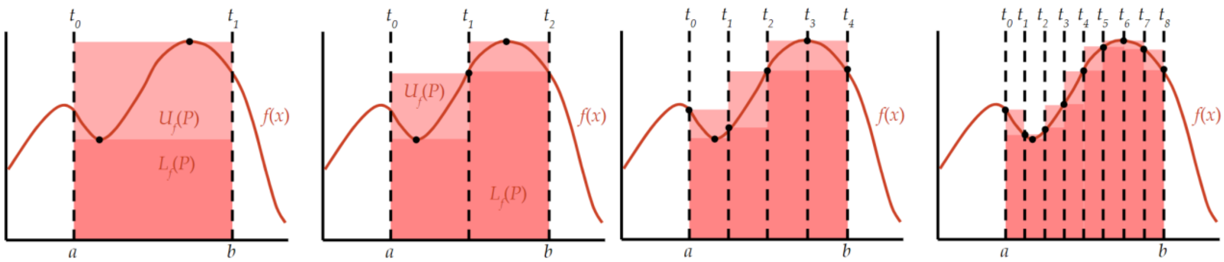
The height of $U_f(P)$, the **upper sum**, is $f(x)$'s max value in P.

The height of $L_f(P)$, the **lower sum**, is $f(x)$'s min value in P.

$$\therefore L_f(P) \le A \le U_f(P)$$



Just like before, as we pick more rectangles in our partition, the approximation becomes more refined.



$f(x)$ is integrable if and only if

$$\boxed{\forall \epsilon > 0, \exists P, U_f(P) - L_f(P) < \epsilon}$$

The intuition is: matter what $\epsilon > 0$ you pick, there'll always be a partition accurate enough that the difference between $U_f(P)$ and $L_f(P)$ is smaller than ϵ. We must pick the partition ourselves.

Partitions are notated like "$0 \to 1 \to 2 \to 3$", which translates to "a partition of 3 rectangles with bases going from $x = 0$ to $x = 1$, $x = 1$ to $x = 2$, and $x = 2$ to $x = 3$".

8.1.3. Proving Integrability

eg. Let $f(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \le 1 \\ 2 & 1 < x \le 2 \\ -1 & 2 < x < 3 \end{cases}$. Prove if $f(x)$ is integrable on $[0,3]$.

$f(x)$ is integrable: $\forall \epsilon > 0, \exists P, U_f(P) - L_f(P) < \epsilon$

Let $\epsilon > 0$

Pick $P: 0 \to l \to (1-l) \to (1+l) \to (2-l) \to (2+l) \to 3$, where $l = \frac{1}{22}\epsilon$

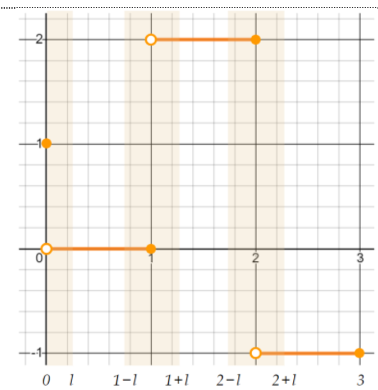
Show $U_f(P) - L_f(P) < \epsilon$

$$\begin{aligned} U_f(P) &= b_1h_1 + b_2h_2 + b_3h_3 + b_4h_4 + b_5h_5 + b_6h_6 \\ &= l(1) + (1-2l)(0) + 2l(2) + (1-2l)(2) + 2l(2) + (1-l)(-1) \end{aligned}$$

$$\begin{aligned} L_f(P) &= b_1h_1 + b_2h_2 + b_3h_3 + b_4h_4 + b_5h_5 + b_6h_6 \\ &= l(0) + (1-2l)(0) + 2l(0) + (1-2l)(2) + 2l(-1) + (1-l)(-1) \end{aligned}$$

$$\begin{aligned} U_f(P) - L_f(P) &= l(1) + 2l(2) + 2l(3) \\ &= 11l \\ &= 11\left(\frac{1}{22}\epsilon\right) \quad \textit{(this is the step where you actually pick } l\textit{)} \\ &= \frac{\epsilon}{2} < \epsilon \end{aligned}$$

■



Where do b_ih_i values come from?

For the rectangle in $0 \to l$,

b is $l - 0 = l$

For $U_f(P)$, $h = \max_{x\in[0,l]} f(x) = 1$

For $L_f(P)$, $h = \min_{x\in[0,l]} f(x) = 0$

Repeat this for all six regions:

$l \to 1-l$, $1-l \to 1+l$, etc.

The strategy here is to pick partitions to <u>isolate regions with discontinuities</u>, surrounding them with $+l$ or $-l$.

- *Our interval is* $[0,3]$ *and our discontinuities are* $x = 0,1,2$.
  - *For* $x = 0$, *we need a rectangle* $0 \to l$
  - *For* $x = 1$, *we need a rectangle* $1 - l \to 1 + l$ *(I've shaded these areas in yellow.)*
  - *For* $x = 2$, *we need a rectangle* $2 - l \to 2 + l$
- *Connect all points to finish our partition: "Pick* $P: 0 \to l \to (1-l) \to (1+l) \to (2-l) \to (2+l) \to 3$*".*
  - *Note the* 3 *at the partition's end. Our partition must cover all of* $[0,3]$.

In each isolated region, we know $U_f(P) \neq L_f(P)$ *is true regardless of* l*'s value, since <u>the rectangle in that region is defined as being centered on a discontinuity</u>.*

- *For* $0 \to l$, $h_{max} = \max_{x\in[0,l]} f(x) = 1$ *and* $h_{min} = \min_{x\in[0,l]} f(x) = 0$, *even as* l *approaches* 0.
- *For* $1 - l \to 1 + l$, $h_{max} = \max_{x\in[1-l,1+l]} f(x) = 2$ *and* $h_{min} = \min_{x\in[1-l,1+l]} f(x) = 0$, *even as* l *approaches* 0.
- *For* $2 - l \to 2 + l$, $h_{max} = \max_{x\in[2-l,2+l]} f(x) = 2$ *and* $h_{min} = \min_{x\in[2-l,2+l]} f(x) = -1$, *even as* l *approaches* 0.
- *Here, the* $U_f(P)$ *rectangle always has a different height than the* $L_f(P)$ *rectangle.*

Between isolated regions, we know $U_f(P) = L_f(P)$ *as there're <u>no discontinuities</u>, so* $U_f(P) - L_f(P) = 0$.

- *For* $l \to 1 - l$, $h_{max} = \max_{x\in[l,1-l]} f(x) = 0$ *and* $h_{min} = \min_{x\in[l,1-l]} f(x) = 0$
- *For* $1 + l \to 2 - l$, $h_{max} = \max_{x\in[l+l,2-l]} f(x) = 2$ *and* $h_{min} = \min_{x\in[l,1-l]} f(x) = 2$
- *For* $2 + l \to 3$, $h_{max} = \max_{x\in[2+l,3]} f(x) = -1$ *and* $h_{min} = \min_{x\in[2+l,3]} f(x) = -1$
- *Here, the* $U_f(P)$ *rectangle always has the same height as the* $L_f(P)$ *rectangle.*

Therefore, when you do $U_f(P) - L_f(P)$ *and simplify, your answer should be* αl. *Pick* $l = \frac{\epsilon}{2\alpha}$ *to get* $\frac{\epsilon}{2} < \epsilon$.

- *If you get the form* $U_f(P) - L_f(P) = \alpha l + \beta$, *then you've chosen your partition wrongly.*

eg. Let $f(x) = -x + 1$. Prove if $f(x)$ is integrable on $[-1,3]$.

$f(x)$ is integrable: $\forall \epsilon > 0, \exists P, U_f(P) - L_f(P) < \epsilon$

Let $\epsilon > 0$

Pick $P: t_0 \to t_1 \to \cdots \to t_n$, where $t_0 = -1$, $t_n = 3$, and all intervals are equally-spaced, with width $b = \frac{1}{n}$

Partition definitions can get wordy.

You could write this instead:
Pick $n = \text{ceil}\frac{4}{\epsilon} + 1 > \text{ceil}\frac{4}{\epsilon} \geq \frac{4}{\epsilon}$
Since $n > \frac{4}{\epsilon}$, *therefore* $\frac{4}{n} < \epsilon$
But that's just a technical detail and isn't too important.

Pick n such that $\frac{4}{n} < \epsilon$ *(leave this blank until the last step)*

Show $U_f(P) - L_f(P) < \epsilon$

Since $f(x)$ is monotonic and decreasing, in any $t_i \to t_{i+1}$, we know $\max_{x\in[t_i,t_{i+1}]} f(x) = f(t_i)$ and $\min_{x\in[t_i,t_{i+1}]} f(x) = f(t_{i+1})$

$$\begin{aligned} U_f(P) &= bf(t_0) + bf(t_1) + \cdots + bf(t_{n-1}) \\ L_f(P) &= bf(t_1) + bf(t_2) + \cdots + bf(t_n) \\ U_f(P) - L_f(P) &= bf(t_0) - bf(t_n) \\ &= bf(-1) - bf(3) \\ &= b(2) - b(-2) \\ &= 4b \\ &= \frac{4}{n} < \epsilon \end{aligned}$$

■

You get $\frac{4}{n}$ *and need it to be* $< \epsilon$. *So you do "Pick* n *such that* $\frac{4}{n} < \epsilon$*".*

If $f(x)$ is not monotonic, divide $f(x)$ into sections that are monotonic and calculate area the same way.

eg. Let $f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ (ie. } x \in \mathbb{Q}) \\ 0 & x \in \mathbb{I} \end{cases}$. Prove if $f(x)$ is integrable on $(0,1)$.

$f(x)$ is integrable: $\forall \epsilon > 0, \exists P, U_f(P) - L_f(P) < \epsilon$

Let $\epsilon > 0$

There are finitely many x where $f(x) \geq \frac{\epsilon}{4}$, $x = c_1, c_2, \dots, c_n$

Pick P by constructing n rectangles around each c_i with width $l = \frac{\epsilon}{4n}$

Show $U_f(P) - L_f(P) < \epsilon$

Since there are infinitely many $x \in \mathbb{I}$ across $(0,1)$, $L_f(P) = 0$

$$U_f(P) = \sum_{\substack{rects \\ around\ c_i}} b_i h_i + \sum_{\substack{rects\ not \\ around\ c_i}} b_i h_i < n\left(\frac{l}{2}\right) + \frac{\epsilon}{4}$$

Take the area of each individual rectangle that we constructed around each c_i.

- *We know $b_i = l$ since we defined it that way*
- *We know $h_i \leq \frac{1}{2}$ because visually, $\max_{x \in [0,1]} f(x) = \frac{1}{2}$*
- $\therefore \sum_{\substack{rects \\ around\ c_i}} b_i h_i \leq \sum_{\substack{rects \\ around\ c_i}} \left(l \cdot \frac{1}{2}\right) = n\left(\frac{l}{2}\right)$

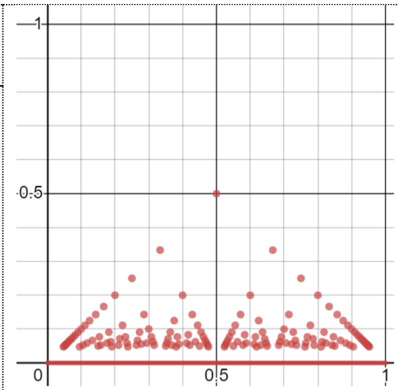
Now take the area of rectangles not around each c_i.

- *We don't know b_i, but we know $\sum b_i \leq 1 - 0 = 1$ (we're working in the interval $(0,1)$, so the rectangle base widths add to that at most)*
- *We know $h_i < \frac{\epsilon}{4}$, because otherwise, any point where $h_i \geq \frac{\epsilon}{4}$ would be in $\{c_1, c_2, \dots, c_n\}$.*
- $\therefore \sum_{\substack{rects\ not \\ around\ c_i}} b_i h_i < \sum_{\substack{rects\ not \\ around\ c_i}} \left(b_i \cdot \frac{\epsilon}{4}\right) = \frac{\epsilon}{4}\left(\sum_{\substack{rects\ not \\ around\ c_i}} b_i\right) = \frac{\epsilon}{4} \cdot 1 = \frac{\epsilon}{4}$

Add these two values up to get $n\left(\frac{l}{2}\right) + \frac{\epsilon}{4}$

$$\begin{aligned} \therefore U_f(P) - L_f(P) &= U_f(P) \\ &= n\left(\frac{l}{2}\right) + \frac{\epsilon}{4} \\ &= \frac{4nl}{4} + \frac{\epsilon}{4} \\ &= \frac{1}{2}nl + \frac{\epsilon}{4} \\ &= \frac{1}{2}n\left(\frac{\epsilon}{2n}\right) + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2} < \epsilon \end{aligned}$$

■



As the points become lower, they get denser. This graph has a gap as I manually inputted every point. I refuse to continue it past $q > 21$.

These are the most complicated partitions. The trick is to realise that even though there're infinitely many "points", they're all approaching a cluster.

This summation notation isn't really proper.

The good news is that no past MAT137 exam in the Old Exam Repository contains any very difficult questions like these.

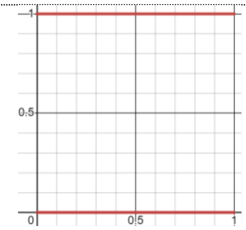
8.1.4. Disproving Integrability

eg. Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{I} \end{cases}$. Prove if $f(x)$ is integrable on [0,1].

$f(x)$ is not integrable: $\exists \epsilon > 0, \forall P, U_f(P) - L_f(P) \geq \epsilon$
Pick $\epsilon = 0.5$
Let P be an arbitrary partition on [0,1]
Show $U_f(P) - L_f(P) \geq \epsilon$
Since there are infinitely many $x \in \mathbb{Q}$ across [0,1], $U_f(P) = 1$
Since there are infinitely many $x \in \mathbb{I}$ across [0,1], $L_f(P) = 0$

$$U_f(P) - L_f(P) = 1 - 0 = 1 \geq 0.5$$

■



Why is this not integrable? Because there're infinitely many "points" AND none of them are approaching any cluster.

In conclusion, integrability has laxer conditions compared to differentiability:

$$f(x) \text{ is bounded} \Leftrightarrow \exists m, M \in \mathbb{R}, \forall x \in \mathbb{R}, m \leq f(x) \leq M$$

$$f(x) \text{ is bounded} \Leftrightarrow \exists m, M \in \mathbb{R}, \forall x \in [a,b], m(b-a) \leq \int_a^b f(x) \leq M(b-a)$$

$$f(x) \text{ is integrable} \Rightarrow f(x) \text{ is bounded}$$

$$f(x) \text{ is integrable on } [a,b] \Rightarrow \forall [c,d] \subseteq [a,b], f(x) \text{ is integrable on } [c,d]$$

$$f(x) \text{ is bounded and monotone} \Rightarrow f(x) \text{ is integrable}$$

$$f(x) \text{ is bounded and continuous} \Rightarrow f(x) \text{ is integrable}$$

$$f(x) \text{ is bounded and discontinuous at finitely many points} \Rightarrow f(x) \text{ is integrable}$$

8.1.5. Definition as Supremum/Infimum

Recall the relevant terms from section 1. Here are their definitions:

Statement	Definition	English Translation
a is an **upper bound** of S	$\forall s \in S, a \geq s$	Everything in S is smaller than a
a is a **lower bound** of S	$\forall s \in S, a \leq s$	Everything in S is larger than a
a is the **supremum** of S	$\forall s \in S, a \geq s$ and $\forall k \in \mathbb{R}, (\forall s \in S, k \geq s) \Rightarrow k \geq a$	a is the smallest upper bound of S
a is the **infimum** of S	$\forall s \in S, a \leq s$ and $\forall k \in \mathbb{R}, (\forall s \in S, k \leq s) \Rightarrow k \leq a$	a is the largest lower bound of S

Note: "S is bounded above by a" means "a is an upper bound of S".

We can alternatively define supremum and infimum as:

$$\forall \epsilon > 0, \exists s \in S, \sup S - \epsilon < s \leq \sup S$$
$$\forall \epsilon > 0, \exists s \in S, \inf S \leq s < \inf S + \epsilon$$

The intuition is that "anything smaller than $\sup S$ will be inside the range of S", and vice versa for infimum.
Lastly, we can define the integral according to supremum and infimum:

$$\boxed{A = \int_a^b f(x) = \inf\{\text{all } U_f(P)\} = \sup\{\text{all } L_f(P)\}}$$

8.1.6. Fundamental Theorem of Calculus (FTC)

My math prof emphasized this as the single most important takeaway of MAT137.

FTC I
Let f be continuous, $a \in \mathbb{R}$.

$$\boxed{\text{Let } F(x) = \int_a^{g(x)} f(t)dt \text{. Then } F'(x) = f\big(g(x)\big) \cdot g'(x)}$$

FTC II
Let f be continuous, $a, b \in \mathbb{R}$.

$$\boxed{\text{Let } F'(x) = f(x). \text{ Then } \int_a^b f(x) = F(b) - F(a)}$$

FTC II lets you to find **definite integrals** through finding the antiderivative of $f(x)$, $F(x)$.

Indefinite integrals, $\int f(x)$, represent a <u>class of $F(x)$</u> (the antiderivative) where $F'(x) = f(x)$.

- There're infinitely many $F(x)$ where $F'(x) = f(x)$.
    - eg. Let $f(x) = 1$. Then $F(x) = x + \alpha \;\Rightarrow\; F'(x) = 1 = f(x)$
- In high school, it's common to compute $\int(1)\,dx = x + C$, where C is the **constant of integration**. Higher-level math almost never uses C because it's misleading: it implies C is <u>a constant</u> while in reality, C is <u>the set of all constants</u>. Still, I believe MAT137 doesn't care if you use C.

If you did integrals in high school, you've probably been using the FTC unaware. You may have been led to believe that derivatives and integrals are "like opposites". This is NOT TRUE. Math profs want you to learn that they are separate concepts, whose computations just happen to be related thanks to the FTC.

For definite integrals, we write $\int_a^b f(x) = [F(x)]_a^b$ instead of $F(b) - F(a)$, since it's compact. There's also this notation, $F(x)|_a^b$, which is trash as it's ambiguous: is $x + x^2|_a^b$ equal to $x + (x^2)|_a^b$ or $(x + x^2)|_a^b$?

8.1.7. Integral Properties

General Theorems	$f(x)$	$\int f(x)$	$f(x)$	$\int f(x)$
$\int_a^b (f+g) = \int_a^b f + \int_a^b g$	$\sin x$	$-\cos x$	$\frac{1}{x}$	$\ln\lvert x\rvert$
$\int_a^b \alpha f = \alpha \int_a^b f$	$\cos x$	$\sin x$	x^n	$\frac{x^{n+1}}{n+1}$
	$\sec x$	$\ln\lvert\tan x + \sec x\rvert$	e^x	e^x
$\int_a^b f = -\int_b^a f$	$-\csc x \cot x = -\frac{\cos x}{\sin^2 x}$	$\frac{1}{\sin x} = \csc x$	$\frac{1}{\sqrt{1-x^2}}$	$(\sin x)^{-1} = \arcsin x$
$\int_a^c f = \int_a^b f + \int_b^c f$	$\sec x \tan x = \frac{\sin x}{\cos^2 x}$	$\frac{1}{\cos x} = \sec x$	$\frac{-1}{\sqrt{1-x^2}}$	$(\cos x)^{-1} = \arccos x$
$\left\lvert\int_a^b f\right\rvert \le \int_a^b \lvert f\rvert$	$-\csc^2 x = -\frac{1}{\sin^2 x}$	$\frac{1}{\tan x} = \cot x$	$\frac{1}{1+x^2}$	$(\tan x)^{-1} = \arctan x$

We didn't do any integral property proofs. They were in no past exams; I doubt they'll be in a future one.

$$f(x) \le g(x) \Rightarrow \int_a^b f(x) \le \int_a^b g(x)$$

$$f(-x) = -f(x) \Rightarrow \int_{-a}^a f(x) = 0$$

$$f(-x) = f(x) \Rightarrow \int_{-a}^a f(x) = 2\int_0^a f(x)$$

Mean Value Theorem (MVT) for Integrals:

$$f(x) \text{ is continuous} \Rightarrow \exists c \in [a,b], \int_a^b f(x) = f(c)(b-a)$$

You can also define some functions as an integral:

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad \arccos x = x\sqrt{1-x^2} + 2\int_x^1 \sqrt{1-t^2}\,dt$$

eg. Let $F(x) = x\sqrt{1-x^2} + 2\int_x^1 \sqrt{1-t^2}\,dt$. Show that $F'(x) = (\arccos x)'$

$$F(x) = x(1-x^2)^{\frac{1}{2}} - 2\int_1^x \sqrt{1-t^2}\,dt$$

$$\begin{aligned} F'(x) &= (1-x^2)^{\frac{1}{2}} + x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) - 2\sqrt{1-(x)^2}\cdot x' \\ &= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} - 2\sqrt{1-x^2} \\ &= \frac{-(1-x^2)}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} \\ &= \frac{-1}{\sqrt{1-x^2}} \\ &= (\arccos x)' \end{aligned}$$

Make sure to swap the integral's upper/lower bounds so that g(x) is the upper bound. Otherwise, the FTC doesn't work.

*Notation-wise, **<u>it's fine to omit "dx"</u>** if it's obvious x is being integrated. If you're using a variable like t, add dt.*

8.2. Computation

8.2.1. General Methods

U-Substitution

Set the equation's most annoying part to u (or anything else).

Do $\frac{du}{dx}$, isolate dx, and substitute it into the equation.

- Your equation must be written completely in terms of u. It should be easier to integrate.

Integrate and replace u with what you used it to substitute.

eg. Evaluate $\int \tan x\, dx$	
$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x} dx \rightarrow (1)\\ &= \int \frac{\sin x}{u} \cdot \left(-\frac{du}{\sin x}\right)\\ &= -\int \frac{du}{u}\\ &= -\ln\lvert u\rvert\\ &= \boxed{-\ln\lvert\cos x\rvert}\end{aligned}$$ (1): The $\cos x$ denominator is annoying, so we set that to u. $$u = \cos x$$ $$\frac{du}{dx} = -\sin x$$ $$dx = -\frac{du}{\sin x}$$	*Lesson 1:* *Set u to the annoying part of the equation. In a fraction, it's usually the denominator.*

eg. Evaluate $\int_{\pi^2}^{1} \frac{\cos\sqrt{x}}{\sqrt{x}} dx$	
$$\begin{aligned}\int_{\pi^2}^{1} \frac{\cos\sqrt{x}}{\sqrt{x}} dx &= \int_{\pi^2}^{1} \frac{\cos u}{u} 2u\, du \rightarrow (1)\\ &= 2\int_{x=\pi^2}^{x=1} \cos u\ du\\ &= 2[\sin u]_{x=\pi^2}^{x=1} \rightarrow (2)\\ &= 2\left[\sin\sqrt{x}\right]_{\pi^2}^{1}\\ &= 2(\sin 1 - \sin\pi)\\ &= \boxed{2\sin 1}\end{aligned}$$ (1): Set u to the denominator, $\sqrt{x}$. $$u = x^{\frac{1}{2}}$$ $$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$$ $$dx = 2u\, du$$ (2): You CANNOT expand this yet. The upper/lower bounds are still defined in terms of x, not u.	*Lesson 2:* *Doing $\frac{du}{dx}$ doesn't always get rid of all the x, so you might have to do more.* *Lesson 3:* *For definite integrals, note what the upper/lower bounds are written in terms of.*

You can define the bounds in terms of u by substituting in $u = x^{\frac{1}{2}}$: $(1)^{\frac{1}{2}} = 1$ and $(\pi^2)^{\frac{1}{2}} = \pi$. But it's more work.

eg. Evaluate $\int_{1/4}^{1/2} \frac{1}{x\ln x} dx$	
$$\begin{aligned}\int_{1/4}^{1/2} \frac{1}{x\ln x} dx &= \int_{1/4}^{1/2} \frac{1}{xu}(x\, du) \rightarrow (1)\\ &= \int_{1/4}^{1/2} \frac{du}{u}\\ &= [\ln\lvert u\rvert]_{1/4}^{1/2}\\ &= [\ln\lvert\ln x\rvert]_{1/4}^{1/2}\\ &= \ln\lvert\ln {}^1\!/_2\rvert - \ln\lvert\ln {}^1\!/_4\rvert\\ &= \boxed{\ln\lvert\ln 2\rvert - \ln\lvert\ln 4\rvert}\end{aligned}$$ (1): This is a special case. $$u = \ln x$$ $$\frac{du}{dx} = \frac{1}{x}$$ $$dx = x\, du$$ *Simplify with* $\ln\frac{1}{\alpha} = -\ln\alpha$	*Lesson 4:* *You can't always set u to be the denominator. Choose strategically to cancel out the x.*

eg. Evaluate $\int \frac{x^2-2x}{x^3-3x^2-5} dx$	
$$\begin{aligned}\int \frac{x^2-2x}{x^3-3x^2-5} dx &= \int \frac{x^2-2x}{u}\left(\frac{du}{3(x^2-2x)}\right)\\ &= \frac{1}{3}\int \frac{du}{u}\\ &= {}^1\!/_3 \ln\lvert u\rvert\\ &= {}^1\!/_3 \ln\lvert x^3-3x^2-5\rvert\end{aligned}$$ Set u = denominator. $$u = x^3 - 3x^2 - 5$$ $$\frac{du}{dx} = 3x^2 - 6x$$ $$dx = \frac{du}{3(x^2-2x)}$$	*Note how limited this method is. What if the numerator is not $x^2 - 2x$? Suffering.*

eg. Evaluate $\int \frac{x}{2x^2-x+2}dx$

$$\int \frac{x}{2x^2-x+2}dx = \int \frac{x}{2\left(x-\frac{1}{4}\right)^2+\frac{15}{8}}dx$$
$$= \int \frac{8x}{16\left(x-{}^1/_4\right)^2+15}dx$$

Set $u = x - \frac{1}{4}$, *then* $x = u + \frac{1}{4}$, $\frac{du}{dx} = 1$ *and* $du = dx$

$$= \int \frac{8\left(u+{}^1/_4\right)}{16(u)^2+15}du$$
$$= 2\int \frac{4u+1}{16u^2+15}du$$
$$= 2\left(\int \frac{4u}{16u^2+15}du + \int \frac{1}{16u^2+15}du\right)$$

$\int \frac{4u}{16u^2+15}du$ *Set* $v = 16u^2+15$, *then* $\frac{dv}{du} = 32u$ *and* $du = \frac{dv}{32u}$ $\int \frac{4u}{v}\cdot\frac{dv}{32u}$ $= \frac{1}{8}\int \frac{dv}{v}$ $= \frac{1}{8}\ln\lvert v\rvert$ $= \frac{1}{8}\ln\lvert 16u^2+15\rvert$ $= \frac{1}{8}\ln\left\lvert 16\left(x-\frac{1}{4}\right)^2+15\right\rvert$ $= \frac{1}{8}\ln\lvert 16x^2-8x+16\rvert$ $= \ln\lvert 2x^2-x+2\rvert$	$\int \frac{1}{16u^2+15}dx$ $= 15\int \frac{1}{\frac{16}{15}u^2+1}dx$ $= 15\int \frac{1}{\left(\frac{4}{\sqrt{15}}u\right)^2+1}dx$ $= 15\arctan\left(\frac{4}{\sqrt{15}}u\right)$ $= 15\arctan\left[\frac{4}{\sqrt{15}}\left(x-\frac{1}{4}\right)\right]$

$$= 2\left[\ln\lvert 2x^2-x+2\rvert + 15\arctan\left[\frac{4}{\sqrt{15}}\left(x-\frac{1}{4}\right)\right]\right]$$
$$= \boxed{2\ln\lvert 2x^2-x+2\rvert + 30\arctan\left[\frac{4}{\sqrt{15}}\left(x-\frac{1}{4}\right)\right]}$$

Suffering.

First thing to realize is that if you set $u = 2x^2-x+2$, *then* $\frac{du}{dx} = 4x-1$, *which fits nothing.*

You can also try $\int \frac{x}{2x^2-x+2}dx$
$= \int \frac{(4x-1)-3x+1}{2x^2-x+2}dx$
$= \int \frac{4x-1}{2x^2-x+2}dx - \int \frac{3x-1}{2x^2-x+2}dx$
As for the fraction with $3x-1$,
$\int \frac{3x-1}{2x^2-x+2}dx$
$= 3\int \frac{x}{2x^2-x+2}dx - \int \frac{1}{2x^2-x+2}dx$
You can try simplifying
$\int \frac{x}{2x^2-x+2}dx =$
$\int \frac{4x-1}{2x^2-x+2}dx - 3\int \frac{x}{2x^2-x+2}dx + \int \frac{1}{2x^2-x+2}dx$
into
$2\int \frac{x}{2x^2-x+2}dx$
$= \int \frac{4x-1}{2x^2-x+2}dx + \int \frac{1}{2x^2-x+2}dx$
and learn to compute $\int \frac{1}{2x^2-x+2}dx$ *in 8.2.3, which…includes completing the square. So you'll have to learn to* <u>*complete the square*</u> *to simplify the denominator no matter what.*

<u>*Lesson 5:*</u> *You may have to u-substitute more than once.*

And remember the formula for arctangent: $\int \arctan x = \frac{1}{x^2+1}$

eg. Evaluate $\int \frac{2x^3+x^2-x-4}{(x-1)^2}dx$

$$\int \frac{2x^3+x^2-x-4}{(x-1)^2}dx$$
$$= \int \left(2x+5+\frac{7x-9}{(x-1)^2}\right)dx$$
$$= \int \left(2x+5+\frac{A}{x-1}+\frac{B}{(x-1)^2}\right)dx$$
$$= \int \left(2x+5+\frac{7}{x-1}-\frac{2}{(x-1)^2}\right)dx$$
$$= \boxed{x^2+5x+7\ln\lvert x-1\rvert+\frac{2}{x-1}}$$

$$\begin{array}{r|l} & 2x \quad +5 \\ \hline x^2-2x+1 & 2x^3+1x^2-1x-4 \\ & -(2x^3-4x^2+2x) \\ \hline & 0 + 5x^2-3x-4 \\ & -(5x^2-10x+5) \\ \hline & 0 \quad +7x-9 \end{array}$$

$$\frac{7x-9}{(x-1)^2} = \frac{A}{x-1}+\frac{B}{(x-1)^2}$$
$$7x-9 = A(x-1)+B$$
$$= x(A)+(-A+B)$$
$$\begin{cases} 7 = A \\ -9 = -A+B \end{cases}$$
$$\therefore A = 7, B = -2$$

<u>*Lesson 6:*</u> *Be prepared to simplify with long division and partial fractions.*

Use U-substitution to solve $\int \frac{2}{(x-1)^2}dx$, *set* $u = x-1$
$\frac{du}{dx} = 1, du = dx$
$\int \frac{2}{(x-1)^2}dx = 2\int \frac{du}{u^2} = -\frac{2}{u}$

Integration by Parts

For an integral of format $\int f(x)g(x)$, set one to u, the other to dv, then use these formulae:

$$\boxed{\int u \cdot dv = uv - \int du \cdot v} \qquad \boxed{\int_a^b u \cdot dv = [uv]_a^b - \int_a^b du \cdot v}$$

I personally prefer internalizing it as this: Let u, v be functions. Let $V = \int(v)$. Then

$$\boxed{\int uv = uV - \int u'V} \qquad \boxed{\int_a^b uv = [uV]_a^b - \int_a^b (u'V)}$$

Usually, you use this on $x^n g(x)$.

- If $g(x) \in \{\ln x, \arctan x, \arccos x, \arcsin x\}$, set $u = g(x)$.
- Otherwise, set $u = x^n$

You may also use this on $f(g(x))$, where you actually treat it like $u = f(g(x))$, $v = 1$.

eg. Evaluate $\int x \sec^2 x \, dx$	
$\int x \sec^2 x \, dx = x \int \sec^2 x \, dx - \int (x)'(\sec^2 x) \, dx$	*We set $u = x, v = \sec^2 x$ and integrate by parts.*
$= x \int \sec^2 x \, dx - \int \sec^2 x \, dx$	
$= (x-1) \int \frac{1}{\cos^2 x} dx$	*Remember $(\tan x)' = \frac{1}{\cos^2 x}$.*
$= \boxed{(x-1)\tan x}$	

eg. Evaluate $\int \arctan \sqrt{x} \, dx$	
$\int (\arctan \sqrt{x})(1) \, dx = x \arctan x - \int (\arctan \sqrt{x})'(x) \, dx$	*We have u, v, where* $u = \arctan \sqrt{x}$, $v = 1$
$= x \arctan x - \int \frac{1}{(\sqrt{x})^2 + 1}(x) \, dx$	*So you do integration by parts. You will need to remember this gimmick.*
$= x \arctan x - \int \frac{x}{x+1} dx$	
$= x \arctan x - \int \frac{x+1-1}{x+1} dx$	*You also do long division to arrive at this same step.*
$= x \arctan x - \int \frac{x+1}{x+1} dx + \int \frac{1}{x+1} dx$	
$= \boxed{x \arctan x - x + \ln\lvert x+1 \rvert}$	

eg. Evaluate $\int e^x \sin x \, dx$	
$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$	*Do integration by parts. You can treat either e^x or $\sin x$ as u or v here.*
$= e^x \sin x - \left(e^x \cos x - \int e^x(-\sin x) \, dx\right)$	
$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$	
$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$	*Add $\int e^x \sin x \, dx$ to both sides of the equation*
$\int e^x \sin x \, dx = \boxed{\frac{1}{2} e^x (\sin x - \cos x)}$	*This is another special case.*

8.2.2. Trigonometric Integrals

Refresher on Trigonometric Identities, Part 2		
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ $\sin x \cos y = {}^{1}/_{2}\,[\sin(x-y) + \sin(x+y)]$ $\sin x \sin y = {}^{1}/_{2}\,[\cos(x-y) - \cos(x+y)]$ $\cos x \cos y = {}^{1}/_{2}\,[\cos(x-y) + \cos(x+y)]$	$\sin^2 x + \cos^2 x = 1$ $1 + \cot^2 x = \csc^2 x$ $\tan^2 x + 1 = \sec^2 x$ *Divide $\sin^2 x + \cos^2 x = 1$ by $\sin^2 x$ or $\cos^2 x$ to get the bottom two formulas*	$\sin 2x = 2 \sin x \cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 1 - 2\sin^2 x$ $= 2\cos^2 x - 1$

Here's a guide of what to u-substitute in different circumstances:

$\sin x \cos^n x$	$\sin^n x \cos x$	$\sin^{odd\,n} x \cos^{even\,n} x$ $\quad$ $\sin^{even\,n} x \cos^{odd\,n} x$	$\sin^{even\,n} x \cos^{even\,n} x$
$\boxed{u = \cos x}$ $\frac{du}{dx} = -\sin x$ $du = -\sin x\,dx$ $\int \sin x \cos^n x\,dx$ $= -\int u^n(-\sin x\,dx)$ $= -\int u^n\,du$	$\boxed{u = \sin x}$ $\frac{du}{dx} = \cos x$ $du = \cos x\,dx$ $\int \sin^n x \cos x\,dx$ $= \int u^n(\cos x\,dx)$ $= \int u^n\,du$	$\int \sin^3 x \cos^4 x$ $= \int \sin x \sin^2 x \cos^4 x$ $= \int \sin x\,(1-\cos^2 x)\cos^4 x$ $= \int \sin x \cos^4 x - \int \sin x \cos^6 x$ … Reduce until there's just one power of $\sin x$ or $\cos x$.	$\cos 2x = 1 - 2\sin^2 x$ $\boxed{\therefore \sin^2 x = \frac{1-\cos 2x}{2}}$ $\cos 2x = 2\cos^2 x - 1$ $\boxed{\therefore \cos^2 x = \frac{1+\cos 2x}{2}}$ $\int \sin^2 x \cos^4 x$ $= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2$ …

Remember this integral: $\int \sec x = \ln\lvert\sec x + \tan x\rvert$

$\sec^2 x \tan^n x$	$\sec^{n\geq 2} x \tan x$	$\sec^{even\,n} x \tan^n x$	$\sec^n x \tan^{odd\,n} x$
$\boxed{u = \tan x}$ $\frac{du}{dx} = \sec^2 x$ $du = \sec^2 x\,dx$ $\int \sec^2 x \tan^n x\,dx$ $= \int u^n(\sec^2 x\,dx)$ $= \int u^n\,du$	$\boxed{u = \sec x}$ $\frac{du}{dx} = \sec x \tan x$ $du = \sec x \tan x\,dx$ $\int \sec^n x \tan x\,dx$ $= \int u^{n-1}(\sec x \tan x\,dx)$ $= \int u^{n-1}\,du$	$\int \sec^4 x \tan^n x$ $= \int \sec^2 x \sec^2 x \tan^n x$ $= \int \sec^2 x\,(\tan^2 x + 1)\tan^n x$ $= \int \sec^2 x \tan^{n+2} x + \int \sec^2 x \tan^n x$ … Reduce until there's a $\sec^2 x$.	$\int \sec^n x \tan^3 x$ $= \int \sec^n x \tan x \tan^2 x$ $= \int \sec^2 x \tan x\,(\sec^2 x - 1)$ $= \int \sec^4 x \tan x - \int \sec^2 x \tan x$ … Reduce until there's a $\tan x$.

Cosecant/cotangent pretty much work the same way as secant/tangent.

$\csc^2 x \cot^n x$	$\csc^{n\geq 2} x \cot x$	$\csc^{even\,n} x \cot^n x$	$\csc^n x \cot^{odd\,n} x$
$\boxed{u = \cot x}$ $\frac{du}{dx} = -\csc^2 x$ $du = -\csc^2 x\,dx$ $\int \csc^2 x \cot^n x\,dx$ $= -\int u^n(-\csc^2 x\,dx)$ $= -\int u^n\,du$	$\boxed{u = \csc x}$ $\frac{du}{dx} = -\csc x \cot x$ $du = -\csc x \cot x\,dx$ $\int \csc^n x \cot x\,dx$ $= -\int u^{n-1}(-\csc x \cot x\,dx)$ $= -\int u^{n-1}\,du$	$\int \csc^4 x \cot^n x$ $= \int \csc^2 x \csc^2 x \cot^n x$ $= \int \csc^2 x\,(\cot^2 x + 1)\cot^n x$ $= \int \csc^2 x \cot^{n+2} x + \int \csc^2 x \cot^n x$ … Reduce until there's a $\csc^2 x$.	$\int \csc^n x \cot^3 x$ $= \int \csc^n x \cot x \cot^2 x$ $= \int \csc^2 x \cot x\,(\csc^2 x - 1)$ $= \int \csc^4 x \cot x - \int \csc^2 x \cot x$ … Reduce until there's a $\tan x$.

Note that not all integral cases are covered; it's because those cases are too hard.

eg. Evaluate $\int \tan^5 x \sec^2 x\,dx$	*These questions with sin/cos, sec/tan, and csc/cot, are fairly common on exams* *I skipped u-substitution steps here to save space. In each case, I set $u = \sec x$.*
$\int \tan^5 x \sec^2 x\,dx = \int \tan x \tan^4 x \sec^2 x\,dx$ $= \int \tan x\,(\sec^2 x - 1)^2 \sec^2 x\,dx$ $= \int \tan x \sec^2 x\,(\sec^4 x - 2\sec^2 x + 1)\,dx$ $= \int \tan x \sec^6 x\,dx - 2\int \tan x \sec^4 x\,dx + \int \tan x \sec^2 x\,dx$ $= \int u^5\,dx - 2\int u^3\,dx + \int u\,dx$ $= \frac{1}{6}u^6 - \frac{1}{2}u^4 + \frac{1}{2}u^2$ $= \boxed{\frac{1}{6}\sec^6 x - \frac{1}{2}\sec^4 x + \frac{1}{2}\sec^2 x}$	

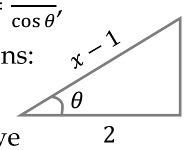
eg. Evaluate $\int \sin 2x \sin 3x \, dx$	*I haven't seen these question types on past exams, probably because it's too based on remembering formulae.*
$\int \sin 2x \sin 3x \, dx = \int \frac{1}{2}[\cos(2x-3x) - \cos(2x+3x)] \, dx$ $= \frac{1}{2}\int [\cos(-x) - \cos(5x)] \, dx$ $= \frac{1}{2}\int [\cos x - \cos(5x)] \, dx$ $= \boxed{\frac{1}{2}\left(\sin x - \frac{1}{5}\sin(5x)\right)}$	*Remember* $\cos(-\theta) = \cos(\theta)$

8.2.3. Trigonometric Substitution

My math prof says there's no practical point to this other than showing off that you can compute complicated integrals. Still, this might be on the exam, so be prepared.

Type 1. $\sqrt{a-x^2}$	**Type 2.** $\sqrt{a+x^2}$	**Type 3.** $\sqrt{x^2-a}$	**Type 4.** $\sqrt{ax^2+bx+c}$
$\sin^2 x + \cos^2 x = 1$ $1 - \sin^2 x = \cos^2 x$ $\boxed{\sqrt{1-(\sin x)^2} = \cos x}$ $\sqrt{a-x^2} = \sqrt{a}\sqrt{1-\left(\frac{x}{\sqrt{a}}\right)^2}$ Set $\boxed{\frac{x}{\sqrt{a}} = \sin\theta}$, then $\sqrt{a}\sqrt{1-(\sin\theta)^2}$ $= \sqrt{a}\cos\theta$ ...	$\tan^2 x + 1 = \sec^2 x$ $1 + \tan^2 x = \sec^2 x$ $\boxed{\sqrt{1+(\tan x)^2} = \sec x}$ $\sqrt{a+x^2} = \sqrt{a}\sqrt{1+\left(\frac{x}{\sqrt{a}}\right)^2}$ Set $\boxed{\frac{x}{\sqrt{a}} = \tan\theta}$, then $\sqrt{a}\sqrt{1-(\tan\theta)^2}$ $= \sqrt{a}\sec\theta$ ...	$\tan^2 x + 1 = \sec^2 x$ $\sec^2 x - 1 = \tan^2 x$ $\boxed{\sqrt{(\sec x)^2 - 1} = \tan x}$ $\sqrt{x^2-a} = \sqrt{a}\sqrt{\left(\frac{x}{\sqrt{a}}\right)^2 - 1}$ Set $\boxed{\frac{x}{\sqrt{a}} = \sec\theta}$, then $\sqrt{a}\sqrt{(\sec\theta)^2 - 1}$ $= \sqrt{a}\tan\theta$ ...	Complete the square. $\sqrt{ax^2+bx+c}$ $= \sqrt{a}\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}}$ $= \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}}$ $= \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}}$ Then do 1/2/3 trig substitution.

I suppose for type 1, you can also set $\frac{x}{a} = \cos\theta$ instead. Do what you prefer.

eg. Evaluate $\int \frac{1}{\sqrt{x^2-2x-3}}$.		
$\int \frac{1}{\sqrt{x^2-2x-3}}$ $= \int \frac{1}{\sqrt{(x-1)^2-4}}$ $= \int \frac{1}{\sqrt{4}\sqrt{\frac{(x-1)^2}{4} - 1}}$ $= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{x-1}{2}\right)^2 - 1}} dx \rightarrow (1)$ $= \frac{1}{2}\int \frac{2\sec\theta\tan\theta}{\sqrt{(\sec\theta)^2 - 1}} d\theta$ $= \int \frac{\sec\theta\tan\theta}{\tan\theta} d\theta$ $= \int \sec\theta \, d\theta$ $= \ln\lvert\tan\theta + \sec\theta\rvert \rightarrow (2)$ $= \ln\left\lvert\frac{\sqrt{x^2-2x-3}}{2} + \frac{x-1}{2}\right\rvert$ $= \boxed{{}^1/_2 \ln\left\lvert\sqrt{x^2-2x-3} + x - 1\right\rvert}$	(1): Set $\frac{x-1}{2} = \sec\theta$ $x = 2\sec\theta + 1$ $\frac{dx}{d\theta} = 2\sec\theta\tan\theta$ $dx = 2\sec\theta\tan\theta \, d\theta$ (2): Since $\frac{x-1}{2} = \sec\theta = \frac{1}{\cos\theta}$, $\cos\theta = \frac{2}{x-1}$, which means:  Since we know $\cos\theta$, we can use it to calculate $\tan\theta$. $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ $= \frac{\sqrt{(x-1)^2 - 2^2}}{2}$ $= \frac{\sqrt{x^2-2x-3}}{2}$	*We have type 3,* $\sqrt{x^2-a}$*, with* $\sqrt{(x-1)^2-4}$ *Differentiate on* θ *Remember that sine, cosine, and tangents are defined as <u>ratios of side lengths</u> of triangles. Since they're ratios, they're related.* *Figure out the adjacent's side length using the Pythagorean Theorem:* $a^2 + b^2 = c^2$ $2^2 + s^2 = (x-1)^2$ $s = \sqrt{(x-1)^2 - 2^2}$

8.2.4. Improper Integrals

Type 1: Infinite Bounds

When one of the integral's bounds → ∞,

$$\boxed{\int_a^\infty f(x) = \lim_{n\to\infty}\int_a^n f(x)}$$

Type 2: Infinite $f(x)$

When $f(x) \to \pm\infty$ at $x = b$,

$$\boxed{\int_a^b f(x) = \lim_{n\to b^-}\int_a^n f(x)}$$

For Type 1, it's a convention in MAT137 to add $\lim\limits_{n\to\infty}$ at the front; MAT157 doesn't do it. It's not a big deal.

eg. Evaluate $\int_1^\infty \frac{1}{x^2}dx$	
$$\begin{aligned}\int_1^\infty \frac{1}{x^2}dx &= \lim_{n\to\infty}\int_1^n \frac{1}{x^2}dx \\ &= \lim_{n\to\infty}[-x^{-1}]_1^n \\ &= \lim_{n\to\infty}\left(-\frac{1}{n}+1\right) \\ &= \boxed{1}\end{aligned}$$	*Type 1 improper integral.* *Since* $-\frac{1}{\infty}\to 0$

eg. Evaluate $\int_0^\infty \frac{1}{(x-1)^3}dx$	*Ensure there's one improper integral type per integral. This is why* $\int_1^\infty \frac{1}{(x-1)^3}$ *splits into* $\int_1^c \frac{1}{(x-1)^3}$ *and* $\int_c^\infty \frac{1}{(x-1)^3}$, *where* $c \in (1,\infty)$

$$\begin{aligned}\int_0^\infty \frac{1}{(x-1)^3} &= \int_0^1 \frac{1}{(x-1)^3} + \int_1^c \frac{1}{(x-1)^3} + \int_c^\infty \frac{1}{(x-1)^3} \\ &= \lim_{n\to1^-}\int_0^n (x-1)^{-3} + \lim_{n\to1^+}\int_n^c (x-1)^{-3} + \lim_{n\to\infty}\int_c^n (x-1)^{-3} \\ &= \lim_{n\to1^-}\left[\frac{-1}{2(x-1)^2}\right]_0^n + \lim_{n\to1^+}\left[\frac{-1}{2(x-1)^2}\right]_n^c + \lim_{n\to\infty}\left[\frac{-1}{2(x-1)^2}\right]_c^n \\ &= \lim_{n\to1^-}\frac{-1}{2(n-1)^2} + \lim_{n\to1^+}\left(\frac{-1}{2(c-1)^2}+\frac{1}{2(n-1)^2}\right) + \lim_{n\to\infty}\left(\frac{-1}{2(n-1)^2}+\frac{1}{2(c-1)^2}\right) \\ &= -\infty - \frac{1}{2(c-1)^2} + \infty - 0 + \frac{1}{2(c-1)^2} \\ &= \boxed{\pm\infty}\ (+\infty \text{ and } -\infty \text{ don't cancel, it diverges})\end{aligned}$$

Sometimes, we can't compute the integral, but we can test if it converges (ie. doesn't diverge).
Let f and g be two continuous functions. Then we can apply:

Comparison Test	**Limit Comparison Test**
If for $x \ge a$, $0 \le f(x) \le g(x)$, $$0 \le \int_a^\infty f(x) \le \int_a^\infty g(x)$$ $\boxed{\int_a^\infty f(x) \text{ diverges} \Rightarrow \int_a^\infty g(x) \text{ diverges}}$ $\boxed{\int_a^\infty g(x) \text{ converges} \Rightarrow \int_a^\infty g(x) \text{ converges}}$	If for $x \ge a$, $f(x) > 0$ and $g(x) > 0$, and $$\lim_{x\to\infty}\frac{f(x)}{g(x)} \neq 0 \neq \infty,$$ Then we can conclude $f(x) \approx g(x)$ and $$\boxed{\int_a^\infty f(x) \text{ converges} \Leftrightarrow \int_a^\infty g(x) \text{ converges}}$$

eg. Show whether $\int_1^\infty \frac{\cos x + 2\arctan x + 10}{x^2}dx$ converges/diverges.	
$$\begin{aligned}\int_1^\infty \frac{\cos x + 2\arctan x + 10}{x^2} &\le \int_1^\infty \frac{(1)+2\left(\frac{\pi}{2}\right)+10}{x^2} \\ &= (\pi+10)\int_1^\infty \frac{1}{x^2} \\ &= \pi + 10\end{aligned}$$ As $\int_1^\infty \frac{(1)+2(1)+10}{x^2} \neq \infty$, it converges, thus $\int_1^\infty \frac{\sin x + 2\cos x + 10}{x^2}$ converges.	*Comparison Test, using:* $-1 \le \cos x \le 1$ $-\frac{\pi}{2} \le \arctan x \le \frac{\pi}{2}$ *We already found* $\int_1^\infty \frac{1}{x^2}$ *two questions above this one.*

eg. Show whether $\int_{10}^{\infty} \frac{\sqrt{x}-6}{3x^2+5x+11} dx$ converges/diverges.	*Limit Comparison Test, with:* $\frac{\sqrt{x}-6}{3x^2+5x+11}$ *vs.* $\frac{\sqrt{x}}{x^2}$
$$\lim_{x\to\infty}\left(\frac{\sqrt{x}-6}{3x^2+5x+11} \div \frac{\sqrt{x}}{x^2}\right) = \lim_{x\to\infty}\left(\frac{\sqrt{x}-6}{\sqrt{x}} \cdot \frac{3x^2+5x+11}{x^2}\right) = \lim_{x\to\infty}\frac{\sqrt{1-\frac{6}{x}}}{\sqrt{1}} \cdot \lim_{x\to\infty}\frac{3+\frac{5}{x}+\frac{11}{x^2}}{1} = 3 \neq 0 \neq \infty$$ $$\therefore \int_{10}^{\infty} \frac{\sqrt{x}-6}{3x^2+5x+11} dx \approx \int_{10}^{\infty} \frac{\sqrt{x}}{x^2} dx$$ $$\int_{10}^{\infty} x^{-\frac{3}{2}} dx = \lim_{n\to\infty}\int_{10}^{n} x^{-\frac{3}{2}} dx = \lim_{n\to\infty}\left[-2x^{-\frac{1}{2}}\right]_{10}^{n} = \lim_{n\to\infty}\left(-\frac{2}{\sqrt{n}}+\frac{2}{\sqrt{10}}\right) = \frac{2}{\sqrt{10}} \neq \infty$$ Since $\int_{10}^{\infty} \frac{\sqrt{x}}{x^2} dx$ converges, $\int_{10}^{\infty} \frac{\sqrt{x}-6}{3x^2+5x+11} dx$ converges.	*Note that I got rid of every term that wasn't the function's highest degree. The function on the right is much easier to calculate than the function on the left.*

<u>P-Series</u>

This is a very important class of functions common in sequences and series. For any $a \in \mathbb{R}$,

$$\boxed{\int_a^{\infty} \frac{1}{x^p}} = \lim_{n\to\infty}\int_a^n x^{-p} = \lim_{n\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_a^n = \lim_{n\to\infty}\left(\frac{n^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right)$$

$p > 1 \Rightarrow$ Converges	(derive from $-p+1 < 0$)
$p \leq 1 \Rightarrow$ Diverges	(derive from $-p+1 > 0$)

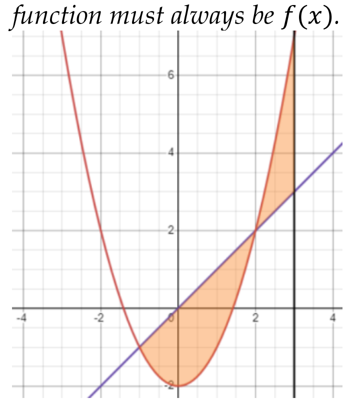
Note that when $p = 1$, the integral becomes $\lim_{n\to\infty}\int_a^n \frac{1}{x} = \lim_{n\to\infty}[\ln x]_a^n = \lim_{n\to\infty}(\ln n - \ln a) = \infty$, which <u>diverges</u>.

8.3. Application

8.3.1. Area

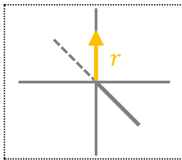
The area of a region under $f(x)$ and above $g(x)$ in interval $[a, b]$ is $\boxed{A = \int_a^b f(x) - g(x)}$

- $A > 0 \Leftrightarrow f(x) > g(x)$
- $A < 0 \Leftrightarrow f(x) < g(x)$
- Nothing changes when $f(x) < 0$ or $g(x) < 0$. The above two rules still hold true.

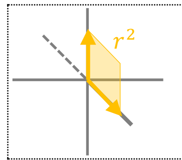
eg. Find the area of all regions bounded by $y = x$, $y = x^2 - 2$, and $x = 3$.	*First find where the two main functions intersect so we know the upper/lower bounds.* *$x = 3$ tells us upper limit is 3.*
$$x = x^2 - 2$$ $$x^2 - x - 2 = 0$$ $$(x+1)(x-2) = 0$$ $$x = -1, 2$$ In $[-1,2]$, $x > x^2 - 2$. In $[2,3]$, $x^2 - 2 > x$. $$\therefore A = \int_{-1}^{2} x - (x^2-2) + \int_{2}^{3} (x^2-2) - x = \int_{-1}^{2} -x^2 + x + 2 + \int_{2}^{3} x^2 - x - 2 = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right]_{-1}^{2} + \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right]_{2}^{3} = \left(-\frac{8}{3}+2+4\right) - \left(\frac{1}{3}+\frac{1}{2}-2\right) + \left(9-\frac{9}{2}-6\right) - \left(\frac{8}{3}-2-4\right) = \frac{10}{3} + \frac{7}{6} - \frac{3}{2} + \frac{10}{3} = \frac{19}{3}$$	*We're finding <u>true area</u>, so in $A = \int_a^b f(x) - g(x)$, the upper function must always be $f(x)$.* 

8.3.2. Volume

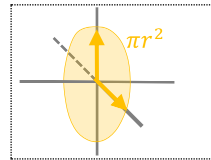
If we take a 1D line of length r,



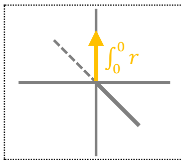
square it,



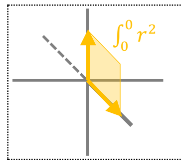
and multiply π, we get a 2D circle:



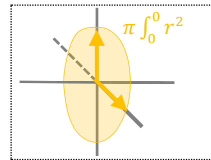
Think of this line as a very thin "slice" of 2D area $\int_0^0 r$,



square it,



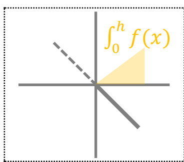
and multiply π for a "3D" shape:



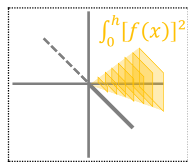
I'm not being "mathematically correct" here; $\int_0^0 f(x) = 0$ and it's not a proper integral. I just want you to intuitively compare a hypothetical infinitesimally-thin "slice" of 2D area to a 1D line.

What if we chose a $f(x)$ and different integrals bounds?

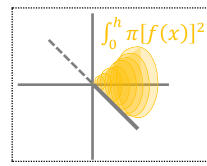
Let $f(x) = \frac{r}{h}x$. If we take the 2D area $\int_0^h f(x)$,



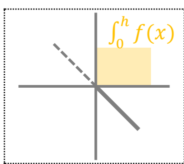
split it into infinitely many "slices" and square each slice,



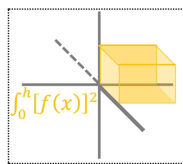
and multiply π, we get a cone:



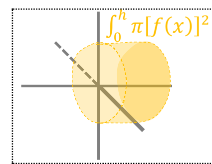
Let $f(x) = r$. If we take the 2D area $\int_0^h f(x)$,



split it into infinitely many "slices" and square each slice,



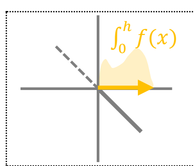
and multiply π, we get a cylinder:



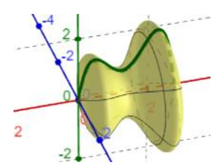
If we try calculating these integrals, we get:

- $\int_0^h \pi[f(x)]^2 = \int_0^h \pi\left(\frac{r}{h}x\right)^2 = \frac{\pi r^2}{h^2}\int_0^h x^2 = \frac{\pi r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h = \boxed{\frac{1}{3}\pi r^2 h}$ which is the volume of a cone.
- $\int_0^h \pi[f(x)]^2 = \int_0^h \pi(r)^2 = \pi r^2 \int_0^h 1 = \pi r^2[x]_0^h = \boxed{\pi r^2 h}$ which is the volume of a cylinder.

It thus follows that if we take the 2D area $\int_0^h f(x)$ for any $f(x)$,

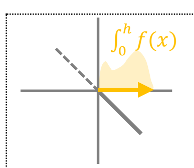


split it into infinite slices, square the slices, and multiply π to the whole thing, we get the shape:

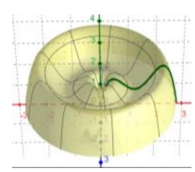


We're taking a 2D area and "rotating" it around the x-axis to produce a 3D shape, the **surface of revolution**. What if we rotated an area around the y-axis instead?

We'd take our 2D area $\int_0^h f(x)$,

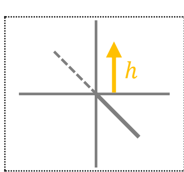


and rotate it around the y-axis to obtain the shape:

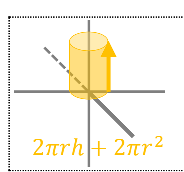


The derivation for the formula of y-axis rotation is trickier to visualize.

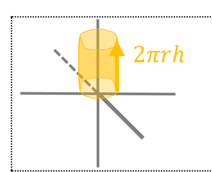
Take a 1D line of length h at $x = r$,



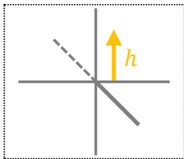
find the surface area of a cylinder at that position,



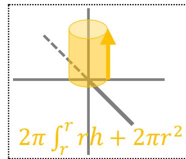
and get rid of the two circles' areas.



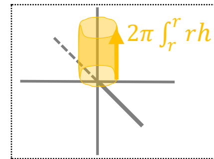
Think of this line as a very thin "slice" of 2D area $\int_r^r h$,



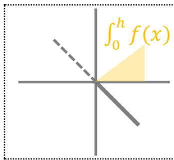
find the surface area of a cylinder at that position,



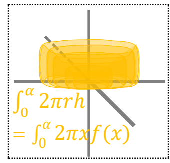
and get rid of the two circles' areas.



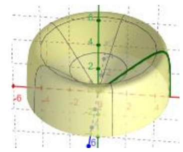
Let $f(x) = cx$. If we take the 2D area $\int_0^\alpha f(x)$,



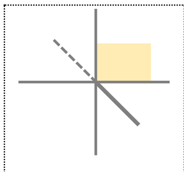
split it into infinitely many "slices" and find surface areas,



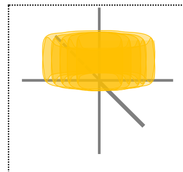
The infinitely many surface areas "add up" to a volume.



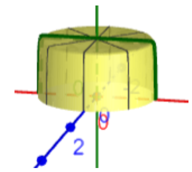
Let $f(x) = r$. If we take the 2D area $\int_0^h f(x)$,



split it into infinitely many "slices" and find surface areas,



The infinitely many surface areas "add up" to a volume.



In conclusion, these are the surface of revolutions of $f(x)$ or $f(y)$ on interval $[a, b]$:

x-axis Rotation	y-axis Rotation
$V = \int_a^b \pi r^2 = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b 2\pi rh = \int_a^b 2\pi x f(x)\,dx$
$V = \int_a^b 2\pi rh = \int_a^b 2\pi y f(y)\,dy$	$V = \int_a^b \pi r^2 = \int_a^b \pi[f(y)]^2\,dy$

Some questions are defined such that you must write them like $f(y)$ and thus use the alternate formulas.

This whole section about finding 3D volume in a 1D Calculus course is weird. My math prof said something like he thinks this content is here "because many people don't go on to take MAT235/237/257, so the department needs to cram as much content as possible in 137 but also make it understandable."

eg. What is the volume of the 3D shape formed by rotating around the y-axis the region bounded by $y = x^2$ and $y = x$?

(From MAT137, August exam 2017)

$$x = x^2$$
$$x(x-1) = 0$$
$$x = 0,1$$

On $[0,1]$, $x > x^2$

$$V = \int_0^1 2\pi x f(x)\,dx$$
$$= 2\pi \int_0^1 x(x - x^2)\,dx$$
$$= 2\pi \int_0^1 (x^2 - x^3)\,dx$$
$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1$$
$$= \pi\left(\frac{1}{3} - \frac{1}{4}\right)$$
$$= \boxed{\frac{\pi}{12}}$$

In questions like these that combine area with volume, treat $x - x^2$ as function on its own and substitute it into $f(x)$.

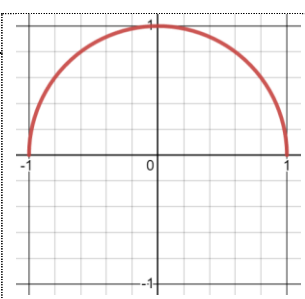
eg. Derive the volume of a sphere using integration.

The equation for a circle of radius r is $x^2 + y^2 = r^2$. Isolate y to get

$$y = \sqrt{r^2 - x^2}$$

The domain of thus function is $[-r, r]$. Let's revolve this around the x-axis.

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_{-r}^r \pi\left(\sqrt{r^2 - x^2}\right)^2 dx$$
$$= \pi \int_{-r}^r (r^2 - x^2)\,dx$$
$$= \pi\left[r^2 x - \frac{1}{3}x^3\right]_{-r}^r$$
$$= \pi\left(r^3 - \frac{1}{3}r^3 + r^3 - \frac{1}{3}r^3\right)$$
$$= \boxed{\frac{4}{3}\pi r^3}$$



This is $y = \sqrt{1 - x^2}$.

9. Sequences and Series

9.1. Sequences

9.1.1. Definition

A sequence $S = \{a_n\}_{n=1}^{\infty}$is a set. It's basically just like $f(x)$ but with a domain of $\mathbb{N}$.

S is increasing $\Leftrightarrow \forall n \in \mathbb{N}, a_{n+1} > a_n$
S is non-decreasing $\Leftrightarrow \forall n \in \mathbb{N}, a_{n+1} \geq a_n$
S is decreasing $\Leftrightarrow \forall n \in \mathbb{N}, a_{n+1} < a_n$
S is non-decreasing $\Leftrightarrow \forall n \in \mathbb{N}, a_{n+1} \leq a_n$

S is bounded above $\Leftrightarrow \exists M \in \mathbb{R}, \forall n \in \mathbb{N}, a_n < M$
S is bounded below $\Leftrightarrow \exists M \in \mathbb{R}, \forall n \in \mathbb{N}, a_n > M$

S converges (to L) $\Leftrightarrow \lim\limits_{n\to\infty} a_n$ exists (= L)
S diverges $\Leftrightarrow \lim\limits_{n\to\infty} a_n$ DNE

S is bounded above and S is non-decreasing $\Rightarrow$ S converges to $\sup S$
S is bounded below and S is non-increasing $\Rightarrow$ S converges to $\inf S$

a_n's initial behaviour is irrelevant to $\lim\limits_{n\to\infty} a_n$. • eg. $a_n = \{203, -0.31, \frac{\pi}{3}, 0.9, 0.99, 0.999, \ldots\}$, $\lim\limits_{n\to\infty} a_n$ still converges at 1.	a_n can be bounded above/below and still diverge. • eg. $a_n = (-1)^n = \{1, -1, 1, -1, \ldots\}$

Growth Rate Hierarchy

$$1 \ll \log_a n \ll \sqrt[a]{n} \ll n \ll n^a \ll a^n \ll n! \ll n^n$$

This relates the convergence of functions:

$$a_n \ll b_n \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

The a's are unique; they can be any constant.

Note: In CS, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0 \Leftrightarrow a_n$ is "Big-O" of b_n.
This has to do with a function's running-time.

You may use the growth rate hierarchy for integrals, sequences, and series, but NOT for computing limits (otherwise, you might lose marks), but it is a handy tool to check your work.

eg. Find the infimum and supremum of $\left\{(-1)^n\left(1-\frac{1}{n}\right) : n \in \mathbb{N}\right\}$

$\left(1-\frac{1}{n}\right)$: $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7} \ldots$
You can do limit and show it converges to 1.
$(-1)^n$: $1, -1, 1, -1, 1, -1, \ldots$
This oscillates endlessly between -1 and 1.

$\therefore$ The infimum of $(-1)^n\left(1-\frac{1}{n}\right)$ is $(-1)(1) = -1$
$\therefore$ The supremum of $(-1)^n\left(1-\frac{1}{n}\right)$ is $(1)(1) = 1$

eg. The sequence $\{a_n\}_{n=1}^{\infty}$ is **eventually bounded above** when $\exists m \in \mathbb{N}$ such that $\{a_n\}_{n=m}^{\infty}$ is bounded above. Prove that if a sequence is eventually bounded above, then it is bounded above.

(From MAT137, April exam 2017)

Assume $\{a_n\}_{n=1}^{\infty}$ is eventually bounded above, meaning
$\exists m \in \mathbb{N}, \{a_n\}_{n=m}^{\infty} = \{a_m, a_{m+1}, a_{m+2}, \ldots\}$ is bounded above
Show $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$ is bounded above.

I'm not sure if this answer is valid since I never did this type of proof in my class. But it makes sense.

Since $\{a_n\}_{n=1}^{m-1}$ is a finite set of numbers, $\max\{a_n\}_{n=1}^{m-1}$ exists
Thus $\{a_n\}_{n=1}^{m-1}$ is bounded above by $\max\{a_n\}_{n=1}^{m-1}$

Think of this kind of like EVT.

We know $\{a_n\}_{n=m}^{\infty}$ is bounded above, meaning $\sup\{a_n\}_m^{\infty}$ exists.

Since $\forall x \in \{a_n\}_{n=1}^{m-1}, x \leq \max\{a_n\}_{n=1}^{m-1}$ and
$\forall x \in \{a_n\}_{n=m}^{\infty}, x \leq \sup\{a_n\}_{n=m}^{\infty}$,
Thus $\forall x \in \{a_n\}_{n=1}^{\infty}, x \in \max\{\max\{a_n\}_{n=1}^{\infty}, \sup\{a_n\}_{n=1}^{m-1}\}$
$\therefore \{a_n\}_{n=1}^{\infty}$ is bounded above by $\max\{\max\{a_n\}_{n=1}^{\infty}, \sup\{a_n\}_{n=1}^{m-1}\}$ ■

Expand the definition of "bounded above". I used different variable names.

9.1.2. Proving Convergence

eg. Prove the recursively-defined sequence $a_1 = 1, a_{n+1} = \frac{1}{2}a_n + 1$ converges to L, and evaluate L.

Rough Work:

Let's assume a_n converges to L, meaning $\lim_{n\to\infty} a_n = L$

$$a_{n+1} = {}^1\!/_2\, a_n + 1$$

$$\lim_{n\to\infty} a_{n+1} = {}^1\!/_2 \lim_{n\to\infty} a_n + 1$$

Since $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_{n+1} = L$,

$$L = {}^1\!/_2\, L + 1$$

$$^1\!/_2\, L = 1$$

$$L = 2$$

If you don't get an answer here, the proof actually gets easier. Just prove that $\lim_{n\to\infty} a_n$ DNE $\Leftrightarrow$ S diverges.

Show S is bounded above, meaning $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, a_n \leq m$

Pick $m = 2$

Proof by induction, let $n \in \mathbb{N}$

<u>Base Case:</u> $n = 1$

$a_1 = 1 \leq 2$

<u>Inductive Step:</u> $a_n \leq 2 \Rightarrow a_{n+1} \leq 2$

Assume $a_n \leq 2$

Show $a_{n+1} \leq 2$

$a_{n+1} = \frac{1}{2}a_n + 1 \leq \frac{1}{2}(2) + 1 = 2$

Show S is non-decreasing, meaning $\forall n \in \mathbb{N}, a_{n+1} \geq a_n$

Proof by induction, let $n \in \mathbb{N}$

<u>Base Case:</u> $n = 1$

$a_2 = \frac{1}{2}a_1 + 1 = \frac{1}{2}(1) + 1 = \frac{3}{2} > 1 = a_1$

<u>Inductive Step:</u> $a_{n+1} \geq a_n \Rightarrow a_{n+2} \geq a_{n+1}$

Assume $a_{n+1} \geq a_n$

Show $a_{n+2} \geq a_{n+1}$

$a_{n+2} = \frac{1}{2}a_{n+1} + 1 \geq \frac{1}{2}a_n + 1 = a_{n+1}$

Since S is bounded above and non-decreasing, it converges at L, meaning $\lim_{n\to\infty} a_n = L$

$$a_{n+1} = {}^1\!/_2\, a_n + 1$$

$$\lim_{n\to\infty} a_{n+1} = {}^1\!/_2 \lim_{n\to\infty} a_n + 1$$

Since $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_{n+1} = L$,

$$L = {}^1\!/_2\, L + 1$$

$$^1\!/_2\, L = 1$$

$$L = 2$$

The goal is to follow the statement

$$\left.\begin{array}{l}\textit{S is bounded above}\\ \textit{S is non-decreasing}\end{array}\right. \Rightarrow \textit{S converges}$$

Prove the premises with induction

This logic is tricky to follow at first, but all other proofs will use this exact same logic.

Copy your rough work here.

Depending on S, you may need to follow this statement instead:

$$\left.\begin{array}{l}\textit{S is bounded below}\\ \textit{S is non-increasing}\end{array}\right. \Rightarrow \textit{S converges}$$

9.2. Series

9.2.1. Definition

A series is the sum of a sequence.

$$\sum_{n=1}^{\infty} a_n$$

It can also be thought of as a sequence of **partial sums** (ie. the sum of a finite part of a sequence)

$$\{s_n\}_{n=1}^{\infty} = \left\{\sum_{i=1}^{n} a_i\right\}_{n=1}^{\infty} = \{a_1, (a_1 + a_2), (a_1 + a_2 + a_3), \dots\}$$

$$\text{Series converges} \Leftrightarrow \{s_n\}_{n=1}^{\infty} \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} a_n = L$$

Just like with sequences,

- Initial behaviour doesn't affect a series' convergence. $\boxed{\sum_{n=\alpha}^{\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=\beta}^{\infty} a_n \text{ converges}}$
- A series can diverge even if it is bounded.

If a series converges, we can do algebra on it: $\displaystyle\sum_{n=a}^{\infty} a_n + \sum_{n=a}^{\infty} b_n = \sum_{n=a}^{\infty}(a_n + b_n)$ $\quad$ $\displaystyle c\sum_{n=a}^{\infty} a_n = \sum_{n=a}^{\infty} ca_n$

9.2.2. Convergence Tests

Usually, series are too difficult to compute; the focus is on testing them for whether they converge/ diverge. Different tests are ideal for different input classes. Examples for these tests are on next page.

Test	Description
Zero Test	$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=k}^{\infty} a_n$ diverges
Integral Test *(P-series)* *(Of the form $\frac{1}{n(\ln n)^a}$)*	When $a_n \geq 0$, decreasing eventually, $\sum_{n=k}^{\infty} a_n \approx \int_k^{\infty} a_n$ $\int_k^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=k}^{\infty} a_n$ converges
Limit Comparison Test *(Rational functions)*	When $a_n, b_n \geq 0$, both decreasing eventually, calculate $\lim_{n\to\infty} \frac{a_n}{b_n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} \neq 0 \neq \infty \Rightarrow \sum_{n=k}^{\infty} a_n \approx \sum_{n=k}^{\infty} b_n$ $\sum_{n=k}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=k}^{\infty} b_n$ converges
Comparison Test *(Trigonometric functions)* *(Logarithmic functions)*	When $a_n \leq b_n$, both decreasing eventually, $\sum_{n=k}^{\infty} a_n \leq \sum_{n=k}^{\infty} b_n$ $\sum_{n=k}^{\infty} b_n$ converges $\Rightarrow \sum_{n=k}^{\infty} a_n$ converges $\quad$ $\sum_{n=k}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=k}^{\infty} b_n$ diverges
Ratio Test *(Factorials)* *(Exponentials)*	When $a_n \geq 0$, decreasing eventually, calculate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ $L < 1 \Rightarrow \sum_{n=k}^{\infty} a_n$ converges $\quad$ $L > 1 \Rightarrow \sum_{n=k}^{\infty} a_n$ diverges $L = 1 \Rightarrow$ inconclusive
Absolute Convergence Test *(Has negative terms)*	$\sum_{n=k}^{\infty} \lvert a_n \rvert$ converges $\Rightarrow \sum_{n=k}^{\infty} a_n$ converges
Alternating Series Test *(Has alternating +/- terms)*	When $a_n \geq 0$, decreasing eventually, $\sum_{n=k}^{\infty} (-1)^n a_n$ converges $\Leftrightarrow \lim_{n\to\infty} a_n = 0$

Every MAT137 exam will ask you to perform convergence tests. There are three possible outcomes:

- A series is **divergent** when $\sum_{n=a}^{\infty} a_n$ diverges.
- A series is **conditionally convergent** when $\sum_{n=a}^{\infty}|a_n|$ diverges but $\sum_{n=a}^{\infty} a_n$ converges.
- A series is **absolutely convergent** when $\sum_{n=a}^{\infty}|a_n|$ converges (which implies $\sum_{n=a}^{\infty} a_n$ also converges).

To save doing extra work, <u>test $|a_n|$ first</u> unless you already know a_n diverges.

- If $|a_n|$ converges, then a_n is absolutely convergent
- If $|a_n|$ diverges, then test a_n. If it converges, then a_n is conditionally convergent

eg. Perform a convergence test on $\sum_{n=420}^{\infty} 1$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1 \neq 0$$
$$\therefore \sum_{n=420}^{\infty} 1 \textbf{ diverges}$$

Zero Test.

eg. Perform a convergence test on $\sum_{n=69}^{\infty} \frac{\sqrt{3n^3+2n}}{n^2-1}$

Let $b_n = \frac{\sqrt{3n^3}}{n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty}\left(\frac{\sqrt{3n^3+2n}}{n^2-1} \div \frac{\sqrt{3n^3}}{n^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{3n^3+2n}}{n^2-1} \cdot \frac{n^2}{\sqrt{3n^3}}\right) \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{3n^3+2n}}{\sqrt{3n^3}} \cdot \frac{n^2}{n^2-1}\right) \\
&= \lim_{n\to\infty}\left(\frac{\sqrt{3+\frac{2}{n^2}}}{\sqrt{3}} \cdot \frac{1}{1-\frac{1}{n^2}}\right) \\
&= \frac{\sqrt{3}}{\sqrt{3}} \cdot \frac{1}{1} = 1 \neq 0 \neq \infty
\end{aligned}$$

$$\begin{aligned}
\therefore \sum_{n=69}^{\infty} a_n &\approx \sum_{n=69}^{\infty} b_n \\
&= \sum_{n=69}^{\infty} \frac{\sqrt{3n^3}}{n^2} \\
&= \sqrt{3}\sum_{n=69}^{\infty} \frac{1}{n^{\frac{1}{2}}} \approx \sqrt{3}\int_{69}^{\infty} \frac{1}{x^{\frac{1}{2}}}
\end{aligned}$$

$\int_{69}^{\infty} \frac{1}{x^{\frac{1}{2}}}$ is part of the p-series, with $p = \frac{1}{2} \leq 1$, thus $\int_{69}^{\infty} \frac{1}{x^{\frac{1}{2}}}$ diverges

Since $\int_{69}^{\infty} \frac{1}{x^{\frac{1}{2}}}$ diverges, $\sum_{n=69}^{\infty} \frac{\sqrt{3n^3}}{n^2}$ diverges

Since $\sum_{n=69}^{\infty} \frac{\sqrt{3n^3}}{n^2}$ diverges, $\sum_{n=69}^{\infty} \frac{\sqrt{3n^3+2n}}{n^2-1}$ **diverges**

Limit Comparison Test.

Pick b_n to be simpler. Get rid of the secondary terms in the numerator and denominator. <u>*b_n should be in the p-series.*</u>

On the exam, convergence test questions don't require you to show any work.

Integral Test.

Having a constant in front of the p-series doesn't change whether it converges/diverges

eg. Perform a convergence test on $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^3}$

Special Case!

$$\sum_{n=1}^{\infty}\left|\frac{1}{n(\ln n)^3}\right| \approx \int_{1}^{\infty}\left|\frac{1}{x(\ln x)^3}\,dx\right|$$

Set $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$ and $dx = x\,du$

$$\begin{aligned}
\int_{1}^{\infty}\left|\frac{1}{x(\ln x)^3}\,dx\right| &= \int_{1}^{\infty}\left|\frac{x}{xu^3}\,du\right| \\
&= \int_{1}^{\infty}\left|\frac{du}{u^3}\right| \\
&= \lim_{n\to\infty}\left[\left|-\frac{1}{2}u^{-2}\right|\right]_1^n \\
&= \lim_{n\to\infty}\left(\frac{1}{2n^2}+\frac{1}{2}\right) \\
&= \frac{1}{2} \neq \infty
\end{aligned}$$

Since $\int_{1}^{\infty}\left|\frac{1}{x(\ln x)^3}\right| dx$ converges, $\sum_{n=1}^{\infty}\left|\frac{1}{n(\ln n)^3}\right|$ converges.

Thus $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^3}$ is **absolutely convergent**.

Integral Test on $|a_n|$.

I'm not checking the tests' preconditions because as long as you have no trig in it, your function is probably positive and eventually decreasing.

Absolute Convergence Test

eg. Perform a convergence test on $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$	
$\sum_{n=1}^{\infty}\left\lvert\frac{\sin n}{n^2}\right\rvert \le \sum_{n=1}^{\infty}\left\lvert\frac{1}{n^2}\right\rvert = \sum_{n=1}^{\infty}\frac{1}{n^2}$	*Comparison Test, using the fact that* $-1 \le \sin n \le 1$.
$\sum_{n=1}^{\infty}\frac{1}{n^2} \approx \int_1^{\infty}\frac{1}{x^2} = \int_1^{\infty}\frac{1}{x^2}$	*Integral Test.*
$\int_1^{\infty}\frac{1}{x^2}$ is part of the p-series, with $p = 2 > 1$, thus $\int_1^{\infty}\frac{1}{x^2}$ converges	
Since $\int_1^{\infty}\frac{1}{x^2}$ converges, $\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\left\lvert\frac{1}{n^2}\right\rvert$ converges	
Since $\sum_{n=1}^{\infty}\left\lvert\frac{1}{n^2}\right\rvert$ converges, $\sum_{n=1}^{\infty}\left\lvert\frac{\sin n}{n^2}\right\rvert$ converges	
Since $\sum_{n=1}^{\infty}\left\lvert\frac{\sin n}{n^2}\right\rvert$ converges, $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ is **absolutely convergent**.	*Absolute Convergence Test*

eg. Perform a convergence test on $\sum_{n=1}^{\infty}(-1)^n \sin\frac{1}{n}$	
$\sum_{n=1}^{\infty}\left\lvert(-1)^n \sin\frac{1}{n}\right\rvert = \sum_{n=1}^{\infty}\sin\frac{1}{n}$	
$\sum_{n=1}^{\infty}\sin\frac{1}{n} \approx \sum_{n=1}^{\infty}\frac{1}{n} \approx \int_1^{\infty}\frac{1}{x}$	*Integral Test.*
$\int_1^{\infty}\frac{1}{x}$ is part of the p-series, with $p = 1 \le 1$, thus $\int_1^{\infty}\sin\frac{1}{x}$ diverges	$\lvert a_n\rvert$ *diverges*
Since $(-1)^n \sin\frac{1}{n}$ is of the form $(-1)^n a_n$ (where $a_n = \sin\frac{1}{n}$),	*Alternating Series Test*
$\lim_{n\to\infty}\sin\frac{1}{n} \approx \lim_{n\to\infty}\frac{1}{n} = 0$	
Thus $\sum_{n=1}^{\infty}(-1)^n \sin\frac{1}{n}$ is **conditionally convergent.**	a_n *converges*

My MAT137 prof used small angle approximation to get $\sum_{n=1}^{\infty}\sin\frac{1}{n} \approx \sum_{n=1}^{\infty}\frac{1}{n}$, but I feel it's still kind of a logical leap from $\lim_{x\to\infty}\sin\frac{1}{x} = \lim_{x\to\infty}\frac{1}{x}$. I don't really know why it follows, but okay.

eg. Perform a convergence test on $\sum_{n=230411}^{\infty}\frac{(n+1)2^{3n}(n)!}{(2n)!}$	
$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert = \lim_{n\to\infty}\left\lvert\frac{((n+1)+1)2^{3(n+1)}(n+1)!}{(2(n+1))!} \div \frac{(n+1)2^{3n}(n)!}{(2n)!}\right\rvert$	*Ratio Test*
$= \lim_{n\to\infty}\left\lvert\frac{(n+2)2^{3n+3}(n+1)!}{(2n+2)!}\cdot\frac{(2n)!}{(n+1)2^{3n}(n)!}\right\rvert$	
$= \lim_{n\to\infty}\left\lvert\frac{n+2}{n+1}\cdot\frac{2^{3n+3}}{2^{3n}}\cdot\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2n+2)!}\right\rvert$	*Remember for factorials:*
$= \lim_{n\to\infty}\left\lvert\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\cdot\frac{2^{3n}\cdot 2^3}{2^{3n}}\cdot\frac{(n+1)n!}{n!}\cdot\frac{(2n)!}{(2n+2)(2n-1)(2n)!}\right\rvert$	$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$ $= 5\cdot(4\cdot 3\cdot 2\cdot 1)$ $= 5\cdot 4!$
$= \lim_{n\to\infty}\left\lvert\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\cdot\frac{2^{3n}\cdot 2^3}{2^{3n}}\right\rvert\cdot\lim_{n\to\infty}\left\lvert\frac{(n+1)n!}{n!}\cdot\frac{(2n)!}{(2n+2)(2n-1)(2n)!}\right\rvert$	*Thus it follows that* $(n+1)! = (n+1)(n)(n-1)\ldots(1)$
$= \lvert 1\cdot 2^3\rvert\cdot\lim_{n\to\infty}\left\lvert\frac{n+1}{(2n+2)(2n-1)}\right\rvert$	$= (n+1)[(n)(n-1)\ldots(1)]$ $= (n+1)n!$
$= 2^3\cdot\lim_{n\to\infty}\left\lvert\frac{1+\frac{1}{n}}{\left(2+\frac{2}{n}\right)(2n-1)}\right\rvert$	
$= 2^3\cdot 0$	
$= 0 < 1$	
Thus $\sum_{n=230411}^{\infty}\frac{(n+1)2^{3n}(n)!}{(2n)!}$ is **absolutely convergent.**	*Absolute Convergence Test*

eg. Perform a convergence test on $\sum_{n=1}^{\infty} \frac{\sqrt{n}\ln n}{(n+1)^2}$

Rough Work:

When you spot a logarithm, ignore it for now. If the series still converges, then the series including the logarithm converges.

$$\sum_{n=1}^{\infty}\left|\frac{\sqrt{n}}{(n+1)^2}\right| = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{(n+1)^2}$$

Since the domain of $\frac{\sqrt{n}}{(n+1)^2}$ *is* $x \geq 0$

Do Limit Comparison Test:

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{(n+1)^2} \approx \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

Do Integral Test and then p-series, realize this converges. Now add the logarithm back.

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^{3/2}}$$

Use the ***Growth Rate Hierarchy****:* $\log_a n \ll \sqrt[a]{n}$*. Do Comparison Test:*

$$\sum_{n=1}^{\infty}\frac{\ln n}{n^{3/2}} \leq \sum_{n=1}^{\infty}\frac{n^{1/a}}{n^{3/2}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}-\frac{1}{a}}}$$

We need to choose a such that $\frac{3}{2}-\frac{1}{a} > 1$ *so we get a converging result in the p-series.*

AKA $a > 2$

Since $\log_a n \ll \sqrt[a]{n}$, $\ln n \ll \sqrt[3]{n}$.

I chose $a = 3$

$$\sum_{n=1}^{\infty}\left|\frac{\sqrt{n}\ln n}{(n+1)^2}\right| = \sum_{n=1}^{\infty}\frac{\sqrt{n}\ln n}{(n+1)^2} \leq \sum_{n=1}^{\infty}\frac{\sqrt{n}\cdot\sqrt[3]{n}}{(n+1)^2}$$
$$= \sum_{n=1}^{\infty}\frac{n^{5/6}}{(n+1)^2}$$

Comparison Test

Let $b_n = \frac{n^{5/6}}{n^2}$.
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{n^{5/6}}{(n+1)^2} \div \frac{n^{5/6}}{n^2}\right)$$
$$= \lim_{n\to\infty}\left(\frac{n^{5/6}}{(n+1)^2}\cdot\frac{n^2}{n^{5/6}}\right)$$
$$= \lim_{n\to\infty}\left(\frac{n^2}{n^2+2n+1}\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{2}{n}+\frac{1}{n^2}}\right)$$
$$= 1 \neq 0 \neq \infty$$

Limit Comparison Test

$$\therefore \sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{\infty} b_n$$
$$= \sum_{n=1}^{\infty}\frac{n^{5/6}}{n^2}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^{7/6}} \approx \int_1^{\infty}\frac{1}{x^{7/6}}$$

$\int_1^{\infty}\frac{1}{x^{7/6}}$ is part of the p-series, with $p = \frac{7}{6} > 1$, thus $\int_1^{\infty}\frac{1}{x^{7/6}}$ converges

Since $\int_1^{\infty}\frac{1}{x^{7/6}}$ converges, $\sum_{n=1}^{\infty}\frac{1}{n^{7/6}}$ converges

Integral Test

Since $\sum_{n=1}^{\infty}\frac{1}{n^{7/6}} = \sum_{n=1}^{\infty}\frac{n^{5/6}}{n^2}$ converges, $\sum_{n=1}^{\infty}\frac{n^{5/6}}{(n+1)^2}$ converges

Since $\sum_{n=1}^{\infty}\frac{n^{5/6}}{(n+1)^2}$ converges, $\sum_{n=1}^{\infty}\frac{\sqrt{n}\ln n}{(n+1)^2} = \sum_{n=1}^{\infty}\left|\frac{\sqrt{n}\ln n}{(n+1)^2}\right|$ converges

Thus $\sum_{n=1}^{\infty}\frac{\sqrt{n}\ln n}{(n+1)^2}$ is **absolutely convergent**.

eg. Perform a convergence test on $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n^{1.4}(\ln n)^2}$

Rough Work:

Do Limit Comparison Test:

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n^{1.4}} \approx \sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n^{1.4}} = \sum_{n=1}^{\infty}\frac{1}{n^{0.9}}$$

Do Integral Test and then p-series, realize this diverges. Now add the logarithm back and use the Growth Rate Hierarchy's $\log_a n \ll n^a$ *plus Comparison Test.*

$$\sum_{n=1}^{\infty}\frac{1}{n^{0.9}(\ln n)^2} \geq \sum_{n=1}^{\infty}\frac{1}{n^{0.9}(n^a)^2} = \sum_{n=1}^{\infty}\frac{1}{n^{0.9+2a}}$$

This means we need to choose a such that $0.9 + 2a \leq 1$ *so we get a diverging p-series.*

Since $(\ln n)^2$ *is on the denominator, if* $\ln n \ll n^a$*, then* $\frac{1}{\ln n} \gg \frac{1}{n^a}$

<table>
<tr><td>
Since $\log_a n \ll n^a$, $\ln n \ll n^{0.05}$.
$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^{1.4}(\ln n)^2} \geq \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^{1.4}(n^{0.05})^2}$
$= \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^{1.5}}$
Let $b_n = \frac{\sqrt{n}}{n^{1.5}}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n+1}}{n^{1.5}} \div \frac{\sqrt{n}}{n^{1.5}}\right)$
$= \lim_{n\to\infty} \left(\frac{\sqrt{n+1}}{n^{1.5}} \cdot \frac{n^{1.5}}{\sqrt{n}}\right)$
$= \lim_{n\to\infty} \left(\frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{n^{1.5}}{n^{1.5}}\right)$
$= \lim_{n\to\infty} \left(\frac{\sqrt{1+1/n}}{\sqrt{1}} \cdot 1\right)$
$= 1 \neq 0 \neq \infty$

$\therefore \sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{\infty} b_n$
$= \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^{1.5}}$
$= \sum_{n=1}^{\infty} \frac{1}{n} \approx \int_1^{\infty} \frac{1}{x}$
$\int_1^{\infty} \frac{1}{x}$ is part of the p-series, with $p = 1 \leq 1$, thus $\int_1^{\infty} \frac{1}{x}$ diverges
Since $\int_1^{\infty} \frac{1}{x}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
Since $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^{1.5}}$ diverges, $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^{1.5}}$ diverges
Since $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^{1.5}}$ diverges, $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^{1.4}(\ln n)^2}$ diverges
</td><td>
Comparison Test

Limit Comparison Test

Integral Test
</td></tr>
</table>

Reminder that when proving $\sum_{n=1}^{\infty} f(x)$ diverges $\Rightarrow \sum_{n=1}^{\infty} g(x)$ diverges only works when $f(x) \leq g(x)$.
Reminder that when proving $\sum_{n=1}^{\infty} f(x)$ converges $\Rightarrow \sum_{n=1}^{\infty} g(x)$ converges only works when $f(x) \geq g(x)$.

9.2.3. Power Series

A power series with a **center of expansion** $x = c$ is a function with the form

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots}$$

The **radius of convergence** $R \geq 0$ is a number where:

- When $|x-c| > R$, $f(x)$ diverges.
- When $|x-c| = R$, $f(x)$ can converge or diverge (depends on the function).
- When $|x-c| < R$, $f(x)$ converges/is "analytic"
    - $f(x)$ is infinitely differentiable in this region; $f^{(n)}(x)$ always has the same R
    - If $f(x)$ converges everywhere, then $R = \infty$

There are two ways to calculate R.

Ratio Test	Root Test
$R = \dfrac{1}{\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert}$	$R = \dfrac{1}{\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert}}$

The **interval of convergence** is the region of x-values where $f(x)$ converges. It can be found by:

- *Slower Method:* Find R with Ratio/Root Test, substitute it into $|x-c| < R$ and $|x-c| = R$
- *Faster Method:* Calculate $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, solve for x, and check cases when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

My MAT137 prof only taught us the faster method and barely even mentioned how to calculate R and do the slower method, so don't use it.

eg. Find the radius and interval of convergence for $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}(x-3)^n$	*This question type, plus convergence tests, are very common on exams,*
$$\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = \lim_{n\to\infty}\left\|\frac{\left((n+1)!\right)^2(x-3)^{n+1}}{\left(2(n+1)\right)!}\cdot\frac{(2n)!}{(n!)^2(x-3)^n}\right\|$$ $$= \lim_{n\to\infty}\left\|\frac{\left((n+1)!\right)^2}{(n!)^2}\cdot\frac{(x-3)^{n+1}}{(x-3)^n}\cdot\frac{(2n)!}{\left(2(n+1)\right)!}\right\|$$ $$= \lim_{n\to\infty}\left\|\frac{(n+1)!}{n!}\cdot\frac{(n+1)!}{n!}\cdot\frac{(x-3)^{n+1}}{(x-3)^n}\cdot\frac{(2n)!}{(2n+2)!}\right\|$$ $$= \lim_{n\to\infty}\left\|\frac{(n+1)n!}{n!}\cdot\frac{(n+1)n!}{n!}\cdot\frac{(x-3)^n(x-3)}{(x-3)^n}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right\|$$ $$= \lim_{n\to\infty}\left\|(n+1)\cdot(n+1)\cdot(x-3)\cdot\frac{1}{(2n+2)(2n+1)}\right\|$$ $$= \lim_{n\to\infty}\left\|(x-3)\frac{\left(1+{}^{1}\!/_{n}\right)\left(1+{}^{1}\!/_{n}\right)}{\left(2+{}^{2}\!/_{n}\right)\left(2+{}^{1}\!/_{n}\right)}\right\|$$ $$= \|x-3\|$$ $$\therefore R = \frac{1}{\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\|} = \frac{1}{\|x-3\|}$$ $$\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = \|x-3\| < 1$$ $$-1 < x-3 < 1$$ $$2 < x < 4$$ When $x = 2$, $$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}((2)-3)^n = \sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}(-1)^n$$ $$\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = \lim_{n\to\infty}\left\|\frac{\left((n+1)!\right)^2(-1)^{n+1}}{\left(2(n+1)\right)!}\cdot\frac{(2n)!}{(n!)^2(-1)^n}\right\|$$ $$= \lim_{n\to\infty}\left\|\frac{\left((n+1)!\right)^2}{(n!)^2}\cdot\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(2n)!}{\left(2(n+1)\right)!}\right\|$$ $$= \lim_{n\to\infty}\left\|\frac{(n+1)!}{n!}\cdot\frac{(n+1)!}{n!}\cdot\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(2n)!}{(2n+2)!}\right\|$$ $$= \lim_{n\to\infty}\left\|\frac{(n+1)n!}{n!}\cdot\frac{(n+1)n!}{n!}\cdot\frac{(-1)^n\cdot(-1)}{(-1)^n}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right\|$$ $$= \lim_{n\to\infty}\left\|-\frac{(n+1)(n+1)}{(2n+2)(2n+1)}\right\|$$ $$= \lim_{n\to\infty}\left\|\frac{\left(1+{}^{1}\!/_{n}\right)\left(1+{}^{1}\!/_{n}\right)}{\left(2+{}^{2}\!/_{n}\right)\left(2+{}^{1}\!/_{n}\right)}\right\|$$ $$= \frac{1}{4} < 1$$ Thus $f(x)$ converges at $x = 2$ When $x = 4$, $$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}(4-3)^n = \sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}(1)^n = \sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$$ Do the same as the above, and you'll find $f(x)$ converges too. $\therefore$ The interval of convergence is $\boxed{2 \le x \le 4}$	*When* $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = 1$, $x = 2,4$ *You must perform a convergence test on* $f(2)$ *and* $f(4)$.

9.2.4. Taylor Series

$\sin x$ sucks to compute with. What if we instead wrote it as a polynomial?

Let $g(x) = \sin x$. Let's pick a point (eg. $x = 0$), and try to create a polynomial $f(x)$ by substituting:

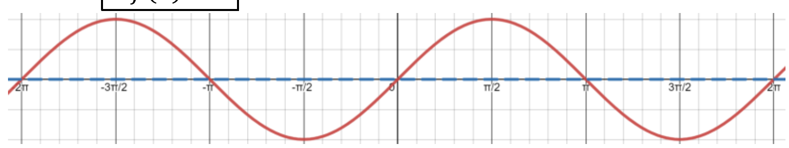
$$f(0) = g(0)$$
$$f'(0) = g'(0)$$
$$f''(0) = g''(0)$$
$$f^{(3)}(0) = g^{(3)}(0)$$
$$\dots$$

Degree 0 Polynomial:

$f(x) = a$	$f(0) = a$	$g(x) = \sin x$	$g(0) = 0$

$$f(0) = g(0)$$
$$a = 0$$
$$\boxed{\therefore f(x) = 0}$$

We get a trash approximation of $g(x)$.



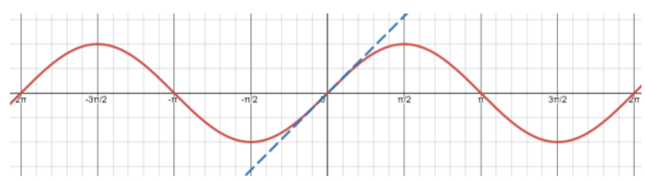
Degree 1 Polynomial:

$f(x) = ax + b$ $f'(x) = a$	$f(0) = b$ $f'(0) = a$	$g(x) = \sin x$ $g'(x) = \cos x$	$g(0) = 0$ $g'(0) = 1$

$$f(0) = g(0) \qquad b = 0 \qquad \therefore f(x) = ax$$

$$f'(0) = g'(0) \qquad a = 1 \qquad \boxed{\therefore f(x) = x}$$

This isn't stellar, but at least it's getting a little bit closer to $g(x)$.



Degree 2 Polynomial:

$f(x) = ax^2 + bx + c$ $f'(x) = 2ax + b$ $f''(x) = 2a$	$f(0) = c$ $f'(0) = b$ $f''(0) = 2a$	$g(x) = \sin x$ $g'(x) = \cos x$ $g''(x) = -\sin x$	$g(0) = 0$ $g'(0) = 1$ $g''(0) = 0$

$$f(0) = g(0) \qquad c = 0 \qquad \therefore f(x) = ax^2 + bx$$

$$f'(0) = g'(0) \qquad b = 1 \qquad \therefore f(x) = ax^2 + x$$

$$f''(0) = g''(0) \qquad 2a = 0 \qquad \boxed{\therefore f(x) = x}$$

I'm not showing another graph because this is still the same function as Degree 1.

Degree 3 Polynomial:

$f(x) = ax^3 + bx^2 + cx + d$ $f'(x) = 3ax^2 + 2bx + c$ $f''(x) = 6ax + 2b$ $f^{(3)}(x) = 6a$	$f(0) = d$ $f'(0) = c$ $f''(0) = 2b$ $f^{(3)}(0) = 6a$	$g(x) = \sin x$ $g'(x) = \cos x$ $g''(x) = -\sin x$ $g^{(3)}(x) = -\cos x$	$g(0) = 0$ $g'(0) = 1$ $g''(0) = 0$ $g^{(3)}(0) = -1$

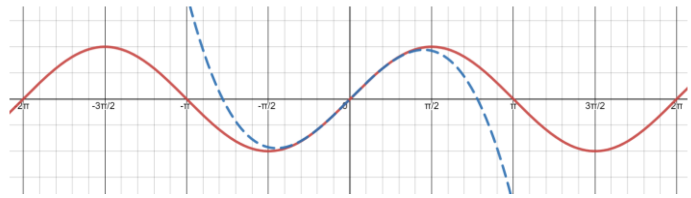
$$f(0) = g(0) \qquad d = 0 \qquad \therefore f(x) = ax^3 + bx^2 + cx$$

$$f'(0) = g'(0) \qquad c = 1 \qquad \therefore f(x) = ax^3 + bx^2 + x$$

$$f''(0) = g''(0) \qquad 2b = 0 \qquad \therefore f(x) = ax^3 + x$$

$$f^{(3)}(0) = g^{(3)}(0) \qquad 6a = -1 \qquad \boxed{\therefore f(x) = -\frac{1}{6}x^3 + x}$$

$f(x)$ is starting to wrap around $g(x)$. It's becoming accurate in a <u>growing area centered around $x = 0$</u>.



Degree 4 Polynomial:

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$	$f(0) = e$	$g(x) = \sin x$	$g(0) = 0$
$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$	$f'(0) = d$	$g'(x) = \cos x$	$g'(0) = 1$
$f''(x) = 12ax^2 + 6bx + 2c$	$f''(0) = 2c$	$g''(x) = -\sin x$	$g''(0) = 0$
$f^{(3)}(x) = 24ax + 6b$	$f^{(3)}(0) = 6b$	$g^{(3)}(x) = -\cos x$	$g^{(3)}(0) = -1$
$f^{(4)}(x) = 24a$	$f^{(4)}(0) = 24a$	$g^{(4)}(x) = \sin x$	$g^{(4)}(0) = 0$

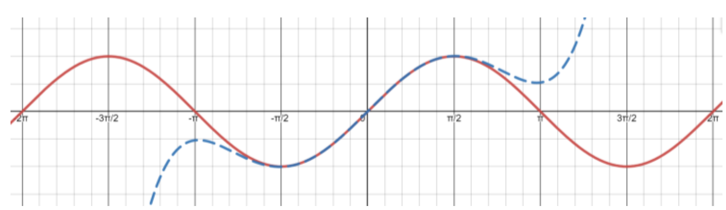
$f(0) = g(0)$ $e = 0$ $\therefore f(x) = ax^4 + bx^3 + cx^2 + dx$	$f'(0) = g'(0)$ $d = 1$ $\therefore f(x) = ax^4 + bx^3 + cx^2 + x$	$f''(0) = g''(0)$ $2c = 0$ $\therefore f(x) = ax^4 + bx^3 + x$
$f^{(3)}(0) = g^{(3)}(0)$ $6b = -1$ $\therefore f(x) = ax^4 - \frac{1}{6}x^3 + x$	$f^{(4)}(0) = g^{(4)}(0)$ $24a = 0$ $\boxed{\therefore f(x) = -\frac{1}{6}x^3 + x}$	

Degree 5 Polynomial:

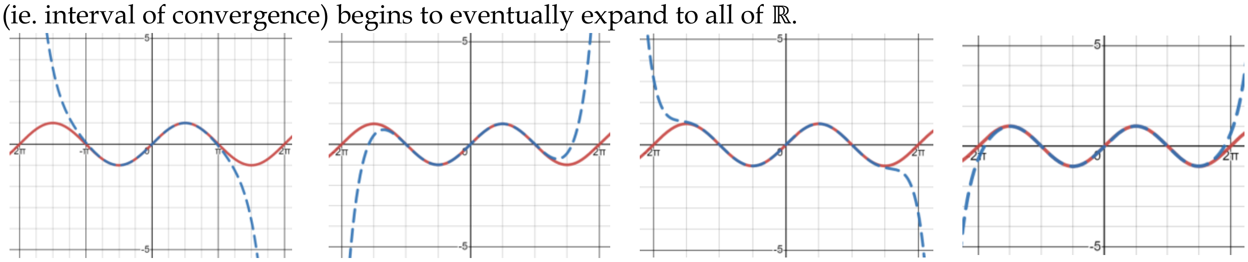
$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$	$f(0) = f$	$g(x) = \sin x$	$g(0) = 0$
$f'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$	$f'(0) = e$	$g'(x) = \cos x$	$g'(0) = 1$
$f''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$	$f''(0) = 2d$	$g''(x) = -\sin x$	$g''(0) = 0$
$f^{(3)}(x) = 60ax^2 + 24bx + 6c$	$f^{(3)}(0) = 6c$	$g^{(3)}(x) = -\cos x$	$g^{(3)}(0) = -1$
$f^{(4)}(x) = 120ax + 24b$	$f^{(4)}(0) = 24b$	$g^{(4)}(x) = \sin x$	$g^{(4)}(0) = 0$
$f^{(5)}(x) = 120a$	$f^{(5)}(0) = 120a$	$g^{(5)}(x) = \cos x$	$g^{(5)}(0) = 1$

$f(0) = g(0)$ $f = 0$ $\therefore f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex$	$f'(0) = g'(0)$ $e = 1$ $\therefore f(x) = ax^5 + bx^4 + cx^3 + dx^2 + x$	$f''(0) = g''(0)$ $2d = 0$ $\therefore f(x) = ax^5 + bx^4 + cx^3 + x$
$f^{(3)}(0) = g^{(3)}(0)$ $6c = -1$ $\therefore f(x) = ax^5 + bx^4 - \frac{1}{6}x^3 + x$	$f^{(4)}(0) = g^{(4)}(0)$ $24b = 0$ $\therefore f(x) = ax^5 - \frac{1}{6}x^3 + x$	$f^{(5)}(0) = g^{(5)}(0)$ $120a = 1$ $\boxed{\therefore f(x) = \frac{1}{120}x^5 - \frac{1}{6}x^3 + x}$

$f(x)$ continues wrapping around the curves of $g(x)$, most accurate around a **center of expansion** at $x = 0$.



As we add more and more polynomials, $f(x)$ becomes a power series. The area of accurate approximation (ie. interval of convergence) begins to eventually expand to all of $\mathbb{R}$.



$f(x)$ is called the Taylor Series of $g(x)$: an infinite series of rising powers that is <u>exactly equal to a function</u>.

The **Taylor Series** of $f(x)$ with center of expansion $x = a$ is a type of power series

$$\boxed{P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

The Taylor Series of $f(x)$ with center of expansion $x = 0$, AKA **Maclaurin Series**, is

$$\boxed{P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n}$$

I recommend committing these to memory:

Maclaurin Series	**Expanded**	**Interval of Convergence**
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$	$x \in \mathbb{R}$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$x \in \mathbb{R}$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$x \in \mathbb{R}$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$\lvert x\rvert < 1$

A Taylor Series approximation is only accurate in its **symmetric** interval of convergence.

- Note that you probably saw the bottom series from high school: "$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ when $-1 < r < 1$"
- Why $|x| < 1$? Because $\lim_{x\to 1^-} \frac{1}{1-x} = \infty$, and Taylor series are symmetric, so it's limited to $|x| < 1$.
- Note that a tangent line is a degree-1 polynomial Taylor Series approximation.

These operations can be performed on a Taylor Series (they don't change the interval of convergence):

$$\int \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \int a_n \qquad \left(\sum_{n=0}^{\infty} a_n\right)' = \sum_{n=0}^{\infty} (a_n)'$$

eg. Find the Maclaurin Series of $\frac{e^x - e^{-x}}{2}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$e^x - e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} - \frac{(-1)^n x^n}{n!}\right)$$

$$= \boxed{\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}(1 - (-1)^n)\right)} \text{ (for } x \in \mathbb{R}\text{)}$$

*If you've ever heard of **hyperbolas**, the function $\frac{e^x - e^{-x}}{2}$ is called **hyperbolic sine**, or "sinh" for short. $\frac{e^x + e^{-x}}{2}$ is "cosh".*

You can write any $e^{f(x)}$ as $\sum_{n=0}^{\infty} \frac{[f(x)]^n}{n!}$. The same applies with the other series.

Check addition of series from 9.2.1.

Remember to include the interval of convergence!

eg. Prove the Small Angle Approximation $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

Substitute the Maclaurin Series

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) = 1$$

Only the first term, 1, has no multiple of x.

eg. Prove Euler's formula: $e^{ix} = \cos x + i \sin x$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$i \sin x = ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots$$

$$\cos x + i \sin x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

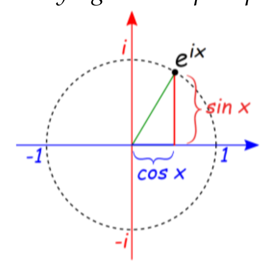
$$= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots$$

$$= e^{ix}$$

Don't be scared! Keep calm and use Taylor Series.

The "…" notation is less formal, but we use it because otherwise, this is really hard to calculate.

This formula is fundamental, studying the complex plane:



eg. Find the Maclaurin Series of $\arctan x$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{for } |x| < 1)$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \quad (\text{for } |-x^2| < 1)$$

Deal with positive through double negative

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad (\text{for } |x| < 1)$$

$|-x^2| < 1 \Rightarrow |x| < 1$

$$\int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Integral/derivative of both sides is a valid mathematical operation.

$$\arctan x = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}$$

Recall $\int \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \int a_n$ *is valid.*

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$$

Recall $\int x^n = \frac{x^{n+1}}{n+1}$. *The* $(-1)^n$ *isn't a power of* x*, so it isn't affected.*

Substitute $x = 0$

$$\arctan 0 = \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1} + C$$

$$0 = 0 + C$$

$$C = 0$$

$$\therefore \arctan x = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}} \quad (\text{for } |x| < 1)$$

You MUST add C every time you integral, because the value of C is a single constant. It's usually 0, but it might not be.

eg. Evaluate $\sum_{n=2}^{\infty} \frac{(4n^2+8n+3)2^n}{n!}$.	
$$\sum_{n=2}^{\infty} \frac{(4n^2+8n+3)2^n}{n!} = \sum_{n=2}^{\infty} \frac{(2n+1)(2n+3)2^n}{n!}$$	*Note the $n!$ denominator. The only series of that form is $\underline{e^x}$.*
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ $$x^3 e^{2x} = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{n!}$$	*Multiply e^{2x} by x^3 to mold the numerator x^n into* $x^{2n} \cdot x^3 = x^{2n+3}$
$$\frac{d}{dx}(x^3 e^{2x}) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{2n+3}}{n!}$$ $$3x^2e^{2x} + 2x^3e^{2x} = \sum_{n=0}^{\infty} \frac{(2n+3)x^{2n+2}}{n!}$$	*Derivative both sides so that x^{2n+3} becomes $(2n+3)x^{2n+2}$*
$$3xe^{2x} + 2x^2e^{2x} = \sum_{n=0}^{\infty} \frac{(2n+3)x^{2n+1}}{n!}$$	*Divide by x on both sides to mold x^{2n+2} into x^{2n+1}*
$$\frac{d}{dx}(3xe^{2x} + 2x^2e^{2x}) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(2n+3)x^{2n+1}}{n!}$$ $$3e^{2x} + 6xe^{2x} + 4xe^{2x} + 4x^2e^{2x} = \sum_{n=0}^{\infty} \frac{(2n+3)(2n+1)x^{2n}}{n!}$$	*Note that the derivative only affects powers of x. $(2n+3)$ and $n!$ stay the same.*
$$e^{2x}(4x^2 + 10x + 3) = \sum_{n=0}^{\infty} \frac{(2n+3)(2n+1)x^{2n}}{n!}$$ Substitute $x = \sqrt{2}$ $e^{2\sqrt{2}}(4(2) + 10\sqrt{2} + 3) = \sum_{n=0}^{\infty} \frac{(2n+3)(2n+1)2^n}{n!}$	*The x^{2n} doesn't match the 2^n that we need. So we choose x such that $x^{2n} = 2^n$, AKA $(x^2)^n = 2^n$, AKA $x = \pm\sqrt{2}$*
$e^{2\sqrt{2}}(11 + 10\sqrt{2}) = \sum_{n=0}^{\infty} \frac{(2n+3)(2n+1)2^n}{n!}$ $= \frac{(2(0)+3)(2(0)+1)2^0}{(0)!} + \frac{(2(1)+3)(2(1)+1)2^1}{(1)!} + \sum_{n=2}^{\infty} \frac{(2n+3)(2n+1)2^n}{n!}$ $= 3 + 30 + \sum_{n=2}^{\infty} \frac{(2n+3)(2n+1)2^n}{n!}$ $$\therefore \sum_{n=2}^{\infty} \frac{(2n+3)(2n+1)2^n}{n!} = \boxed{e^{2\sqrt{2}}(11 + 10\sqrt{2}) - 33}$$	*We need:* $\sum_{n=2}^{\infty} a_n = a_2 + a_3 + a_4 + \cdots$ *We have:* $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$ *So it follows that* $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \sum_{n=2}^{\infty} a_n$

eg. Evaluate the infinite repeating decimal 0.3121212 ... using Taylor series.	
$$\begin{aligned} 0.3121212\ldots &= 0.3 + 0.012 + 0.00012 + 0.0000012 + \cdots \\ &= 0.3 + 12(0.001 + 0.00001 + 0.0000001 + \cdots) \\ &= 0.3 + 1.2(0.01 + 0.0001 + 0.000001 + \cdots) \\ &= 0.3 + 1.2\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^n \\ &= 0.3 + 1.2\left[\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n - \left(\frac{1}{100}\right)^0\right] \\ &= 0.3 + 1.2\left[\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n - 1\right] \end{aligned}$$ Since $-1 < \frac{1}{100} < 1$, $\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n = \frac{1}{1-\frac{1}{100}} = \frac{100}{99}$ $$\begin{aligned} 0.3 + 1.2\left[\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n - 1\right] &= 0.3 + 1.2\left(\frac{100}{99} - 1\right) \\ &= \frac{3}{10} + 12 \cdot \frac{1}{10} \cdot \frac{1}{99} \\ &= \frac{3}{10} + \frac{4}{330} = \boxed{\frac{103}{330}} \end{aligned}$$	*Since* $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$ *and* $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ *it follows that* $\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$ *and thus* $\sum_{n=1}^{\infty} a_n = a_0 - \sum_{n=0}^{\infty} a_n$

eg. Let $F(x) = (1+x^4)\sin(2x^2)$. Evaluate $F^{(42)}(0)$.	*(From MAT137, April exam 2017)*
$F(x) = (1+x^4)\sin(2x^2) = \sin(2x^2) + x^4\sin(2x^2)$ $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ $\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x^2)^{2n+1}$ $= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} x^{4n+2}$ $x^4\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} x^{4n+6}$	*I do not recommend differentiating $F(x)$ 42 times.*
$4n+2 = 42$ $n = 10$ $\frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} x^{4n+2} = \frac{(-1)^{10} \cdot 2^{21}}{21!} x^{42} = \frac{2^{21}}{21!} x^{42}$ After differentiating 42 times, the term becomes $42!\frac{2^{21}}{21!}$ $4n+6 = 42$ $n = 9$ $\frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} x^{4n+6} = \frac{(-1)^{9} \cdot 2^{19}}{19!} x^{42} = -\frac{2^{19}}{19!} x^{42}$ After differentiating 42 times, the term becomes $-42!\frac{2^{19}}{19!}$	*Find the terms in the series of $\sin(2x^2)$ and $x^6\sin(2x^2)$ of the format __x^{42}.* *After differentiating 42 times, these terms will be the <u>only constants</u>. $F^{(42)}(0)$ will equal those two constants.*
$\therefore F^{(42)}(0) = \boxed{42!\left(\frac{2^{21}}{21!} - \frac{2^{19}}{19!}\right)}$	*Usually, you won't have to simplify any further than this.*

eg. Find the Taylor Series expansion for $f(x) = 1 - x^2 + x^3$ at $a = 2$.	*(From MAT137, August exam 2017)*
$f(x) = 1 - x^2 + x^3$ $f(2) = 1 - 2^2 + 2^3 = 5$ $f'(x) = -2x + 3x^2$ $f'(2) = -2(2) + 3(2)^2 = 8$ $f''(x) = -2 + 6x$ $f''(2) = -2 + 6(2) = 10$ $f^{(3)}(x) = 6$ $f^{(3)}(2) = 6$ $f^{(4)}(x) = 0$ $f^{(4)}(2) = 0$	
$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ $= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$ $= \frac{f(2)}{0!} + \frac{f'(2)}{1!}(x-2)^1 + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \cdots$ $= 5 + 8(x-2) + 5(x-2)^2 + (x-2)^3 + \frac{0}{3!}(x-2)^4 + \frac{0}{3!}(x-2)^5 \ldots$ $= \boxed{5 + 8(x-2) + 5(x-2)^2 + (x-2)^3}$	If you differentiate past $f^{(4)}(x)$, everything is 0. $f^{(n)}(x) = 0$ when $x \geq 4$.

eg. Find the Taylor Series expansion for $f(x) = e^x$ at $a = 1$.
The Maclaurin Series for e^x is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$. This makes sense as $f^{(n)}(0) = 1$ for all n. We know $f^{(n)}(1) = e$ for all n, thus $\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \boxed{\sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n}$

9.2.5. Telescoping Series

A telescoping series is a type of series where all adjacent terms cancel out. Set $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$.

eg. Evaluate $\sum_{n=0}^{\infty}[\arctan n - \arctan(n+1)]$
$\sum_{n=0}^{\infty}[\arctan n - \arctan(n+1)] = \lim_{N\to\infty} \sum_{n=0}^{N}[\arctan n - \arctan(n+1)]$ $= \lim_{N\to\infty}[(\arctan 0 - \arctan 1) + (\arctan 1 - \arctan 2) + \cdots + (\arctan(N-1) - \arctan N)]$ $= \lim_{N\to\infty}[\arctan 0 - \arctan N] = 0 - \frac{\pi}{2} = \boxed{-\frac{\pi}{2}}$

10. Resources

MAT137 Playlist

Videos of all MAT137 topics by the late Alfonso Gracia-Saz, who taught the course for a long time. These videos are clear and most importantly, concise.

MAT137 Lecture Notes

A MAT137 textbook by Tyler Holden. It's clear and contains detailed explanations, but I believe he teaches in UTM and some covered topics are not as relevant/big a focus.

MAT137 Past Practice Questions and Assignments (with Answers!)

These questions can be difficult, but serve as good exam preparation.

Old Exam Repository

Contains past MAT137 exams. It requires you have a UTORid to login. All questions from Artsci MAT137 exams are already in *Questions from MAT137*.

Essence of Calculus

An excellent Youtube playlist by 3Blue1Brown with very pretty illustrations/animations of calculus concepts. Builds great visual intuition for concepts like:

- Epsilon-Delta Definition of Limit
- Definition of Integral
- Definition of Taylor Series
- Derivative Techniques (Product Rule, Chain Rule, Implicit Differentiation)