Notation and Convention

Vectors	$x \in \mathbb{R}^n$ is a shortform for $(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.
Functions	For $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = \big(f_1(x), \dots, f_m(x)\big) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}$
Combined Points	$\big(x, f(x)\big) \in \mathbb{R}^{n+m}$ is a shortform for $\big(x_1, \dots, x_n, f_1(x), \dots, f_m(x)\big) \in \mathbb{R}^{n+m}$
Standard Basis	e_i is a unit vector where the i-th component is 1 and other components are 0.
Subsets	$\underline{S}$ means any/some subset of $\underline{S}$. $\underline{S}_{\text{open}}$ means any/some open subset of $\underline{S}$. $\underline{S}_i$ helps differentiate between different subsets of $\underline{S}$.

$\mathbb{R} \to \mathbb{R}^n$ (Vector-valued functions of a real variable / Parametric curves)

Parametric Functions: Maps with a function in each of its output dimensions.

- eg. $\gamma: [0,2\pi) \to \mathbb{R}^3$ where $\gamma(t) = (\cos t, \sin t, t)$ AKA $\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$ corresponds to a helix
- eg. $\gamma: \mathbb{R} \to \mathbb{R}^2$ where $\gamma(t) = (t, t^2)$ AKA $\gamma(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ corresponds to $y = x^2$

Trace: The image of a parametric function.
Curve: The trace of a continuous parametric function

We commonly interpret the domain as **time** & do kinematics.

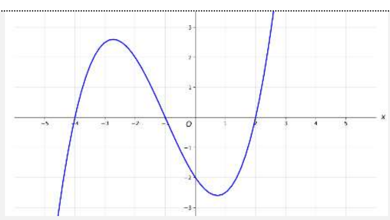
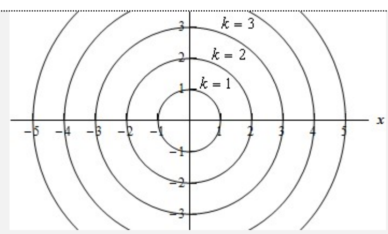
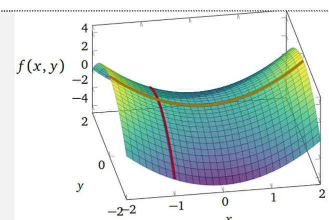
Position (Vector)	Velocity (Vector)	Instantaneous Speed (Scalar)
$\gamma(t)$	$\gamma'(t) = \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$	$\lVert\gamma'(t)\rVert = \lim_{h\to 0} \frac{\lVert\gamma(t+h) - \gamma(t)\rVert}{\lvert h\rvert} = \lvert\gamma'(t)\rvert$

For instantaneous speed:

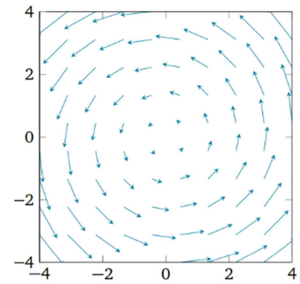
- Add *absolute value* $|h|$ – otherwise, as numerator ≥ 0, h flips sign and there's a jump discontinuity at $h = 0$.
- You can also use the MAT223 definition of norm/magnitude/length of a vector, $\lVert\vec{x}\rVert = \sqrt{x_1^2 + \cdots + x_n^2}$

Unit Tangent Vector	Unit Normal Vector
$T(x) = \frac{\gamma'(x)}{\lVert\gamma'(x)\rVert}$ Length of 1. Shows vector direction, like a tangent line.	$N(x) = \frac{T'(x)}{\lVert T'(x)\rVert}$ Length of 1. Orthogonal to tangent vector.

$\mathbb{R}^n \to \mathbb{R}$ (Real-valued functions / Scalar fields / Scalar functions / Potentials)

Graph	K-Level Set	Slice
Plot all points in f's domain and for their corresponding codomain outputs, use an extra dimension. $\{(x, f(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$	Find points satisfying $f(x) = k$ and plot them in $\mathbb{R}^n$. Repeat for many values of k. $\{x \in \mathbb{R}^n : f(x) = k\}$	Set 1+ variable in f equal to something, plot the remaining variables of f. $\{(y, f(\alpha, y)) \in \mathbb{R}^2 : (\alpha, y) \in \mathbb{R}^2\}$
 The graph of the above $f: \mathbb{R}^1 \to \mathbb{R}^1$ is in $1 + 1 = 2$ dimensions.	 K-level set of $f(x, y) = x^2 + y^2$	 Green horizontal curve is a y-slice Red vertical curve is an x-slice
For multi-dimensional outputs, add a new dimension for each output: $\{(x, f(x)) \in \mathbb{R}^{n+m} : x \in \mathbb{R}^n\}$ where $x \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^m$	Basically the **preimage**, $f^{-1}(\{k\})$. Level sets in $\mathbb{R}^2$ form **contour lines**. For level sets that use continuous values, do **heat maps**.	"x-slice at α" means you set $x = \alpha$ and graph y against $f(\alpha, y)$

$\mathbb{R}^n \to \mathbb{R}^n$ (Vector fields)



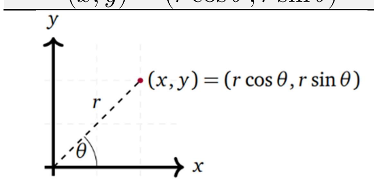
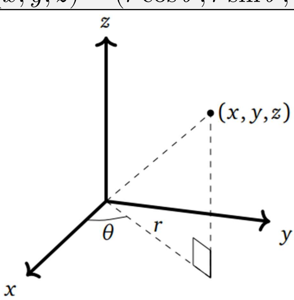
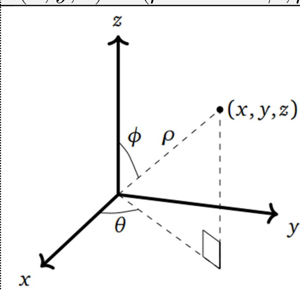
Think about these like more complex **transformations** in linear algebra.

Vector Field: A function of form $f: \mathbb{R}^n \to \mathbb{R}^n$. To visualize, plot all $x \in \mathbb{Z}^n$; for each x, plot a vector $f(x)$ whose origin/tail begins on x

- Can get visually messy; people often scale down magnitude, or use colour

Coordinate Transformation: A continuous transformation $f: A \to B$ that makes a coordinate system.

- "A and f form a coordinate system for B"
- "Point b in B-space is point a in A-space", or $(b_1, \dots, b_n)_B = (a_1, \dots, a_n)_A$, like a change of basis

Polar Coordinates	Cylindrical Coordinates	Spherical Coordinates
$T(r,\theta) = (r\cos\theta, r\sin\theta)$	$T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$	$T(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$
$(x,y) = (r\cos\theta, r\sin\theta)$	$(x,y,z) = (r\cos\theta, r\sin\theta, z)$	$(x,y,z) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$
 $r =$ radius $\theta =$ direction *Notice that* $x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$		 $\rho =$ radius $\theta =$ polar angle/longitude $\phi =$ azimuthal angle/latitude

Polar Transformation Domain:

- Restrict domain to $\{(r,\theta) \in \mathbb{R}^2 : r > 0, 0 \le \theta < 2\pi\} \cup \{(0,0)\}$ for invertibility
- $(0,\infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ maps bijectively to $\mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \le 0\}$. Here, $r = \sqrt{x^2+y^2}$, $\theta = \arctan(\frac{y}{x})$

$\mathbb{R}^n \to \mathbb{R}^m$

Manifold: A non-linear k-dimensional surface in $\mathbb{R}^n$

When $n < m$

Parametric Form: Writing sets as **images** of f

$$S = \operatorname{img}(f) = \{f(x) \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

For a hollow sphere of radius 1:

$$f(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$
$$S = \{f(\theta,\phi) \in \mathbb{R}^3 : (\theta,\phi) \in \mathbb{R}^2\} = \operatorname{img}(f)$$

Explicit Form: Writing sets as **graphs** of $f: \mathbb{R}^n \to \mathbb{R}^m$

$$S = \{(x, f(x)) \in \mathbb{R}^{n+m} : x \in \mathbb{R}^n\}$$

For a hollow sphere of radius 1:

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : z = \sqrt{1-x^2-y^2}\right\} \cup \left\{(x,y,z) \in \mathbb{R}^3 : z = -\sqrt{1-x^2-y^2}\right\}$$

We can only write the top/bottom half alone because we can't convert $x^2+y^2+z^2 = 1$ into a function.

When $n > m$

Implicit Form: Writing sets as **preimages** of $f(\{\alpha\})$

For a hollow sphere of radius 1:

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

We're finding the preimage $f^{-1}(\{1\})$.

Projections: Squishing sets into smaller dimensions by ignoring components of the vector

- **i-th Coordinate Map:** $\pi_i: \mathbb{R}^n \to \mathbb{R}$ where
  $$\pi_i(x) = x_i$$
  Ignores all dimensions except for one.
- **i-th Coordinate Plane Projection:** The function $\Pi_i: \mathbb{R}^n \to \mathbb{R}^{n-1}$ where
  $$\Pi_i(x) \to (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$
  Ignores one dimension.

Writable in Explicit Form $\Rightarrow$ Writable in Parametric Form and Implicit Form

Proving Writability in Explicit Form: Write it in explicit form.

Proving Non-Writability in Explicit Form: Assume otherwise. Find $a \ne b$ where $f(x) = a$, $f(x) = b$ (contradiction).

Topology

Let $a =$ center, $r =$ radius. Then

Open Ball	Closed Ball	Sphere
$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$	$\overline{B_r(a)} = \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$	$\partial B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| = r\}$
In 2D, a disk with no edges	In 2D, a disk with edges	In 2D, a hollow circle

Punctured: A ball with no center point, like $B(a, r) \setminus \{a\}$
Solid: Something with volume (eg. closed ball, cubes), as opposed to outlines of objects (eg. sphere)

Open Rectangle	Closed Rectangle	Hypercube
$R_{\text{open}} = \prod_{i=1}^{n}(a_i, b_i)$	$R_{\text{closed}} = \overline{R_{\text{open}}} = \prod_{i=1}^{n}[a_i, b_i]$	$[a, b]^n$
In 2D, a rectangle with no edges	In 2D, a rectangle with edges	In 2D, a rectangle. The unit hypercube is $[0,1]^n$

Interior Point: Of $A \subseteq \mathbb{R}^n$, point $p \in \mathbb{R}^n$ if $\exists \epsilon > 0, B(p, \epsilon) \subseteq A$ "A small enough p-centered ball is in A"	**Interior (A°, $\text{int}(A)$):** Set of interior points of A. ➢ $A^\circ \subseteq A$ ➢ $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ ➢ $A^\circ \cap B^\circ = (A \cap B)^\circ$ ➢ $A^\circ \times B^\circ = (A \times B)^\circ$
Boundary Point: Of $A \subseteq \mathbb{R}^n$, point $p \in \mathbb{R}^n$ if $\forall \epsilon > 0, B(p, \epsilon) \cap A \neq \emptyset$ and $B(p, \epsilon) \cap A^c \neq \emptyset$ "Any p-centered ball overlaps A and A^c"	**(Topological) Boundary (∂A):** Set of boundary points of A ➢ A° and ∂A are disjoint
Limit Point: Of $A \subseteq \mathbb{R}^n$, point $p \in \mathbb{R}^n$ if $\forall \epsilon > 0, \exists a \in B(p, \epsilon) \setminus \{p\}, a \in A$ "Any punctured p-centered ball overlaps A" • All interior points are limit points • All boundary points are limit points	**Closure ($\bar{A}$, $\text{cl}(A)$):** Set of A and all limit points, $A \cup A^*$ ➢ $A \subseteq \bar{A}$ ➢ $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$ ➢ $\text{cl}(A \cap B) = \text{cl}(A) \cap \text{cl}(B)$ ➢ $\text{cl}(A \times B) = \text{cl}(A) \times \text{cl}(B)$ ➢ $\bar{A} = A^\circ \cup \partial A$

➢ $\mathbb{Q}$ is really weird! $\mathbb{Q}^\circ = \emptyset$, $\partial\mathbb{Q} = \mathbb{R}$, $\overline{\mathbb{Q}} = \mathbb{R}$

Sequences

Sequence (x_k, $\{x(k)\}_{k=k_0}^{\infty}$): In $\mathbb{R}^n$, a function $f: \mathbb{Z}^{\text{has a min}} \to \mathbb{R}^n$ (usually, assume domain is $\mathbb{N}^+$)
Converges ($\lim_{k\to\infty} x(k) = p$, $x(k) \to p$): To $p \in \mathbb{R}^n$, the sequence $x(k)$ if

$$\forall \epsilon > 0, \exists \delta \in \mathbb{N}, \forall k \in \mathbb{N}, k \geq \delta \Rightarrow \|x(k) - p\| < \epsilon$$

- $x(k) \to p \Leftrightarrow \forall \epsilon > 0, \exists k_0 \in \mathbb{N}^+, \{x(k)\}_{k=k_0}^{\infty} \subseteq B_\epsilon(p)$
- $x(k) \to p \Leftrightarrow x_i(k) \to p$

Subsequence: Of sequence $x(k)$, the sequence $x(m(k))$, where $m: \mathbb{Z} \to \mathbb{Z}$ is strictly increasing

- $x(k) \to p \Rightarrow$ all subsequences of $x(k)$ converge to p

p is a **limit point** of A $\Leftrightarrow$ A sequence in $A \setminus \{p\}$ converges to p
p is an **interior point** of A $\Leftrightarrow$ All sequences converging to p are eventually in A (ie. $\exists k_0 \in \mathbb{N}^+, \{x(k)\}_{k=k_0}^{\infty} \subseteq A$)
p is a **boundary point** of A $\Leftrightarrow$ A sequence in A and a sequence in A^c converge to p

Sets

Open: Set $A \subseteq \mathbb{R}^n$ if all its points are interior points.

- A is open $\Leftrightarrow A = A^\circ \Leftrightarrow A \cap \partial A = \emptyset$
- A° is open
- Open intervals & balls are open

Closed: Set $A \subseteq \mathbb{R}^n$ if all limit points are in it.

- A is closed $\Leftrightarrow A = \bar{A} \Leftrightarrow \partial A \subseteq A$
- $\bar{A}$ is closed
- Closed intervals & balls are closed

Let O be an open set, C be a closed set. Let $\star_{\text{finite}}$ mean $\star$ is performed finitely many times.

Necessarily Open			*Sometimes Open*		
$O \cap_{\text{finite}} O$	$O \times_{\text{finite}} O$	$O \cup O$	$O \cap O$	eg. $\bigcap_{\epsilon>0}(-\epsilon,\epsilon) = \{0\}$	$\bigcap_{n=1}^{\infty}\left(0{,}1+\frac{1}{n}\right) = (0{,}1]$
Necessarily Closed			*Sometimes Closed*		
$C \cup_{\text{finite}} C$	$C \times_{\text{finite}} C$	$C \cap C$	$C \cup C$	eg. $\bigcup_{0<\epsilon<1}[-\epsilon,\epsilon] = (-1{,}1)$	$\bigcup_{n=1}^{\infty}\left[0{,}1-\frac{1}{n}\right] = [0{,}1)$

- A is open $\Leftrightarrow A^c$ is closed
- A is **clopen** $\Leftrightarrow A$ is open and closed $\Leftrightarrow A \in \{\emptyset, \mathbb{R}^n\}$

	Open	*Not Open*
Closed	$\emptyset$, $\mathbb{R}^n$	$[a,b]$, $\overline{B_\epsilon(p)}$, $\{(x,y) \in \mathbb{R}^2, x \geq 0\}$, $\mathbb{R} \times \mathbb{Z}$, $\mathbb{Z}^n$, $\partial B_\epsilon(p)$
Not Closed	(a,b), $B_\epsilon(p)$, $\{(x,y) \in \mathbb{R}^2, x > 0\}$	$\mathbb{Q}$, $[a,b)$, $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, $\overline{B_\epsilon(p)} \setminus p$

Compact: Set $A \subseteq \mathbb{R}^n$ if all sequences in A have a subsequence converging to some $p \in A$,

- $B \subseteq A$ is closed $\Rightarrow B$ is compact

Bounded: Set $A \subseteq \mathbb{R}^n$ if it's surroundable by a ball; otherwise, it is **unbounded**.

$$\exists r > 0, A \subseteq B_r(0)$$

- **Bolzano-Weierstrass Theorem:** A is compact $\Leftrightarrow A$ is closed and bounded.

Let S be a compact set. Let $\star_{\text{finite}}$ mean $\star$ is performed finitely many times.

Necessarily Compact			*Sometimes Compact*		
$S \cup_{\text{finite}} S$	$S \cap_{\text{finite}} S$	$S \times S$	$S \cup S$	eg. $\bigcap_{\epsilon>0}(-\epsilon,\epsilon) = \{0\}$	$\bigcap_{n=1}^{\infty}(0{,}1+\frac{1}{n}) = (0{,}1]$

Limits and Continuity

Because we don't want to consider <u>one-sided limits</u> (edge cases), we assume:

- Domain is open
- a is an interior point of the domain

Limit ($\lim_{x\to a} f(x) = L$): Of function $f\colon \underline{\mathbb{R}^n} \to \mathbb{R}^m$, at a limit point a of $\mathbb{R}^n$, when

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \underline{\mathbb{R}^n}, \quad 0 < \|x-a\| < \delta \Rightarrow \|f(x) - L\| < \epsilon$$

$$\forall \epsilon > 0, \exists \delta > 0, \quad x \in B_\delta(a) \cap \mathbb{R}^n \setminus \{a\} \Rightarrow f(x) \in B_\epsilon(L)$$

$$\forall \{x(k)\} \subseteq \underline{\mathbb{R}^n} \setminus \{a\}, \quad x(k) \to \underline{\mathbb{R}^n} \Rightarrow f(x(k)) \to L$$

$\lim\limits_{\|x\|\to\infty} \ldots$	$\ldots, \|x\| > \delta \Rightarrow \cdots$ $\ldots, x(k) \to \infty \Rightarrow \cdots$
$\ldots = \infty$	$\ldots \Rightarrow \|f(x)\| > \epsilon$ $\ldots \Rightarrow f(x(k)) \to \infty$

- $\lim_{x\to a}[\alpha f(x) + \beta g(x)] = \alpha \lim_{x\to a} f(x) + \beta \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x) g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

- $\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a} f_i(x) = L_i$
- **Squeeze theorem** works the same, but only for $f\colon \mathbb{R}^n \to \mathbb{R}$

Proving Limit Exists: Use the $\epsilon - \delta$ definition

eg. Prove $\lim_{(x,y)\to(2,3)}(xy+3y) = 15$

Show $\forall\epsilon > 0, \exists\delta > 0, \forall x \in \mathbb{R}^2, 0 < \|(x,y)-(2{,}3)\| < \delta \Rightarrow |xy+3y-15| < \epsilon$
Let $\epsilon > 0$, pick $\delta = \min\{1, \frac{\epsilon}{9}\}$
Let $x \in \mathbb{R}^2$
Assume $0 < \|(x,y)-(2{,}3)\| < \delta$

- Then $|x-2| = \sqrt{(x-2)^2} \le \sqrt{(x-2)^2+(y-3)^2} = \|(x,y)-(2{,}3)\| < \delta$
- Then $|y-3| = \sqrt{(y-3)^2} \le \sqrt{(x-2)^2+(y-3)^2} = \|(x,y)-(2{,}3)\| < \delta,$

Show $|xy+3y-15| < \epsilon$

$$\begin{aligned} |xy+3y-15| &= |xy-2y+5y-15| \\ &= |y(x-2)+5(y-3)| \\ &\le |y(x-2)|+|5(y-3)| \\ &= |y||x-2|+5|y-3| \\ &< |y|\delta+5\delta \\ &= 4\delta+5\delta \\ &= 9\delta \\ &\le 9\cdot\frac{\epsilon}{9} \\ &= \epsilon \end{aligned}$$

Since $\delta \le 1$, and $|y-3| < \delta$, then

$$\begin{gathered} |y-3| < \delta \le 1 \\ -1 < y-3 < 1 \\ 2 < y < 4 \\ \therefore |y| < 4 \end{gathered}$$

Since $\delta \le \frac{\epsilon}{9}$

Intermediate Approximation: Technique to break multidimensional function into parts. Useful in limit proofs

- $|xy-6| = |xy-2y+2y-6| \le |y(x-2)+2(y-3)| \le |y||x-2|+2|y-3|$
- In this case, $f(x,y) = xy$, so intermediate function is $g(x) = f(2,y) = 2y$

Proving Limit Doesn't Exist: Prove it's infinity, negate the sequential definition, or consider limits along lines

eg. Prove $\lim_{(x,y)\to(0,0)}\frac{x^{137}y^{223}}{\|(x,y)\|^{360}}$ DNE.

Let $f(x,y) = \frac{x^{137}y^{223}}{\|(x,y)\|^{360}}$.
Show $\forall L \in \mathbb{R}, \exists\{x(k)\} \subseteq \mathbb{R}^n \setminus \{a\}, x(k) \to (0{,}0)$ and $f(x(k)) \nrightarrow L$
Let $L \in \mathbb{R}$

Consider $x_1(k) = (k^{-1}, k^{-1})$: Since $k^{-1} \to 0$, then $x_1(k) \to (0{,}0)$

$$f(x_1(k)) = \frac{k^{-137}\cdot k^{-223}}{\sqrt{1/k^2+1/k^2}^{360}} = \frac{1}{k^{360}\left(\sqrt{2/k^2}\right)^{360}} = \frac{1}{k^{360}(2/k^2)^{180}} = \frac{1}{2^{180}}$$

Therefore $f(x_1(k)) \to \frac{1}{2^{180}}$

Consider $x_2(k) = (0, k^{-0.5})$: Since $k^{-0.5} \to 0$, then $x_2(k) \to (0{,}0)$

$$f(x_2(k)) = \frac{k^{-223}}{\sqrt{1/k^{0.5}}^{360}} = \frac{1}{k^{223}\cdot k^{-90}} = \frac{1}{k^{133}}$$

Therefore $f(x_2(k)) \to 0$

If $L = 0$, then pick $x(k) = x_1(k)$, so $x_1(k) \to (0{,}0)$ and $f(x_1(k)) \to \frac{1}{2^{180}} \ne 0$
If $L \ne 0$, then pick $x(k) = x_2(k)$, so $x_2(k) \to (0{,}0)$ and $f(x_2(k)) \to 0 \ne L$

eg. Prove $\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}$ DNE.

Along the line $(x,y) = (t,t)$, we have $f(t,t) = \frac{t^2-t^2}{t^2+t^2} = 0$ when $t \ne 0$, so $\lim_{t\to0} f(t,t) = \lim_{t\to0} 0 = 0$
Along the line $(x,y) = (0,t)$, we have $f(0,t) = \frac{t^2}{t^2} = 1$ when $t \ne 0$, so $\lim_{t\to0} f(0,t) = \lim_{t\to0} 1 = 1$
Since $(t,t) \to (0{,}0)$ and $(0,t) \to (0{,}0)$ but $\lim_{t\to0} f(t,t) \ne \lim_{t\to0} f(0,t)$, so $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Apparently, this isn't enough; we need some sort of contradiction?

Continuous: Function $f\colon \underline{\mathbb{R}^n} \to \mathbb{R}^m$ at point $a \in \underline{\mathbb{R}^n}$ if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \underline{\mathbb{R}^n}, \qquad \|x-a\| < \delta \Rightarrow \|f(x) - f(a)\| < \epsilon$$
$$\forall \{x(k)\} \in \underline{\mathbb{R}^n}, \qquad x(k) \to a \Rightarrow f\big(x(k)\big) \to f(a)$$
$$\forall \epsilon > 0, \exists \delta > 0, \qquad x \in \underline{\mathbb{R}^n} \cap B_\delta(a) \Rightarrow f(x) \in B_\epsilon\big(f(a)\big)$$
$$\forall \epsilon > 0, \exists \delta > 0, \qquad f\big(\underline{\mathbb{R}^n} \cap B_\delta(a)\big) \subseteq B_\epsilon\big(f(a)\big)$$

Basically $\lim_{x\to a} f(x) = f(a)$ but we allow $x = a$ so that isolated points are continuous.

- $f(x)$ is continuous at $a \Leftrightarrow f_i(x)$ is continuous at a
- **Composition**, **scalar product**, and **dot product** of continuous functions are continuous
  - g is continuous at $\lim_{x\to a} f(x) \Rightarrow \lim_{x\to a} g\big(f(x)\big) = g(\lim_{x\to a} f(x))$
- **Linear transformations** are continuous
- **Polynomials** in n variables are continuous in $\mathbb{R}^n$
  - **Monomial:** In n variables, formally, a function $x_1^{a_1} \cdots x_n^{a_n}$ for some $n \in \mathbb{N}$
  - **Polynomial:** In n variables, formally, a linear combination of monomials
- f continuous on $S_{\text{open}} \Rightarrow f^{-1}(S_{\text{open}})$ is open
- f continuous on $S_{\text{closed}} \Rightarrow f^{-1}(S_{\text{closed}})$ is closed
- f continuous on $S_{\text{compact}} \Rightarrow f(S_{\text{compact}})$ is compact

Path-Connected: Set $S \subseteq \mathbb{R}^n$ if $\forall x, y \in S, \exists \gamma\colon [a,b] \to \mathbb{R}^n, \gamma(a) = x, \gamma(b) = y, \operatorname{img}(\gamma) \subseteq S$

- f continuous on $S_{\text{path connected}} \Rightarrow f(S_{\text{path connected}})$ is path-connected

Convex: Set $S \subseteq \mathbb{R}^n$ if $\forall x, y \subseteq \mathbb{R}^n$, the line segment between x, y is in S

- S is convex $\Rightarrow S$ is path-connected

Global Maximum:	Of $f\colon \underline{\mathbb{R}^n}_1 \to \mathbb{R}$, point $p \in \mathbb{R}^n$ if $\forall x \in \underline{\mathbb{R}^n}_2, f(p) \geq f(x)$
Global Minimum:	Of $f\colon \underline{\mathbb{R}^n}_1 \to \mathbb{R}$, point $p \in \mathbb{R}^n$ if $\forall x \in \underline{\mathbb{R}^n}_2, f(p) \leq f(x)$
Local Maximum:	Of $f\colon \underline{\mathbb{R}^n}_1 \to \mathbb{R}$, point $p \in \mathbb{R}^n$ if $\exists \delta > 0, \forall x \in \underline{\mathbb{R}^n}_2 \cap B_\delta(p), f(p) \geq f(x)$
Local Minimum:	Of $f\colon \underline{\mathbb{R}^n}_1 \to \mathbb{R}$, point $p \in \mathbb{R}^n$ if $\exists \delta > 0, \forall x \in \underline{\mathbb{R}^n}_2 \cap B_\delta(p), f(p) \leq f(x)$

Intermediate Value Theorem (IVT):	f is continuous on $[a,b] \Rightarrow f([a,b])$ is path-connected
Extreme Value Theorem (EVT):	Let $A \subseteq \mathbb{R}^n$ be compact (and non-empty), $f\colon A \to \mathbb{R}$, f is continuous $\Rightarrow \exists \max_{x\in A} f(x), \min_{x\in A} f(x)$ Let $A \subseteq \mathbb{R}^n$ is closed and unbounded, $f\colon A \to \mathbb{R}$, $f(x) \to -\infty$ as $\|x\| \to \infty \Rightarrow \exists \max_{x\in A} f(x)$ $f(x) \to \infty$ as $\|x\| \to \infty \Rightarrow \exists \min_{x\in A} f(x)$

Derivatives

Note: Derivatives only work for inner points of sets.

Unary Derivative: Rate of change for $f: \mathbb{R} \to \mathbb{R}^n$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

- f is differentiable at $a \Leftrightarrow f_i$ is differentiable at a
- Only interior points of the domain are differentiable.
- $f'(x)$ exists $\Leftrightarrow f_i'(x)$ exists

- $f'(x)g'(x) = f'(x)g(x) + g'(x)f(x)$
- $f'(x)\cdot g'(x) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$
- $[f(g(x))]' = f'\big(g(x)\big)g'(x)$

Linear Approximation: Of f at a, function $\ell: \mathbb{R} \to \mathbb{R}^n$ where $\ell(x) = f(a) + f'(a)(x-a)$

$$f'(a) \approx \frac{f(x)-f(a)}{x-a} \quad \text{(when } x \approx a\text{)}$$
$$f'(a)(x-a) \approx f(x) - f(a)$$
$$f(x) \approx f(a) + f'(a)(x-a) = \ell(x)$$

Physics	$f'(t)$ is instantaneous velocity at time t, position $f(t)$
Geometry	$f'(a)$ is direction vector of the tangent line $f(a) + f'(a)h$
Analysis	$f(x) \approx \ell(x)$ when $x \approx a$
Linear Algebra	f is differentiable at $a \Leftrightarrow \exists L: \mathbb{R} \to \mathbb{R}^m$ linear, $L(h) = f'(a)h$ and $\lim_{h\to 0} \frac{f(a+h)-f(a)-L(h)}{h} = 0$ *(multidimensional case)* $\Leftrightarrow \exists L: \mathbb{R}^n \to \mathbb{R}^m$ linear, $L(h) = Df(a)h$ and $\lim_{h\to\vec{0}} \frac{f(a+h)-f(a)-L(h)}{\lVert h\rVert} = \vec{0}$ $\Leftrightarrow$ the differential of f at a exists $f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$ $0 = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h} - f'(a)\right)$ $= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h} - \frac{f'(a)h}{h}\right)$ $= \lim_{h\to 0}\frac{f(a+h)-f(a)-L(h)}{h}$ **Differential:** Of $f: \mathbb{R}^n \to \mathbb{R}^m$ at a, linear map $df_a: \mathbb{R}^n \to \mathbb{R}^m$, $df_a(h) = L(h) \overset{\text{if } n=1}{=} f'(a)h$ ➢ $df_a(h) \approx f(a+h) - f(a)$ ➢ $d(g\circ f)_a(h) \overset{\text{if } f,g \text{ differentiable}}{=} dg_{f(a)}\big(df_a(h)\big)$ ➢ $d(\alpha f + \beta g)_a(h) \overset{\text{if } f,g \text{ differentiable}}{=} \alpha\cdot df_a(h) + \beta\cdot dg_a(h)$

Partial Derivative: Rate of change for $f: \mathbb{R}^n \to \mathbb{R}^m$ in the direction of an axis/standard basis.

$$\frac{\partial f(x)}{\partial x_i} = \lim_{h\to 0} \frac{f(x+he_i)-f(x)}{h} = \left(\frac{\partial f_1(x)}{\partial x_i}, \dots, \frac{\partial f_m(x)}{\partial x_i}\right)$$

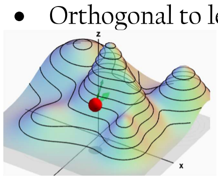
- $\frac{\partial f(x)}{\partial x_i} = \partial_i f(x) = \partial_{x_i} f(x) = D_{e_i} f(x) = D_i f(x)$
- Like 1D derivatives, but treat all but 1 variable as constant.

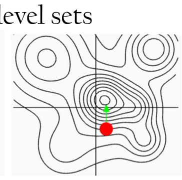
- $\frac{\partial}{\partial x_i}[f(x)+g(x)] = \frac{\partial}{\partial x_i} f(x) + \frac{\partial}{\partial x_i} g(x)$
- $\frac{\partial}{\partial x_i}[f(x)g(x)] = \frac{\partial}{\partial x_i}[f(x)]g(x) + \frac{\partial}{\partial x_i}[g(x)]f(x)$
- $\frac{\partial}{\partial x_i}[f(x)\cdot g(x)] = \frac{\partial}{\partial x_i}[f(x)]\cdot g(x) + \frac{\partial}{\partial x_i}[g(x)]\cdot f(x)$

Gradient: Of function $f: \mathbb{R}^n \to \mathbb{R}$, the vector and direction of steepest ascent.

$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix}$$

- Orthogonal to level sets





Jacobian: The gradient, but for a function $f: \mathbb{R}^n \to \mathbb{R}^m$; the $m \times n$ matrix

$$Df(x) = \frac{\partial f}{\partial x} = \left[\frac{\partial f_j}{\partial x_i}\right]_{i,j} = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) & \cdots & \frac{\partial}{\partial x_n} f(x) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x) & \cdots & \frac{\partial}{\partial x_n} f_1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_m(x) & \cdots & \frac{\partial}{\partial x_n} f_m(x) \end{bmatrix}$$

- $Df(x) = f'(x) = Jf(x) = J_f(x) = \mathrm{Jac}_f(x)$
- For $f: \mathbb{R} \to \mathbb{R}^m$, $f'(x) = Df(x) = \nabla f(x)^{\mathrm{T}}$
- $D(g\circ f)(a) \overset{\text{if } f,g \text{ differentiable}}{=} Dg\big(f(a)\big)Df(a)$

Directional Derivative: Rate of change for $f\colon \underline{\mathbb{R}^n} \to \mathbb{R}^m$ in the direction of $\vec{v} \in \mathbb{R}^n$, $D_{\vec{v}}f\colon \underline{\mathbb{R}^n} \to \mathbb{R}^m$, where

$$D_{\vec{v}}f(\vec{x}) = \lim_{h\to 0}\frac{f(\vec{x}+h\vec{v})-f(\vec{x})}{h} \overset{\text{if } f \text{ differentiable}}{=} \sum_{i=1}^{n} v_i \frac{\partial f(\vec{x})}{\partial x_i} = \vec{v}\cdot\nabla f(\vec{x}) = df_{\vec{x}}(\vec{v}) = Df(\vec{x})\vec{v}$$

- For $\mathbb{R}^n \to \mathbb{R}$, the gradient is characterized by:
  - $\operatorname{argmax}_{v\in\mathbb{R}^n,\|v\|=1} D_v f(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|}$
  - $\operatorname{argmin}_{v\in\mathbb{R}^n,\|v\|=1} D_v f(x) = -\frac{\nabla f(x)}{\|\nabla f(x)\|}$
  - $\max_{v\in\mathbb{R}^n,\|v\|=1} D_v f(x) = \|\nabla f(x)\|$
  - $\min_{v\in\mathbb{R}^n,\|v\|=1} D_v f(x) = -\|\nabla f(x)\|$
- $D_v f(x) = \nabla_v f(x)$
- $D_v f(x)$ exists $\Leftrightarrow D_v f_i(x)$ exists
- All $D_v f$ exist $\Rightarrow$ All $\partial_i f$ exist
  - Not other way around. Try $f(x,y) = \sqrt{xy}$ at (0,0)
- $\partial_i f$ is a special case of $D_v f$ where $v \in \{e_1, \dots, e_n\}$

Continuously Differentiable (C^1): A stronger condition on differentiability for function $f\colon \underline{\mathbb{R}^n} \to \mathbb{R}^m$.

$$f \text{ is } C^1 \text{ at } a \Leftrightarrow \text{All } \frac{\partial f}{\partial x_i} \text{ are continuous on } a \in \underline{\mathbb{R}^n} \text{ where } S \text{ is open}$$

$$f \text{ is } C^1 \Leftrightarrow \text{All } \frac{\partial f}{\partial x_i} \text{ are continuous}$$

- f is $C^1 \Leftrightarrow$ All $\frac{\partial f}{\partial x_i}$ continuous????
- f is $C^1 \Leftrightarrow f_i$ is C^1
- f is C^1 at $a \Rightarrow f$ differentiable at $a \Rightarrow f$ continuous at a
- f, g are $C^1 \Rightarrow f \circ g$ is C^1
- Linearity, dot/scalar product, quotients of C^1 functions are C^1

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eg. Give a differentiable function that isn't C^1.

Let $g(x) = x^2 \sin\frac{1}{x}$. Consider $f(x) = \begin{cases} g(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

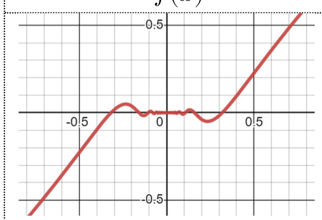
Then $f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Notice $g(x)$ exists but $g'(0)$ DNE but exists everywhere else.

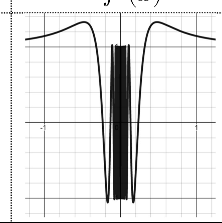
Notice $f(x)$ exists and $f'(0) = 0$.

So $f'(x)$ exists everywhere but is discontinuous at $x = 0$.

$f(x)$



$f'(x)$



Second-Order Partial Derivative: Of $f\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}^m$, the value $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f$

- **Pure:** A 2nd order derivative where $i = j$
- **Mixed:** A 2nd order derivative where $i \neq j$
- $\partial_i \partial_j f = f_{ji} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f = \frac{\partial^2}{\partial x_i \partial x_j} f$

Twice Continuously Differentiable (C^2): Function $f\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}^m$, if all $\partial_i \partial_j f$ are continuous

- **Clairaut's Theorem:** f is $C^2 \Rightarrow \partial_i \partial_j f = \partial_j \partial_i f$

Hessian: Of C^2 function $f\colon \mathbb{R}^n \to \mathbb{R}$ at a, the symmetric $n \times n$ matrix

$$Hf(a) = [\partial_i \partial_j f(a)]_{i,j} = \begin{bmatrix} \partial_1\partial_1 f(a) & \cdots & \partial_1 \partial_n f(a) \\ \vdots & \ddots & \vdots \\ \partial_n \partial_1 f(a) & \cdots & \partial_n \partial_n f(a) \end{bmatrix} = T$$

- Mostly just encodes useful information about second derivatives

k-times Continuously Differentiable (C^k): Function $f\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}^m$, if k-th order partials are continuous

- **Generalized Clairaut's Theorem:** f is $C^k \Rightarrow \partial_{i_1} \cdots \partial_{i_k} f = \partial_{j_1} \cdots \partial_{j_k} f$ for all re-orderings $j_1, \dots, j_k$ of $i_1, \dots, i_k$

Smooth (C^∞): Function $f\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}^m$, if $\forall k \in \mathbb{N}$, its k-th order partials are continuous

- Polynomials are C^∞

Tangent Spaces and Manifolds

Tangent: Vector $v \in \underline{\mathbb{R}^n}$ to set $\underline{\mathbb{R}^n}$ at point $p \in \underline{\mathbb{R}^n}$ iff $\exists I \subseteq \mathbb{R}$ open, $\exists \gamma: I \to \mathbb{R}^n$ differentiable,

- $0 \in I$
- $\gamma(0) = p$
- $\gamma(I) \subseteq \underline{\mathbb{R}^n}$
- $\gamma'(0) = v$

"There's a particle moving along $\underline{\mathbb{R}^n}$ ($\gamma(I) \subseteq \underline{\mathbb{R}^n}$) through p ($\gamma(0) = p$) with velocity v ($\gamma'(0) = v$)"

Tangent Space (T_pS): Of $S \subseteq \mathbb{R}^n$ at $p \in S$, the set of tangent vectors to S at p.

- $0 \in T_pS$

"All possible velocities for a particle moving along S through p"

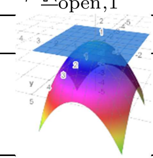
Tangent Plane: Of $S \subseteq \mathbb{R}^n$ at $p \in S$, the tangent space of S translated to p.

$$p + T_pS = \{p + v : v \in T_pS\}$$

Let $f: \underline{\mathbb{R}^k}_{\text{open},1} \to \mathbb{R}^n$ be differentiable and $S \subseteq \mathbb{R}^{k+n}$ be its graph. Let $p = (a, f(a)) \in S$. Then...

$\gamma: \underline{\mathbb{R}}_{\text{open},2} \to \mathbb{R}^{n+k}$ is differentiable and $\gamma(\underline{\mathbb{R}}_{\text{open},2}) \subseteq S \Leftrightarrow \gamma(t) = \big(g(t), f(g(t))\big)$ for some $g: \mathbb{R} \to \underline{\mathbb{R}^k}_{\text{open},1}$

$T_pS = \{(v, df_a(v)) \in \mathbb{R}^{n+k} : v \in \mathbb{R}^k\}$ is a k-dimensional subspace of $\mathbb{R}^{n+k}$



eg. Consider $f: \mathbb{R}^2 \to \mathbb{R}^1$ where $f(x, y) = -x^2 - y^2$.
So $n = 2, m = 1$.
Then for any $p \in \mathbb{R}^2$, T_pS is a 2-dimensional subspace of $\mathbb{R}^3$ (ie. a plane passing the origin).

***k*-dimensional Smooth Manifold:** AKA smooth surface for $k = 2$, smooth curve for $k = 1$...

Set $S \subseteq \mathbb{R}^{n+k}$ at $p \in S$...	Set $S \subseteq \mathbb{R}^{n+k}$ if...
• iff $\exists U \subseteq \mathbb{R}^{n+k}$ open, $p \in U$, $S \cap U$ is a graph of C^1 $f: \underline{\mathbb{R}^k}_{\text{open}} \to \mathbb{R}^n$ • iff $\exists \epsilon > 0$, $S \cap B_\epsilon(p)$ is a graph of C^1 $f: \underline{\mathbb{R}^k}_{\text{open}} \to \mathbb{R}^n$ • if S is a graph of C^1 $f: \underline{\mathbb{R}^k}_{\text{open}} \to \mathbb{R}^n$	• $\forall p \in S$, S is a k-dimensional smooth manifold at p • $\forall p \in S$, $T_p(S)$ is a k-dimensional subspace of $\mathbb{R}^{n+k}$

- "Any set in $\mathbb{R}^{n+k}$ that is a graph of a C^1 function with an open domain in $\mathbb{R}^k$"
- S is a k-dimensional smooth manifold at $p \in S \Rightarrow T_p(S)$ is a k-dimensional subspace of $\mathbb{R}^n$
- S is a k-dimensional smooth manifold $\Rightarrow \partial S = \emptyset$

eg. Prove $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a 1-dimensional smooth manifold at $p = (1,0)$.

Show $\exists U \subseteq \mathbb{R}^2$ open, $p \in U$, $S \cap U$ is the graph of $f: V \to \mathbb{R}$ where f is C^1, $V \subseteq \mathbb{R}$ open
Pick $U = \{(x, y) \in \mathbb{R}^2 : x > 0\} \subseteq \mathbb{R}^2$, which is open, then $p \in U$
Pick $V = (-1,1) \subseteq \mathbb{R}$
Pick $f: V \to \mathbb{R}$ where $f(x) = \sqrt{1 - x^2}$, which is C^1 on V
Then

$$\begin{aligned} S \cap U &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x > 0\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : x = \sqrt{1 - y^2}, y \in (-1,1)\right\} \\ &= \{(f(y), y) \in \mathbb{R}^2 : y \in V\} \end{aligned}$$

Thus $S \cap U$ is the graph of f.

eg. Prove $S = \{(x, y) \in \mathbb{R}^2 : x^2 = y^3\}$ is not a 1-dimensional smooth manifold at $p = (0,0)$.

Suppose otherwise, that $\exists U \subseteq \mathbb{R}^2$ open, $p \in U$, $S \cap U$ is the graph of $f: V \to \mathbb{R}$ where f is C^1, $V \subseteq \mathbb{R}$ open

Case 1: $S \cap U = \{(x, f(x)) \in \mathbb{R}^2 : x \in V\}$

$$\begin{aligned} S \cap U &= \{(x, y) \in \mathbb{R}^2 : x^2 = y^3, x \in V\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : y = \sqrt[3]{x^2}, x \in V\right\} \\ &= \left\{\big(x, f(x)\big) \in \mathbb{R}^2 : x \in V\right\} \end{aligned}$$

So $f(x) = \sqrt[3]{x^2}$. Since $p = (0,0) \in S \cap U$, then $0 \in V$. But $f(x)$ is not C^1 at 0, a contradiction.

Case 2: $S \cap U = \{(f(y), y) \in \mathbb{R}^2 : y \in V\}$

Since $(0,0) \in U$ and U is open, $\exists 0 < \epsilon < 1$, $B_{2\epsilon}(0,0) \subseteq U$
Then $(\pm\epsilon^3, \epsilon^2) \in S$ and $(\pm\epsilon^3, \epsilon^2) \in U$
(since $(\pm\epsilon^3)^2 = (\epsilon^2)^3$ and $\|(\pm\epsilon^3, \epsilon^2)\| = \sqrt{2\epsilon^6} \leq 2\epsilon$)
Then $(\pm\epsilon^3, \epsilon^2) \in S \cap U = \{(f(y), y) \in \mathbb{R}^2 : y \in V\}$
Therefore $f(\epsilon^2) = \pm\epsilon^3$, which is a contradiction.

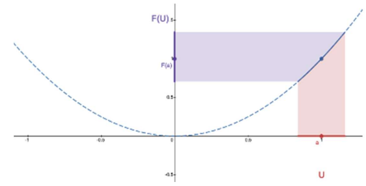
Inverse and Implicit Functions

Global Inverse: Of $f: \underline{\mathbb{R}^n}_1 \to \underline{\mathbb{R}^n}_2$, function $f^{-1}: \underline{\mathbb{R}^n}_2 \to \underline{\mathbb{R}^n}_1$ such that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

(Global) Diffeomorphism: Invertible function $f: \underline{\mathbb{R}^n}_1 \to \underline{\mathbb{R}^n}_2$ with an open domain/codomain if f, f^{-1} are C^1

- f is a diffeomorphism $\Leftrightarrow$ f^{-1} is a diffeomorphism
- f is a diffeomorphism $\Rightarrow$ $f(S)$ and S have the same open/closed/compact/path-connectedness
- f is a diffeomorphism $\Rightarrow$ $Df^{-1}\big(f(a)\big) = [Df(a)]^{-1}$
- eg. $f(x) = x^3$ not diffeomorphism as $f^{-1}(x) = x^{\frac{1}{3}}$, not C^1 at 0 since $f'(0) = 0$ and $f^{-1\prime}(0) = \infty$.



Local Diffeomorphism: Function $f: \underline{\mathbb{R}^n}_{\text{open},} \to \underline{\mathbb{R}^n}_{\text{open},}$ at a iff

$\exists U \subseteq \underline{\mathbb{R}}_{\text{open},1}$ open, $a \in U$, $f|_U: U \to f(U)$ is a diffeomorphism

- $f|_U$ is just fancy notation for f with domain U, codomain $f(U)$
- **Local Inverse:** The function $f^{-1}: f(U) \to U$
- $f: \underline{\mathbb{R}^n}_{\text{open}} \to \underline{\mathbb{R}^n}_{\text{ope}}$ is a global diffeo. $\Rightarrow \forall a \in \underline{\mathbb{R}^n}_{\text{open},}$, f is a local diffeo. at a
- $f: \underline{\mathbb{R}^n}_{\text{open},1} \to \underline{\mathbb{R}^n}_{\text{open},2}$ is a local diffeo. at a and is C^1 $\Leftrightarrow$ $Df(a)$ is invertible (ie. det $Df(a) \neq 0$)

---

eg. Give a local diffeomorphism at every point in its domain that isn't a global diffeomorphism.

Consider $f: \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (e^x \cos y\,, e^x \sin y)$

Intuitively: From the sin and cos, f isn't invertible on $\mathbb{R}^2$, so f^{-1} doesn't exist, so f is not a global diffeo.
But if $y's$ domain is restricted to a length of $< \pi$, the f is C^1 and invertible, making it a local diffeo. everywhere.

Formally: $Df(x, y) = [\partial_1 f(x, y) \quad \partial_2 f(x, y)] = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$, whose determinant is $e^{2x} \neq 0$, therefore $Df(x, y)$ is invertible everywhere, thus f is a local diffeomorphism everywhere.
But f is not a global diffeo., as $f(x, y) = (e^x \cos y\,, e^x \sin y) = (e^x \cos(y + 2\pi)\,, e^x \sin(y + 2\pi)) = f(x, y + 2\pi)$, so f is not one-to-one, and thus not invertible, and thus not a diffeo.

---

eg. Prove $f(x) = x^2$ is not a local diffeomorphism at $x = 0$.

Show $\forall U \subseteq \mathbb{R}$ open, $0 \in U \Rightarrow f|_U: U \to f(U)$ is not a diffeomorphism.
Let $U \subseteq \mathbb{R}$ be open, assume $0 \in U$.
Then by definition of open, $\exists \epsilon > 0, B_{2\epsilon}(0) = (-2\epsilon, 2\epsilon) \subseteq U$.
Then $f(\epsilon) = f(-\epsilon) = \epsilon^2$, meaning f is not one-to-one, and thus not invertible, and thus not a diffeomorphism.

---

eg. Prove $f(x) = \sqrt[3]{x}$ is local diffeomorphism at $x \neq 0$.

Let $x \neq 0$
Show $\exists U \subseteq \mathbb{R}$ open, $x \in U$, $f|_U: U \to f(U)$ is a diffeomorphism.
Consider $U = (\frac{1}{2}x, 2x)$, which is open. $x \in U$ holds too.
We know $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ is continuous on $x \neq 0$, so it is continuous on U, so $f|_U$ is C^1 on its domain.
Pick $f^{-1}(x) = x^3$, the global inverse of f, so it's invertible on $f(U)$. It's a polynomial so it's C^1 everywhere, including $f(U)$. Thus $F|_U$ is a diffeomorphism.

Locally Defined: Variable y, near $(a, b) \in U \subseteq \mathbb{R}^{n+k}$, by C^1 function $f: U \to \mathbb{R}^k$ with $f(x, y) = 0$, if

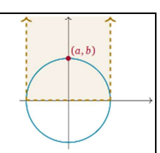
$\exists V \subseteq \mathbb{R}^n$ open, $\exists W \subseteq \mathbb{R}^k$ open, $\exists \phi: V \to W$,

- $a \in V$
- $V \times W \subseteq U$
- $b \in W$
- ϕ is C^1
- $\forall (x, y) \in V \times W, f(x, y) = 0 \Leftrightarrow y = \phi(x)$

AKA...

$\{(x, y) \in V \times W: f(x, y) = 0\} = \{\big(x, \phi(x)\big): x \in V\}$

eg. $f(x, y) = x^2 + y^2 - 1 = 0$ locally defines y as a C^1 function of x near (0,1), but not x as a C^1 function of y near (0,1).



"You can write $x^2 + y^2 - 1 = 0$ as a local $g(x) = y = \sqrt{1 - x^2}$, which passes through (0,1), so $f(x, y) = \big(x, g(x)\big)$ around (0,1). You can't do this for $f(y)$."

Let A be $k \times n$ matrix, B be $k \times k$ matrix, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$

$$B \text{ is invertible} \Leftrightarrow [A|B]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ globally defines } y \text{ as a } C^1 \text{ function of } x$$

Let $f: \underline{\mathbb{R}^{n+k}}_{\text{open}} \to \mathbb{R}^k$ be C^1 and non-constant, $(a,b) \in \underline{\mathbb{R}^{n+k}}_{\text{open}}$.

Assume $f(x,y) = 0$ defines y locally as a C^1 function $\phi: V \to W$ of x near (a,b)

If $k = 1$...	If $k > 1$...
$\frac{\partial f}{\partial x_i}(v,\phi(v)) + \frac{\partial f}{\partial y}(v,\phi(v))\frac{\partial \phi}{\partial x_i}(v) = 0$	$\frac{\partial f}{\partial x}(v,\phi(v)) + \frac{\partial f}{\partial y}(v,\phi(v))D\phi(v) = 0$

Implicit Function Theorem:

Let $f: \underline{\mathbb{R}^{n+k}}_{\text{open}} \to \mathbb{R}^k$ be C^1, $(a,b) \in \underline{\mathbb{R}^{n+k}}_{\text{open}}$.

If $k = 1$...	If $k > 1$...
Assume $f(a,b) = 0, \frac{\partial f}{\partial y}(a,b) \neq 0$	Assume $f(a,b) = 0, \frac{\partial f}{\partial y}(a,b) = \left[\frac{\partial f_i}{\partial y_j}(a,b)\right]_{i,j}$ is invertible

$$f(x,y) = 0 \text{ defines } y \text{ locally as a } C^1 \text{ function } \phi: \mathbb{R}^n \to \mathbb{R}^k \text{ of } x \text{ near } (a,b)$$

eg. $(x,y,z) = (1,-4,3)$ solves $x^2 + \sin(x+y+z) = 1$. Can x be expressed locally as a C^1 function of (y,z)?

Define $f: \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = x^2 + \sin(x,y,z) - 1$.
Since f is the sum of a polynomial (C^1), a constant (C^1), and a sine function (C^1), f is C^1.
Also, $f(1,-4,3) = 1 + \sin(0) - 1 = 0$
Also, $\frac{\partial f}{\partial x} = 2x + \cos(x+y+z)$, so $\frac{\partial f}{\partial x}(1,-4,3) = 2 + \cos(0) = 3 \neq 0$
Therefore, by implicit function theorem, $f(x,y,z) = 0$ defines x locally as a C^1 function of (y,z) near $(1,-4,3)$.

Let $f: \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}$ be C^1, $p \in S = f^{-1}(\{0\})$,

$$\nabla f(p) \neq 0 \text{ for any } p \in S \Rightarrow S \text{ is a } (n-1)\text{-D smooth manifold at } p$$

"Any steepness in contours mean smooth manifold at that point"

$$\nabla f(p) \cdot v = 0 \Leftrightarrow v \in \mathbb{R}^n \text{ is a tangent vector of } S \text{ at } p$$
$$T_pS = \{v \in \mathbb{R}^n : \nabla f(p) \cdot v = 0\}$$
$$p + T_pS = \{x \in \mathbb{R}^n : \nabla f(p) \cdot (x-p) = 0\}$$

"Tangent vectors to level sets of f are orthogonal to ∇f"

eg. Give a f where $\nabla f(p) = 0$ and S is a $(n-1)$-D smooth manifold at p.

Consider C^1 function $f: \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = (y-x^2)^2$.

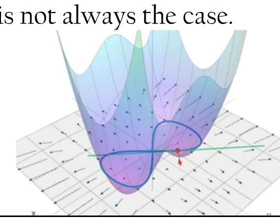
$$f^{-1}(\{0\}) = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$$
$$\nabla f(x,y) = (4x(y-x^2), 2(y-x^2))$$

Note that $f^{-1}(\{0\}) = \{(x,g(x)) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ is a graph of C^1 function $g(x) = x^2$, so $f^{-1}(\{0\})$ is a 1-D smooth manifold at $(0,0)$
Note that $\nabla f(0,0) = (0,0)$.

eg. C^1 function $f: \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = x^4 - x^2 + y^2$ has gradient $\nabla f(x,y) = (4x^3 - 2x, 2y)$.

For $p = (1,0)$: $\nabla f(p) = (2,0) \neq 0$, so $S = f^{-1}(\{0\})$ is a 1-D smooth manifold at p.

For $p = (0,0)$: $\nabla f(p) = (0,0)$. S is not a 1-D smooth manifold at p, but this is not always the case.



Rank:	Of matrix A, the dimension of basis formed by column/row vectors *(ie. highest # of linearly indep. vectors)*
Null Space:	Of matrix A, every vector that gets mapped to 0 under A

Let $f: \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}^k$ be C^1, $p \in S = f^{-1}(\{0\})$

$$\text{rank}(Df(p)) = k \Rightarrow S \text{ is a } (n-k)\text{-D smooth manifold at } p$$
$$T_pS = \{v \in \mathbb{R}^n : DF(p)v = 0\}$$
$$= \text{null}(df_p)$$

eg. Consider C^1 function $f: \mathbb{R}^3 \to \mathbb{R}^2$ with $f(x,y,z) = (x^2 + y^2 + z^2 - 16, (y-2)^2 + z^2 - 4)$. Find at what points is $S = f^{-1}(\{0\})$ a 1-D smooth manifold. Note that $Df(x,y,z) = \begin{bmatrix} 2x & 2y & 2z \\ 0 & 2y-4 & 2z \end{bmatrix}$.

Find all p where $\operatorname{rank}\big(Df(p)\big) = k = 2$ (ie. 2 of 3 column vectors are linearly independent).
Recall a matrix is invertible iff column vectors are linearly independent, so check for non-zero determinants of:

$$\begin{bmatrix} 2x & 2y \\ 0 & 2y-4 \end{bmatrix}, \begin{bmatrix} 2x & 2z \\ 0 & 2z \end{bmatrix}, \begin{bmatrix} 2y & 2z \\ 2y-4 & 2z \end{bmatrix}$$

The above is just combinations of the 3 column vectors. We get three simple equations:

$$\begin{aligned} 2x(2y-4) &\neq 0, & 4xz &\neq 0, & 4yz - 2z(2y-4) &\neq 0 \\ x(y-2) &\neq 0, & xz &\neq 0, & z &\neq 0 \end{aligned}$$

If any is true, then rank is 2. Consider when all of them are false, meaning $z = 0$ and $(x = 0 \text{ or } y = 2)$.
If $y = 2$, then $f(x, 2, 0) = (x^2 - 12, -4)$
Since $(x^2 - 12, -4) \neq (0,0)$ for any x value, then it's not in $f^{-1}(\{0\})$

If $x = 0$, then $f(0, y, 0) = (y^2 - 16, (y-2)^2 - 4)$
Since $(y^2 - 16, (y-2)^2 - 4) = (0,0)$ for $y = 4$, then it's in $f^{-1}(\{0\})$

So when $p = (x,y,z) \neq (0,4,0)$, we know $\operatorname{rank}\big(Df(p)\big) = k$ and thus f is a 1-D smooth manifold at p.
We don't know what happens at $p = (0,4,0)$.

Optimization

Local Extreme Value Theorem: For $f\colon \underline{\mathbb{R}^n} \to \mathbb{R}$, interior point $a \in \underline{\mathbb{R}^n}$,

$$(a \text{ is a local extremum of } f) \text{ and } (f \text{ is differentiable at } a) \Rightarrow \nabla f(a) = 0$$

Critical Point: For $f\colon \underline{\mathbb{R}^n} \to \mathbb{R}$, point $a \in \underline{\mathbb{R}^n}$ if $\nabla f(a) = 0$ or $\nabla f(a)$ DNE

- a is a local extremum of $f \Rightarrow a$ is a boundary point of $\mathbb{R}^n$ or a is a critical point of f
- Critical points must be interior points

Saddle Point: For $f\colon \underline{\mathbb{R}^n} \to \mathbb{R}$, point $a \in \underline{\mathbb{R}^n}$ if $\nabla f(a) = 0$ and a is not a local extremum of f

If the domain is bounded (and f continuous), conclude a maximum/minimum exists

Find critical points on A° using $\nabla f = 0$

Parametrize the boundary ∂A and find critical points using $\nabla \gamma = 0$. Check edges of γ too. Or Lagrange.

If the domain is unbounded (and f continuous),

Check $\lim_{\|x\|\to\infty} f(x) = \pm\infty$ to see if a minimum/maximum exists

Lagrange Multiplier Theorem: Useful function for optimizing f on any number of level sets of g.

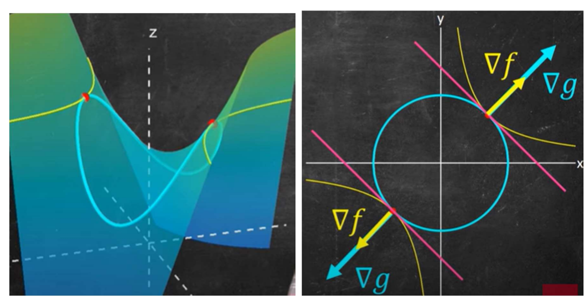
Single-Function Case	Multi-Function Case
Let $f\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}$ differentiable	Let $f\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}$ differentiable
Let $g\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}$ be C^1	Let $g_i\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}$ be C^1
Let $S = g^{-1}(\{k\}) = \left\{x \in \underline{\mathbb{R}^n}_{\text{open}} : g(x) = k\right\}$	Let $S = \left\{x \in \underline{\mathbb{R}^n}_{\text{open}} : g_i(x) = k_i \text{ for all } i\right\}$
Assume $\nabla g(x) \neq 0$ for all $x \in S$	Assume $\nabla g_i(x)$ are linearly independent for all $x \in S$
f has local extremum on S at a $\Rightarrow \exists \lambda \in \mathbb{R}, \nabla f(a) = \lambda \nabla g(a)$	f has local extremum on S at a $\Rightarrow \exists \lambda \in \mathbb{R}, \nabla f(a) = \sum_{i=1}^{n} \lambda_i \nabla g_i(a)$

We optimize $f(x)$ with a restriction $\boxed{g(x) = k}$.

At optimal areas, $f(x)$'s level set skims $g(x)$'s level set.

Since level sets are orthogonal to gradients, then we know ∇f and ∇g point in the same direction, but they may have different magnitudes.

Thus $\boxed{\nabla f = \lambda \nabla g}$, where λ is the **Lagrange multiplier**. Solve the two boxed equations, the Lagrange system.



eg. Find the maximum and minimum of $f\colon \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x_1 + \cdots + x_n$ on unit sphere $x_1^2 + \cdots + x_n^2 = 1$.

f is the sum of polynomials, so it is differentiable.

Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be $g(x) = x_1^2 + \cdots + x_n^2$, which is the sum of polynomials so it is C^1.

$\nabla g(x) = (2x_1, \dots, 2x_n)$, which is non-zero when $x_1^2 + \cdots + x_n^2 = 1$ (ie. on $g^{-1}(\{1\})$)

The unit sphere is compact and f is continuous, thus by extreme value theorem, $\exists \max_{x\in S} f(x)$ and $\exists \min_{x\in S} f(x)$.

We then solve the system $\begin{cases} \nabla f(x) = \lambda \nabla g(x) \\ g(x) = 1 \end{cases} = \begin{cases} (1, \dots, 1) = \lambda(2x_1, \dots, 2x_n) \\ x_1^2 + \cdots + x_n^2 = 1 \end{cases} = \begin{cases} 1 = 2\lambda x_i \\ x_1^2 + \cdots + x_n^2 = 1 \end{cases}$

The top simplifies to $x_i = \frac{1}{2\lambda}$, which, when plugged in the bottom, gets $\frac{n}{(2\lambda)^2} = 1$, which solves to $\lambda = \pm\frac{\sqrt{n}}{2}$.

Plugging this gets $x_i = \pm\frac{1}{\sqrt{n}}$, so the solutions are $x = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right), \left(\frac{1}{-\sqrt{n}}, \dots, \frac{1}{-\sqrt{n}}\right)$.

$$f\left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right) = \frac{n}{\sqrt{n}} = \sqrt{n}, \qquad f\left(\frac{1}{-\sqrt{n}}, \dots, \frac{1}{-\sqrt{n}}\right) = \frac{n}{-\sqrt{n}} = -\sqrt{n}$$

These two values are the maximum and the minimum.

Approximations

Unknown $f(x)$, Known $f(p)$	Unknown $D_v f(x)$, Known $v, f(x)$	Nonlinear $f(x)$, known $f(a)$
Find v such that $p+v=x$ $\begin{aligned} f(x) &\approx f(p)+D_v f(p) \\ &= f(p)+df_p(v) \\ &= f(p)+f'(p)v \end{aligned}$	Look where v points, find a h, p so that $p=x+hv$ $\begin{aligned} D_v f(x) &\approx \frac{f(x+hv)-f(x)}{h} \\ &= \frac{f(p)-f(x)}{h} \end{aligned}$	$f(x) \approx f(a)+Df(a)(x-a)$

Mean Value Theorem: Let $f: \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}$ differentiable. If $\underline{\mathbb{R}^n}_{\text{open}}$ contains line segment L from a to b,

$$\exists c \in L, f(b)-f(a)=\nabla f(c)\cdot(b-a)$$

Let $f, g: \underline{\mathbb{R}^n}_{\text{open,path connected}} \to \mathbb{R}^m$ be C^1, then

- $Df(x)$ is zero matrix $\Leftrightarrow$ f is constant function
- $Df(x)=Dg(x) \Rightarrow \exists c \in \mathbb{R}^m, f(x)=g(x)+c$

Multi-Index: A vector α where α_i is the frequency of ∂_i in a monomial or partial derivative of a C^k function f.

$$\partial^\alpha f = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f, \qquad x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$$

- **Degree (deg):** The non-negative integer $|\alpha|=\alpha_1+\cdots+\alpha_n$
  - $|\alpha| \geq N+1 \Rightarrow \lim_{x\to 0}\frac{x^\alpha}{\|x\|^N}=0$
- **Factorial:** The positive integer $\alpha! = \alpha_1!\cdots\alpha_n!$
- Let $f: \mathbb{R}^n \to \mathbb{R}$ where $f(x)=x^\beta$, then
  - $\alpha=\beta \Rightarrow \partial^\alpha f(x)=\alpha!$
  - $\alpha\neq\beta \Rightarrow \partial^\alpha f(0)=0$
  - $|\alpha|>|\beta| \Rightarrow \partial^\alpha f(x)=0$
- $P: \mathbb{R}^n \to \mathbb{R}$ is a polynomial and $\deg(P)\leq k \Rightarrow P(x)=\sum_{\alpha\in\mathbb{N}^n,|\alpha|\leq k}\frac{\partial P^\alpha(0)}{\alpha!}x^\alpha$

For $\alpha=(0,1,2), \beta=(1,1,0), \gamma=(1,3,2)$, note that $|\gamma|>|\alpha|>|\beta|$.

Show $\partial^\alpha x^\alpha = \alpha!$	Show $\lvert\alpha\rvert>\lvert\beta\rvert \Rightarrow \partial^\alpha x^\beta=0$	Show $\lvert\alpha\rvert\neq\lvert\gamma\rvert \Rightarrow \partial^\alpha f(x)=0$
$\begin{aligned} \partial^\alpha(x,y,z)^\alpha &= \partial^{(0,1,2)}(x,y,z)^{(0,1,2)} \\ &= \frac{\partial}{\partial y}\frac{\partial^2}{\partial z^2}yz^2 \\ &= 2 \\ &= 0!\cdot 1!\cdot 2! \\ &= \alpha! \end{aligned}$	$\begin{aligned} \partial^\alpha(x,y,z)^\beta &= \partial^{(0,1,2)}(x,y,z)^{(1,1,0)} \\ &= \frac{\partial}{\partial y}\frac{\partial^2}{\partial z^2}xy \\ &= 0 \end{aligned}$	$\begin{aligned} \partial^\alpha(x,y,z)^\gamma &= \partial^{(0,1,2)}(x,y,z)^{(1,3,2)} \\ &= \frac{\partial}{\partial y}\frac{\partial^2}{\partial z^2}xy^3z^2 \\ &= 6xy^2 \end{aligned}$ $\therefore \partial^\alpha f(0)=0$

Let $P(x,y,z)=(x,y,z)^\beta=(x,y,z)^{(1,1,0)}$.

$$\begin{aligned} P(x,y,z) &= \sum_{\alpha\in\mathbb{N}^3,|\alpha|\leq 2}\frac{\partial P^\alpha(0)}{\alpha!}(x,y,z)^\alpha \\ &= \frac{\partial P^{(1,1,0)}(0)}{(1,1,0)!}(x,y,z)^{(1,1,0)} + \frac{\partial P^{(1,0,1)}(0)}{(1,0,1)!}(x,y,z)^{(1,0,1)} + \frac{\partial P^{(0,1,1)}(0)}{(0,1,1)!}(x,y,z)^{(0,1,1)} \\ &\quad + \frac{\partial P^{(0,0,2)}(0)}{(0,0,2)!}(x,y,z)^{(0,0,2)} + \frac{\partial P^{(0,2,0)}(0)}{(0,2,0)!}(x,y,z)^{(0,2,0)} + \frac{\partial P^{(2,0,0)}(0)}{(2,0,0)!}(x,y,z)^{(2,0,0)} \\ &= xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}P(0) + xz\frac{\partial}{\partial x}\frac{\partial}{\partial z}P(0) + yz\frac{\partial}{\partial y}\frac{\partial}{\partial z}P(0) + \frac{1}{2}\left[z^2\frac{\partial^2}{\partial z^2}P(0) + y^2\frac{\partial^2}{\partial y^2}P(0) + x^2\frac{\partial^2}{\partial x^2}P(0)\right] \\ &= xy \end{aligned}$$

The summation makes every xyz combination where $|\alpha|\leq k$, and $\frac{\partial P^\alpha(0)}{\alpha!}=1$ iff the xyz combination is exactly that of P's.

***N*-th Order Approximation:** Of $f: \mathbb{R}^n \to \mathbb{R}$ at $a \in \mathbb{R}^n$, function $g: \mathbb{R}^n \to \mathbb{R}$ if

$$\lim_{x\to a}\frac{f(x)-g(x)}{\|x-a\|^N}=0$$

N-th Taylor Polynomial: At $a \in \mathbb{R}^n$, of $f: \underline{\mathbb{R}^n} \to \mathbb{R}$ that is C^N on some $B_\epsilon(a)$,

$$P_N(x) = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \le N} \frac{\partial^\alpha f(a)}{\alpha!}(x-a)^\alpha$$
$$= \sum_{k=0}^{N} \sum_{\alpha \in \mathbb{N}^n, \alpha_1 + .. + \alpha_n = k} \frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^k f(a)}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}} (x_1 - a_1)^{\alpha_1} \cdots (x_n - a_n)^{\alpha_n}$$

- P is N-th Taylor polynomial of f at $a \Leftrightarrow \deg(P) \le N$ and $\left(\forall \alpha \in \mathbb{N}^n, |\alpha| \le N \Rightarrow \partial^\alpha f(a) = \partial^\alpha P(a)\right)$
  - "Any partial derivatives of f and P must match up until they add up to N"
- **Taylor's Theorem:** Let $f: \mathbb{R}^n \to \mathbb{R}$ be C^{N+1} on some $B_\epsilon(a)$ where $a \in \mathbb{R}^n$. Then
  P is N-th Taylor polynomial of f at $a \Leftrightarrow \deg(P) \le N$ and P is an Nth order approximation of f at a

$$P_0(x) = f(a)$$
$$P_1(x) = f(a) + \nabla f(a) \cdot (x-a)$$
$$P_2(x) = f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2}(x-a)^{\mathrm{T}} Hf(a)(x-a)$$

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eg. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be C^∞ and its 3rd Taylor polynomial at $a = (0,2,0)$ is

$$P(x,y,z) = 1 - 2x + 3(y-2) - 7z + x^2 - x(y-2) - 3xz + (y-2)^2 - 6(y-2)z + 2z^2 + 3x^3 + 7xz^2$$

Find $\nabla f(a)$ and $Hf(a)$.

$\nabla f(a) = \left(\partial_1 f(a), \cdots, \partial_3 f(a)\right)$, which corresponds to $|\alpha| = 1$

$Hf(a) = \begin{bmatrix} \partial_1\partial_1 f(a) & \cdots & \partial_1\partial_3 f(a) \\ \vdots & \ddots & \vdots \\ \partial_3\partial_1 f(a) & \cdots & \partial_3\partial_3 f(a) \end{bmatrix}$, which corresponds to $|\alpha| = 2$

α	$\alpha!$	P's α Term	P's α Coefficient	$\partial^\alpha P$	$\partial^\alpha P(a)$
(0,0,0)	1	1	1	$P(x,y,z)$	1
(1,0,0)	1	$-2x$	-2	$-2 + 2x - (y-2) - 3z + 9x^2 + 7z^2$	-2
(0,1,0)	1	$3(y-2)$	3	$3 - x + 2(y-2) - 6z$	3
(0,0,1)	1	$-7z$	-7	$-7 - 3x - 6(y-2) + 4z + 14xz$	-7
(1,1,0)	1	$-x(y-2)$	-1	-1	-1
(1,0,1)	1	$-3xz$	-3	$-3 + 14z$	-3
(0,1,1)	1	$-6(y-2)z$	-6	-6	-6
(2,0,0)	2	x^2	1	$2 + 18x$	2
(0,2,0)	2	$(y-2)^2$	1	2	2
(0,0,2)	2	$2z^2$	2	$4 + 14x$	4

We don't count the 2 in $(y-2)$ as a coefficient because it's part of the y offset in order to match $a = (0,2,0)$.

Note that $\underline{\partial^\alpha P(a) = \alpha! \times P\text{'s } \alpha \text{ Coefficient}}$. This is because repeated derivatives create factorials.

Since P is a 3rd Taylor polynomial, $\partial^\alpha P(a) = \partial^\alpha f(a)$ for $|\alpha| \le 3$. So we can calculate

$$\nabla f(a) = \begin{bmatrix} \partial_1 f(a) \\ \partial_2 f(a) \\ \partial_3 f(a) \end{bmatrix} = \begin{bmatrix} \partial^{(1,0,0)} f(a) \\ \partial^{(0,1,0)} f(a) \\ \partial^{(0,0,1)} f(a) \end{bmatrix} = \begin{bmatrix} \partial^{(1,0,0)} P(a) \\ \partial^{(0,1,0)} P(a) \\ \partial^{(0,0,1)} P(a) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -7 \end{bmatrix}$$

$$Hf(a) = \begin{bmatrix} \partial_1\partial_1 f(a) & \partial_1\partial_2 f(a) & \partial_1\partial_3 f(a) \\ \partial_2\partial_1 f(a) & \partial_2\partial_2 f(a) & \partial_2\partial_3 f(a) \\ \partial_3\partial_1 f(a) & \partial_3\partial_2 f(a) & \partial_3\partial_3 f(a) \end{bmatrix} = \begin{bmatrix} \partial^{(2,0,0)} P(a) & \partial^{(1,1,0)} P(a) & \partial^{(1,0,1)} P(a) \\ \partial^{(1,1,0)} P(a) & \partial^{(0,2,0)} P(a) & \partial^{(0,1,1)} P(a) \\ \partial^{(1,0,1)} P(a) & \partial^{(0,1,1)} P(a) & \partial^{(0,0,2)} P(a) \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -6 \\ -3 & -6 & 4 \end{bmatrix}$$

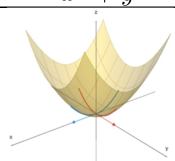
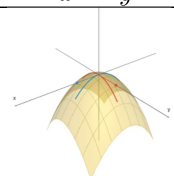
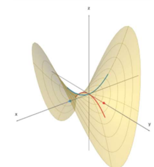
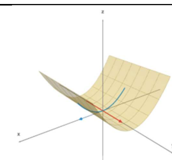
Orthonormal: Vectors x, y if they are unit vectors (ie. $\|x\| = \|y\| = 1$) and orthogonal (ie. $x \cdot y = 0$).

- Every square symmetric matrix has an orthonormal basis of eigenvectors

Quadratic Form:

At $a \in \mathbb{R}^n$, of $f: \mathbb{R}^n \to \mathbb{R}$ that is C^2 on some $B_\epsilon(a)$, function $q: \mathbb{R}^n \to \mathbb{R}$, $q(v) = v^{\mathrm{T}} Hf(a)$	Associated with symmetric square matrix A, function $q: \mathbb{R}^n \to \mathbb{R}$, $q(v) = v^{\mathrm{T}} A v$

- This comes from the last part of $P_2(x) = f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2}(x-a)^{\mathrm{T}} Hf(a)(x-a)$, whose behaviour determines what happens at critical points (ie. the $\mathbb{R}^n$ generalization of $f''(x)$)
- We use $P_2(x)$ to approximate $f(x)$ because it's easier.
- If $\nabla f(a) = 0$, then $P_2(x) = f(a) + \frac{1}{2} q(x-a)$
  - $q > 0$ on $\mathbb{R}^n \setminus \{0\} \Rightarrow P_2$ has a global minimum at a
  - $q < 0$ on $\mathbb{R}^n \setminus \{0\} \Rightarrow P_2$ has a global maximum at a
  - $q > 0$ and $q < 0$ on $\mathbb{R}^n \setminus \{0\} \Rightarrow P_2$ has no global extremum at a

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
$q_1(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2$	$q_2(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - y^2$	$q_3(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 - y^2$	$q_4(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2$
 $q_1(x,y) > 0$ for $(x,y) \neq 0$ $\lambda = 1$	 $q_2(x,y) < 0$ for $(x,y) \neq 0$ $\lambda = -1$	 $q_3(x,y) > 0, q_3(x,y) < 0$ $\lambda = 1, -1$	 Neither $\lambda = 1,0$

- $q(v) = \lambda \|v\|^2$ if λ, v are eigenvalue/eigenvector pairs of A
- $\max_{v \in \partial B_1(0)} q(v)$ and $\min_{v \in \partial B_1(0)} q(v)$ are the largest/smallest eigenvalues of A

- If $\nabla f(a) = 0$ and $f\colon \mathbb{R}^n \to \mathbb{R}$ is C^3 on some $B_\epsilon(a)$,
  - $Hf(a)$ eigenvalues $> 0 \Rightarrow a$ is local minimum
  - $Hf(a)$ eigenvalues $< 0 \Rightarrow a$ is local maximum
  - $Hf(a)$ eigenvalues > 0 and $< 0 \Rightarrow a$ is a saddle point
  - $Hf(a)$ eigenvalues ≥ 0 or $\leq 0 \Rightarrow$ inconclusive
- If $\nabla f(a) = 0$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^3 on some $B_\epsilon(a)$,
  - $f_{xx}(a) f_{yy}(a) - f_{xy}(a)^2 > 0$ and $f_{xx}(a) > 0 \Rightarrow a$ is local minimum
  - $f_{xx}(a) f_{yy}(a) - f_{xy}(a)^2 > 0$ and $f_{xx}(a) < 0 \Rightarrow a$ is local maximum
  - $f_{xx}(a) f_{yy}(a) - f_{xy}(a)^2 < 0 \Rightarrow a$ is a saddle point

***k*-th Iterated Directional Derivative:** Of C^k function $f\colon \mathbb{R}^n \to \mathbb{R}$, the map

$$D_h^k f = \underbrace{D_h(D_h(\cdots(D_h}_{k \text{ times}} f)))$$

- $D_h^2 f(p) = D_h(D_h f(p)) = \sum_{i=1}^n h_i^2 (\partial_i^2 f)(p) + \sum_{i=1}^n \sum_{j=i+1}^n 2 h_i h_j (\partial_i \partial_j f)(p)$
- $\frac{D_h^k f(a)}{k!} = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha$
- $P_N(a+h) = \sum_{k=0}^N \frac{D_h^k f(a)}{k!}$

***N*-th Remainder:** Of f at a, the function $R_N(x) = f(x) - P_N(x)$

- f is C^{N+1} on $\underline{\mathbb{R}^n}_{\text{open,contai}\quad \text{line segment from } a \text{ to } a+h} \Rightarrow \exists \xi \in L, R_N(a+h) = \frac{D_h^{N+1} f(\xi)}{(N+1)!}$
- Q is polynomial in n variables, $\deg(Q) \leq N \Rightarrow \left(Q = 0 \Leftrightarrow \lim_{x \to 0} \frac{Q(x)}{\|x\|^N} = 0\right)$

Integrals

Rectangle: In $\mathbb{R}^n$, set $R = [a_1, b_1] \times \dots \times [a_n, b_n]$
Length: Of rectangle $R = [a, b] \subseteq \mathbb{R}$, length$(R) = b - a$
Area: Of rectangle $R = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$, area$(R) = (b_1 - a_1)(b_2 - a_2)$
Volume: Of rectangle $R = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$, vol$(R) = (b_1 - a_1) \cdots (b_n - a_n)$

	$\mathbb{R}$	$\mathbb{R}^n$
Partition *(of a rectangle)*	Finite set $P = \{x_0, \dots, x_k\}$ where $\{a, b\} \subseteq P \subseteq [a, b]$	Tuple of sets $P = (P_1, \dots, P_n)$ where $P_i = \left\{x_0^{(i)}, \dots, x_{k_i}^{(i)}\right\}$, $\{a_i, b_i\} \subseteq P_i \subseteq [a_i, b_i]$
Index Set *(of a partition)*	Set of numbers $I = \{1, \dots, k\}$	Set of tuples $I = \{(i_1, \dots, i_n) \in \mathbb{N}^n : 1 \le i_1 \le k_1, \dots, 1 \le i_n \le k_n\}$
Subrectangles *(of a partition)*	Set $\{R_1, \dots, R_k\}$ where $R_i = [x_{i-1}, x_i]$	Set $\{R_i : i \in I\}$ where $R_i = R_{(i_1, \dots, i_n)} = \left[x_{i_1-1}^{(1)}, x_{i_1}^{(1)}\right] \times \dots \times \left[x_{i_n-1}^{(n)}, x_{i_n}^{(n)}\right]$

- By convention, $a = x_0 < x_1 < \dots < x_{k-1} < x_k = b$
- $i \ne j \Rightarrow (R_i \cap R_j)^\circ = \emptyset$ "Subrectangles don't overlap"
- $R = \bigcup_{i \in I} R_i$ "Rectangle R is made of union of all subrectangles"
- $\text{vol}(R) = \sum_{i \in I} \text{vol}(R_i)$ "Volume of R is the sum of subrectangle volumes"
- $S \subseteq \bigcup_{i=1}^k R_i \Rightarrow \text{vol}(S) \le \sum_{i=1}^k \text{vol}(R_i)$ "Subsets of subrects have less volume than the subrects"

Refinement: Of partition P, partition P' if each $P_i \subseteq P_i'$
- Let P and refinement P' have subrectangles R_{i_1}, R'_{i_2}. For all $i_1 \in I_1$...
  - $\exists i_2 \in I_2, R'_{i_2} \subseteq R_{i_1}$ "Every subrect of P in has a subrect in the refinement"
- Let $I_2^{i_1} = \left\{i_2 \in I_2 : R'_{i_2} \subseteq R_{i_1}\right\}$ "All indices of R_{i_1}'s subrects in I_2"
  - $R_{i_1} = \bigcup_{i_2 \in I_2^{i_1}} R'_{i_2}$ "R_{i_1} = union of its subrects"
  - $\text{vol}(R_{i_1}) = \sum_{i_2 \in I_2^{i_1}} \text{vol}\left(R'_{i_2}\right)$ "volume of R_{i_1} = sum of volume of its subrects"
  - $I_2^{i_1} \cap I_2^{i_1'} = \emptyset$ "R_{i_1} and $R_{i_1'}$'s subrects are disjoint"
  - $I_2 = \bigcup_{i_1 \in I_1} I_2^{i_1}$ "Indices of refinement is the indices of all of R_{i_1}'s subrects"
- P'' is refinement of P' is a refinement of $P \Rightarrow P''$ is a refinement of P
- Common Refinement: Of partitions P', P'', partition P where $P_i = P_i' \cup P_i''$
  - The common refinement of P', P'' is a refinement of both P', P''
  - Refinements of both P', P'' are refinements of the common refinement of P', P''

Diameter: Of rectangle R, the largest distance between two points in it. $\max_{x_1, x_2 \in R}\left(\text{dist}(x_1, x_2)\right)$
Regular: Partition P, if for each P_j, its subintervals $[x_{i-1}, x_i]$ have the same length; $x_i = a + \frac{b-a}{k} i$ for $0 \le i \le k$
Norm: Of partition P, notated $\|P\|$, is the largest diameter between all subrectangles
- $\forall \delta > 0, \exists P, \|P\| < \delta$ "Partitions can get arbitrarily small"

***P*-Upper Sum:** Of bounded $f : R \to \mathbb{R}$,
$U_P(f) = \sum_{i \in I} \sup_{x \in R_i} f(x)\, \text{vol}(R_i)$
***P*-Lower Sum:** Of bounded $f : R \to \mathbb{R}$,
$L_P(f) = \sum_{i \in I} \inf_{x \in R_i} f(x)\, \text{vol}(R_i)$

- $U_P(f + g) \le U_P(f) + U_P(g)$
- $f \le g \Rightarrow U_P(f) \le U_P(g)$
- $U_P(\alpha f) = \alpha U_P(f)$
- $U_P(-f) = -L_P(f)$

$$\boxed{L_P(f) \le L_{P'}(f) \le \sup_P L_P(f) = \underline{I_R(f)} \le \overline{I_R(f)} = \inf_P U_{P'}(f) \le U_{P'}(f) \le U_P(f)}$$

Subdivided: Rectangle $S \subseteq R$, by partition P, if it's a union of some of R's subrectangles

- $\exists P$ subdividing any subset rectangles $S_1, \dots, S_k$ of R
- P subdivides $S_1, \dots, S_k \Rightarrow P'$ subdivides every $S_1, \dots, S_k$

Indicator Function: Of set $S \subseteq \mathbb{R}^n$, the functions

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \qquad \chi_S f(x) = \begin{cases} f(x) & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Integrable: The bounded function f, iff...

$f: R \to \mathbb{R}$	$f: S_{\text{bounded}} \to \mathbb{R}$
$\forall \epsilon > 0, \exists P, U_P(f) - L_P(f) < \epsilon$ $\underline{I_R}(f) = \overline{I_R}(f)$ ("Darboux Integrable")	$\chi_S f: \mathbb{R}^n \to \mathbb{R}$ is integrable on a $R \supseteq S$
$\int_R f dV = \underline{I_R}(f) = \overline{I_R}(f)$	$\int_S f dV = \int_R \chi_S f dV$

Riemann Sum: Of bounded $f: R \to \mathbb{R}$ and sample points $x_i^* \in R_i$,

$$S_P^*(f) = \sum_{i \in I} f(x_i^*) \text{vol}(R_i)$$

$$S_P^*(\alpha f + \beta g) = \alpha S_P^*(f) + \beta S_P^*(g)$$
$$f \le g \Rightarrow S_P^*(f) \le S_P^*(g)$$

- $\int_R f dV = \lim_{N \to \infty} S_{P_N}^*(f)$ for partitions P_N where $x_N^* \in R_N$ and $N \to \infty \Rightarrow \|P_N\| \to 0$

Uniformly Continuous: Function $f: \underline{\mathbb{R}^n} \to \mathbb{R}^m$ if $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in \underline{\mathbb{R}^n}, \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon$

- f is uniformly continuous $\Rightarrow f$ is continuous
- f is uniformly continuous on S $\Rightarrow f$ is uniformly continuous on any subset of S
- f is continuous on compact domain $\Rightarrow f$ is uniformly continuous
- $f: R \to \mathbb{R}$ is continuous on R $\Rightarrow f$ is integrable on R

Zero Volume/Jordan Measure Set: Set $S \subseteq \mathbb{R}^n$ if $\forall \epsilon > 0, \exists R_1, \dots, R_k, S \subseteq \bigcup_{i=1}^k R_i, \sum_{i=1}^k \text{vol}(R_i) < \epsilon$

- Zero volume is preserved under subsets, finite unions, closure
- S has zero volume $\Rightarrow S$ is bounded and $S^\circ = \emptyset$ *(Converse false: $\mathbb{Q} \cap [0,1]$)*
- f, S is bounded, S has zero volume $\Rightarrow f$ is integrable on S and $\int_S f dV = 0$
- f, S is bounded, $f = 0$ on S except on a zero-volume subset $\Rightarrow f$ is integrable on S and $\int_S f dV = 0$
- **Sard's Theorem:** $k < n, R \subseteq \underline{\mathbb{R}^k}_{\text{open}}, f: \underline{\mathbb{R}^k}_{\text{open}} \to \mathbb{R}^n$ is C^1 $\Rightarrow f(R)$ has zero Jordan measure

Jordan Measurable: A bounded set S where ∂S has zero Jordan measure

- S is Jordan measurable $\Leftrightarrow \chi_S$ is integrable on all rectangles containing S
- Jordan measurability is preserved under closure, interior, boundary, finite union/intersection
- S has zero volume $\Rightarrow S$ is Jordan measurable
- S is Jordan measurable and discontinuities of $f: S \to \mathbb{R}$ have zero Jordan measure $\Rightarrow f$ is integrable on S
- S is Jordan measurable, $f, g: S \to \mathbb{R}$ bounded, $f = g$ except on zero-volume set $\Rightarrow f$ integrable on S iff g integrable on S, then $\int_S f dV = \int_S g dV$
- S is Jordan measurable, compact, $f: S \to [0, \infty), f \ge 0 \Rightarrow T = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}: x \in S, 0 \le y \le f(x)\}$ is compact, Jordan measurable, $\int_S f dV = \text{vol}(T)$

Jordan Measure (Volume/Area): Of Jordan-measurable set $S \subseteq \mathbb{R}^n$, $\text{vol}(S) = \int_R \chi_S dv$ for some $R \supseteq S$

- S has zero Jordan measure $\Leftrightarrow \text{vol}(S) = 0$
- R_1, R_2 contain $S \Rightarrow \int_{R_1} \chi_S dv = \int_{R_2} \chi_S dv$
- $S_1 \subseteq S_2 \Rightarrow \text{vol}(S_1) \le \text{vol}(S_2)$
- $\text{vol}(S_1 \cup S_2) = \text{vol}(S_1) + \text{vol}(S_2) - \text{vol}(S_1 \cap S_2)$
- f is a linear map $\Rightarrow \text{vol}\big(f(R)\big) = |\det f| \text{vol}(R)$

Assumption	Conclusion	Assumption	Conclusion
$S_{\text{JM}} \in \mathbb{R}^n$	$\int_S \alpha dV = \alpha \text{vol}(S)$	$f, g: S_B \to \mathbb{R}$, bounded, integrable on S	$\int_S (\alpha f + \beta g) dV = \alpha \int_S f dV + \beta \int_S f dV$
$f, g: S_B \to \mathbb{R}$, bounded, integrable on S	$\int_S fg dV \leq \sqrt{\int_S f^2 dV} \sqrt{\int_S g^2 dV}$	$f: R_1 \to \mathbb{R}$, bounded, $R_1 = R_2 \cup R_3$, $R_2 \cap R_3 = \emptyset$	f integrable on R_1 $\Leftrightarrow f$ integrable on R_2, R_3, $\int_{R_1} fg dV = \int_{R_2} f dV + \int_{R_3} f dV$
$f, g: S_B \to \mathbb{R}$, bounded, integrable on S, $f \leq g$	$\int_S f dV \leq \int_S g dV$	$f: S_1 \to \mathbb{R}$, bounded, $S_1 = S_2 \cup S_3$, $S_2 \cap S_3 = \emptyset$	f integrable on S_2, S_3 $\Rightarrow f$ integrable on S_1, $\int_{S_1} fg dV = \int_{S_2} f dV + \int_{S_3} f dV$
$f: S_B \to \mathbb{R}$, bounded, integrable on S	$\left\lvert \int_S f dV \right\rvert \leq \int_S \lvert f \rvert dV$	$f: R \to \mathbb{R}$, bounded, integrable on R, $\lVert P_N \rVert \to 0$ as $N \to \infty$, $x_i^* \in R_i$ of P_i,	$\int_R fg dV = \lim_{N \to \infty} S_{P_N}^*(f)$

Average Value: On Jordan-measurable $S \subseteq \mathbb{R}^n$ with non-zero volume, of f integrable on S, value

$$\text{avg}_S f = \frac{1}{\text{vol}(S)} \int_S f dV$$

Integral Mean Value Theorem: For compact, path-connected, Jordan measurable $S \subseteq \mathbb{R}^n$, function f continuous on S, $\exists p \in S$, $\int_S f dV = f(p)\text{vol}(S)$ (ie. $f(p) = \text{avg}_S f$ for $\text{vol}(S) \neq 0$)

- For $p \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$ continuous on open set containing p, $f(p) = \lim_{\epsilon \to 0^+} \left(\text{avg}_{B_\epsilon(p)} f\right)$

Mass: Of bounded object $S \subseteq \mathbb{R}^n$ with density function $f: S \to [0, \infty)$, $m = \int_S f dV \in \mathbb{R}$

Average Density: Of bounded object $S \subseteq \mathbb{R}^n$ with density function $f: S \to [0, \infty)$, $\rho = \frac{1}{\text{vol}(S)} \int_S f dV \in \mathbb{R}$

Center of Mass: Of bounded object $S \subseteq \mathbb{R}^n$ with density function $f: S \to [0, \infty)$, $\overline{x} = \frac{1}{m} \int_S x f(x) dV \in \mathbb{R}^n$

- **Centroid:** The geometrical center. Equal to center of mass if density function is constant

Sample Space (Ω): An arbitrary non-empty set of all possible outcomes
Event Space (Σ): A collection of subsets of Ω where each subset $A \subseteq \Sigma$ is an **event**.
Probability Function ($\mathbb{P}$): A function $\mathbb{P}: \Sigma \to [0{,}1]$ representing the probability that some $A \in \Sigma$ occurs

- **Probability Density Function:** Of probability function $\mathbb{P}$, the function $f: \Omega \to [0, \infty)$
- **Uniform:** Probability function $\mathbb{P}$ if f is constant; that is, $f(x) = \frac{1}{\text{vol}(\Omega)}$. Thus, $\mathbb{P}(A) = \frac{\text{vol}(A)}{\text{vol}(\Omega)}$

Continuous Probability Space: The triple $(\Omega, \Sigma, \mathbb{P})$

- If $\Omega \subseteq \mathbb{R}^2$ is Jordan measurable, $\Sigma = \{A \subseteq \Omega: A \text{ is Jordan measurable}\}$, then
  - $\Omega \in \Sigma$
  - $A \in \Sigma \Rightarrow \Omega \setminus A \in \Sigma$
  - $A_1, \dots, A_N \in \Sigma \Rightarrow \mathrm{A}_1 \cup \dots \cup A_N \in \Sigma$
  - $\mathbb{P}(\mathrm{A}) = \int_A f dV$, where $f: \Omega \to [0, \infty)$, $f \geq 0$, f is continuous on Ω except for a zero volume set
    - $\mathbb{P}(\Omega) = 1$
    - $0 \leq \mathbb{P}(A) \leq 1$
    - $\mathbb{P}(A_1 \cup \dots \cup A_N) = \sum_{i=1}^N \mathbb{P}(A_i)$ if $A_i \cap A_j = \emptyset$

x-Slice: Of $f: [a,b] \times [c,d] \to \mathbb{R}$ at α, function $f^x: [c,d] \to \mathbb{R}$ of form $f^x(y) = f(\alpha, y)$
y-Slice: Of $f: [a,b] \times [c,d] \to \mathbb{R}$ at α, function $f^y: [a,b] \to \mathbb{R}$ of form $f^y(x) = f(x, \alpha)$
x-Slice: Of $f: [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$ at α, function $f^x: [c,d] \times [e,f] \to \mathbb{R}$ of form $f^x(y,z) = f(\alpha, y, z)$
y-Slice: Of $f: [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$ at α, function $f^y: [a,b] \times [e,f] \to \mathbb{R}$ of form $f^y(x,z) = f(x, \alpha, z)$
z-Slice: Of $f: [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$ at α, function $f^z: [a,b] \times [c,d] \to \mathbb{R}$ of form $f^z(x,y) = f(x, y, \alpha)$
(x,y)-Slice: Of $f: [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$, function $f^{x,y}: [e,f] \to \mathbb{R}$ by $f^{x,y}(z) = f(x,y,z)$ for fixed $x, y \in [a,b] \times [c,d]$
(x,z)-Slice: Of $f: [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$, function $f^{x,z}: [c,d] \to \mathbb{R}$ by $f^{x,z}(y) = f(x,y,z)$ for fixed $x, z \in [a,b] \times [e,f]$
(y,z)-Slice: Of $f: [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$, function $f^{y,z}: [a,b] \to \mathbb{R}$ by $f^{y,z}(x) = f(x,y,z)$ for fixed $y, z \in [c,d] \times [e,f]$

- $f: R \to \mathbb{R}$ is bounded $\Rightarrow$ f's slices are bounded
- $f: R \to \mathbb{R}$ is continuous $\Rightarrow$ f's slices are continuous

Iterated Double Integrals: Of bounded $f: [a,b] \times [c,d] \to \mathbb{R}$, values $\int_a^b \int_c^d f(x,y)\,dy\,dx$ and $\int_c^d \int_a^b f(x,y)\,dx\,dy$

- f's x-slices integrable on $[c,d]$ and $\int_c^d f(x,y)\,dy$ integrable on $[a,b]$ $\Rightarrow$ $\int_a^b \int_c^d f(x,y)\,dy\,dx$ exists
- f's y-slices integrable on $[a,b]$ and $\int_a^b f(x,y)\,dx$ integrable on $[c,d]$ $\Rightarrow$ $\int_c^d \int_a^b f(x,y)\,dx\,dy$ exists

Iterated Triple Integrals: Of bounded $f: [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$, any ordering of $\int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$

Iterated n-Fold Integrals: Of bounded $f: [a_1,b_1] \times \cdots \times [a_n,b_n] \to \mathbb{R}$, any ordering of $\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x)\,dx_n \cdots dx_1$

Fubini's Theorem:

For bounded $f: [a,b] \times [c,d] \to \mathbb{R}$ and $f^x: [c,d] \to \mathbb{R}$ both integrable on their domain, $$\int_a^b \int_c^d f(x,y)dy\,dx = \iint_{[a,b]\times[c,d]} f dA$$ If f is continuous, same results: $$\int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy$$	For bounded $f: [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$, $f^{(x,y)}: [e,f] \to \mathbb{R}$, and $f^x: [c,d] \times [e,f] \to \mathbb{R}$ all integrable on their domain, $$\int_a^b \int_c^d \int_e^f f dz\,dy\,dx = \iiint_{[a,b]\times[c,d]\times[e,f]} f dV$$ If f is also continuous, All iterated integrals exists, equal to $\int_a^b \int_c^d \int_e^f f dz\,dy\,dx$
For bounded $f: R \to \mathbb{R}$ and its slices integrable on its domain, every ordering of the n-fold integral exists is equal, $$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x)\,dx_n \cdots dx_1 = \int_R f dV$$ If f is also continuous, this is also true.	For bounded $f: R \times [a,b] \to \mathbb{R}$, the slices $f^t: R \to \mathbb{R}$ with $f^t(x) = f(x,t)$ integrable on their domains, the function $\int_R f^t dV$ is integrable on $[a,b]$ and $$\int_{R\times[a,b]} f dV = \int_a^b \left(\int_R f^t dV \right) dt$$

x-Simple: Set $S \subseteq \mathbb{R}^2$ if $\exists f: [a,b] \to \mathbb{R}, g: [a,b] \to \mathbb{R}$ continuous, $S = \{(x,y) \in \mathbb{R}^2 : x \in [a,b], y \in [f(x), g(x)]\}$

- Fubini's theorem holds $\Rightarrow$ $\iint_S f dA = \int_a^b \int_{f(x)}^{g(x)} f(x,y)\,dy\,dx$

y-Simple: Set $S \subseteq \mathbb{R}^2$ if $\exists f: [c,d] \to \mathbb{R}, g: [c,d] \to \mathbb{R}$ continuous, $S = \{(x,y) \in \mathbb{R}^2 : y \in [c,d], x \in [f(y), g(y)]\}$

- Fubini's theorem holds $\Rightarrow$ $\iint_S f dA = \int_c^d \int_{f(y)}^{g(y)} f(x,y)\,dx\,dy$

Define S so that we have invertible $g: S \to g(S)$ defined by

$g(r,\theta) = (r\cos\theta\,, r\sin\theta)$	$g(r,\theta,z) = (r\cos\theta\,, r\sin\theta\,, z)$	$g(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi\,, \rho\sin\theta\sin\phi\,, \rho\cos\phi)$
If $f: g(S) \to \mathbb{R}$ integrable on $g(S)$,	If $f: g(S) \to \mathbb{R}$ integrable on $g(S)$,	If $f: g(S) \to \mathbb{R}$ integrable on $g(S)$,
$F(r,\theta) = f\big(g(r,\theta)\big)\lvert r\rvert$ is integrable on S and $$\iint_{g(S)} f dA = \iint_S F dA$$	$F(r,\theta,z) = f\big(g(r,\theta,z)\big)\lvert r\rvert$ is integrable on S and $$\iiint_{g(S)} f dV = \iiint_S F dV$$	$F(\rho,\theta,\phi) = f\big(g(\rho,\theta,\phi)\big)\lvert\rho^2\sin\phi\rvert$ is integrable on S and $$\iiint_{g(S)} f dV = \iiint_S F dV$$

Change of Variables: Let $g: \underline{\mathbb{R}^n}_{\text{open},1} \to \underline{\mathbb{R}^n}_{\text{open},}$ be diffeomorphism, $S \subseteq \underline{\mathbb{R}^n}_{\text{open},1}$ be compact, Jordan measurable. f integrable on $g(S) \Leftrightarrow f\big(g(|\det Dg|)\big)$ integrable on S, then

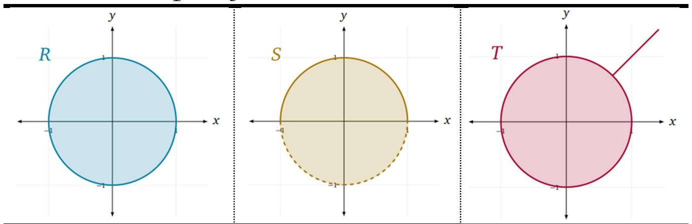
$$\int_{g(S)} f dV = \int_S (f \circ g)|\det Dg|\, dV$$

$$\int \cdots \int_{g(S)} f dx_1 \cdots dx_n \overset{\text{Fubini holds}}{=} \int \cdots \int_S f\big(g(u)\big)|\det Dg(u)| du_1 \cdots du_n$$

- $g: \underline{\mathbb{R}}_{\text{ope}} \to \underline{\mathbb{R}}_{\text{open},2}$ is C^1, increasing, f integrable on $[g(a), g(b)] \Rightarrow \int_{g(a)}^{g(b)} f(x)dx = \int_a^b f\big(g(u)\big)g'(u)dx$
- $g: \underline{\mathbb{R}}^n_{\text{open},} \to \underline{\mathbb{R}}^n_{\text{open},}$ is diffeomorphism, $S \subseteq \underline{\mathbb{R}}^n_{\text{open},}$ is compact $\Rightarrow \text{vol}(g(\text{S})) = \int_S |\det Dg| dV$
- $g: \underline{\mathbb{R}}^n_{\text{open},1} \to \underline{\mathbb{R}}^n_{\text{open},2}$ is C^1, invertible, $\det Dg(x) \neq 0 \Rightarrow g$ is a diffeomorphism

Vector Calculus

Regular Region: Set $S \subseteq \mathbb{R}^n$ if S is compact, Jordan measurable, and $\overline{S^\circ} = S$ *"A bounded open set with a filled boundary"*

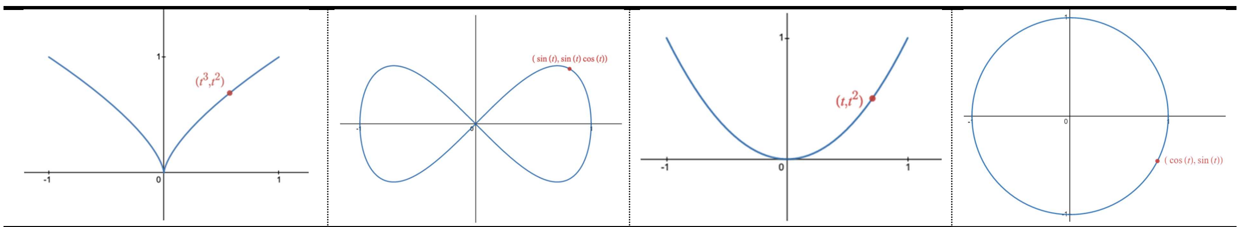


Regular	Not regular	Not regular

	1-Variable	*2-Variable*
Parameterization	Of set $S \subseteq \mathbb{R}^n$, a continuous function $\gamma: [a,b] \to \mathbb{R}^n$, $\gamma([a,b]) = S$	Of set $S \subseteq \mathbb{R}^3$, a continuous function $\gamma: U \to \mathbb{R}^3$ with path-connected and regular $U \subseteq \mathbb{R}^2$, $\gamma(U) = S$
Regular *The parameterization is "smooth" and "monotonic"*	γ is C^1 and $\gamma' \neq 0$ on (a,b) **Piecewise Regular:** If γ is regular except at finitely many points	γ is C^1 and $\{\partial_1\gamma(p), \partial_2\gamma(p)\}$ is linearly independent on U°
Simple *The parameterization's path doesn't "overlap" itself*	γ is one-to-one on (a,b) **Closed:** If $\gamma(a) = \gamma(b)$	γ is one-to-one, except possibly along ∂U

Simple Regular Parameterization (SRP): A parameterization that is simple and regular.

- f is SRP of $S \subseteq \mathbb{R}^2 \Rightarrow S$ is 1D smooth manifold at $\gamma(x)$ for all $x \in (a,b)$
- f is SRP of $S \subseteq \mathbb{R}^3 \Rightarrow S$ is 2D smooth manifold at $\gamma(x)$ for all $x \in U^\circ$



$\gamma(t) = (t^3, t^2)$ is simple, piecewise regular	$\gamma(t) = (\sin t, \sin t \cos t)$ is regular, not simple	$\gamma(t) = (t, t^2)$ is simple, regular. $\gamma(t) = (t^3, t^6)$ is simple, piecewise regular	$\gamma(t) = (\cos t, \sin t)$ is simple, regular, closed

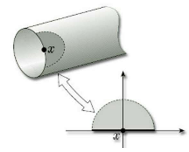
Curve: Set $S \subseteq \mathbb{R}^n$ if there's a 1-variable SRP of it.	**Surface:** In $\mathbb{R}^3$, set $S \subseteq \mathbb{R}^3$ if there's a 2-variable SRP of it.
Curves are 1D smooth manifolds everywhere except possibly at $x = a, b$	Surfaces are 2D smooth manifolds everywhere except possibly at U°
Piecewise: Curve $S \subseteq \mathbb{R}^n$ if S is the union of finitely-many distinct curves S_i where $S_i \cap S_j$ is a finite set	**Piecewise:** Surface $S \subseteq \mathbb{R}^3$ if S can be made by "gluing" together finitely many distinct surfaces S_i along their ∂U_i
Closed: A curve, if the SRP of it is closed	**Closed:** A piecewise surface, if its relative boundary $\partial S = \emptyset$

- S is a curve/surface $\Rightarrow S$ is a piecewise curve/surface $\Rightarrow S$ is compact
- Curves/surface are not smooth manifolds everywhere due to boundaries not being smooth.
- Smooth manifolds are not always curves/surfaces because they are not bounded

Relative Boundary Point: Of piecewise surface $S \subseteq \mathbb{R}^3$, point $p \in S$ if

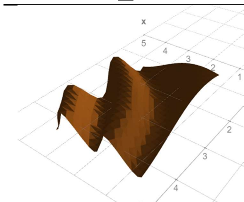
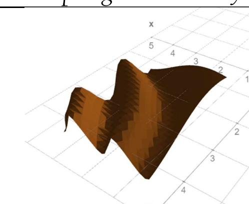
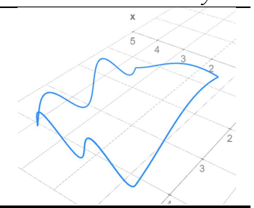
- $\exists U_{\text{open}} \subseteq \mathbb{R}^2$
- $\exists V_{\text{open}} \subseteq \mathbb{R}^3$
- $\exists \phi\colon U \cap \mathbb{R} \times \mathbb{R}_{\geq 0} \to V \cap S$
- $p \in V$
- ϕ is invertible
- ϕ, ϕ^{-1} continuous
- $\phi^{-1}(p) = (k, 0)$ for some $k \in \mathbb{R}$

"If the region around $p \in \mathbb{R}^3$ can be mapped to $\mathbb{R} \times \mathbb{R}_{\geq 0}$ by a continuous ϕ^{-1}, with p on the x-axis."



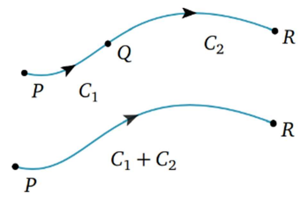
Relative Boundary: Of piecewise surface $S \subseteq \mathbb{R}^3$, the set of relative boundary points, notated ∂S

- Topological boundary finds everything that locally forms a "plane" in $\mathbb{R}^3$.
- Relative boundary finds everything that locally forms a "line" in $\mathbb{R}^3$.

$S \subseteq \mathbb{R}^3$	*Topological Boundary*	*Relative Boundary*
		

Reparameterization: Of SRP $\gamma_2\colon [c,d] \to \mathbb{R}^n$, SRP $\gamma_1\colon [a,b] \to \mathbb{R}^n$ if $\exists \phi\colon [a,b] \to [c,d]$, ➢ ϕ is continuous and invertible ➢ ϕ is C^1 and $\phi' \neq 0$ on (a,b) ➢ $\gamma_1 = \gamma_2 \circ \phi$	**Reparameterization:** Of SRP $\gamma_2\colon V \to \mathbb{R}^3$, SRP $\gamma_1\colon U \to \mathbb{R}^3$ if $\exists \phi\colon U \to V$, ➢ ϕ is continuous and invertible ➢ ϕ is C^1 and $\det D\phi \neq 0$ on U° ➢ $\gamma_1 = \gamma_2 \circ \phi$
Same Orientation: A reparameterization, if $\phi' > 0$ on (a,b)	**Same Orientation:** A reparameterization, if $\det D\phi > 0$ on U°
Opposite Orientation: A reparameterization, if $\phi' < 0$ on (a,b)	**Opposite Orientation:** A reparameterization, if $\det D\phi < 0$ on U°
Unit Normal: Of parameterization $\gamma\colon [a,b] \to \mathbb{R}^2$, $-\frac{T'(t)}{\lVert T'(t)\rVert} \left(\text{where } T(t) = \frac{\gamma'(t)}{\lVert \gamma'(t)\rVert}\right)$	**Unit Normal:** Of parameterization $\gamma\colon U \to \mathbb{R}^3$, the C^1 $\frac{\partial_1\gamma \times \partial_2\gamma}{\lVert \partial_1\gamma \times \partial_2\gamma\rVert}$ (defined on U°)
Unit Normal: Of oriented closed curve $S \subseteq \mathbb{R}^2$, continuous function $n\colon [a,b] \to \mathbb{R}^2$ where ➢ $n(t) \cdot T(t) = 0$ ➢ $\det[n(t) \quad T(t)] > 0$	**Unit Normal:** Of oriented surface $S \subseteq \mathbb{R}^3$, continuous function $n\colon S \to S^2$ where for all $(u,v) \in U^\circ$, $(n \circ \gamma)(u,v) = \frac{(\partial_1\gamma \times \partial_2\gamma)(u,v)}{\lVert(\partial_1\gamma \times \partial_2\gamma)(u,v)\rVert}$
If γ_1, γ_2 have the same orientation, $\frac{\gamma_1'(t)}{\lVert\gamma_1'(t)\rVert} = \frac{\gamma_2'(\phi(t))}{\lVert\gamma_2'(\phi(t))\rVert}$ If γ_1, γ_2 have opposite orientations, $\frac{\gamma_1'(t)}{\lVert\gamma_1'(t)\rVert} = -\frac{\gamma_2'(\phi(t))}{\lVert\gamma_2'(\phi(t))\rVert}$ for $t \in [a,b], \phi(t) \in [c,d]$	If γ_1, γ_2 have the same orientation, $\frac{(\partial_1\gamma_1 \times \partial_2\gamma_1)(u,v)}{\lVert(\partial_1\gamma_1 \times \partial_2\gamma_1)(u,v)\rVert} = \frac{(\partial_1\gamma_2 \times \partial_2\gamma_2)(\phi(u,v))}{\lVert(\partial_1\gamma_2 \times \partial_2\gamma_2)(\phi(u,v))\rVert}$ If γ_1, γ_2 have opposite orientations, $\frac{(\partial_1\gamma_1 \times \partial_2\gamma_1)(u,v)}{\lVert(\partial_1\gamma_1 \times \partial_2\gamma_1)(u,v)\rVert} = -\frac{(\partial_1\gamma_2 \times \partial_2\gamma_2)(\phi(u,v))}{\lVert(\partial_1\gamma_2 \times \partial_2\gamma_2)(\phi(u,v))\rVert}$ for $(u,v) \in U^\circ, \phi(u,v) \in V^\circ$
Oriented Curve (S): Set of same-oriented 1-variable reparameterizations of each other.	**Oriented Surface (S):** Set of same-oriented 2-variable reparameterizations of each other.
Oppositely-Oriented Curve ($-S$): Set of oppositely-oriented 1-variable reparameterizations of each other.	**Oppositely-Oriented Surface ($-S$):** A set of oppositely-oriented 2-variable reparameterizations of each other.
Concatenation ($S_1 + S_2$): Of oriented curves S_1, S_2, the set of continuous $\gamma\colon [a,c] \to \mathbb{R}^n$ where $\gamma\vert_{[a,b]}, \gamma\vert_{[b,c]}$ are parameterizations of S_1, S_2 for some $b \in (a,c)$.	**Concatenation ($S_1 + S_2$):** Of piecewise oriented surfaces S_1, S_2, the surface formed by "gluing" together their relative boundaries.
Piecewise Oriented Curve: The concatenation of finitely many oriented curves	**Piecewise Oriented Surface:** The concatenation of finitely many oriented surfaces

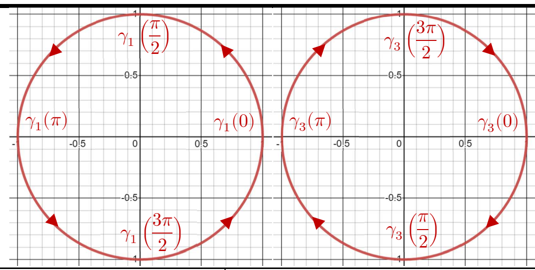
- S is a curve/surface $\Rightarrow$ S is a piecewise curve/surface
- S is an oriented curve/surface $\Rightarrow$ S is an oriented piecewise curve/surface

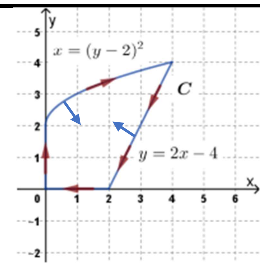


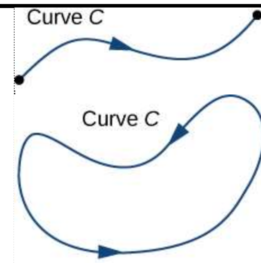
- $\phi: (a, b) \to (c, d)$ and $\phi: U^\circ \to V^\circ$ are diffeomorphisms
- SRPS are reparameterizations of themselves
- SRPS are reparameterizations of reparameterizations of them

Cross Product: Of vectors $u, v \in \mathbb{R}^3$, value $u \times v = \|u\|\|v\| \sin\theta\, n$, where θ is the angle between u, v, and n is the unit normal to u, v. Returns a vector orthogonal to u and v with magnitude $\|u\|\|v\| \sin\theta$.

- For us, use $u \times v = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$. You may substitute functions into u and v.
- Cross-product follows same rules as multiplication, but $u \times v = -v \times u$.







$\gamma_1(t) = (\cos t, \sin t)$, $\gamma_2(t) = (\cos 2t, \sin 2t)$, $\gamma_3(t) = (\cos t, -\sin t)$	γ_1, γ_2: clockwise orientation γ_3: counter-clockwise orientation γ_1, γ_2: same orientation γ_1, γ_3: opposite orientation	Closed simple regular piecewise oriented curve. Normals (blue) point left of tangents (red)	Oriented curve (top) and closed oriented curve (bottom)

Arc Length: Of curve $S \subseteq \mathbb{R}^n$ parameterized by $\gamma: [a, b] \to \mathbb{R}^n$,

$$\ell(S) = \int_a^b \|\gamma'(t)\| dt$$

"Image the curve as a string being stretched to a straight line; the length of that string."

Surface Area: Of surface $S \subseteq \mathbb{R}^3$ parameterized by $\gamma: U \to \mathbb{R}^3$,

$$A(S) = \iint_U \|\partial_1 \gamma \times \partial_2 \gamma\| dA$$

"Image the surface as any other Euclidean geometrical shape; the surface area of that shape"

- $\ell(S) = \sup_P \left\{\sum_{i=1}^k \|\gamma(t_i) - \gamma(t_{i-1})\|\right\}$ for all partitions $P = \{t_0, \dots, t_k\}$ of $[a, b]$
- **Arc Length Parameter:** Of parameterization γ, function $s: [a, b] \to [0, \infty)$ with $s(t) = \int_a^t \|\gamma'(x)\| dx$
- **Parameterized by Arc Length:** Parameterization γ if $\|\gamma'(t)\| = 1$ for $a < t < b$
    - γ is parameterized by arc length $\Leftrightarrow$ $s(t) = t - a$
    - To parameterize γ by arc length, you have $\gamma\left(s^{-1}(t)\right)$ for $t \in [s(a), s(b)]$

Line Integral: Over oriented curve $S \subseteq \mathbb{R}^n$ parameterized by $\gamma: [a, b] \to \mathbb{R}^n$ with unit tangent vector $T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$,

Of bounded $f: S \to \mathbb{R}$, the integral

$$\int_S f ds = \int_a^b f(\gamma(t))\|\gamma'(t)\|dt$$

Arclength Element: The infinitesimal length of the curve, ds

Of $F: S \to \mathbb{R}^n$, the integral

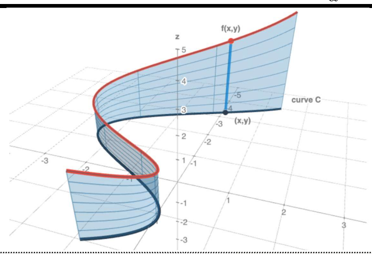
$$\int_S F \cdot T ds = \int_a^b F(\gamma(t)) \cdot T(t)\|\gamma'(t)\|dt$$
$$= \int_a^b F(\gamma(t)) \cdot \gamma'(t)dt$$

If F, G continuous,

- $\int_{-S} F \cdot T ds = -\int_S F \cdot T ds$
- $\int_S (\alpha F + \beta G) \cdot T ds = \alpha \int_S F \cdot T ds + \beta \int_S G \cdot T ds$
- $\int_{S_1 + S_2} F \cdot T ds = \int_{S_1} F \cdot T ds + \int_{S_2} F \cdot T ds$

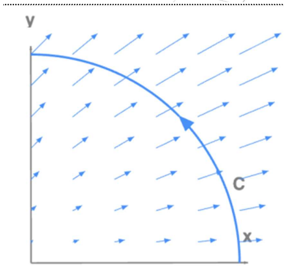
(assuming $S_1 + S_2$ is an oriented surface)

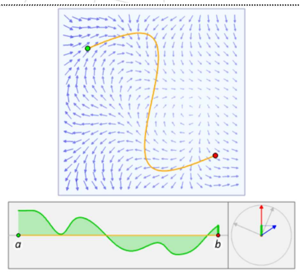
- Shorthand notation includes $\int_a^b F(\gamma(t)) \cdot \gamma'(t)dt = \int_S F \cdot d\gamma = \int_S F \cdot dr = \int_S F_1 dx_1 + \cdots + F_n dx_n$



Line integrals of f are like 1D integrals, but the x-axis is S. If the curve S is like a string and we stretch it into a straight line, it's equivalent to a 1D integral.

We convert ds (tiny movements across arclength of S) into $\|\gamma'(t)\|dt$ (tiny movements across x-axis).





Line integrals of F are best-imagined in physics, as energy released/work done (F) by travelling some path S.

When moving against/with the arrows, $F(\gamma(t)) \cdot \gamma'(t)$ is more negative/positive. We move t through the parameterization, and find the final area.

$\gamma'(t)$ measures how much we move with/against arrows.

Scalar Surface Integral: Over surface $S \subseteq \mathbb{R}^3$ parameterized by $\gamma: U \to \mathbb{R}^3$, of bounded $f: \mathbb{R}^3 \to \mathbb{R}$,

$$\iint_S f dS = \iint_U (f \circ \gamma)\|\partial_1\gamma \times \partial_2\gamma\| dA$$

Surface Element: The infinitesimal area of a small piece of the surface, dS

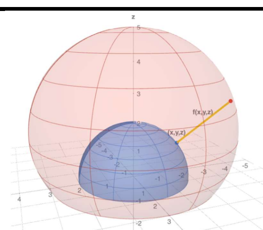
Surface Integral: Over oriented surface $S \subseteq \mathbb{R}^3$ parameterized by $\gamma: U \to \mathbb{R}^3$, of $F: S \to \mathbb{R}^3$,

$$\iint_S F \cdot n dS = \iint_U (F \circ \gamma) \cdot n\|\partial_1\gamma \times \partial_2\gamma\| dA$$
$$= \iint_U (F \circ \gamma) \cdot (\partial_1\gamma \times \partial_2\gamma) dA$$

If F, G continuous,

- $\iint_{-S} F \cdot n dS = -\iint_S F \cdot n dS$
- $\int_S (\alpha F + \beta G) \cdot n dS = \alpha \iint_S F \cdot n dS + \beta \iint_S G \cdot n dS$
- $\iint_{S_1 + S_2} F \cdot n dS = \iint_{S_1} F \cdot n dS + \iint_{S_2} F \cdot n dS$

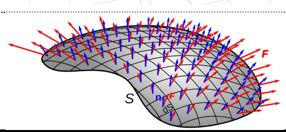
(assuming $S_1 + S_2$ is an oriented surface)



Scalar surface integrals can't be visualized well. They're like double integrals of f, but the rectangle is S, a bumpy terrain. We stretch S into a rectangle and integrate.

We convert dS (tiny unit of area of the surface) into $\|\partial_1\gamma \times \partial_2\gamma\| dA$ (a tiny unit of area of a rectangle).

The picture interprets blue as S, the line as $f(x)$ for some point, and the volume of the yellow as the surface integral. This only works for convex S and nonzero f.



Surface integrals are best thought of in physics as the sum of the components of all forces F that point directly towards/away from S, like electromagnetic flux.

$(\partial_1\gamma \times \partial_2\gamma)$ measures the component of forces normal to S.

Invariance Theorems: For reparameterizations γ_1, γ_2, A is integrable on $B \Leftrightarrow C$ is integrable on D. If so, E.

	A	B	C	D	E
Arc Length	$\|\gamma_2'(t)\|$	$[a,b]$	$\|\gamma_2'\|$	$[c,d]$	$\int_a^b \|\gamma_1'(t)\|dt = \int_c^d \|\gamma_2'(t)\|dt$
Line Integral	$(f\circ\gamma_1)\|\gamma_1'\|$	$[a,b]$	$(f\circ\gamma_2)\|\gamma_2'\|$	$[c,d]$	$\int_a^b f(\gamma_1(t))\|\gamma_1'(t)\|dt = \int_c^d f(\gamma_2(t))\|\gamma_2'(t)\|dt$
	$(F\circ\gamma_1)\cdot\gamma_1'$	$[a,b]$	$(F\circ\gamma_2)\cdot\gamma_2'$	$[c,d]$	$\int_a^b F(\gamma_1(t))\cdot\gamma_1'(t)dt = \int_a^b F(\gamma_2(t))\cdot\gamma_2'(t)dt$
Surface Area	$\|\partial_1\gamma_1\times\partial_2\gamma_1\|$	U	$\|\partial_1\gamma_2\times\partial_2\gamma_2\|$	V	$\iint_U \|\partial_1\gamma_1\times\partial_2\gamma_1\|dA = \iint_V \|\partial_1\gamma_2\times\partial_2\gamma_2\|dA$
Scalar Surface Integral	$(f\circ\gamma_1)\|\partial_1\gamma_1\times\partial_2\gamma_1\|$	U	$(f\circ\gamma_2)\|\partial_1\gamma_2\times\partial_2\gamma_2\|$	V	$\iint_U (f\circ\gamma_1)\|\partial_1\gamma_1\times\partial_2\gamma_1\|dA = \iint_V (f\circ\gamma_2)\|\partial_1\gamma_2\times\partial_2\gamma_2\|dA$
Surface Integral	$(F\circ\gamma_1)\cdot(\partial_1\gamma_1\times\partial_2\gamma_1)$	U	$(F\circ\gamma_2)\cdot(\partial_1\gamma_2\times\partial_2\gamma_2)$	V	$\iint_U (F\circ\gamma_1)\cdot(\partial_1\gamma_1\times\partial_2\gamma_1)dA = \iint_V (F\circ\gamma_2)\cdot(\partial_1\gamma_2\times\partial_2\gamma_2)dA$

Fundamental Theorem of Line Integrals (FTLI): The line integral of ∇f depends only on endpoints $\gamma(a), \gamma(b)$.

- Oriented piecewise curve $S \subseteq \underline{\mathbb{R}^n}_{\text{open}}$
- Parameterized by $\gamma\colon [a,b] \to \mathbb{R}^n$
- C^1 function $f\colon \underline{\mathbb{R}^n}_{\text{open}} \to \mathbb{R}$

$$\Rightarrow \quad \int_S \nabla f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$$

Conservative: On open set $U \subseteq \mathbb{R}^n$, function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ if $F = \nabla f$ for some $f\colon U \to \mathbb{R}$

- f is the **potential function/scalar potential** of F
- *"A vector field where FTLI applies – given start/end points, line integral is the same regardless of the path, 'path-independent'"*
- If F continuous, U path-connected, the following are equivalent:
  - $F = \nabla f$ on U for some C^1 $f\colon \mathbb{R}^n \to \mathbb{R}$
  - $\int_{S_1} F\cdot d\gamma = \int_{S_2} F\cdot d\gamma$ for oriented piecewise curves $S_1, S_2 \subseteq U$ with the same start/end points
  - $\int_S F\cdot d\gamma = 0$ for closed piecewise curves $S \subseteq U$

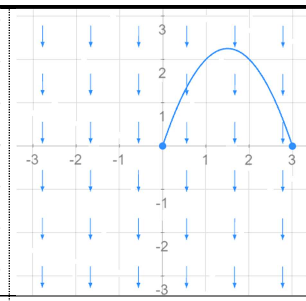
eg. A conservative vector field

$$f(x,y) = -0.5y$$
$$F(x,y) = (0,-0.5)$$
$$\therefore F(x,y) = \nabla f(x,y)$$

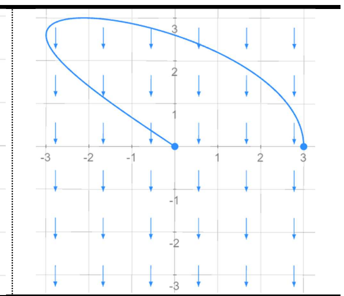
Then for any $\gamma\colon [a,b] \to \mathbb{R}^2$,

$$\int_S F\cdot d\gamma = f(\gamma(b)) - f(\gamma(a)) = f(3,0) - f(0,0) = 0$$

$\gamma(t) = (t,0)$



$\gamma(t) = (t, -t^2 + 3t)$



$\gamma(t) = \left(-3\sin\frac{3}{2}t, 3\sin t\right)$

Circulation: Of $F\colon S \to \mathbb{R}^n$ around simple closed oriented curve $S \subseteq \mathbb{R}^n$ where $n \in \{2,3\}$, line integral

$$\oint_S F\cdot T ds = \int_S F\cdot T ds = \int_a^b F(\gamma(t))\cdot\gamma'(t)dt$$

"The speed at which points move along S"

- Think of F physics-wise as a velocity field, and the integral as the sum of the velocities moving parallel (T) to the path of S
- The circle in $\oint$ means S is closed
- If S is not closed, this is called **flow**
- Exact same equation as a <u>line integral</u>

Flux: Of $F\colon S \to \mathbb{R}^n$ for simple closed oriented curve $S \subseteq \mathbb{R}^n$ where $n \in \{2,3\}$, the integral

$$\oint_S F\cdot n ds = \int_S F\cdot n ds = \int_a^b F(\gamma(t))\cdot n(t)\|\gamma'(t)\|dt$$

"The speed at which points move away from/toward S"

- Think of F physics-wise as a velocity field, and the integral as the sum of the velocities moving perpendicular (n) to the path of S
- C oriented clockwise $\Rightarrow$ inward flux
- C oriented counter-clockwise $\Rightarrow$ outward flux
- Exact same equation as a <u>surface integral</u>

Curl: Of $F: \mathbb{R}^2 \to \mathbb{R}^2$, continuous function

$$\text{curl}(F) = \partial_1 F_2 - \partial_2 F_1$$

"How counterclockwise the points around a p move"

- "Curl is infinitesimal circulation on a single point"
- F is C^1 on open set containing $p \Rightarrow$
  $(\text{curl } F)(p) = \lim_{\epsilon \to 0^+} \frac{1}{\text{area}(\ _\epsilon(p))} \oint_{\partial B_\epsilon(p)} (F \cdot T) ds$
  (where $B_\epsilon(p)$ is oriented counterclockwise)

Divergence: Of $F: \mathbb{R}^2 \to \mathbb{R}^2$, continuous function

$$\text{div}(F) = \partial_1 F_1 + \partial_2 F_2$$

"How away from/towards the points around a p move"

- "Divergence is infinitesimal flux on a point"
- F is C^1 on open set containing $p \in \mathbb{R}^3 \Rightarrow$
  $(\text{div } F)(p) = \lim_{\epsilon \to 0^+} \frac{1}{\text{vol}(B_\epsilon(p))} \oint_{\partial B_\epsilon(p)} F \cdot n ds$
  ($\partial B_\epsilon(p)$ is oriented counterclockwise)

Curl: Of $F: \mathbb{R}^3 \to \mathbb{R}^3$, continuous function

$$\text{curl}(F) = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$$

"Assume n points up; how counterclockwise points around p move"

- Let F, G be C^1, $f: \underline{\mathbb{R}^3} \to \mathbb{R}$ be C^2,
  - $\max\{(\text{curl } F)(p) \cdot n\} = \|(\text{curl } F)(p)\|$
  - $\min\{(\text{curl } F)(p) \cdot n\} = -\|(\text{curl } F)(p)\|$
  - $\text{curl}(\alpha F + \beta G) = \alpha \text{ curl}(F) + \beta \text{ curl}(G)$
  - $\text{curl}(fF) = f \text{ curl}(G) + (\nabla f) \times G$
  - $\text{curl}(F \times G) = \sum_{i=1}^3 G_i \partial_i F + (\text{div } G) F + \sum_{i=1}^3 F_i \partial_i G + (\text{div } F) G$
- Let F be C^2, $f: \mathbb{R}^n \to \mathbb{R}$,
  - $\text{curl}(\nabla f) = (0,0,0)$
  - $\text{div}(\text{curl}(F)) = 0$

Divergence: Of $F: \mathbb{R}^n \to \mathbb{R}^n$, continuous function

$$\text{div}(F) = \partial_1 F_1 + \cdots + \partial_n F_n$$

"How away from/towards the points around a p move"

- **Source:** Point $p \in \mathbb{R}^n$ if $(\text{div } F)(p) > 0$
- **Sink:** Point $p \in \mathbb{R}^n$ if $(\text{div } F)(p) < 0$
- **Sourceless:** Point $p \in \mathbb{R}^n$ if $(\text{div } F)(p) = 0$
- Let F, G be C^1, $f: \underline{\mathbb{R}^3} \to \mathbb{R}$ be C^2,
  - $\text{div}(\alpha F + \beta G) = \alpha \text{ div}(F) + \beta \text{ div}(G)$
  - $\text{div}(fF) = (\nabla f) \cdot F + f \text{ div}(F)$
  - $\text{div}(\nabla f) = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f$

Laplacian: Operator $\Delta f = \text{div}(\nabla f) = \nabla \cdot (\nabla f) = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f$

- For $F: \mathbb{R}^3 \to \mathbb{R}^3$ C^1 on some open convex $U \subseteq \mathbb{R}^3$,
  - $\text{curl}(F) = 0$ on $U \quad \Rightarrow \quad F = \nabla f$ for some C^2 $f: U \to \mathbb{R}$
  - $\text{div}(F) = 0$ on $U \quad \Leftrightarrow \quad F = \text{curl}(G)$ for some C^2 $G: U \to \mathbb{R}^3$???

Irrotational/Curl-Free: On an open set $U \subseteq \mathbb{R}^n$, C^1 function $F: \mathbb{R}^n \to \mathbb{R}^n$ if $\partial_i F_j = \partial_j F_i$

- F is irrotational $\Leftrightarrow \text{curl}(F) = 0$
- F is conservative on $U \Rightarrow F$ is irrotational on U
- F is irrotational on convex $U \Rightarrow F$ is conservative on U

Jordan Curve Theorem: Any simple closed curve $S \subseteq \mathbb{R}^2$ divides $\mathbb{R}^2$ into Ω and $\mathbb{R}^2 \setminus \Omega$, where

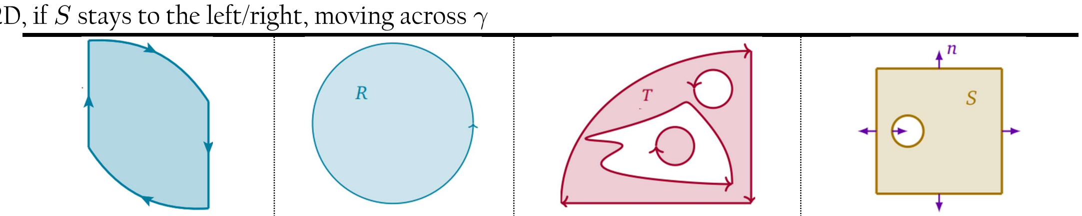
- Ω is open, bounded, Jordan measurable, and $\partial\Omega = S$
- $\mathbb{R}^2 \setminus \Omega$ is unbounded

Simply Connected Domain: Open, path-connected set $D \subseteq \mathbb{R}^2$ if for all simple closed curves $S \subseteq D, \Omega \subseteq D$.

- *"There are no holes inside D"*
- F is irrotational on simply connected $D \Rightarrow F$ is conservative on D
- 3D extension – open, path-connected set $D \subseteq \mathbb{R}^2$ if for all simple closed curves $S \subseteq D$, $S = \partial E$ (relative boundary) for some $E \subseteq D$. Some exceptions – curves can't be knots or self-intersections

Positively/Negatively Oriented: The boundary ∂S of regular region $S \subseteq \mathbb{R}^2$ or $S \subseteq \mathbb{R}^3$, interpreted as a closed piecewise curve, if, equivalently,

- n points away from/towards S
- In 2D, if S stays to the left/right, moving across γ



Negatively-oriented	Positively-oriented	Negatively-oriented	Positively-oriented
Simply connected	Simply connected	Not simply connected	Not simply connected

**Assume sets are open*

Green's Theorem: For $F: \mathbb{R}^2 \to \mathbb{R}^2$ C^1 on regular region $S \subseteq \mathbb{R}^2$, if ∂S is positively-oriented closed piecewise curve,

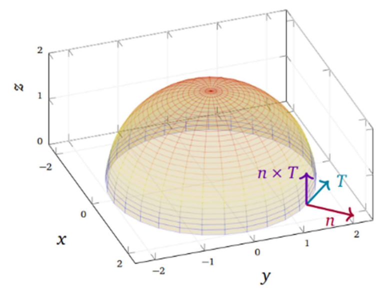
$$\oint_{\partial S} (F \cdot T) ds = \iint_S \text{curl}(F) dA \qquad \text{“Circulation over } \partial S \text{ is the sum of curls of points in } S\text{”}$$

$$\oint_{\partial S} (F \cdot n) ds = \iint_S \text{div}(F) dA \qquad \text{“Flux over } \partial S \text{ is the sum of divergence of points in } S\text{”}$$

Divergence Theorem: For $F: \mathbb{R}^3 \to \mathbb{R}^3$ C^1 on regular region $S \subseteq \mathbb{R}^3$, if ∂S is positively-oriented closed piecewise surface,

$$\oiint_{\partial S} (F \cdot n) dS = \iiint_S \text{div}(F) dV$$

Stokes Orientation: If, moving along relative boundary ∂S of oriented surface S and assuming the normal n points away from S, if S is on the left of T.



Stokes' Theorem: For $F: \mathbb{R}^3 \to \mathbb{R}^3$ C^1 on open set containing surface $S \subseteq \mathbb{R}^3$, if ∂S is a closed piecewise curve with Stokes orientation,

$$\oint_{\partial S} (F \cdot T) ds = \iint_S (\text{curl } F) \cdot n dS$$

“Circulation along ∂S is the surface integral/flux of curl over S”

“Circulation along ∂S is the sum of circulations at single points”

- S is closed $\Rightarrow \partial S = \emptyset \Rightarrow \oint_{\partial S} (F \cdot T) ds = 0$