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# Notes – MAT223 (Linear Algebra)

$$\vec{w} = \begin{bmatrix}1\\1\end{bmatrix}$$
is a linear combination of
$$\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}, \vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$$
when there is a solution to
$$a\vec{u} + b\vec{v} = \vec{w}$$
$$a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$
$$\begin{cases} a + 2b = 1 \\ 2a + 3b = 1 \end{cases}$$

**Linear Equation:**
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

**Consistent:** System has 1+ solutions
**Inconsistent:** System has 0 solutions

$\mathbb{R}^1 = \mathbb{R} =$ 1D Space (number line)
$\mathbb{R}^2 =$ 2D Space
$\mathbb{R}^3 =$ 3D Space
$\mathbb{R}^n =$ n-D Space

For some = $\exists$ (there exists)
For any = $\forall$ (for all)

$$\begin{bmatrix}x\\y\end{bmatrix} = [x, y] = (x, y) = \langle x, y\rangle = \left\langle\begin{matrix}x\\y\end{matrix}\right\rangle$$

**Equations vs. Set-Builder Notation**

$$S = \{(x, y) : y = -x + 4\}$$
is equivalent to
$$y = -x + 4$$
$$S = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$$
is equivalent to
$$y = mx + b$$

"A set of all possible $(x, y)$ points in which $y = -x + 4$"

*Note:* Think of a set of infinite points as a line, and a set of infinite lines as a plane.

**2D Line**

$l = \{\vec{x} \in \mathbb{R}^2 \mid \vec{x} = t\vec{d} + \vec{p}$ for some $t \in \mathbb{R}\}$ (set-builder notation)
$\vec{x} = t\vec{d} + \vec{p}$ (vector form, v.1)
$\begin{bmatrix}x\\y\end{bmatrix} = t\begin{bmatrix}d_1\\d_2\end{bmatrix} + \begin{bmatrix}p_1\\p_2\end{bmatrix}$ (vector form, v.2)

Why 'for some'? No *single* vector in the set $l$ satisfies $\vec{x} = t\vec{d} + \vec{p}$ 'for all' values of $t$ simultaneously.

$\vec{d} =$ direction vector
$t =$ parameter variable
Think of $\begin{bmatrix}d_1\\d_2\end{bmatrix}$ as $slope = \frac{d_2}{d_1}$
Think of $\begin{bmatrix}p_1\\p_2\end{bmatrix}$ as a point that vector crosses
*Note:* the same vector can be expressed with different $\vec{d}/t$ values.

**Skew:** Two direction vectors that intersect
**Colinear:** Two points on the same line
**Overdetermined:** System # of equations > # of unknown variables
**Underdetermined:** System # of equations < # of unknown variables

**Customizing Restrictions**

$R = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p}$ for some $t \geq 0\}$ — ray
$S = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p}$ for some $t \in [0, 2]\}$ — line segment
$C = \{\vec{x} : \|\vec{x}\| = 1\}$ — circle (radius 1, centered at origin)
$U = \{\vec{x} : \vec{x} = t\vec{e_1} + s\vec{e_2}$ for some $t, s \in [0, 1]\}$ — unit square (polygon, 2d area)
$P = \left\{\vec{x} : \vec{x} = t\begin{bmatrix}2\\1\end{bmatrix} + s\begin{bmatrix}-1\\1\end{bmatrix}\right.$ for some $\left. t \in [0, 1], s \in [-1, 1]\right\}$ — parallelogram (polygon, 2d area)

Let $\vec{w} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}$ (ie. a linear combination of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$)
$\vec{w}$ is a **non-negative** linear combination of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_2}$ if $a_1, a_2, \dots, a_n \geq 0$.
$\vec{w}$ is a **convex** linear combination of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_2}$ if $a_1, a_2, \dots, a_n \geq 0$ and $a_1 + a_2 + \cdots + a_n = 1$.

- *Note:* the set of $\vec{v_1}$ and $\vec{v_2}$'s convex linear combinations form a straight line between $\vec{v_1}$ and $\vec{v_2}$

**3D Line**

$l = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = t\vec{d} + \vec{p}$ for some $t \in \mathbb{R}\}$
$\vec{x} = t\vec{d} + \vec{p}$
$$\begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix} + \begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}$$

**3D Plane**

$\mathcal{P} = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = t\vec{d_1} + s\vec{d_2} + \vec{p}$ for some $t, s \in \mathbb{R}\}$
$\vec{x} = t\vec{d_1} + s\vec{d_2} + \vec{p}$
$$\begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}d_{1_1}\\d_{1_2}\\d_{1_3}\end{bmatrix} + s\begin{bmatrix}d_{2_1}\\d_{2_2}\\d_{2_3}\end{bmatrix} + \begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}$$

**Span:** The set containing all linear combinations of the vectors in a set of vectors *V*. The set of every single point that a set of vectors can "reach" when combined together in any way.

$$\text{span } V = \{\vec{v} : \vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n} \quad \text{for some} \quad \vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V, \quad a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$

*Example:* $\vec{u} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\-2\end{bmatrix}$. Find span$\{\vec{u}, \vec{v}\}$.

$$\text{span}\{\vec{u}, \vec{v}\} =$$
$$\{\vec{x} : \vec{x} = a\vec{u} + b\vec{v} \text{ for some } a, b \in \mathbb{R}\}$$
$$a\vec{u} + b\vec{v} = \vec{x}$$
$$a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$
$$\begin{cases} a + b = x \\ a - 2b = y \end{cases}$$
$$a = \frac{1}{3}(2x + y), b = \frac{1}{3}(x - y)$$

$a, b$ can be set based on the $(x, y)$ coordinates of the point we're testing if it's a linear combination. Thus, span$\{\vec{u}, \vec{v}\} = \mathbb{R}^2$ (all of 2D space)

*Example:* $\vec{u} = \begin{bmatrix}-1\\2\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\-2\end{bmatrix}$. Find span$\{\vec{u}, \vec{v}\}$.

$$a\vec{u} + b\vec{v} = \vec{x}$$
$$a\begin{bmatrix}-1\\2\end{bmatrix} + b\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$
$$\begin{cases} -a + b = x \\ 2a - 2b = y \end{cases}$$
$$0 = 2x + y$$

Convert $y = -2x$ to vector form

$$\text{span}\{\vec{u}, \vec{v}\} = \begin{bmatrix}x\\y\end{bmatrix}$$
$$= t\begin{bmatrix}1\\-2\end{bmatrix}$$
$$= t\vec{v} \text{ for some } t \in \mathbb{R}^2$$
$$= \text{span}\{\vec{v}\} \in \mathbb{R}^2$$

*Note:* A single vector's span is always a single line.

*Example:* $S = \{(x, y) \in \mathbb{R}^2 : y = x - 4\}$. Find span $S$.

Note that $\begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix} \in S$

All LCs of $\left\{\begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix}\right\} \subseteq$ All LCs of all $\vec{v} \in S =$ span $S$

Since span $\left\{\begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix}\right\} = \mathbb{R}^2$,

$$\therefore \text{span } S = \mathbb{R}^2$$

*Note:* The span of a line not touching the origin is $\mathbb{R}^2$.

**Rewriting Spans as Vectors**

The span of a vector,

$$\text{span}\{\vec{v_1}\} = \{\vec{v} : \vec{v} = a_1\vec{v_1} \text{ for some } a_1 \in \mathbb{R}\},$$

can be re-written as

$$\text{span}\{\vec{d}\} = \{\vec{x} : \vec{x} = t\vec{d} \text{ for some } t \in \mathbb{R}\},$$

which is equivalent to

$$l = \{\vec{x} : \vec{x} = t\vec{d} + \vec{0} \text{ for some } t \in \mathbb{R}\},$$

which is a **line** that passes the origin.

The span of two vectors,

$$\text{span}\{\vec{d_1}, \vec{d_2}\} = \{\vec{v} : \vec{v} = a_1\vec{v_1} + a_2\vec{v_2} \text{ for some } a_1, a_2 \in \mathbb{R}\},$$

can be re-written as

$$\text{span}\{\vec{d_1}, \vec{d_2}\} = \{\vec{x} : \vec{x} = t\vec{d_1} + s\vec{d_2} \text{ for some } t, s \in \mathbb{R}\},$$

which is equivalent to

$$\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d_1} + s\vec{d_2} + \vec{0} \text{ for some } t, s \in \mathbb{R}\},$$

which is a **plane** that passes the origin.

**Set Addition**

$$A = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}, B = \{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}\} \Rightarrow A + B = \{\vec{x} : \vec{x} = \vec{a} + \vec{b} \text{ for some } \vec{a} \in A, \vec{b} \in B\}$$

*Example:* $C = \{\vec{x} : \|\vec{x}\| = 1\}$ and $X = \{\vec{e_1}, 2\vec{e_2}\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right\}$. Find $C + X$.

$$C + X = \left\{\vec{x} + \begin{bmatrix}1\\0\end{bmatrix} : \|\vec{x}\| = 1\right\} \cup \left\{\vec{x} + \begin{bmatrix}0\\2\end{bmatrix} : \|\vec{x}\| = 1\right\},$$

which can be written as

$$C + X = \left\{\vec{x} + \vec{v} : \|\vec{x}\| = 1, \vec{v} = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right\}.$$

*C* is a set referring to a circle (r = 1, center = origin).

*X* is a set referring to two vectors $\vec{e_1}$ and $2\vec{e_2}$

*C*+*X* is a set referring to two circles,

- Circle 1 ($r$ = 1, center = origin + $\vec{e_1}$)
- Circle 2 ($r$ = 1, center = origin + $2\vec{e_2}$)

**Rewriting Vectors as Translated Spans**

Since span$\{\vec{d}\} = t\vec{d} + \vec{0}$, for a line/plane/something $\vec{x}$,

$$\vec{x} = (a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}) + \vec{p}$$
$$= \text{span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\} + \{\vec{p}\}$$

**Homogenous:** A vector equation/system of equations with variables $a_1, a_2, \ldots, a_n$ of the form

$$a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n} = \vec{0}$$

**Trivial:** A linear combination $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}$ where all coefficients $a_1, a_2, \ldots a_n = 0$.

**Non-Trivial:** A linear combination $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}$ where 1+ coefficients aren't 0.

## Linear Independence and Dependence – Two Equivalent Definitions

| Geometric Definition | Algebraic Definition |
| --- | --- |
| $\vec{w} = a\vec{u} + b\vec{v} \Rightarrow \vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$<br>$\vec{w} = a\vec{u} + b\vec{v} \Rightarrow \text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}$<br><br>**Linearly Independent:** In a set of vectors, if all vectors contribute to the span. ($\vec{u}$, $\vec{v}$)<br>**Linearly Dependent:** In a set of vectors, if a vector doesn't contribute to the span. ($\vec{w}$)<br><br>If one of the vectors is a linear combination of the others. Or, if some point can be represented as 2+ unique linear combinations. | $\vec{w} = \begin{bmatrix}1\\2\end{bmatrix}, \vec{u} = \begin{bmatrix}-1\\-2\end{bmatrix}, \vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$<br>$1\vec{w} + 1\vec{u} + 0\vec{v} = \vec{0}, \therefore \vec{w}, \vec{u}$ are linearly dependent<br><br>**Linearly Independent:** In a set of vectors, if there's no non-trivial linear combination of them equalling 0.<br>**Linearly Dependent:** In a set of vectors, if there's a non-trivial linear combination of them equalling 0.<br><br>If the only solution to homogenous equation<br>$a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n} = \vec{0}$<br>Is $a_1, a_2, \ldots, a_n = 0$ (trivial), the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are linearly independent. |

## Reduced Row Echelon Form (RREF)

*REF*

$$\begin{bmatrix} 1 & a & b & c & w \\ 0 & 3 & d & e & x \\ 0 & 0 & 1 & f & y \\ 0 & 0 & 0 & 1 & z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & a & 0 & x \\ 0 & 2 & b & 0 & y \\ 0 & 0 & 0 & 1 & z \end{bmatrix}$$

*RREF*

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & w \\ 0 & \mathbf{1} & 0 & 0 & x \\ 0 & 0 & \mathbf{1} & 0 & y \\ 0 & 0 & 0 & \mathbf{1} & z \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} & 0 & a & 0 & x \\ 0 & \mathbf{1} & b & 0 & y \\ 0 & 0 & 0 & \mathbf{1} & z \end{bmatrix}$$

**Leading Ones/Pivots:** A row's first number, which = 1

Identity Matrix: Notated $I_{n\times n}$ or just $I$, a square matrix where main diagonal numbers = 1, everything else = 0

Pivot Column: Columns with leading ones
Free Variable Columns: Columns without leading ones

*Example:* Find the complete solution to

$$\begin{cases} 3w - 6x + 6y + 4z = -5 \\ 3v - 7w + 8x - 5y + 8z = 9 \\ 3v - 9w + 12x - 9y + 6z = 15 \end{cases}$$

Turn into matrix, row reduce to

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Exists free variable columns, thus infinite solutions.

Introduce **free variables** (placeholders) so $\begin{cases} x = t \\ y = s \end{cases}$.

$$\begin{cases} v - 2t + 3s = -24 \\ w - 2t + 2s = -7 \\ z = 4 \end{cases}$$

Isolate the main variables, convert to vector form:

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ for some } t, s \in \mathbb{R}$$

*Note:* Think of solutions as where things intersect. Can be an entire line/plane, not just a single triplet of x/y/z values (a single point or "unique solution")

*Example:* Are $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ linearly dependent?

Convert to matrix, row reduce to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 5/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 3 is a free variable column, infinite solutions, and thus linearly dependent. If no free variables columns, linearly independent.

**Maximal Linearly Independent Subset (MLIS):** Largest linearly independent subset of a vector set

*Get rid of the free variable column in the above example, resulting two vectors are its MLIS

*Note:* If you are in $\mathbb{R}^n$ and have $>n$ vectors in a set, they are linearly dependent.

**Dot Product** – Two Equivalent Definitions

| Geometric Definition | Algebraic Definition |
|---|---|
| $\vec{a}\cdot\vec{b} = \lVert\vec{a}\rVert\lVert\vec{b}\rVert\cos\theta$<br>Where $\vec{a}$ and $\vec{b}$ are two vectors and $\theta$ ($0 \le \theta \le \pi$), is the smallest angle between them | $\vec{a}\cdot\vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ |

**Combining Definitions** (try deriving them as exercises!)

| Angle Between Vectors | Length of a Vector | |
|---|---|---|
| $\theta = \arccos\left(\dfrac{u_1v_1 + u_2v_2 + \cdots + u_nv_n}{\lVert\vec{u}\rVert\lVert\vec{v}\rVert}\right)$ | $\lVert\vec{v}\rVert = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ | $\lVert\vec{v}\rVert^2 = \vec{v}\cdot\vec{v}$ |

*Example:* Plane $\mathcal{P}$ passes $A = (1,0,0)$, $B = (0,1,0), C = (0,0,1)$. Find vector and normal form.

First, vector form. Since $\mathcal{P}$'s a plane, two direction vectors are needed.

$$\overrightarrow{d_1} = A - B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{d_2} = B - C = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

*Remember:* the same vector can be written with different $d_1, d_2$.

$$\therefore \mathcal{P} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Next, normal form. Solve $\vec{n}\cdot d_1 = 0$ and $\vec{n}\cdot d_2 = 0$, where $\vec{n} = (x,y,z)$.

$$\begin{cases} 1x - 1y + 0z = 0 \\ 0x + 1y - 1z = 0 \end{cases}$$

Get equation of normal vector, $x - 2y + z = 0$. Find a point it passes, $\vec{n} = (1,1,1)$.

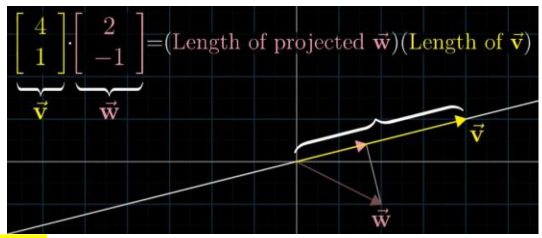
$$\therefore \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 0$$

*Note:* You can convert back to vector form by just doing the above equation.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x - 1 \\ y \\ z \end{bmatrix}$$

$$= x + y + z - 1 = 0$$

You get the original plane's equation, $x + y + z = 1$. Convert to vector form by finding $\overrightarrow{d_1}, \overrightarrow{d_2}$, and $\vec{p}$, via finding points the plane passes through.




**Direction:** $\vec{u}$ points in direction of $\vec{v}$ if $\exists k \in \mathbb{R}, \vec{u} = k\vec{v}$ (if $\vec{u}$ is a multiple of $\vec{k}$). Positive direction $\Leftrightarrow k > 0$

**Orthogonal:** Essentially, perpendicular, forms a right angle. The vectors $\vec{u}, \vec{v}$, when $\vec{u}\cdot\vec{v} = 0$.

- *Note:* Perpendicularity is the orthogonality of classical geometric objects.

**Normal Vector:** A non-zero (not $\vec{0}$) vector orthogonal to all direction vectors of a line/plane/hyperplane $\vec{x}$

**Normal Form:** Form of a normal vector of $\vec{x}$, expressed as the solution to $\vec{n}\cdot(\vec{x} - \vec{p}) = 0$ where $\vec{n} \ne 0$.

- *Note:* The equation basically says $\vec{n}$ and $\vec{x}$ are perpendicular. $-\vec{p}$ gets rid of any translations of $\vec{x}$.

**Hyperplane:** A set of all normal vectors to a line / plane / hyperplane $\vec{x}$. The set of solutions to normal form.

- *Note:* Hyperplanes are neither hyper nor planes. In $\mathbb{R}^n$, hyperplanes are always in $\mathbb{R}^{n-1}$.

**Zero Matrix:** Matrix where everything is 0.

**Size/Shape/Dimensions:** # rows × # columns. $(i,j)$ entry means the $i$-th row, $j$-th column.

**Diagonal Matrix:** All non-diagonals = 0

**Upper Triangular Matrix:** Everything below diagonal = 0
**Lower Triangular Matrix:** Everything above diagonal = 0

**Symmetric Matrix:** Square matrix if $\forall i,j, (i,j) = (j,i)$
**Skew/Anti-Symmetric Matrix:** If $\forall i,j, (i,j) = -(j,i)$

**Matrix Multiplication:** Requires dimensions L $= a \times b$, R $= b \times c$. Results in X $= a \times c$. Watch 3Blue1Brown's video on this for good intuition!

$$\begin{bmatrix} a_1 & b_1 & \cdots & z_1 \\ a_2 & b_2 & \cdots & z_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & b_n & \dots & z_n \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \cdots & \omega_1 \\ \alpha_2 & \beta_2 & \cdots & \omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \beta_n & \dots & \omega_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1\alpha_1 + b_1\alpha_2 + \cdots + z_1\alpha_n & a_1\beta_1 + b_1\beta_2 + \cdots + z_1\beta_n & \cdots & a_1\omega_1 + b_1\omega_2 + \cdots + z_1\omega_n \\ a_2\alpha_1 + b_2\alpha_2 + \cdots + z_2\alpha_n & a_2\beta_1 + b_2\beta_2 + \cdots + z_2\beta_n & \cdots & a_2\omega_1 + b_2\omega_2 + \cdots + z_2\omega_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n\alpha_1 + b_n\alpha_2 + \cdots + z_n\alpha_n & a_n\beta_1 + b_n\beta_2 + \cdots + z_n\beta_n & \dots & a_n\omega_1 + b_n\omega_2 + \cdots + z_n\omega_n \end{bmatrix}$$

It's hard to read, so instead, <u>think of it as a bunch of dot products</u> between single rows and single columns. You can do dot products between single rows and single columns (see specific case below).

$$= \begin{bmatrix} \text{row } 1_L \cdot \text{col } 1_R & \text{row } 1_L \cdot \text{col } 2_R & \cdots & \text{row } 1_L \cdot \text{col } n_R \\ \text{row } 2_L \cdot \text{col } 1_R & \text{row } 2_L \cdot \text{col } 2_R & \cdots & \text{row } 2_L \cdot \text{col } n_R \\ \vdots & \vdots & \ddots & \vdots \\ \text{row } m_L \cdot \text{col } 1_R & \text{row } m_L \cdot \text{col } 2_R & \dots & \text{row } m_L \cdot \text{col } n_R \end{bmatrix}$$

**Specific Case:** LM $= a \times b$, RM $= b \times 1$

$$\begin{bmatrix} a_1 & b_1 & \cdots & z_1 \\ a_2 & b_2 & \cdots & z_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & b_n & \dots & z_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + x_2 \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \cdots + x_n \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1x_1 + b_1x_2 + \cdots + z_nx_n \\ a_2x_1 + b_2x_2 + \cdots + z_nx_n \\ \cdots \\ a_nx_1 + b_2x_2 + \cdots + z_nx_n \end{bmatrix}$$

which can be rewritten as:

$$\begin{bmatrix} \vec{a} & \vec{b} & \cdots & \vec{z} \end{bmatrix} \vec{x}$$
$$= \left[ x_1\vec{a} + x_2\vec{b} + \cdots + x_n\vec{z} \right]$$

*Note:* If LM has 1 row, it's dot product

$$\begin{bmatrix} a & b & \cdots & c \end{bmatrix} \vec{x}$$
$$= [x_1a + x_2b + \cdots + x_nz]$$

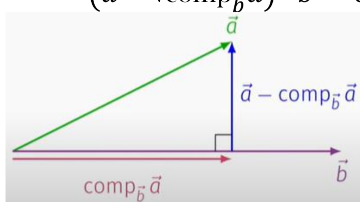
which can be rewritten as...

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_2 \end{bmatrix} \vec{b}$$
$$= [a_1b_1 + a_2b_2 + \cdots + a_nb_n]$$
$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

It's not the exact same since the scalar's in a $1 \times 1$ matrix. But it does show that <u>we can flip the left matrix so it's vertical</u>.

**Component:** The component of $\vec{a}$ in direction $\vec{b}$ is a vector from $\vec{0}$ to $\vec{b}$'s closest point to $\vec{a}$. Written as $\text{vcomp}_{\vec{b}}\vec{a}$.

- $\left(\vec{a} - \text{vcomp}_{\vec{b}}\vec{a}\right) \cdot \vec{b} = 0$




$$\text{vcomp}_{\vec{b}}\vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\right)\vec{b}$$

**Projection:** The projection of $\vec{v}$ onto set $X$ is a vector from $\vec{0}$ to $X$'s closest point to $\vec{v}$. Written as $\text{proj}_X\vec{v}$.

- $\left(\vec{v} - \text{proj}_X\vec{v}\right) \cdot X = 0$

Projection can be found...

- Visually
- Using the above equation
- Calculating the minimum of $\|\vec{v} - X\|$ when $X$ can be expressed as a vector equation.
- With the theorem $\text{proj}_{\text{span}\{\vec{v}\}}\vec{u} = \text{vcomp}_{\vec{v}}\vec{u}$, $(\vec{v} \neq 0)$

*Note:* Projection will be undefined if $X$ is an open set (eg. an interval or 2D shape)

**Subspace:** A subset $\mathcal{V} \subseteq \mathbb{R}^n$, where for $\vec{u}, \vec{v} \in \mathcal{V}, k \in \mathbb{R}$,

(1) $\vec{u} + \vec{v} \in \mathcal{V}$ (2) $k\vec{u} \in \mathcal{V}$ (3) $\vec{0} \in \mathcal{V}$

*Note:* Every subspace is a span. Every span is a subspace.
$\mathcal{V}$ is subspace $\Leftrightarrow \mathcal{V} =$ span $X$ for some set $X$.

**Trivial Subspace:** $\{\vec{0}\} \subseteq \mathbb{R}^n$. Always a subspace of $\mathbb{R}^n$ along with $\mathbb{R}^n$ itself.

*Example:* $\mathcal{V} \subseteq \mathbb{R}^2$ is the complete solution to $x + 2y = 0$. Prove $\mathcal{V}$ is a subspace.

Let $\vec{u}, \vec{v} \in \mathcal{V}, k \in \mathbb{R}$. By definition, this is true:

$$\begin{cases} u_1 + 2u_2 = 0 \\ v_1 + 2v_2 = 0 \end{cases}$$

(1) Prove $\vec{u} + \vec{v} \in \mathcal{V}$

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

And $\vec{u} + \vec{v}$'s coordinates satisfy $\mathcal{V}$'s equation.

$$\begin{aligned} &(u_1 + v_1) + 2(u_2 + v_2) \\ &= (u_1 + 2u_2) + (v_1 + 2v_2) = 0 \end{aligned}$$

(2) Prove $k\vec{u} \in \mathcal{V}$

$$k\vec{u} = \begin{bmatrix} ku_1 \\ ku_2 \end{bmatrix}$$

And $k\vec{u}$'s coordinates satisfy $\mathcal{V}$'s equation.

$$\begin{aligned} &ku_1 + 2(ku_2) \\ &= k(u_1 + 2u_2) = 0 \end{aligned}$$

(3) Prove $\vec{0} \in \mathcal{V}$

$$0 + 2(0) = 0$$

**Basis:** A basis for subspace $\mathcal{V}$ is a linearly independent vector set $\mathcal{B}$ where span $\mathcal{B} = \mathcal{V}$.

- AKA any linearly independent set whose span = a given subspace. There're infinitely many.

**Dimension:** The # of elements in a basis for subspace $\mathcal{V}$. Written as dim $\mathcal{V}$

*Example:* $A = \{\vec{x}: x_1 + 2x_2 - x_3 = 0, x_1 + 6x_4 = 0\}$. Find a basis for and dimension of $A$.

Find the complete solution to the system, which is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -6 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$A = \text{span}\left\{\begin{bmatrix} 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\}$, which is a linearly independent set with two elements, so $\dim A = 2$.

**Equivalent Notations**

$$\begin{cases} x + 2y + z = 3 \\ x - y - z = -2 \end{cases}$$

$$\begin{bmatrix} x + 2y + z \\ x - y - z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$A$ = coefficient matrix

$\vec{x}$ = column vector of variables

$\vec{b}$ = column vector of constants

**Matrix Multiplication – Column Interpretation**

The thing we've been doing the whole time.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} + z\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

**Matrix Multiplication - Row Interpretation**

Let $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}$ = the rows of the coefficient matrix.

$$\vec{r_1} = [1 \quad 2 \quad 1]$$
$$\vec{r_2} = [1 \quad -1 \quad -1]$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \vec{r_1} \\ \hline \vec{r_2} \end{bmatrix}\vec{v} = \begin{bmatrix} \vec{r_1} \cdot \vec{v} \\ \vec{r_2} \cdot \vec{v} \end{bmatrix}$$

*Example:* Find all vectors orthogonal to $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Find $\vec{x}$ that satisfies $\vec{a} \cdot \vec{x} = 0$ and $\vec{b} \cdot \vec{x} = 0$.

Use the row-interpretation matrix equation

$$\begin{bmatrix} \vec{a} \\ \hline \vec{b} \end{bmatrix}\vec{x} = \begin{bmatrix} \vec{a} \cdot \vec{x} \\ \vec{b} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$$

Row-reduce $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and solve the above equation to get

$$\vec{x} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

*Example:* Find a normal vector (in normal form) of hyperplane $Q$, $\vec{x} = t\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + r\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

The normal vector has to be orthogonal to $d_1, d_2, d_3$.

Use the row-interpretation matrix equation

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \vec{0}$$

Row-reduce and solve to get

$\vec{x} = t\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, AKA $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) = 0$

*Note:* You can still solve it without a matrix equation, so I don't see the point of the above.

$\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \vec{x} = 0$ and $\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \vec{x} = 0$ have equations

$$\begin{cases} x + y + z = 0 \\ x + 2y + z = 0 \end{cases}$$

Solution is $y = 0, x + z = 0$. If $x = -1$, you get $z = -x = 1$ and the same vector as above.

**Standard Basis:** In $\mathbb{R}^n$, the set $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \dots, \overrightarrow{e_n}\}$, where

$$\overrightarrow{e_1} = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \overrightarrow{e_2} = \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, \overrightarrow{e_n} = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

**Three Equivalent Notations**

$$\vec{v} = \begin{bmatrix}2\\3\end{bmatrix} \qquad \vec{v} = 2\overrightarrow{e_1} + 3\overrightarrow{e_2} \qquad \vec{v} = \begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{E}}$$

Where $\mathcal{E} = \{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ is the standard basis. The vectors in $\mathcal{E}$ define 'length' of one unit in our made-up grid.

*Example:* Let $\mathcal{E} = \{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ be $\mathbb{R}^2$'s standard basis, $\mathcal{C} = \{\overrightarrow{c_1}, \overrightarrow{c_2}\}$, $\overrightarrow{c_1} = \overrightarrow{e_1} + \overrightarrow{e_2}, \overrightarrow{c_2} = 3\overrightarrow{e_2}$, $\vec{v} = 2\overrightarrow{e_1} - \overrightarrow{e_2}$. Find $[\vec{v}]_{\mathcal{E}}, [\vec{v}]_{\mathcal{C}}$.

Write out the question more clearly.

$$\vec{v} = \begin{bmatrix}2\\-1\end{bmatrix}_{\mathcal{E}}, \overrightarrow{c_1} = \begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{E}}, \overrightarrow{c_2} = \begin{bmatrix}0\\3\end{bmatrix}_{\mathcal{E}}$$

Must find $\vec{v} = \begin{bmatrix}2\\-1\end{bmatrix}_{\mathcal{C}}$

$$x\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{E}} + y\begin{bmatrix}0\\3\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}2\\-1\end{bmatrix}_{\mathcal{E}}$$

$$\begin{cases} x = 2 \\ x + 3y = -1 \end{cases}$$

Solve to get $x = 2, y = -1$, therefore

$$\vec{v} = \begin{bmatrix}2\\-1\end{bmatrix}_{\mathcal{C}}$$

*Note:* $\{\overrightarrow{e_1}, \overrightarrow{e_2}\}, \{\overrightarrow{-e_1}, \overrightarrow{-e_2}\}$ is right-handed.

$\{-\overrightarrow{e_1}, \overrightarrow{e_2}\}, \{\overrightarrow{e_1}, \overrightarrow{-e_2}\}, \{\overrightarrow{e_2}, \overrightarrow{e_1}\}$ is left-handed.

$$\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}_{\mathcal{B}} = a_1\overrightarrow{b_1} + a_2\overrightarrow{b_2} + \dots + a_n\overrightarrow{b_n}$$

Representation of $\vec{v}$ in basis $\mathcal{B} = \{\overrightarrow{b_1}, \overrightarrow{b_2}, \dots \overrightarrow{b_n}\}$ of subspace $\mathcal{V}$:

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} \text{ where}$$

$$\vec{v} = a_1\overrightarrow{b_1} + a_2\overrightarrow{b_2} + \dots + a_n\overrightarrow{b_n}$$

**Real Object vs. Representation of an Object**

| | |
|---|---|
| $\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{E}}$ | True vector, real-life object |
| $[\vec{v}]_{\mathcal{E}} = \begin{bmatrix}2\\3\end{bmatrix}$ | Artificial construct, meaningless |

**Orthonormal:** $\mathcal{B} = \{\overrightarrow{b_1}, \overrightarrow{b_2}, \dots, \overrightarrow{b_n}\}$, if all vectors are orthogonal.

**Right-Handed/Positively Oriented:** $\mathcal{B} = \{\overrightarrow{b_1}, \overrightarrow{b_2}, \dots, \overrightarrow{b_n}\}$, if all vectors can be rotated to align with standard basis vectors.

- "Can be continuously transformed to the standard basis while remaining linearly independent throughout."
- In $\mathbb{R}^2$, basically if $\overrightarrow{b_2}$ points counterclockwise of $\overrightarrow{b_1}$

**Left-Handed/Negatively Oriented:** In $\mathcal{B} = \{\overrightarrow{b_1}, \overrightarrow{b_2}\}$, if vectors have to be flipped on an axis to align with standard basis vectors.

*Example:* Is $T\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}2x^2\\y+4\end{bmatrix}$ linear?

Let $\vec{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}, \vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}, k \in \mathbb{R}$.

(1) Prove $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

$$T(\vec{u} + \vec{v}) = \begin{bmatrix}2(u_1 + v_1)^2\\(u_2 + v_2) + 4\end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix}2(u_1)^2 + 2(v_1)^2\\(u_2 + 4) + (v_2 + 4)\end{bmatrix}$$

Two sides do not match, not linear.

(2) Prove $T(k\vec{v}) = kT(\vec{v})$

$$T(k\vec{v}) = \begin{bmatrix}2(kv_1)^2\\(kv_2) + 4\end{bmatrix} = \begin{bmatrix}2k^2v_1^2\\kv_2 + 4\end{bmatrix}$$

**Linear Transformation:** A function $T: V \to W$, where $V$ and $W$ are subspaces, when $\forall \vec{u}, \vec{v} \in V, k \in \mathbb{R}$,

(1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

(2) $T(k\vec{v}) = kT(\vec{v})$

*Note:* Same notations, $T: \mathbb{R}^n \to \mathbb{R}^m, T(\vec{x}) = M\vec{x}$, and $T\vec{x}$

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear, then...

- $T(\vec{0}) = \vec{0}$
- $T$ moves lines to lines/points
- $T$ moves parallel lines to parallel lines/points
- $T$ moves subspaces to subspaces

**Composition:** Let $f: A \to B, g: B \to C$. Composition of $g$ and $f$, $g \circ f$, is the function $h: A \to C$ where $h(x) = g \circ f(x) = g(f(x))$.

$$kT(\vec{v}) = k\begin{bmatrix} 2(v_1)^2 \\ (v_2)+4 \end{bmatrix} = \begin{bmatrix} 2kv_1^2 \\ kv_2 + 4k \end{bmatrix}$$

Two sides do not match, not linear.

*Note:* Use counterexamples too.

**Image/Range:** Set of all of transformation $T: V \to W$'s outputs on set $X$, $T(X)$.

$\text{range}(T) = \{\vec{v} \in W : \vec{v} = T\vec{x} \text{ for some } \vec{x} \in V\}$
$\forall T: \mathbb{R}^n \to \mathbb{R}^m$, $\text{range}(T) \subseteq \mathbb{R}^m$ is subspace.

**Rank:** $\text{rank}(T) = \dim(\text{image}(T))$

- Rank 0 $\Rightarrow \text{range}(T) = \vec{0}$
- Rank 1 $\Rightarrow \text{range}(T) = \text{line}$

*Example:* Find matrix $M$ for $T\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ x \end{bmatrix}$.

Since $T: \mathbb{R}^2 \to \mathbb{R}^2$, let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Get example inputs $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Since $M$ is a matrix for $T$, $T\vec{x} = M\vec{x}$ for all $\vec{x}$.

| $T\vec{x} = M\vec{x}$ | $T\vec{x} = M\vec{x}$ | Result is $M$ |
|---|---|---|
| $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = M\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = M\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ | $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ |
| $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ | $= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. |
| $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ | $\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | |

**Null Space/Kernel:** Set of all vectors that become $\vec{0}$ under transformation $T: V \to W$.

$\text{null}(T) = \{\vec{x} \in V : T\vec{x} = 0\}$
$\forall T: \mathbb{R}^n \to \mathbb{R}^m$, $\text{null}(T) \subseteq \mathbb{R}^m$ is subspace.

**Nullity:** $\text{nullity}(T) = \dim(\text{null}(T))$

**Fundamental Subspace:** Three special subspaces of any matrix $M$:

1) **Row Space:** span of $M$'s rows, $\text{row}(M)$
2) **Column Space:** span of $M$'s columns, $\text{col}(M)$
3) **Null Space:** set of solutions to $M\vec{x} = \vec{0}$

**Transpose:** A matrix with rows/columns swapped, notated $M^T$

Solutions to $M\vec{x} = \vec{b}$ are $\text{null}(M) + \{\vec{p}\}$ where $\vec{p}$ = one solution to the above equation

Linearly independent $\Leftrightarrow \text{null}(M) = \{\vec{0}\}$

$\dim(\text{row}(M)) = \dim(\text{col}(M))$
$\text{row}(M) = \text{col}(M^T)$
$\text{range}(T) = \text{col}(M)$
$\text{rank}(M) = \text{rank}(M^T)$
$\text{rank}(M) + \text{nullity}(M) = \#$ columns $M$
\# of pivots + \# of non-pivots = \# columns
$\text{rank}(T) + \text{nullity}(T) = \dim(\text{domain of } T)$
$\text{rank}(M) = \#$ pivots in $\text{rref}(M)$
$\text{nullity}(M) = \#$ free variable columns in $\text{rref}(M)$

**Induced Transformation:** A linear transformation $T_M: \mathbb{R}^n \to \mathbb{R}^m$ is induced by matrix $M$ when $[T_M\vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}}$.

- $\mathcal{E}, \mathcal{E}' = \mathbb{R}^m, \mathbb{R}^n$'s standard bases
- $[T\vec{v}]_{\mathcal{B}} = [T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}}$, $[T]_{\mathcal{B}}$ is the representation of $T$ in basis $\mathcal{B}$.

*Example:* Let $\mathcal{P}$ be plane $x+y+z=0$, $T: \mathbb{R}^3 \to \mathbb{R}$ is projection on $\mathcal{P}$. Find $\text{range}(T)$, $\text{rank}(T)$, $\text{null}(T)$, $\text{nullity}(T)$.

$T$ is projection onto $\mathcal{P} \Rightarrow \text{range}(T) \subseteq \mathcal{P}$.
$\forall \vec{x} \in \mathcal{P}, T(\vec{x}) = \vec{x} \quad \Rightarrow \mathcal{P} \subseteq \text{range}(T)$
$\therefore \text{range}(T) = \mathcal{P}$

$\mathcal{P}$ is a plane $\Rightarrow \dim(\mathcal{P}) = 2$
$\therefore \text{rank}(T) = \dim(\text{range}(T)) = \dim(\mathcal{P}) = 2$

$T$ is projection onto $\mathcal{P} \Rightarrow \forall \vec{n}, \vec{n} \cdot \mathcal{P} = 0 \Rightarrow T(\vec{n}) = \vec{0}$

$\therefore \text{null}(T) = \{\text{all } \vec{n}\} \cup \{\vec{0}\} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

$\text{null}(T)$ is a span/line $\Rightarrow \dim(\text{null}(T)) = 1$
$\therefore \text{nullity}(T) = \dim(\text{null}(T)) = 1$

*Example:* $M = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}$. Find null, row, column space.

$\text{null}(M)$ is solution to $M\vec{x} = \vec{0}$, $\text{span}\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}$

$\text{row}(M) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}\right\}$

$\text{col}(M) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \end{bmatrix}\right\} = \mathbb{R}^2$

*Example:* $T$ is induced by $M$, let $\vec{v} = 3\vec{e_1} - 3\vec{e_3}$. Find $T(\vec{v})$.

$$[T_M\vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -12 \\ 12 \end{bmatrix}_{\mathcal{E}}$$
$$\therefore T(\vec{v}) = -12\vec{e_1} + 12\vec{e_2}$$

**Identity Function:** Function $\text{id}: X \to X$ where $\text{id}(x) = x$ for all $x \in X$ (ie. it does nothing).

- Corresponding matrix is identity matrix $I$

*Example:* Prove $T: \mathbb{R}^2 \to \mathbb{R}^2$ projection onto x-axis isn't invertible.

Method 1: Prove $T$ isn't one-to-one
$\exists \vec{u}, \vec{v} \in \mathbb{R}, T(\vec{u}) = T(\vec{v})$ and $\vec{u} \neq \vec{v}$

Pick $\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}, \vec{v} = \begin{bmatrix}2\\0\end{bmatrix}$

$\vec{u} = \begin{bmatrix}1\\0\end{bmatrix} \neq \begin{bmatrix}2\\0\end{bmatrix} = \vec{v}$

$T(\vec{u}) = \vec{0} = T(\vec{v})$

Method 2: Prove $T$ isn't onto
$\text{image}(f) \neq \text{codomain}(f)$

$\text{image}(f) =$ the x-axis
$\text{codomain}(f) = \mathbb{R}^2$

*Example:* Find inverse of $M = \begin{bmatrix}3 & 6\\2 & 1\end{bmatrix}$.

$MM^{-1} = I$

$\begin{bmatrix}3 & 6\\2 & 1\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$\begin{bmatrix}3a+6c & 3b+6d\\2a+c & 2b+d\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Get $a = -\frac{1}{9}, c = \frac{2}{9}, b = \frac{2}{3}, d = -\frac{1}{3}$,

$\therefore M^{-1} = \frac{1}{9}\begin{bmatrix}-1 & 6\\2 & -3\end{bmatrix}$

Or solve with elementary matrices:

$\begin{bmatrix}3 & 6\\2 & 1\end{bmatrix} \to \begin{bmatrix}-9 & 0\\2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Matrices corresponding to each step:

$E_1 = \begin{bmatrix}1 & -6\\0 & 1\end{bmatrix}, E_2 = \begin{bmatrix}-\frac{1}{9} & 0\\0 & 1\end{bmatrix} E_3 = \begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}$

Plug into $M^{-1} = E_3E_2E_1$

**Inverse Function:** $f$ is invertible if there exists inverse of $f$, $g: Y \to X$, where $f \circ g = g \circ f = \text{id}$.

- $g = f^{-1}$.
- invertible $\Leftrightarrow$ onto & one-to-one $\Leftrightarrow$ $\text{rref}(A) = I$
- Inverse of matrix $A$ is matrix $B = A^{-1}$ where $AB = BA = I$

**One-to-One/Injective:** $f: X \to Y$, if $\forall a, b \in \mathbb{R}, f(a) = f(b) \Rightarrow a = b$

**Onto/Surjective:** $f: X \to Y$, if $\text{image}(f) = \text{codomain}(f)$

- *Note:* Domain is $X$, codomain is $Y$, image is the set of every actual output value of $f$.

Given function $T: \mathbb{R}^n \to \mathbb{R}^m$ with corresponding matrix $M_{a \times b}$,

- onto $\Leftrightarrow$ $\text{range}(T) = \mathbb{R}^m \Leftrightarrow \text{rank}(T) = m \Leftrightarrow \text{rank}(M) = a$
- one-to-one $\Leftrightarrow$ $\text{nullity}(T) = 0 \Leftrightarrow \text{nullity}(M) = 0$
- invertible $\Leftrightarrow n = m \Leftrightarrow a = b \Leftrightarrow \text{rref}(M) = I$

**Elementary Matrix:** An identity matrix but after one elementary row operation (multiplication, adding multiples of rows, swapping rows).

- invertible $\Leftrightarrow E_k \dots E_2E_1M = I \Leftrightarrow M = E_n \dots E_2E_1$

**Change of Basis Matrix:** Converts any $\vec{x} \in \mathbb{R}^n$ from base $\mathcal{A}$ to $B$.

- Notated $M[\vec{x}]_{\mathcal{A}} = [\vec{x}]_{\mathcal{B}}$
- $M = [\mathcal{B} \leftarrow \mathcal{A}]$ is a matrix converting from $\mathcal{A}$ to $\mathcal{B}$
- invertible $\Leftrightarrow M$ is a change of basis matrix

**Similar Matrix:** Matrices $A$ and $B$, if they're the same linearly transformation but in different bases.

- Notated $A \sim B$
- $A \sim B \Leftrightarrow \exists X, A = XBX^{-1}$

*Example:* $\mathcal{A} = \left\{\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{E}}, \begin{bmatrix}1\\-1\end{bmatrix}_{\mathcal{E}}\right\}, \mathcal{B} = \left\{\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}, \begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}\right\}$, $[\vec{x}]_{\mathcal{A}} = \begin{bmatrix}2\\-3\end{bmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$ and $M = [\mathcal{B} \leftarrow \mathcal{A}]$.

$$\vec{x} = 2\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{E}} - 3\begin{bmatrix}1\\-1\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}-1\\5\end{bmatrix}_{\mathcal{E}}$$

$$\vec{x} = a\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}} + b\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}} = \begin{bmatrix}-1\\5\end{bmatrix}_{\mathcal{E}}$$

$\begin{cases}2a + 5b = -1\\ a + 3b = 5\end{cases}$, solved to $a = -28, b = 11$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}-28\\11\end{bmatrix}$$

Find $\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}-2\\1\end{bmatrix}_{\mathcal{E}}, \begin{bmatrix}1\\-1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}8\\-3\end{bmatrix}_{\mathcal{E}}$, combine the results into a matrix

$$M[\mathcal{B} \leftarrow \mathcal{A}] = \begin{bmatrix}-2 & 8\\1 & -3\end{bmatrix}$$

*Example:* $\mathcal{B} = \left\{\begin{bmatrix}2\\-3\end{bmatrix}_{\mathcal{E}}, \begin{bmatrix}5\\-7\end{bmatrix}_{\mathcal{E}}\right\}$, $\mathcal{S}: \mathbb{R}^2 \to \mathbb{R}^2$ stretches in $\overrightarrow{b_1}$ direction by a factor of 2, reflects vectors in $\overrightarrow{b_2}$ direction. Find $[\mathcal{S}]_{\mathcal{E}}$ and $[\mathcal{S}]_{\mathcal{B}}$.

$\mathcal{S}\overrightarrow{b_1} = 2\overrightarrow{b_1}, \mathcal{S}\overrightarrow{b_2} = -\overrightarrow{b_2}$, therefore $[\mathcal{S}]_{\mathcal{B}} = \begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$

We know $[\mathcal{S}]_{\mathcal{E}} = [\mathcal{E} \leftarrow \mathcal{B}][\mathcal{S}]_{\mathcal{E}}[\mathcal{B} \leftarrow \mathcal{E}]$

$$[\mathcal{E} \leftarrow \mathcal{B}] = \begin{bmatrix}2 & 5\\-3 & -7\end{bmatrix}$$

$$[\mathcal{B} \leftarrow \mathcal{E}] = [\mathcal{E} \leftarrow \mathcal{B}]^{-1} = \begin{bmatrix}-7 & -5\\3 & 2\end{bmatrix}$$

$$\therefore [\mathcal{S}]_{\mathcal{E}} = \begin{bmatrix}2 & 5\\-3 & -7\end{bmatrix}\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-7 & -5\\3 & 2\end{bmatrix} = \begin{bmatrix}-43 & -30\\63 & 44\end{bmatrix}$$

**Unit n-cube:** Cube in $\mathbb{R}^n$ with side lengths corresponding to basis vectors and volume 1.

- $C_N = \{\vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} a_i \vec{e_i}$ for some $a_1, \dots, a_n \in [0,1]\} = [0,1]^n$
- $C_N = \{\vec{x} : \vec{x} = t\vec{e_1} + s\vec{e_2}$ for some $t, s \in [0,1]\}$ (2D unit square)

**Determinant:** Oriented volume of the unit n-cube in a basis. Notated $\det(M)$ or $|M|$.

- $\det(S \circ T) = \det(S_M T_M) = \det(S)\det(T) = \det(S_M)\det(T_M)$
- invertible $\Leftrightarrow \det(T) \neq 0$
- positively-oriented basis $\Leftrightarrow \det(T) > 0$

$\mathbb{R}^2$: $\det(M) = ad - bc$
$\mathbb{R}^3$: $\det(M) = aei + bfg + cdh - gec - hfa - idb$
$\mathbb{R}^n$: $\det(M) = \lambda_1 \times \lambda_2 \times \lambda_3 \times \dots$

- Multiply row by $k$: $\det(M) = k$
- Swap rows: $\det(M) = -1$
- Adding multiple of rows: $\det(M) = 1$

**Eigenvector:** A vector $\vec{v}$ for $T$ where $T_M\vec{v} = \lambda\vec{v}$. Has a corresponding **eigenvalue**, $\lambda$.

- They want us to find it by doing $(T_M - \lambda I)\vec{v} = 0$, setting $E_\lambda = T_M - \lambda I$.
- Eigenvector exists $\Leftrightarrow \text{null}(E_\lambda) \neq \vec{0}$
- Eigenvectors aren't $\vec{0}$ by definition
- 0 is an eigenvalue $\Leftrightarrow$ not invertible

**Characteristic Polynomial:** $\text{char}(M) = \det(E_\lambda) = \det(M - \lambda I)$

- Solve $\text{char}(M) = 0$ for eigenvalues
- Solve $\text{null}(M - \lambda I)$ for eigenvectors

*Example:* Find $\det\left(\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\right)$.

No easy determinant formula for $4 \times 4$. Row-reduce to diagonals (but not further).

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Determinant stays same: adding multiples of rows changes nothing. Calculate remaining determinant: $1 \times 3 \times -1 \times 4 = -12$

*Example:* Find eigenvectors/values of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

<u>Method 1.</u> My way

Expand, solve $T_M\vec{v} = \lambda\vec{v}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{cases} x + 2y = \lambda x \\ 3x + 2y = \lambda y \end{cases}$$
$$\begin{cases} (\lambda - 1)x = 2y \\ 3x = (\lambda - 2)y \end{cases}$$
$$\frac{\lambda - 1}{3} = \frac{2}{\lambda - 2}$$
$$(\lambda - 1)(\lambda - 2) = 6$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda + 1)(\lambda - 4) = 0$$
$$\lambda = -1, 4$$

Substitute back into system, get eigenvectors:

$$x + 2y = (-1)x$$
$$\therefore y = -x$$
$$x + 2y = (4)x$$
$$y = {}^3/_2\, x$$

<u>Method 2.</u> Their way

Solve $\text{char}(M) = 0$

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{bmatrix}\right)$$
$$= (2 - \lambda)(1 - \lambda) - 6$$
$$= \lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda + 1)(\lambda - 4) = 0$$
$$\lambda = -1, 4$$

Now find $\text{null}(\text{char}(M))$

$\text{null}(M - \lambda I)$

$$= \text{null}\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= \text{null}\left(\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\right)$$
$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}$$

$2x + 2y = 0 \Rightarrow y = -x$

Repeat for the $\lambda = 4$, get $y = {}^3/_2\, x$.

**Diagonalization:** For a matrix $M$, a similar diagonal matrix $D$ where $M = PDP^{-1}$.

- $D$ is diagonalizable $\Leftrightarrow$ $P$ is a "change-of-basis matrix for a basis of eigenvectors"

**Eigenspace:** For a matrix $M$, $\text{null}(M - \lambda I)$ for all $\lambda$

**Geometric Multiplicity:** Dimension of eigenspace

**Algebraic Multiplicity:** # of $x - \lambda$ in $\text{char}(M)$

- geometric mult$(\lambda) \leq$ algebraic mult$(\lambda)$
- $\sum$ geometric mult$(\lambda) = n \Leftrightarrow$ $\forall\lambda$, algebraic mult$(\lambda) =$ geometric mult$(\lambda) \Leftrightarrow$ $n \times n$ matrix $M$ is diagonalizable

*Example:* Is $\begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}$ diagonalizable?

$$\text{char}\left(\begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 5 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix}\right)$$
$$= (\lambda - 5)(\lambda - 2), \therefore\ \lambda = 5, 2$$
$$\text{null}(M - \lambda_1 I) = \text{null}\left(\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$
$$\text{null}(M - \lambda_2 I) = \text{null}\left(\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$$

$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$ is eigenvector basis, diagonalizable

*Example:* Diagonalize $M = \begin{bmatrix} 1 & 2 & 5 \\ -11 & 14 & 5 \\ -3 & 2 & 9 \end{bmatrix}$.

Find eigenvectors and eigenvalues.

$$\text{char}(M) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 5 \\ -11 & 14-\lambda & 5 \\ -3 & 2 & 9-\lambda \end{bmatrix}\right)$$

$$= (1\text{-}\lambda)(14\text{-}\lambda)(9\text{-}\lambda) + (2)(5)(\text{-}3) + (5)(\text{-}11)(2)\text{-}(\text{-}3)(14\text{-}\lambda)(5)\text{-}(2)(5)(1\text{-}\lambda)\text{-}(9\text{-}\lambda)(\text{-}11)(2)$$

$$= -\lambda^3 + 24\lambda^2 - 176\lambda + 384$$

$$= -(\lambda - 4)(\lambda - 8)(\lambda - 12) = 0$$

$$\therefore \lambda = 4,8,12$$

$\text{null}(M - \lambda I)$

$$= \text{null}\left(\begin{bmatrix} -3 & 2 & 5 \\ -11 & 10 & 5 \\ -3 & 2 & 5 \end{bmatrix}\right)$$

$$\begin{bmatrix} -3 & 2 & 5 \\ -11 & 10 & 5 \\ -3 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{cases} -3x + 2y + 5z = 0 \\ -11x + 10y + 5z = 0 \\ -3x + 2y + 5z = 0 \end{cases}$$

Get $y = x$, span$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$

$$= \text{null}\left(\begin{bmatrix} -7 & 2 & 5 \\ -11 & 6 & 5 \\ -3 & 2 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} -7 & 2 & 5 \\ -11 & 6 & 5 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{cases} -7x + 2y + 5z = 0 \\ -11x + 6y + 5z = 0 \\ -3x + 2y + z = 0 \end{cases}$$

Get $x = y = z$, span$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

$$= \text{null}\left(\begin{bmatrix} -11 & 2 & 5 \\ -11 & 2 & 5 \\ -3 & 2 & -3 \end{bmatrix}\right)$$

$$\begin{bmatrix} -11 & 2 & 5 \\ -11 & 2 & 5 \\ -3 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{cases} -11x + 2y + 5z = 0 \\ -11x + 2y + 5z = 0 \\ -3x + 2y - 3z = 0 \end{cases}$$

Get $3x = y = 3z$, span$\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\right\}$

$$\therefore D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1/2 & -1/2 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1/2 & -1/2 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

*Example:* Find examples of all combinations of invertible/diagonalizable matrices.

<u>Invertible, Diagonalizable</u>

$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\det(M) = 1 \neq 0$, thus $M$ is invertible

$\lambda = 1,2 \neq 0$, thus $M$ is invertible

(2 unique $\lambda$) $= n$, thus $M$ is diagonalizable

algebraic $\text{mult}(1) = 1$, geometric $\text{mult}(1) = 1$

algebraic $\text{mult}(2) = 1$, geometric $\text{mult}(2) = 1$

Thus diagonalizable

<u>Invertible, Not Diagonalizable</u>

$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\det(M) = 1 \neq 0$, thus $M$ is invertible

$\lambda = 1 \neq 0$, thus $M$ is invertible

(1 unique $\lambda$) $\neq n$, thus $M$ is not diagonalizable

algebraic $\text{mult}(1) = 2$, geometric $\text{mult}(1) = 1$

$1 \neq 2$, not diagonalizable

| Not Invertible, Diagonalizable | Not Invertible, Not Diagonalizable |
|---|---|
| $M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ | $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ |
| $\det(M) = 0$, thus $M$ is not invertible | $\det(M) = 0$, thus $M$ is not invertible |
| $\lambda = 0,1$ thus $M$ is not invertible | $\lambda = 0$ thus $M$ is not invertible |
| | |
| 2 unique $\lambda) = n$, thus $M$ is diagonalizable | (1 unique $\lambda) \neq n$, thus $M$ is not diagonalizable |
| algebraic mult$(0) = 1$, geometric mult$(1) = 1$ | algebraic mult$(0) = 2$, geometric mult$(0) = 1$ |
| algebraic mult$(1) = 1$, geometric mult$(2) = 1$ | $0 \neq 1$, not diagonalizable |
| Thus diagonalizable | |

<u>A matrix's determinant is the product of all eigenvalues.</u>

Let $M =$ a matrix, $\lambda_1, \lambda_2, \lambda_3 \ldots =$ its eigenvectors. Then the following relation is true

$$\text{char}(M) = \det(M - \lambda I) = \cdots (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots$$

Set $\lambda = 0$, then

$$\det(M) = \lambda_1 \lambda_2 \ldots$$

<u>A triangular matrix's eigenvalues are its diagonals.</u>

To prove it, just do the math with arbitrary variables, it works out.

<u>Positive orientation = det(M) > 0</u>

<u>n x n matrix is invertible = rank(M) = n, nullity(M) = 0, det(M) != 0, rref(M) = I, rows/cols linearly indep, 0 not eigenvalue</u>

<u>Elementary row operations change column space, not row space</u>

<u>A *n* x *n* matrix is diagonalizable if it has *n* eigenvalues.</u> This way, there're *n* eigenvectors, and they're linearly independent, meaning they form an "eigenvector basis of $R^n$".

- If there're under *n* eigenvalues, one of them has a double solution for char(M) (ie. you get a situation like $(1 - \text{lambda})^2 = 0$), and not all eigenvectors would be linearly independent then

<u>Uses of diagonalization:</u>

$$\begin{aligned} M^n &= (PDP^{-1})^n \\ &= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \ldots \\ &= PD(P^{-1}P)D(P^{-1}P)D(P^{-1} \ldots)DP^{-1} \\ &= PD^nP^{-1} \\ &= P \begin{bmatrix} d_1^n & 0 & \cdots \\ 0 & d_2^n & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix} P^{-1} \end{aligned}$$

**Recommended Watch List:**

★ 3Blue1Brown – Essence of linear algebra
*Must watch series. Beautiful visuals, gives very strong intuitive understanding of concepts.*

https://textbooks.math.gatech.edu/ila/

Jason Siefken – MAT223 Playlists
*Professor's weekly videos to watch. Pretty good, a little slow, recommend watching at 1.5 speed.*