Subdivision Surfaces

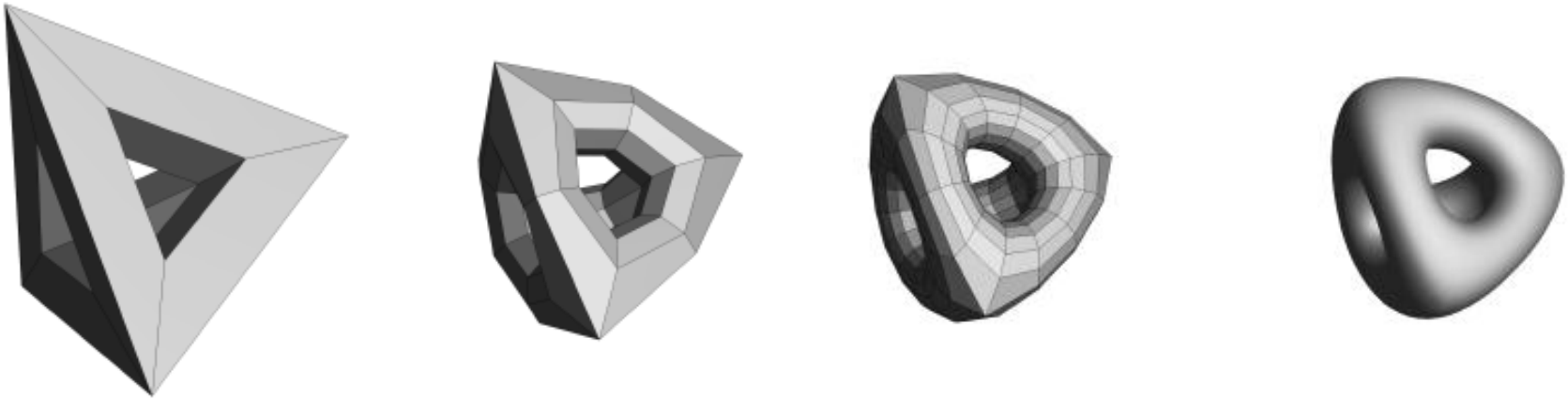
Michael Kazhdan

(600.357 / 600.457)

Subdivision for Modeling and Animation, Zorin et al. 2000

Subdivision Surfaces

- Coarse mesh & subdivision rule
  - Define smooth surface as limit of sequence of refinements



(a) (b) (c) (d)

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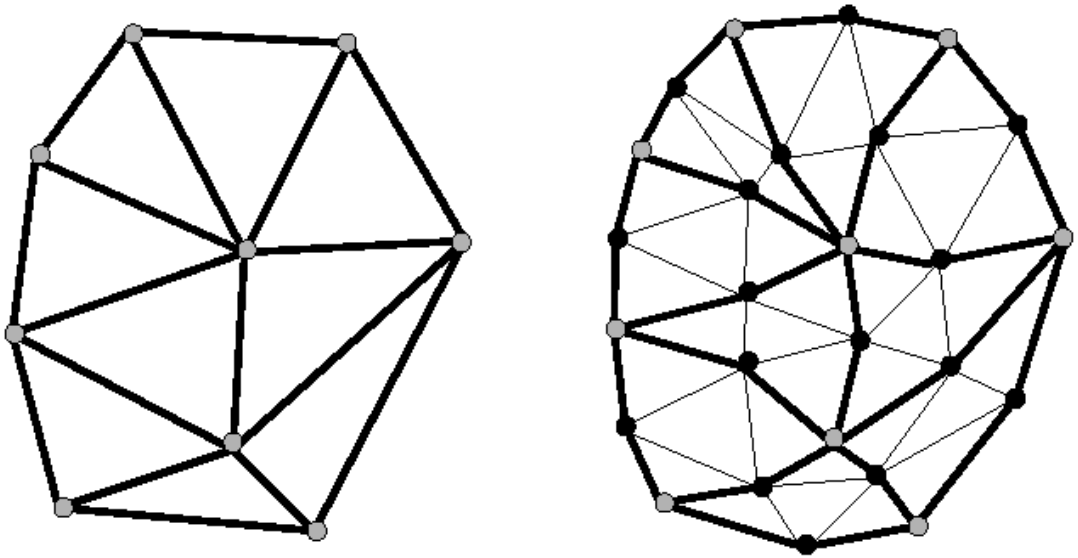
Course Notes

Key Questions

- How to subdivide the mesh?
  - Aim for properties like smoothness
- How to store the mesh?
  - Aim for efficiency of implementing subdivision rules



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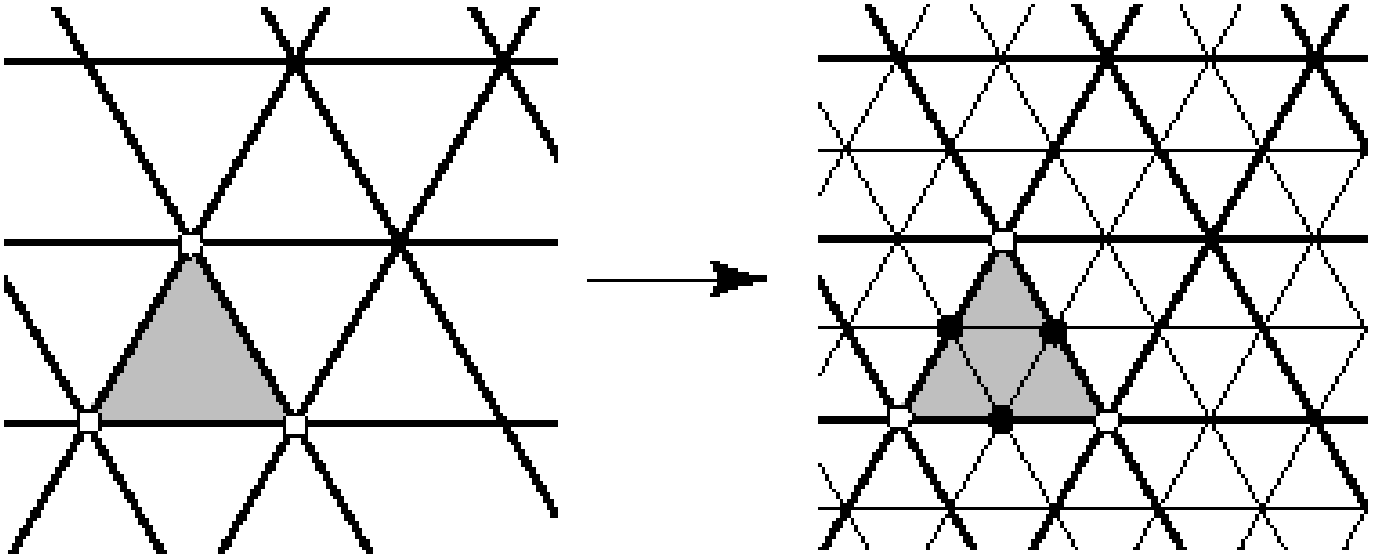
General Subdivision Scheme

- How to subdivide the mesh?
  - Two parts:
    - » Refinement:
      - – Add new vertices and connect (topological)
    - » Smoothing:
      - – Move vertex positions (geometric)

Loop Subdivision Scheme

- How to subdivide the mesh?
  - Refinement:
    - » Subdivide each triangle into 4 triangles by splitting each edge and connecting new vertices



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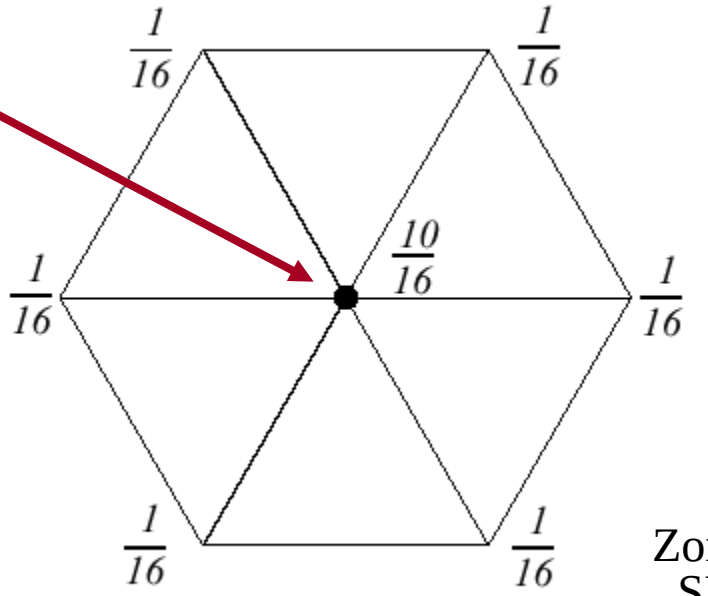
Loop Subdivision Scheme

- How to subdivide the mesh:
  - Refinement
  - Smoothing:
    - » Existing Vertices: Choose *new* location as weighted average of *original* vertex and its neighbors

Existing vertex being moved from one level to the next



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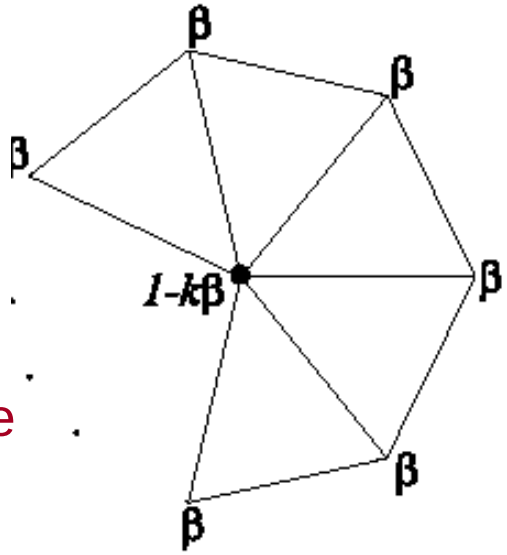
Loop Subdivision Scheme

- General rule for moving existing *interior vertices*:



What about vertices that have more
Or less than 6 neighboring faces?

New_position = (1 - $k\beta$)original_position + sum(β * *each_original_vertex*)

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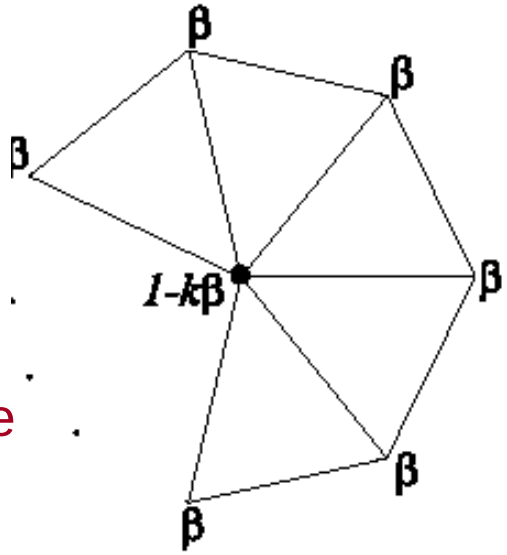
Loop Subdivision Scheme

- General rule for moving existing *interior vertices*:



What about vertices that have more
Or less than 6 neighboring faces?

New

rtex)

$0 \leq \beta \leq 1/k$:

• As β increases, the contribution from adjacent vertices plays a more important role.

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Where do existing vertices move?

- How to choose β?
  - Analyze properties of limit surface
  - Interested in continuity of surface and smoothness
  - Involves calculating eigenvalues of matrices
    - » Original Loop

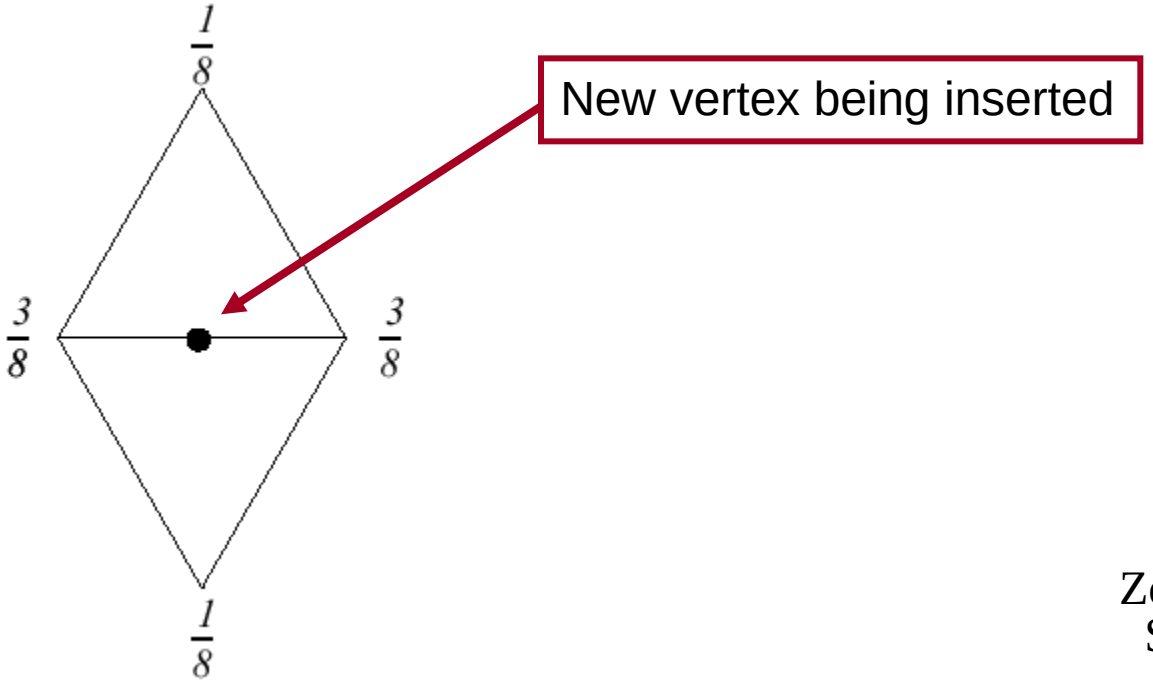
$$\beta = \frac{1}{k}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{k}\right)^2\right)$$

    - » Warren

$$\beta = \begin{cases} \frac{3}{8k} & n > 3 \\ \frac{3}{16} & n = 3 \end{cases}$$

Loop Subdivision Scheme

- How to subdivide the mesh:
  - Refinement
  - Smoothing:
    - » Inserted Vertices: Choose location as weighted average of *original* vertices in local neighborhood



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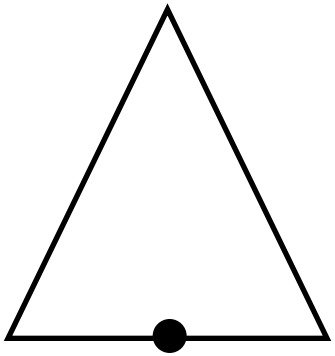
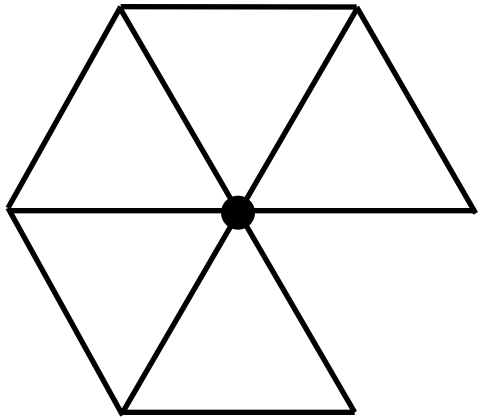
Boundary Cases?

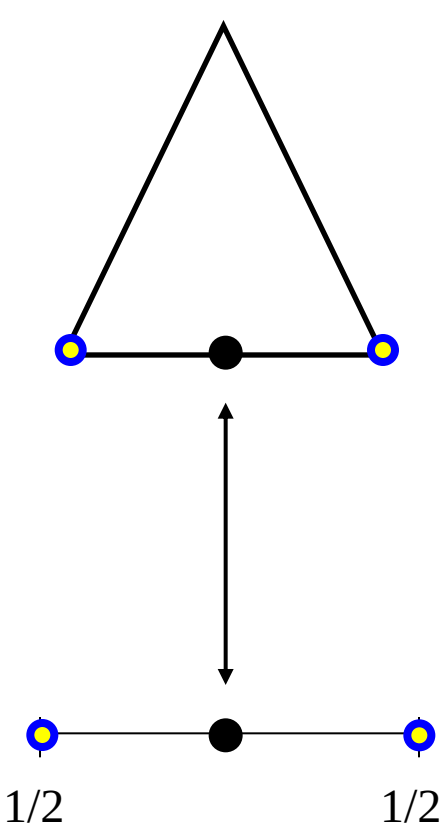
- What about *extraordinary vertices* and *boundary edges?*:
  - Existing vertex adjacent to a missing triangle
  - New vertex bordered by only one triangle

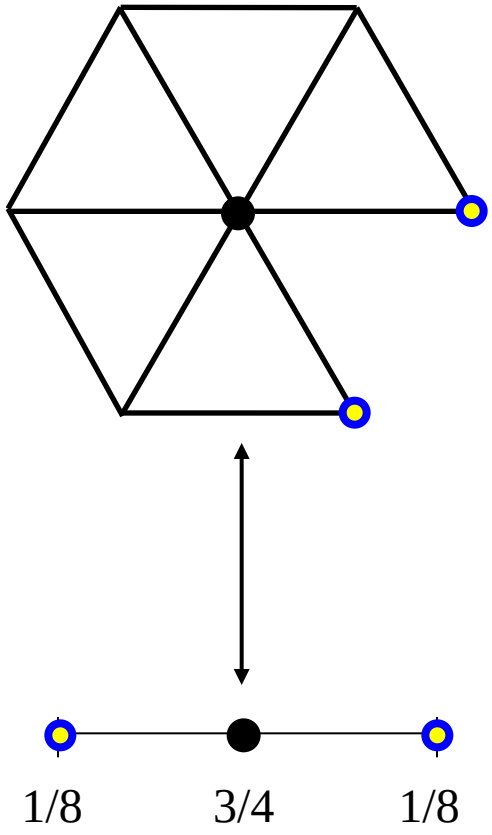




Boundary Cases?

- Rules for *extraordinary vertices* and *boundaries*:





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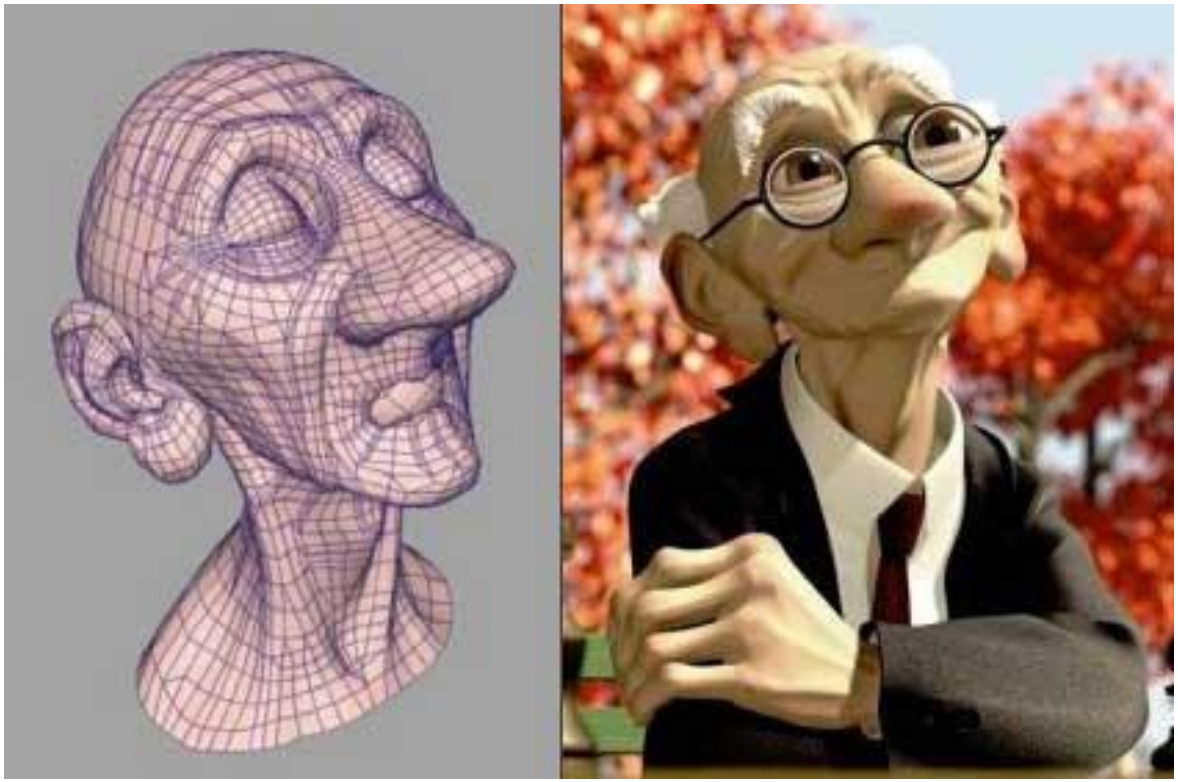
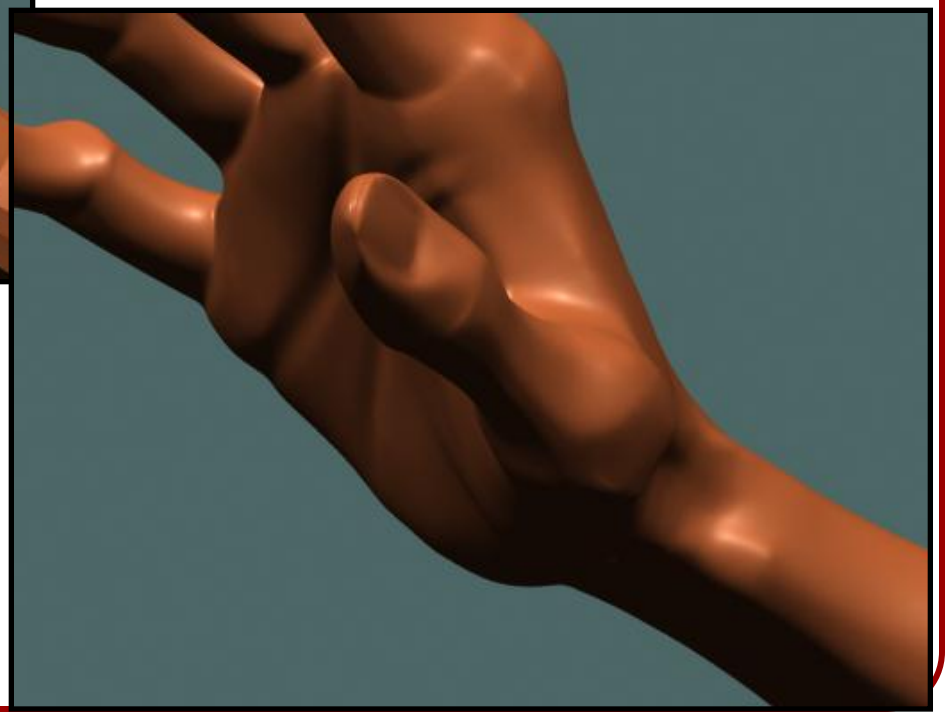
Loop Subdivision Scheme



Pixar

Loop Subdivision Scheme



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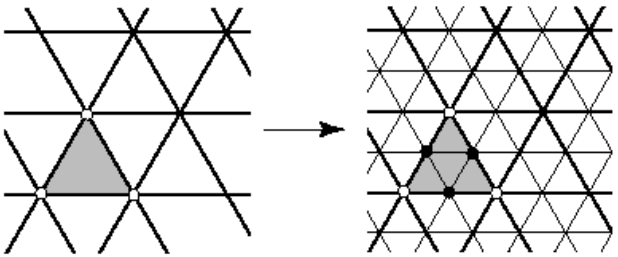
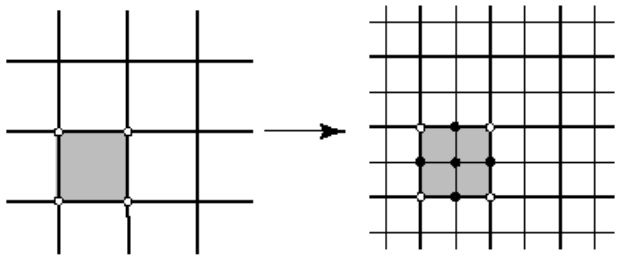
Subdivision Schemes

- There are different subdivision schemes
  - Different methods for refining topology
  - Different rules for positioning vertices
    - » Interpolating versus approximating



Face split for triangles



Face split for quads

Face split		
	Triangular meshes	*Quad. meshes*
Approximating	Loop (C^2)	Catmull-Clark (C^2)
Interpolating	Mod. Butterfly (C^1)	Kobbelt (C^1)

Vertex split
Doo-Sabin, Midedge (C^1)
Biquartic (C^2)

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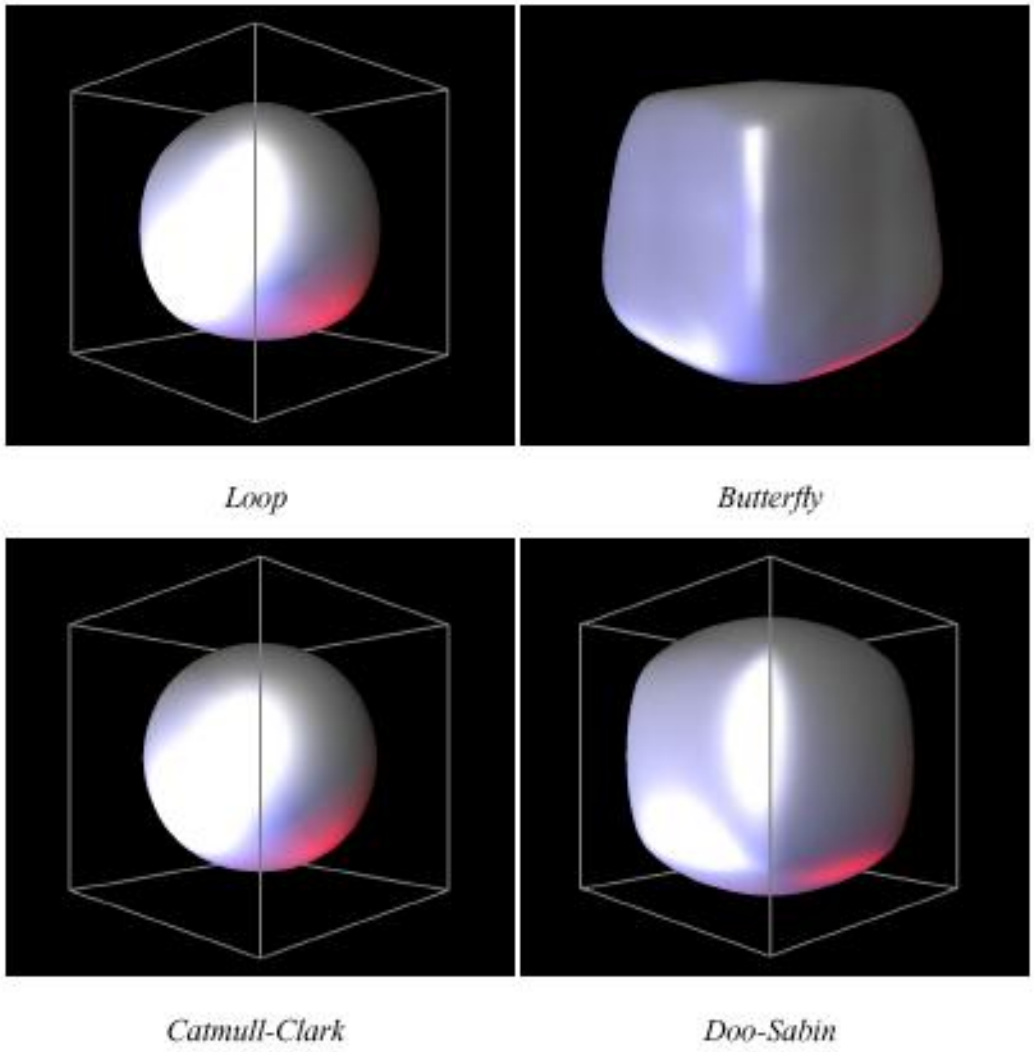
Subdivision Schemes



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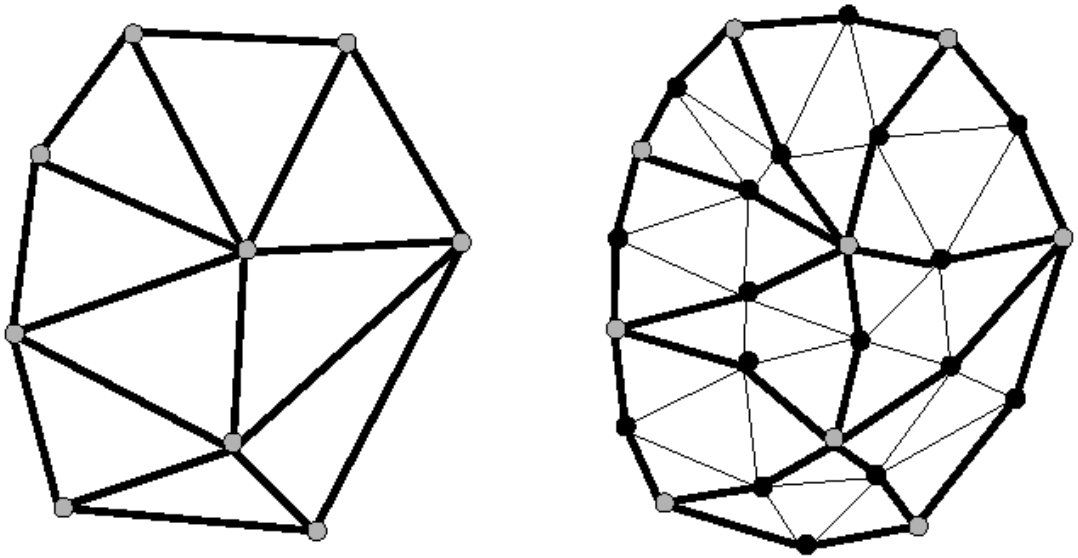
Key Questions

- How to refine the mesh?
  - **Aim for properties like smoothness**
- **How to store the mesh?**
  - Aim for efficiency for implementing subdivision rules



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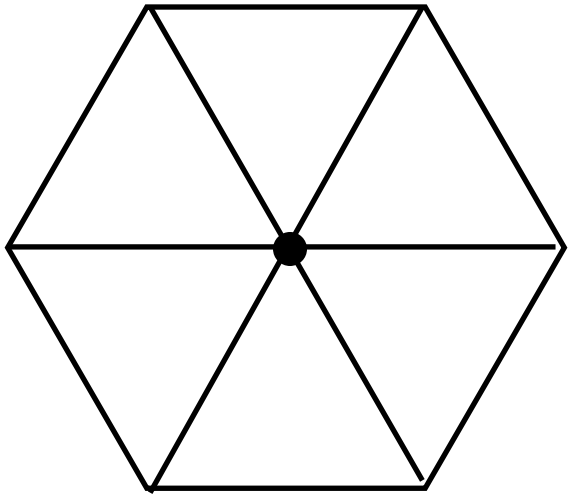
Subdivision Smoothness

To determine the smoothness of the subdivision:

- Repeatedly apply the subdivision scheme
- Look at the neighborhood in the limit.



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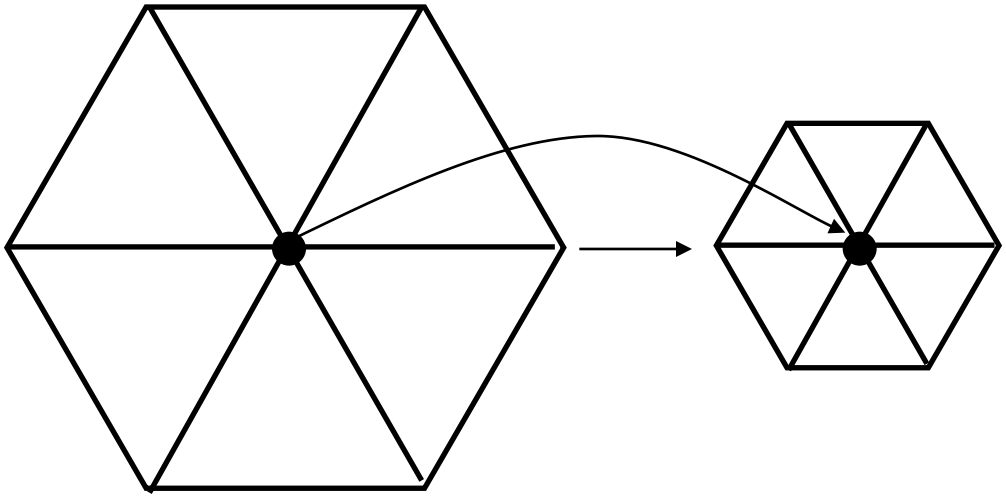

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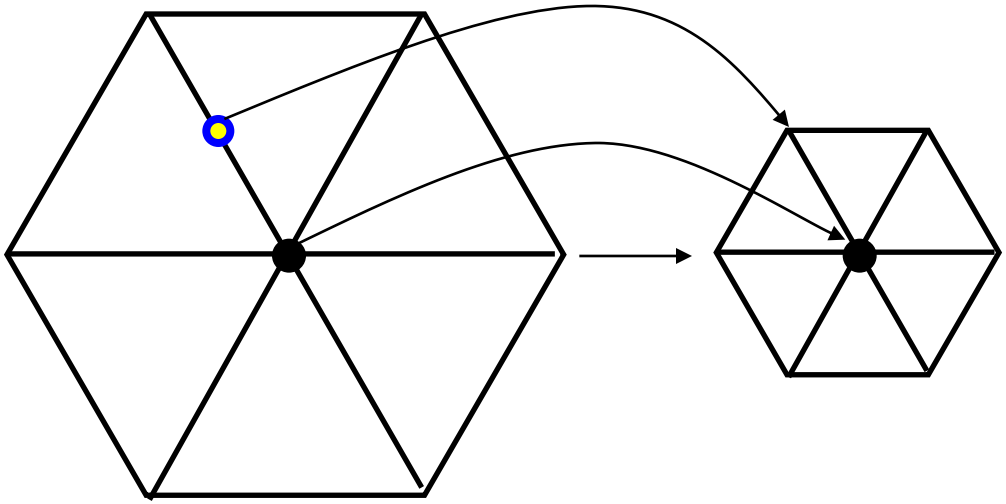

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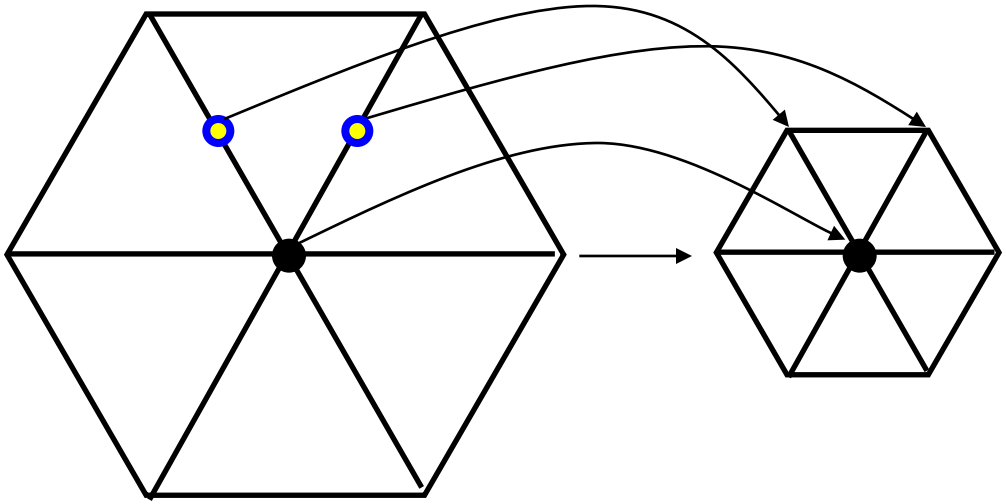

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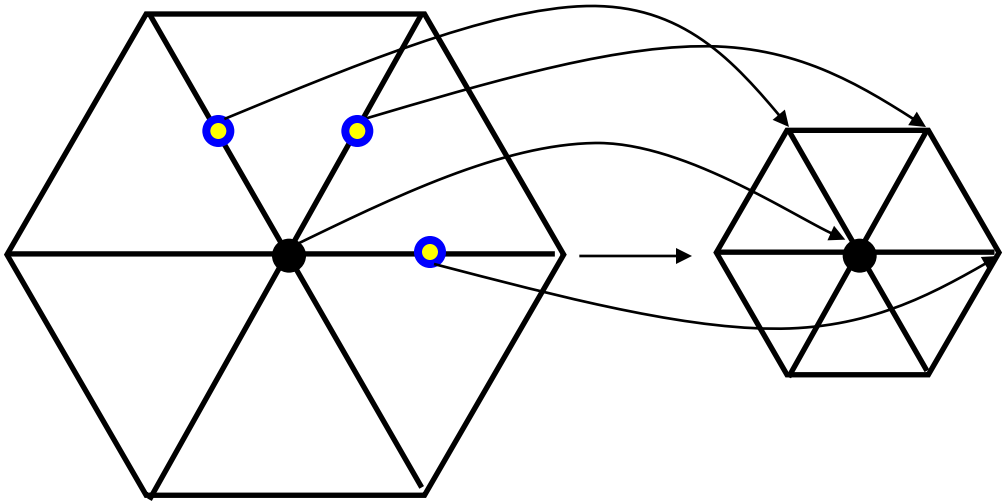

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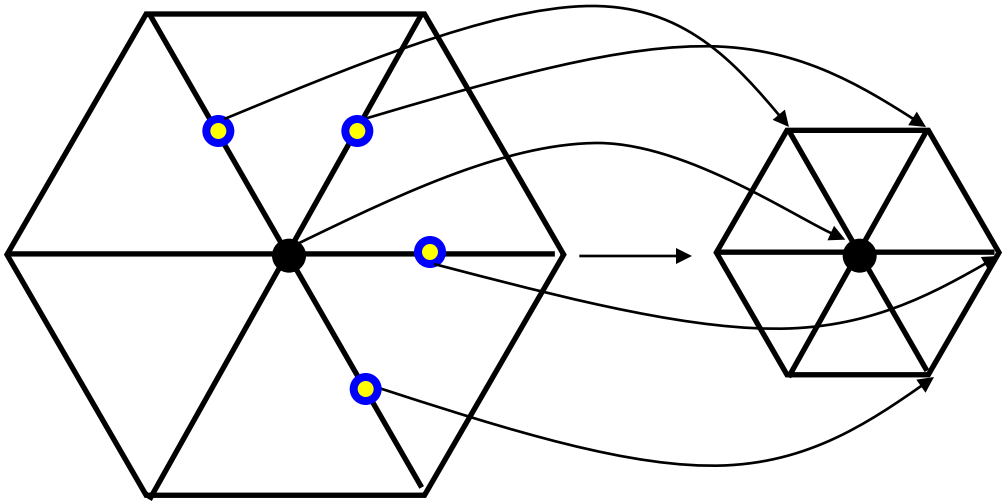

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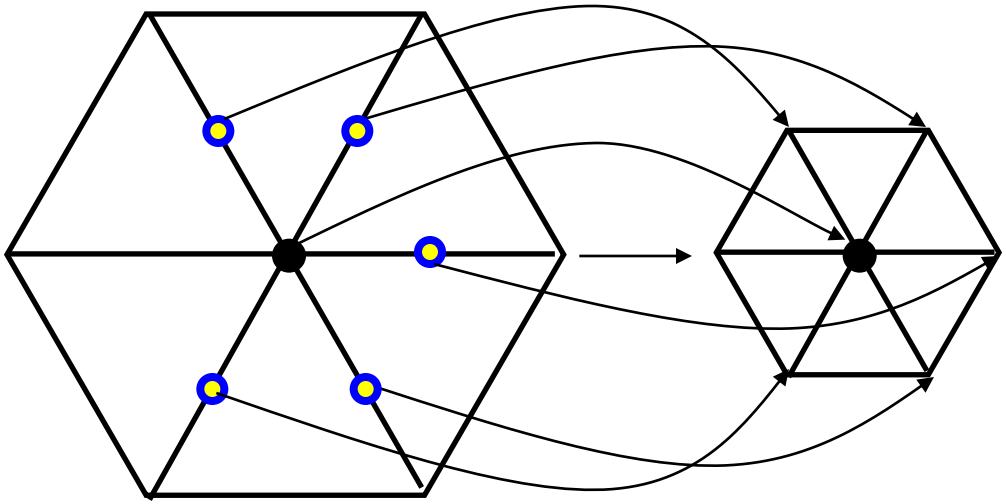

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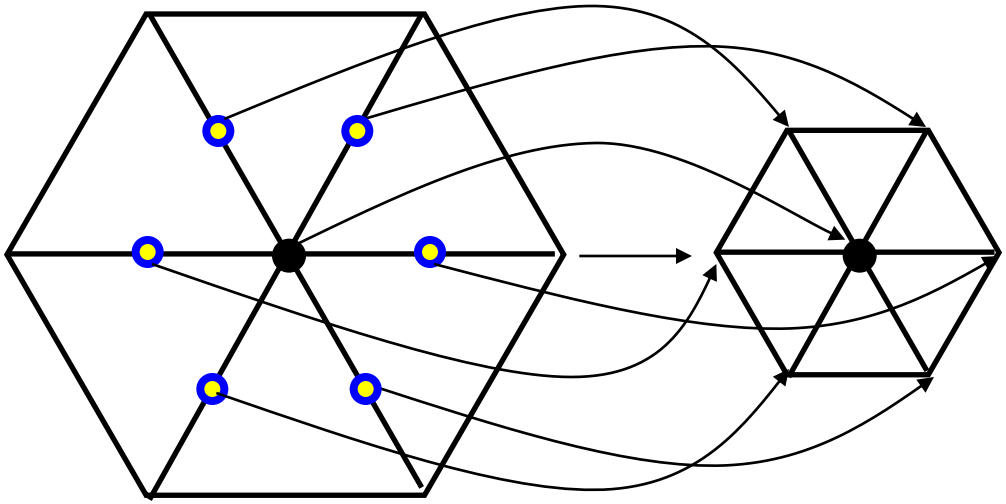

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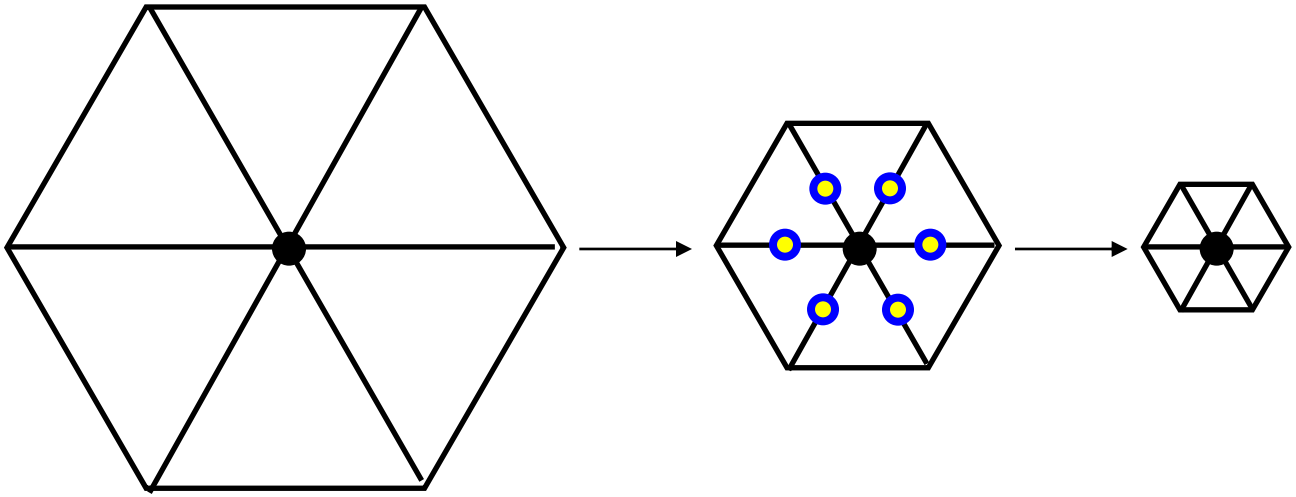

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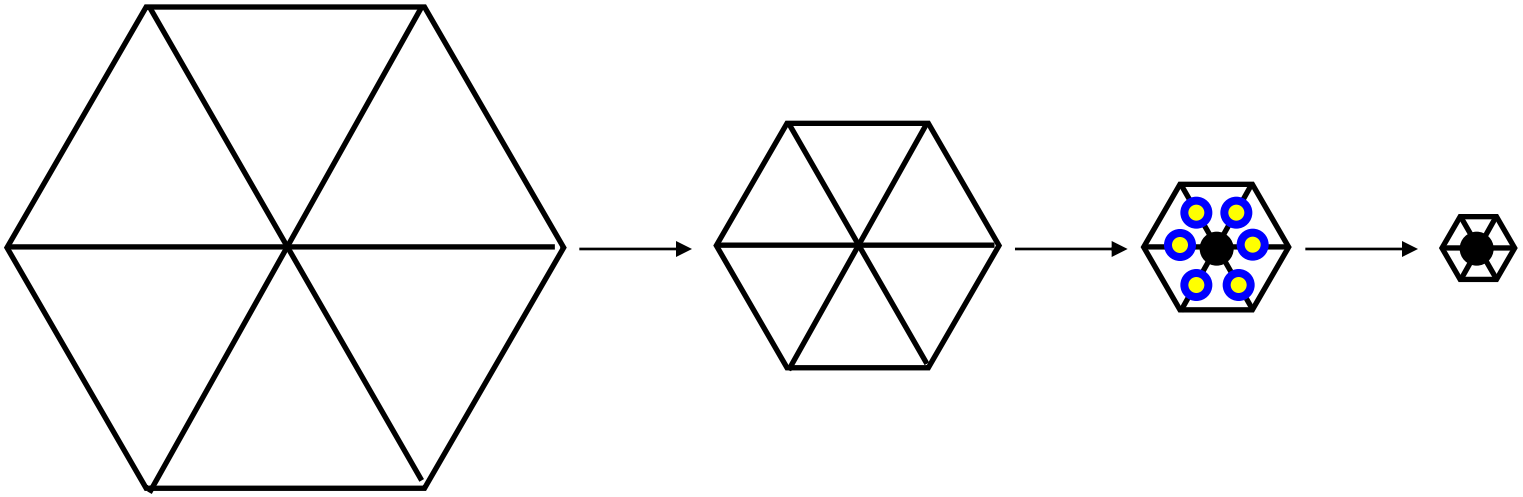

Subdivision Smoothness

To determine the smoothness of the subdivision:

- Repeatedly apply the subdivision scheme
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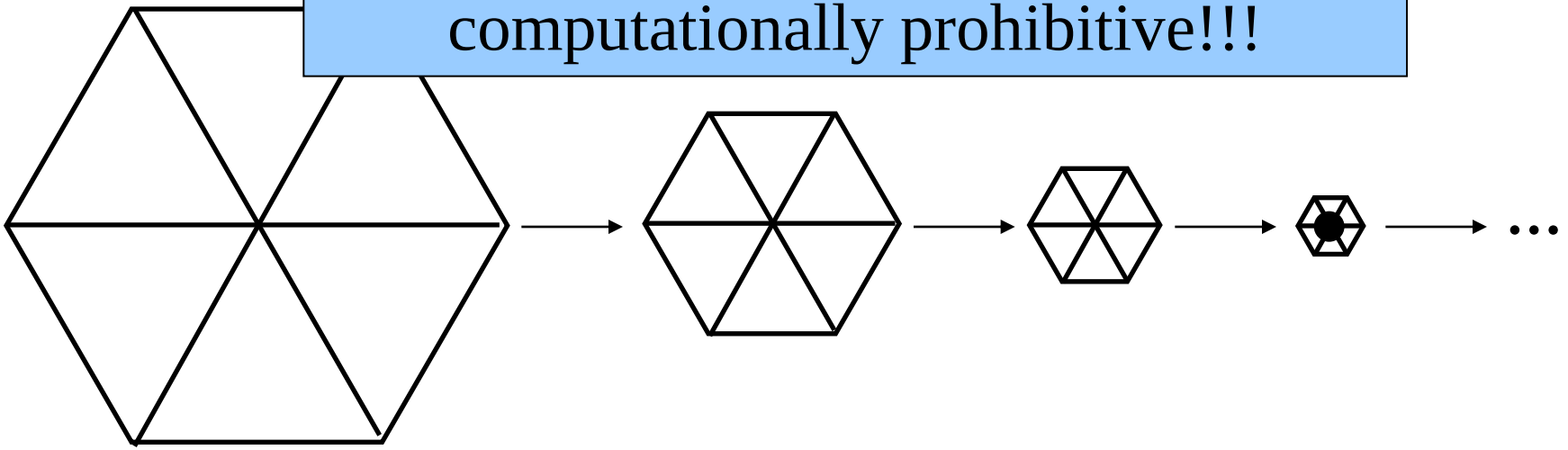
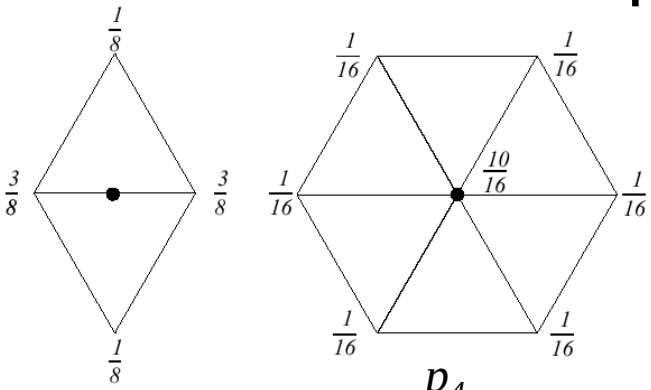
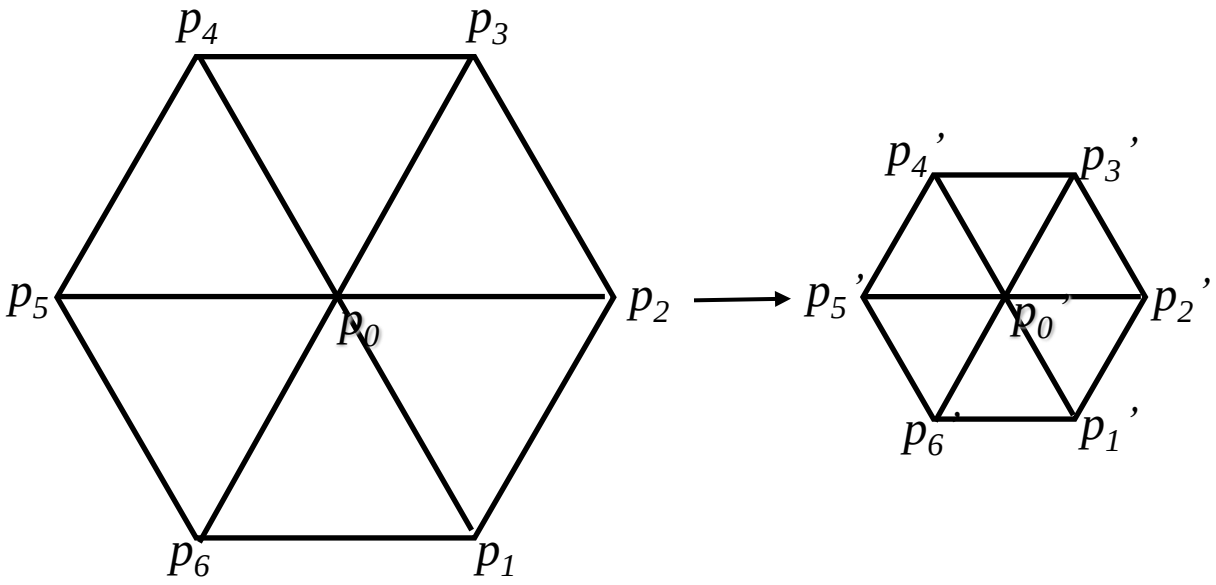
Computing infinitely many iterations is computationally prohibitive!!!



Subdivision Matrix

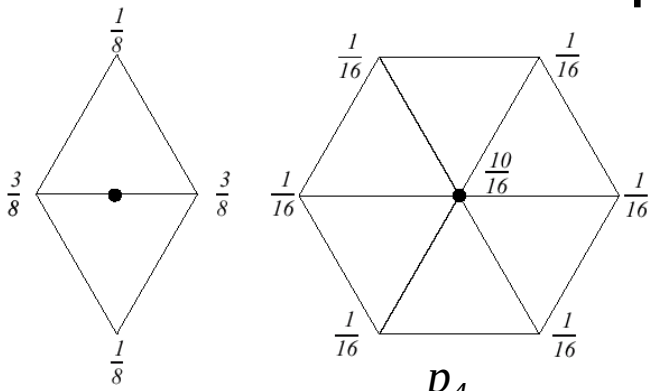
- Compute the new positions/vertices as a linear combination of previous ones.





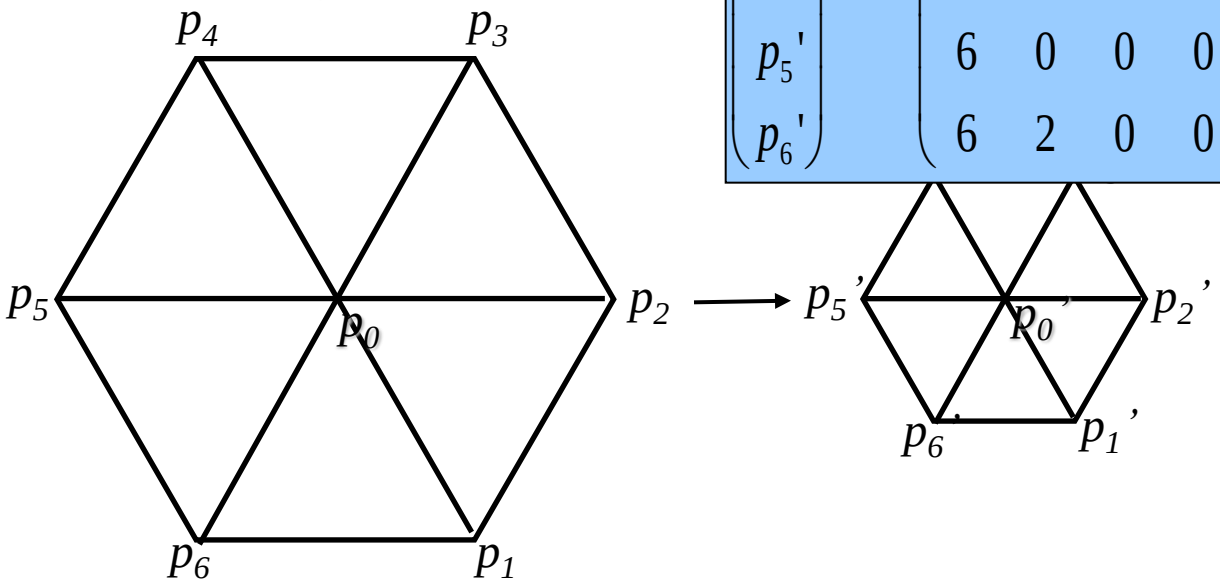
Subdivision Matrix

- Compute the new position
  combination of previous o



Subdivision Matrix

$$\begin{pmatrix} p_0' \\ p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \\ p_6' \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 6 & 2 & 0 & 0 & 0 & 2 \\ 6 & 2 & 6 & 2 & 0 & 0 & 0 \\ 6 & 0 & 2 & 6 & 2 & 0 & 0 \\ 6 & 0 & 0 & 2 & 6 & 2 & 0 \\ 6 & 0 & 0 & 0 & 2 & 6 & 2 \\ 6 & 2 & 0 & 0 & 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$



Subdivision Matrix

- Compute the new positions/vertices as a linear combination of previous ones.
- To find the limit position of p_0, repeatedly apply the subdivision matrix.

$$\begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_2^{(n)} \\ p_3^{(n)} \\ p_4^{(n)} \\ p_5^{(n)} \\ p_6^{(n)} \end{pmatrix} = \left[\frac{1}{16} \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 0 & 0 & 0 & 6 \\ 2 & 6 & 2 & 6 & 0 & 0 & 0 \\ 2 & 0 & 6 & 2 & 6 & 0 & 0 \\ 2 & 0 & 0 & 6 & 2 & 6 & 0 \\ 2 & 0 & 0 & 0 & 6 & 2 & 6 \\ 2 & 6 & 0 & 0 & 0 & 6 & 2 \end{pmatrix} \right]^n \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$

Subdivision Matrix

- Compute the new positions/vertices as a linear combination of previous ones.
- To find the limit position of p_0, repeatedly apply the subdivision matrix.
- Use eigen-value decomposition to compute the n^{th} power of the matrix efficiently.

$$\begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_2^{(n)} \\ p_3^{(n)} \\ p_4^{(n)} \\ p_5^{(n)} \\ p_6^{(n)} \end{pmatrix} = \left[\frac{1}{16} \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 0 & 0 & 0 & 6 \\ 2 & 6 & 2 & 6 & 0 & 0 & 0 \\ 2 & 0 & 6 & 2 & 6 & 0 & 0 \\ 2 & 0 & 0 & 6 & 2 & 6 & 0 \\ 2 & 0 & 0 & 0 & 6 & 2 & 6 \\ 2 & 6 & 0 & 0 & 0 & 6 & 2 \end{pmatrix} \right]^n \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$

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- Compute the new positions/vertices as a linear combination of previous ones.
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- Use eigen-value decomposition to

$$\begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p^{(n)} \end{pmatrix} \quad \left[\begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 0 & 0 & 0 & 6 \\ 2 & 6 & 2 & 6 & 0 & 0 & 0 \end{pmatrix}^{n} \right] \begin{pmatrix} p_0 \\ p_1 \\ p \end{pmatrix}$$

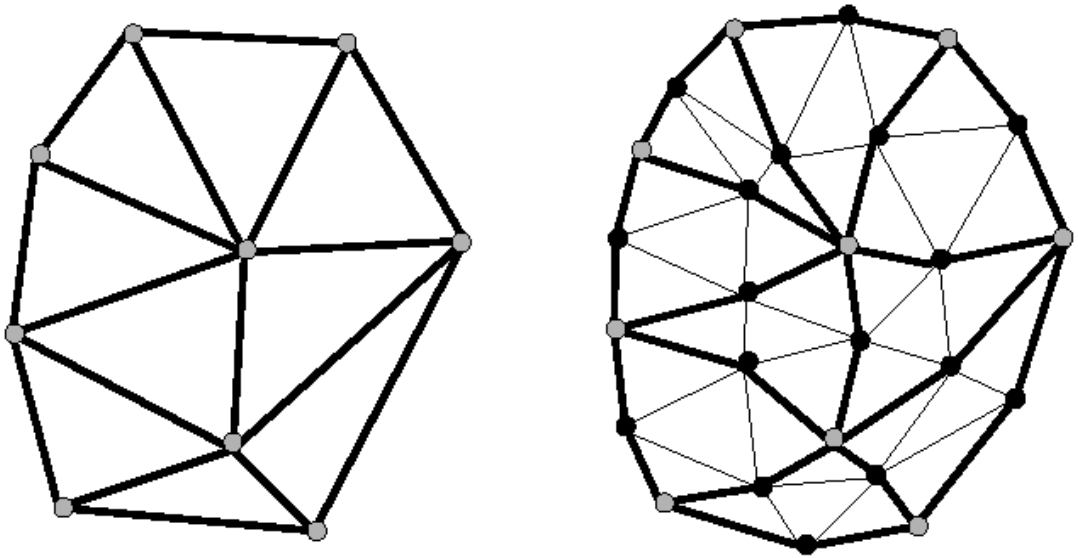
If, after a change of basis we have $M=A^{-1}DA$, where D is a diagonal matrix, then:

$$M^n=(A^{-1}DA)\cdots(A^{-1}DA)=A^{-1}D^nA,$$

Since D is diagonal, raising D to the n-th power just amounts to raising each of the diagonal entries of D to the n-th power.

Key Questions

- How to refine the mesh?
  - Aim for properties like smoothness
- **How to store the mesh?**
  - Aim for efficiency for implementing subdivision rules



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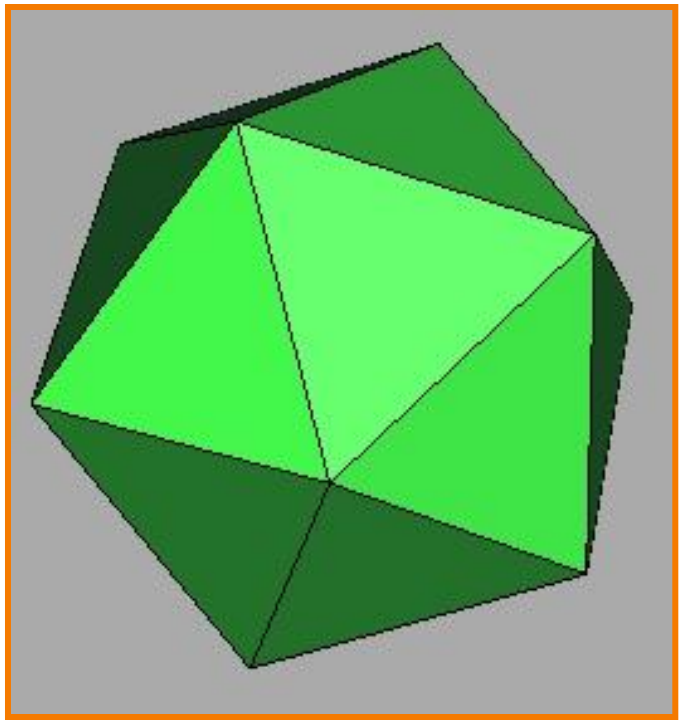
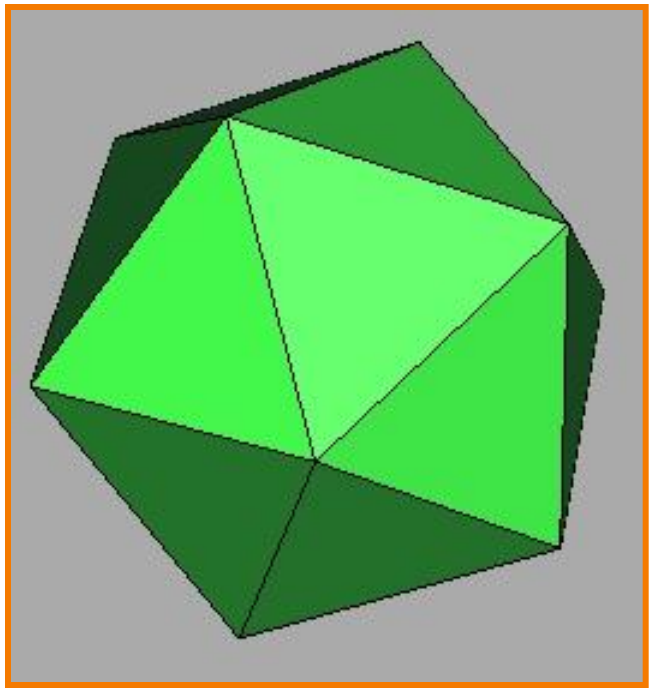
Polygon Meshes

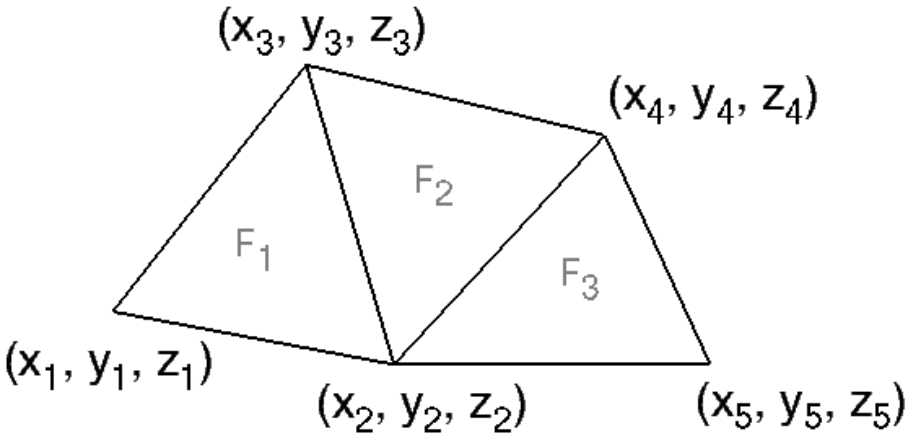
- Mesh Representations
  - Independent faces
  - Vertex and face tables
  - Adjacency lists
  - Winged-Edge



Independent Faces

- Each face lists vertex coordinates

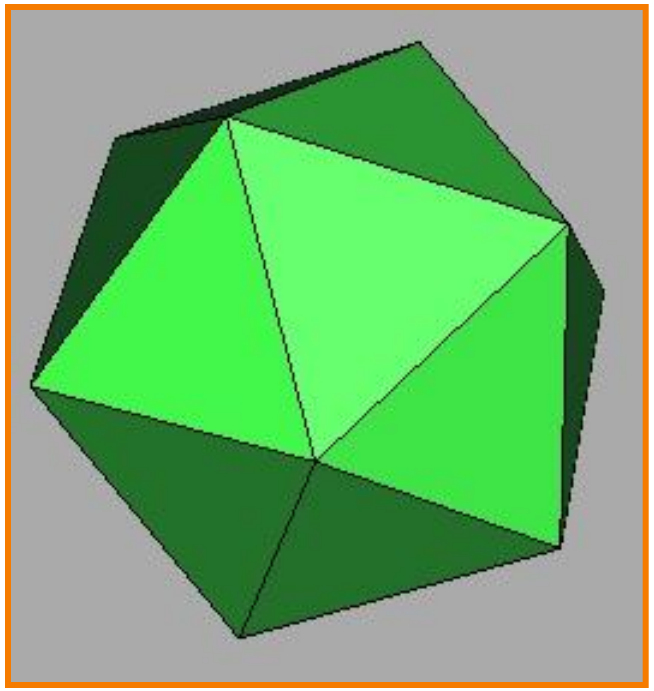


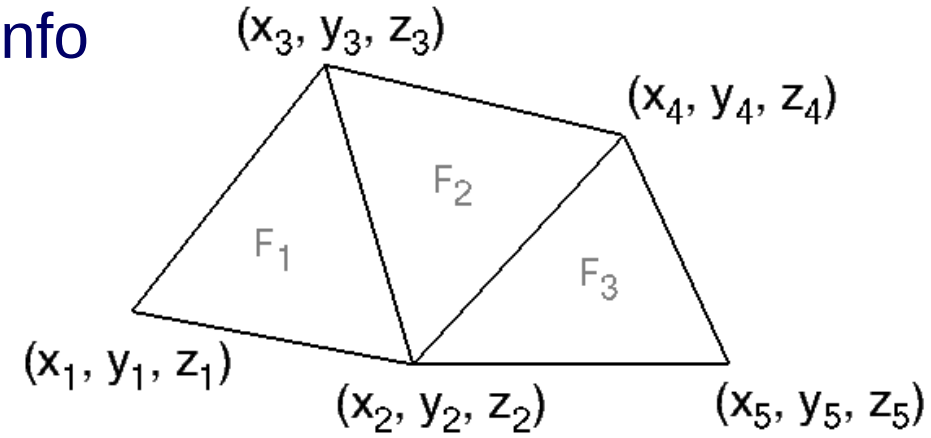


FACE TABLE	
F_1	(x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3)
F_2	(x_2, y_2, z_2) (x_4, y_4, z_4) (x_3, y_3, z_3)
F_3	(x_2, y_2, z_2) (x_5, y_5, z_5) (x_4, y_4, z_4)

Independent Faces

- Each face lists vertex coordinates
  - Redundant vertices
  - No vertex-adjacency info





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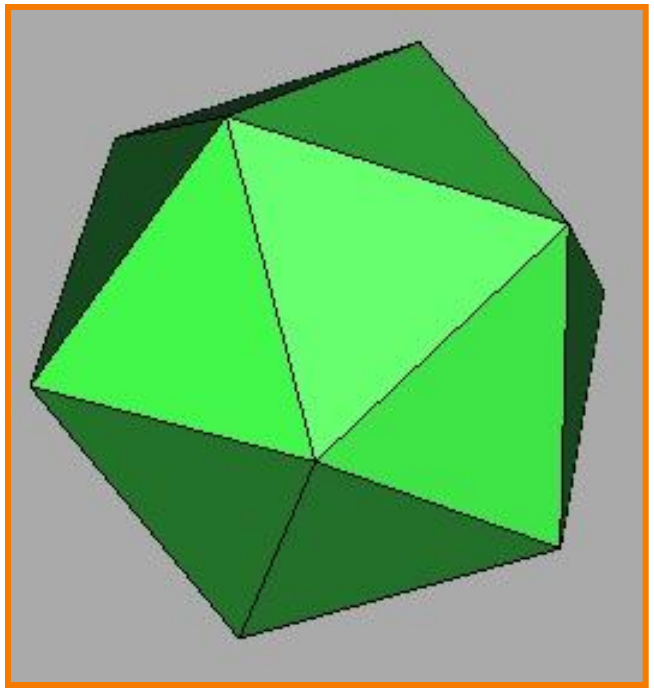
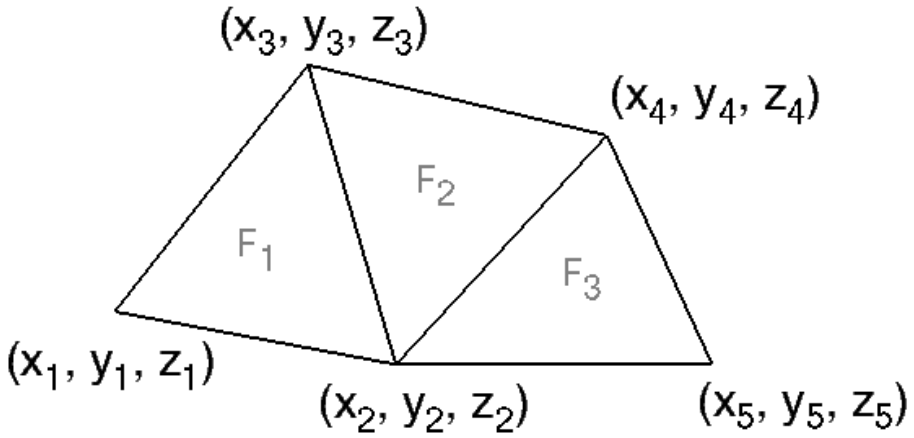
Vertex and Face Tables

- Each face lists vertex references



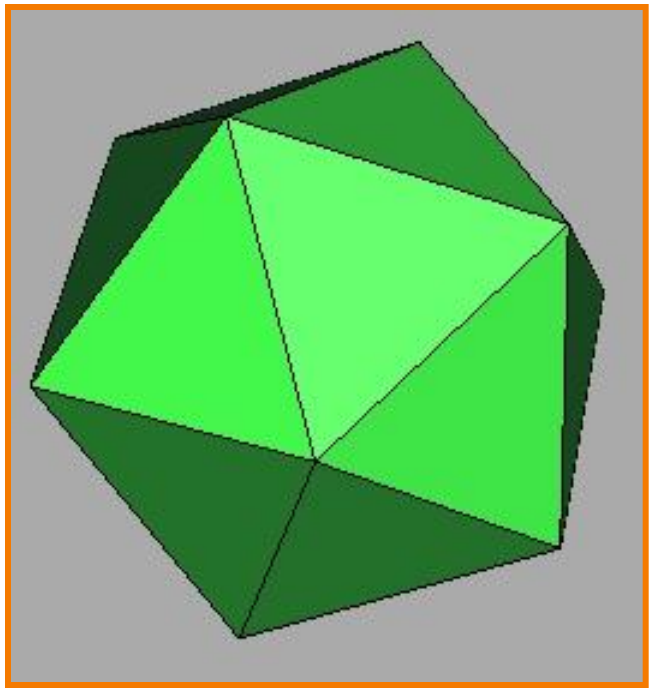


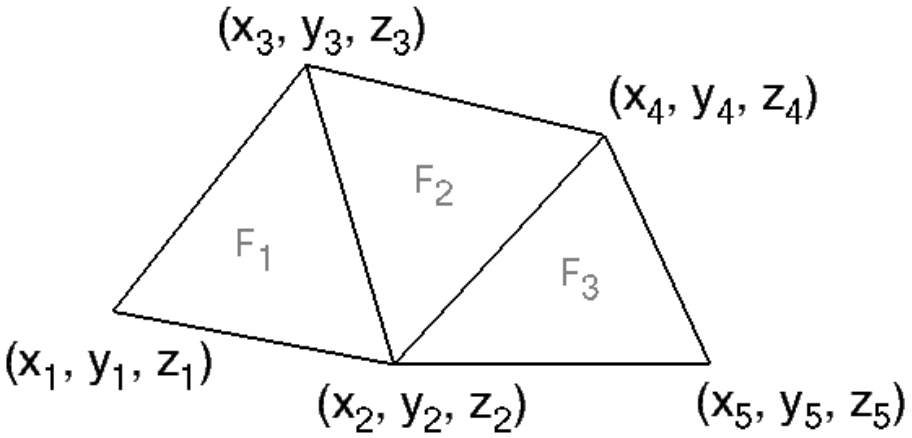
VERTEX TABLE			
V_1	X_1	Y_1	Z_1
V_2	X_2	Y_2	Z_2
V_3	X_3	Y_3	Z_3
V_4	X_4	Y_4	Z_4
V_5	X_5	Y_5	Z_5

FACE TABLE			
F_1	V_1	V_2	V_3
F_2	V_2	V_4	V_3
F_3	V_2	V_5	V_4

Vertex and Face Tables

- Each face lists vertex references
  - ✓ Shared vertices



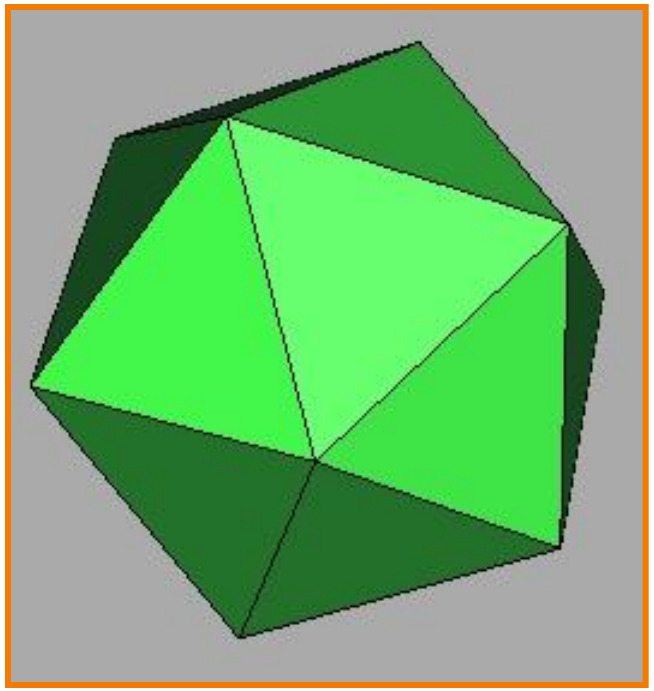


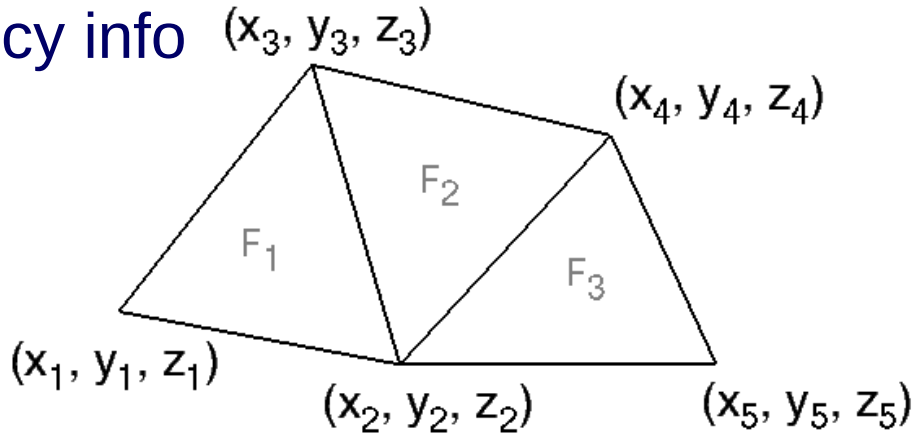
VERTEX TABLE			
V_1	X_1	Y_1	Z_1
V_2	X_2	Y_2	Z_2
V_3	X_3	Y_3	Z_3
V_4	X_4	Y_4	Z_4
V_5	X_5	Y_5	Z_5

FACE TABLE			
F_1	V_1	V_2	V_3
F_2	V_2	V_4	V_3
F_3	V_2	V_5	V_4

Vertex and Face Tables

- Each face lists vertex references
  - ✓ Shared vertices
  - ✗ Still no vertex-adjacency info



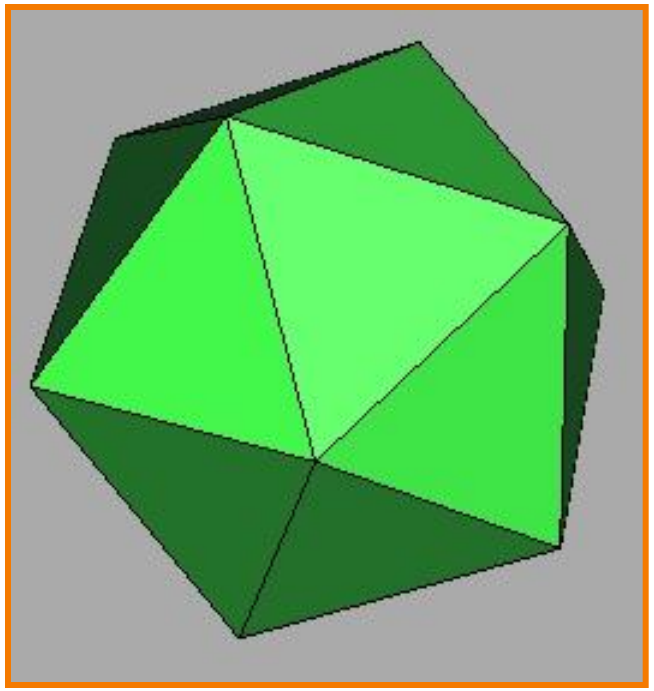


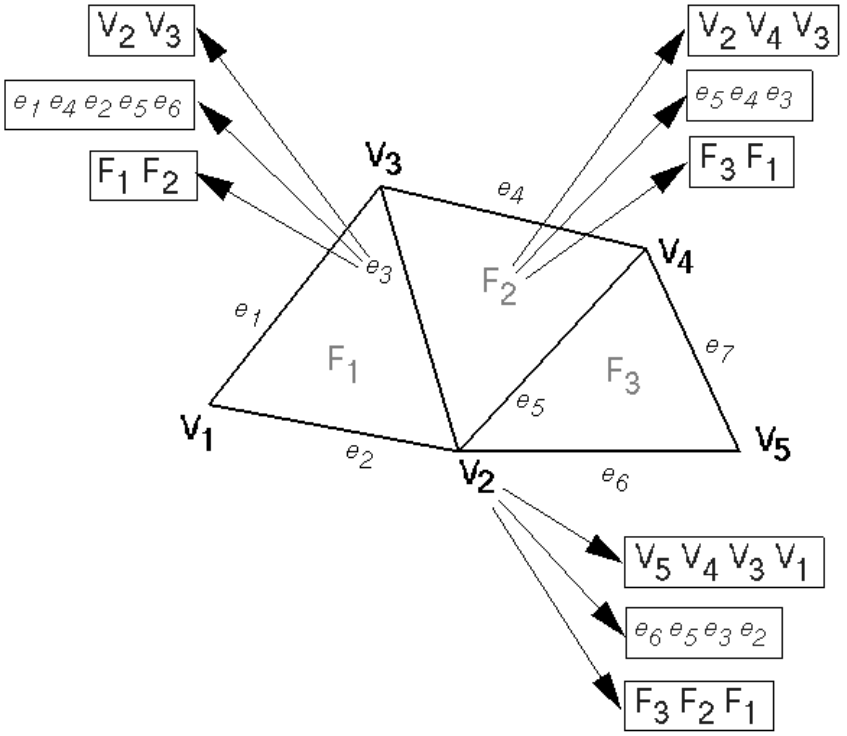
VERTEX TABLE			
V_1	X_1	Y_1	Z_1
V_2	X_2	Y_2	Z_2
V_3	X_3	Y_3	Z_3
V_4	X_4	Y_4	Z_4
V_5	X_5	Y_5	Z_5

FACE TABLE			
F_1	V_1	V_2	V_3
F_2	V_2	V_4	V_3
F_3	V_2	V_5	V_4

Adjacency Lists

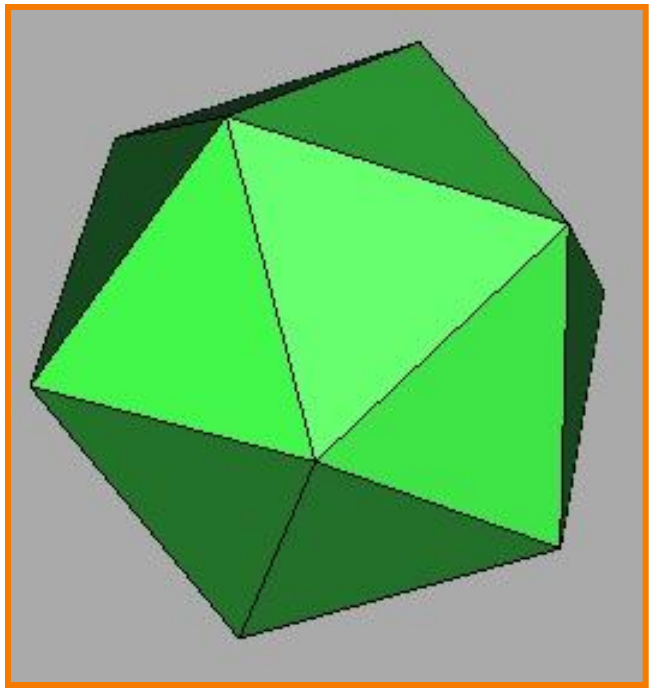
- Store all vertex, edge, and face adjacencies

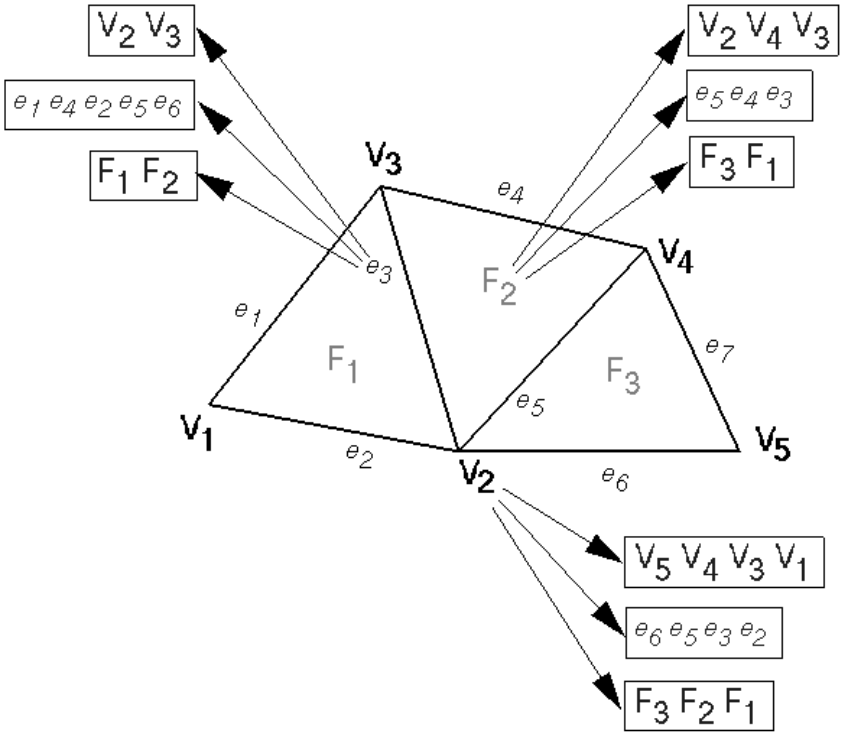




Adjacency Lists

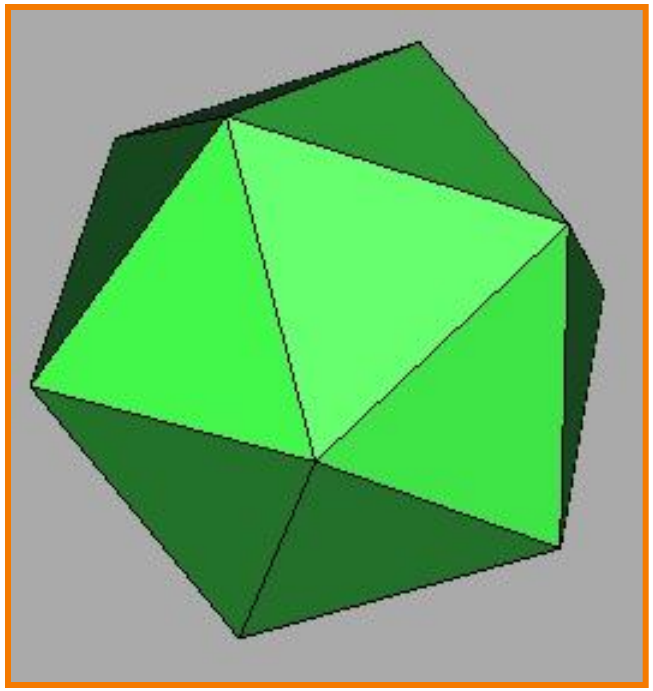
- Store all vertex, edge, and face adjacencies
  - ✓ Efficient adjacency traversal

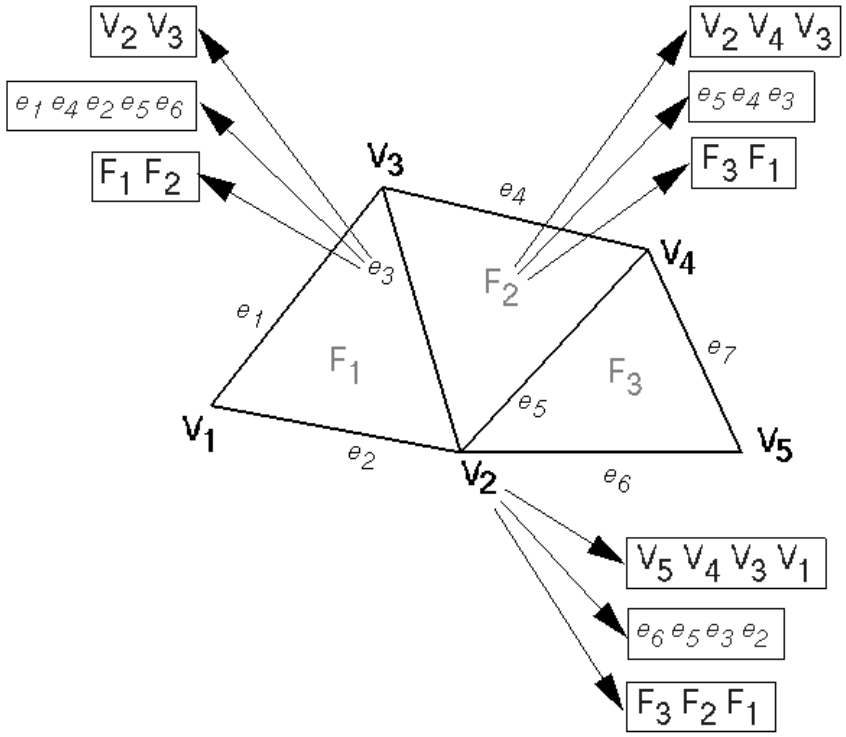




Adjacency Lists

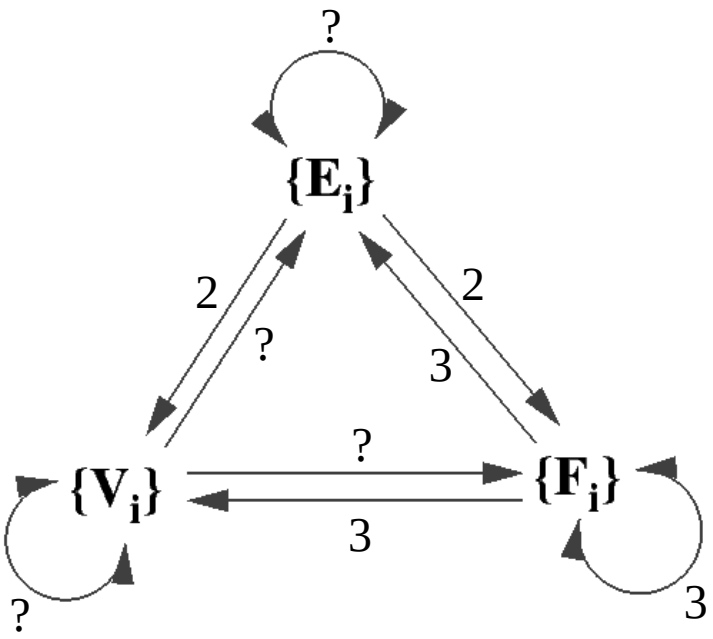
- Store all vertex, edge, and face adjacencies
  - ✓ Efficient adjacency traversal
  - ✗ Extra storage
  - ✗ Variable size arrays

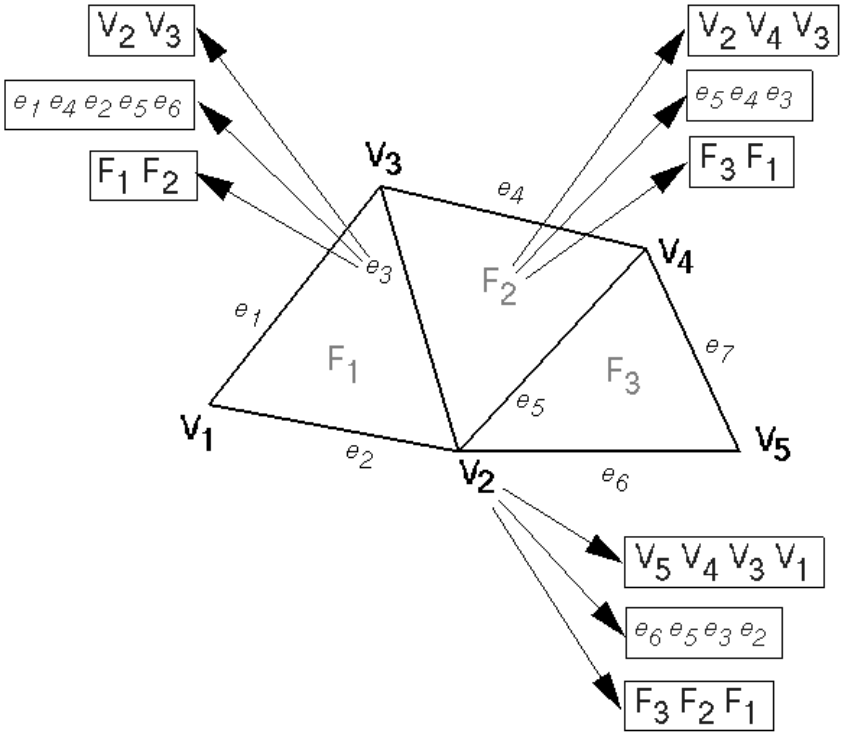




Partial Adjacency Lists

- Can we store only some adjacency relationships and derive others?





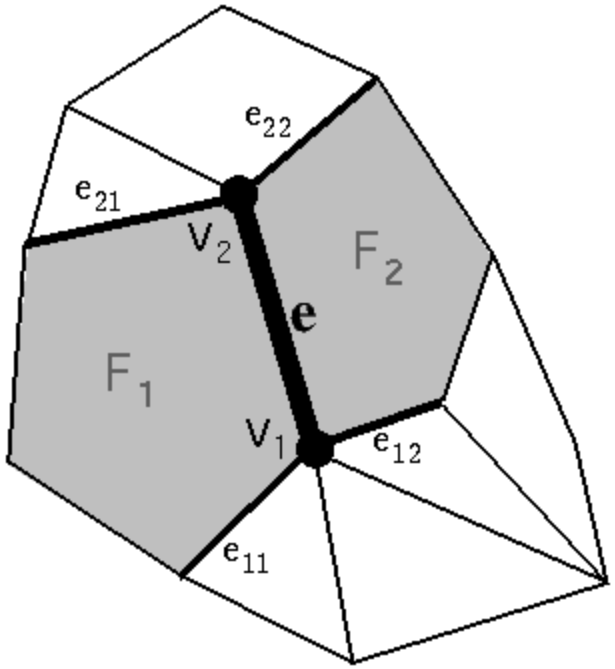
Winged Edge

- Adjacency encoded in edges
  - All adjacencies in O(1) time
  - Little extra storage (fixed records)
  - Arbitrary polygons

Each edge stores:
- 4 “wing” edges
- 2 vertices
- 2 faces

Each face stores:
- 1 edge

Each vertex stores:
- 1 edge



Winged Edge

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  - All adjacencies in O(1) time
  - Little extra storage (fixed records)
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- 2 vertices
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Each face stores:
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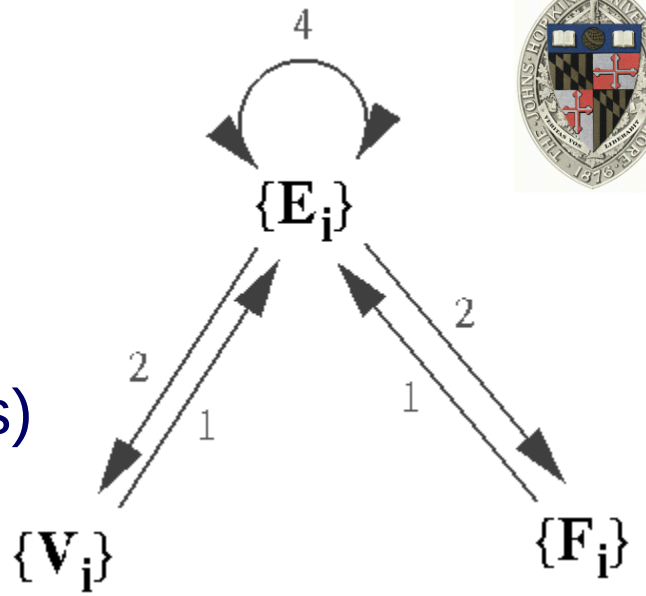
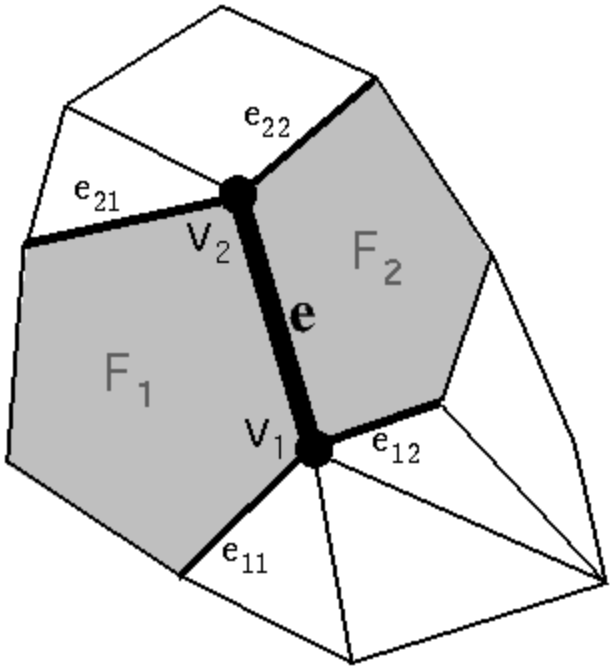
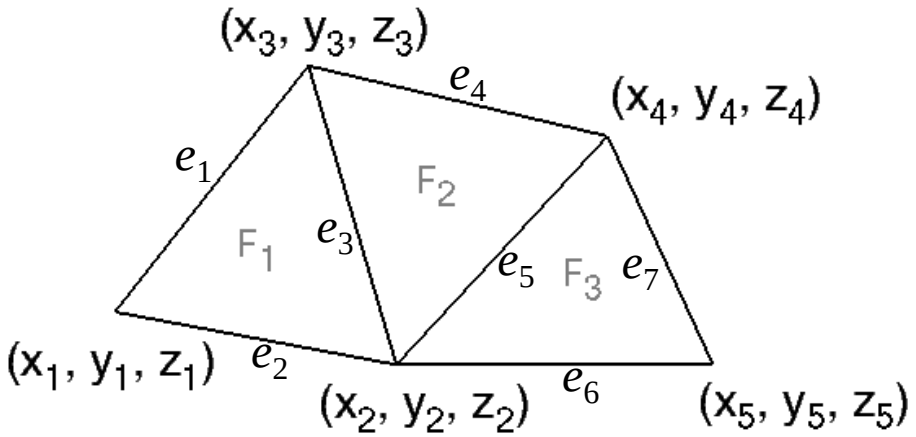




Winged Edge

- Example:



VERTEX TABLE

V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE

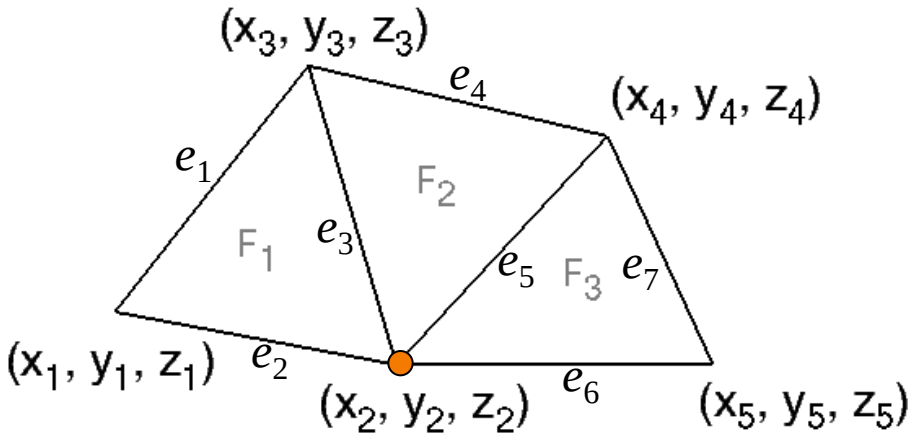
					S		F	
	S	E	L	R	CCW	CW	CW	CCW
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE

F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Example: Find CCW edges adjacent to V_2.



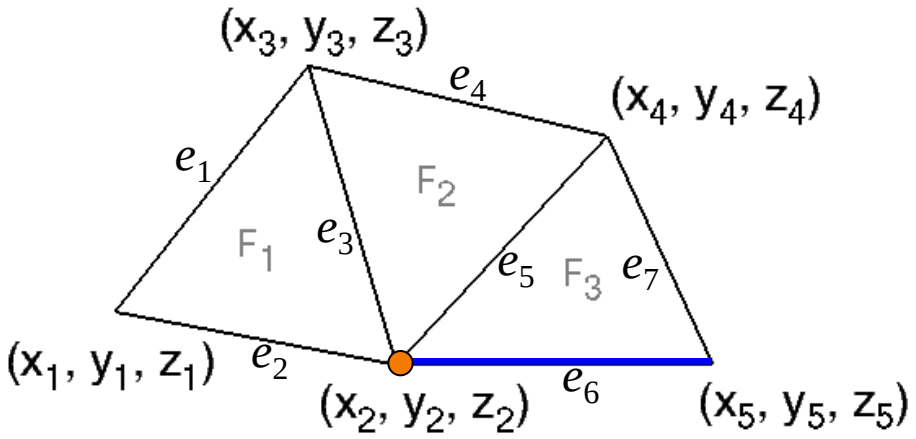
VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	CCW	CW	CW	CCW
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Example: Find CCW edges adjacent to V_2.



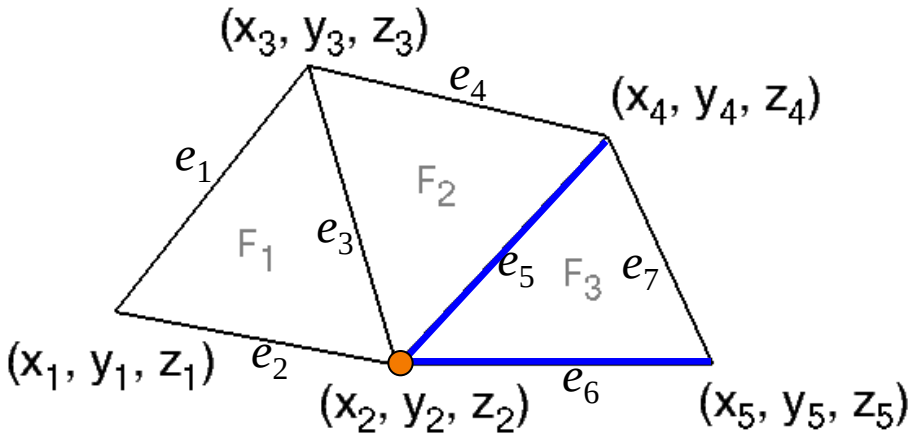
VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	CCW	CW	CW	CCW
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Example: Find CCW edges adjacent to V_2.



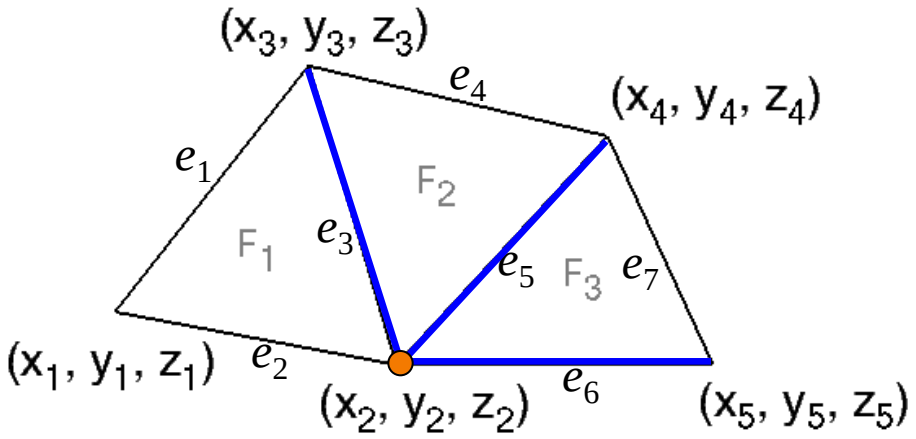
VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	CCW	CW	CW	CCW
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Example: Find CCW edges adjacent to V_2.



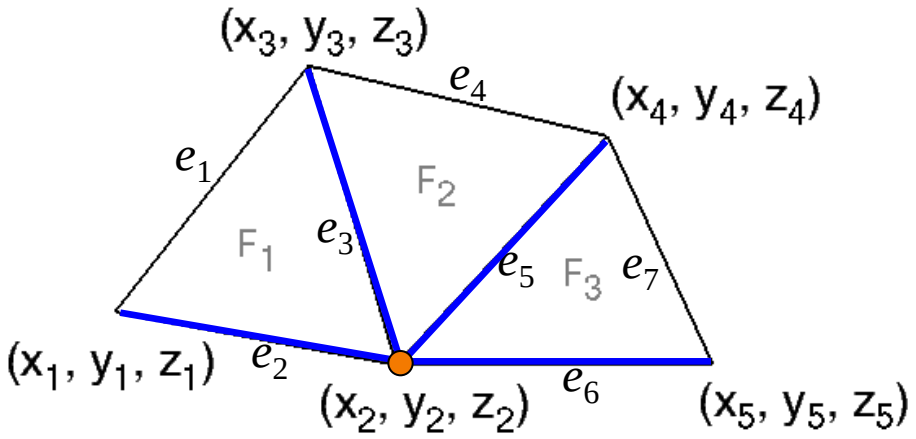
VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	CCW	CW	CW	CCW
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

FACE TABLE	
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Example: Find CCW edges adjacent to V_2.



VERTEX TABLE				
V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	CCW	CW	CW	CCW
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

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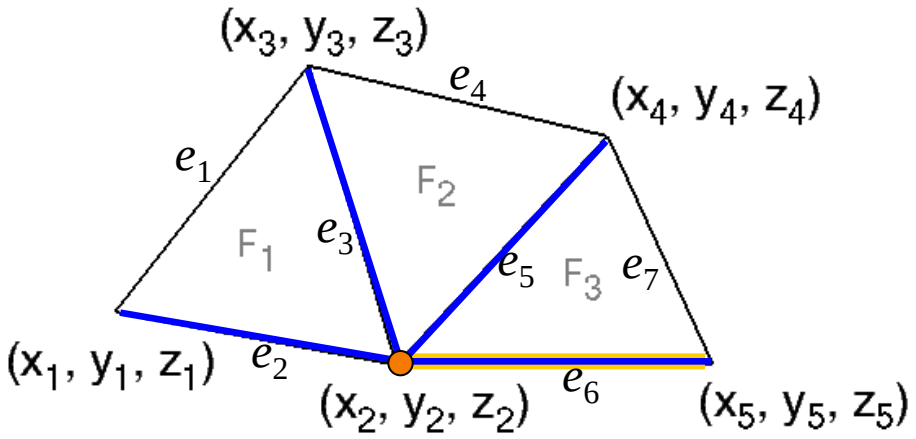
F_1	e_1
F_2	e_3
F_3	e_5

Winged Edge

- Example: Find CCW edges adjacent to V_2.



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V_1	X_1	Y_1	Z_1	e_1
V_2	X_2	Y_2	Z_2	e_6
V_3	X_3	Y_3	Z_3	e_3
V_4	X_4	Y_4	Z_4	e_5
V_5	X_5	Y_5	Z_5	e_6

EDGE TABLE					S		E	
	S	E	L	R	CCW	CW	CW	CCW
e_1	V_1	V_3		F_1	e_2	e_2	e_4	e_3
e_2	V_1	V_2	F_1		e_1	e_1	e_3	e_6
e_3	V_2	V_3	F_1	F_2	e_2	e_5	e_1	e_4
e_4	V_3	V_4		F_2	e_1	e_3	e_7	e_5
e_5	V_2	V_4	F_2	F_3	e_3	e_6	e_4	e_7
e_6	V_2	V_5	F_3		e_5	e_2	e_7	e_7
e_7	V_4	V_5		F_3	e_4	e_5	e_6	e_6

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F_3	e_5