Assignment 1

A1 a. Given the complex numbers $z_1 = 1+i$ and $z_2 = 1-i$, Compute.

$$\left(\frac{z_1}{z_2}\right)^{16} + \left(\frac{z_2}{z_1}\right)^{8}$$

$$= \left(\frac{1+i}{1-i}\right)^{16} + \left(\frac{1-i}{1+i}\right)^{8}$$

$$= \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^{16} + \left[\frac{(1-i)(1-i)}{(1+i)(1-i)}\right]^{8}$$

$$= \left[\frac{1+i+i+i^2}{1+i-i-i^2}\right]^{16} + \left[\frac{1-i-i+i^2}{1-i+i-i^2}\right]^{8}$$

$$= \left(\frac{2i}{2}\right)^{16} + \left(-\frac{2i}{2}\right)^{8}$$

$$= (i)^{16} + (-i)^{8}$$

$$= 1+1$$

$$= 2$$

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b). Using $z\bar{z} = |z|^2$

let $z = |z_1 + z_2|^2$

$\bar{z} = |z_1 - z_2|^2$

$\therefore |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

$= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)}$

$= z_1\bar{z} + z_1\bar{z}_2 + z_2\bar{z} + z_2\bar{z} + z_1\bar{z} - z_1 z_2 - \bar{z}_1\bar{z}_2 + z_2\bar{z}_2$

$= (z_1\bar{z}_2 + z_1\bar{z}_1) + (z_1\bar{z}_2 - z_1\bar{z}_2) + (z_2\bar{z}_1 - z_2\bar{z}) + z_2 z_2 + z_2\bar{z}$

$= 2(z_1\bar{z}_1) + 2\bar{z}_2 z_2$

$= 2|z_1|^2 + 2|z_2|^2$ shown.

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C. $z = x + iy$

$\bar{z} = x - iy$

$x = z - iy$

Substitute in $\bar{z}$, $\bar{z} = z - iy - iy$

$\bar{z} = z - 2iy$

$\bar{z} - z = -2iy \Rightarrow -\bar{z} + z = 2iy$

$\frac{\bar{z} - z}{-2i} = y \Rightarrow \frac{z - \bar{z}}{2i} = y$

$\therefore x = \frac{z + \bar{z}}{2}$

Substitute in $2x + y = 5$

$\Rightarrow 2\left[\frac{z + \bar{z}}{2}\right] + \left(\frac{z - \bar{z}}{2i}\right) = 5$

$\Rightarrow 2i(z + \bar{z}) + (z - \bar{z}) = 5(2i)$

$\Rightarrow 2i(z + \bar{z}) + (z - \bar{z}) = 10i$

$\Rightarrow -2(z + \bar{z}) + i(z - \bar{z}) = -10$ since $i^2 = -$
$i^2 = -1$

Hence $2(z + \bar{z}) - i(z - \bar{z}) = 10$

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A2a. $p(z) = z^4 - 3z^3 + rz^2 + sz + t$

$p(2) = 0$

$p(2) = 0$

$2^4 - 3(2)^3 + r(2)^2 + s(2) + t = 0$

$16 - 24 + 4r + 2s + t = 0$

$4r + 2s + t = 8 \quad \ldots \quad (i)$

Since $1+2i$ is a root then $1-2i$ is also a root

$[z - (1+2i)][z - (1-2i)] = z^2 - 2z + 5$

$$\begin{array}{r} z^2 - z + (r-7) \\ z^2 - 2z + 5 \;\big)\; z^4 - 3z^3 + rz^2 + sz + t \\ -\left(z^4 - 2z^3 + 5z^2\right) \\ \hline -z^3 + (r-5)z^2 + sz \\ -\left(-z^3 + 2z^2 - 5z\right) \\ \hline (r-7)z^2 + (s+5)z + t \\ -\; (r-7)z^2 - 2(r-7)z + 5(r-7) \\ \hline \end{array}$$

$(s+5)z + 2(r-7)z = 0$

$z(s + 2r - 9) = 0$

$s + 2r = 9 \quad \ldots \quad (ii)$

$t - 5(r-7) = 0$

$t - 5r = -35 \quad \ldots \quad (iii)$

Combining equation iii) and (i)

$t = 5r - 3s$

$4r + 2s + 5r - 3s = 8$

$9r + 2s = 43$ . . . . (iv)

Combining (iv) and (ii)

$$\begin{pmatrix} 9r + 2s = 43 \\ 2r + s = 9 \end{pmatrix} \begin{matrix} \times 2 \\ \times 9 \end{matrix}$$

$18r + 4s = 86$

$18r + 9s = 81$

$-5s = 5$

$s = -1$

$9r + 2(-1) = 43$

$9r = 45$

$r = 5$

$t = 5(5) - 35$

$= -10$

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2b. $z^2 + (2i-3)z + (5-i) = 0$

$$\frac{-(2i-3) \pm \left[(2i-3)^2 - 4(1)(5-i)\right]^{1/2}}{2(1)}$$

$$= \frac{-2i+3\left[4i^2 - 12i + 9 - 20 + 4i\right]^{1/2}}{2(1)} = \frac{-2i+3i\left[-4-12i+9-20+4i\right]^{1/2}}{2}$$

$$= \frac{-2i+3i\left[-15-8i\right]^{1/2}}{2}$$

$r = \left[\sqrt{(-15)^2 + (-8)^2}\right]^{1/2}$

let $a+bi = \sqrt{-15-8i}$

$(a+bi)^2 = -15-8i$

$a^2 + 2abi - b^2 = -15 - 8i$

$2ab = -8$

$b = -\frac{4}{a}$

$a^2 - b^2 = -15$

$a^2 - \left(-\frac{4}{a}\right)^2 = -15$

$a^4 + 15a^2 - 16 = 0$

let $x = a^2 \Rightarrow x^2 + 15x - 16 = 0 \Rightarrow x = 1$ or -16

but $x = a^2$, $\Rightarrow 1 = a^2 \Rightarrow a = 1$

$x = a^2 \Rightarrow -16 = a^2 \Rightarrow a = 4i$

$b = -4$ or $b = -\frac{4}{4i} = -\frac{4}{4i} \times \frac{-i}{-i} = -i \Rightarrow b = -i$

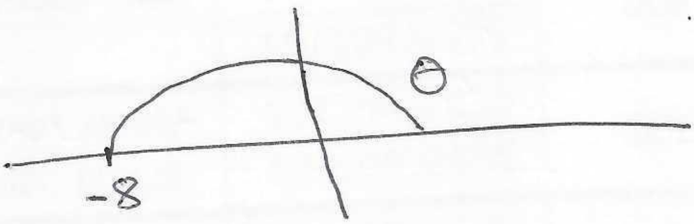
$$\frac{3-2i}{2} \pm \frac{1-4i}{2}$$

$$\frac{4-6i}{2} \text{ or } \frac{2+2i}{2}$$

$\Rightarrow 2-3i$ or $1+i$.

A3a) find all the cube roots of -8.

$$-8 = -8 + 0i$$



$\theta = \pi$

$r = \sqrt{(-8)^2 + 0^2} = \sqrt{64} = 8$

$(-8)^{1/3} = \sqrt[3]{8}\left[\cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right]$

$k=0 \quad 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i$

$k=1 \quad 2\left[\cos\pi + i\sin\pi\right] = 2((-1)+0) = -2$

$k=2 \quad 2\left[\cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)\right] + i\sin\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)]$

$= 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] = 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$

$= 1 - \sqrt{3}i$

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3b) $\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + 3/8$

Proof

$(2i\sin\theta)^4 = \left(z - \frac{1}{z}\right)^4$

$16\sin^4\theta = z^4 + 4z^3\left(\frac{-1}{z}\right) + 6z^2\cdot\left(\frac{-1}{z}\right)^2 + 4z\cdot\left(\frac{-1}{z}\right) + \left(\frac{-1}{z}\right)$

$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$= 2\cos 4\theta - 8\cos 2\theta + 6$

$\sin^4\theta = \frac{2}{16}\cos 4\theta - \frac{8}{16}\cos 2\theta + \frac{6}{16}$

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$= \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$ SHOWN

A4a) $a+2b-2c=0$ -- ①

$b-a=0$ -- ②

$c-1$ -- ③

$c=1$

$b=a$

$a+2b-2=0$

$b+2b=2$

$3b=2$ since $b=a$

$b=\frac{2}{3} \Rightarrow a=2/3 \quad c=1$

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C). $|A\times B| = |A||B| \sin\theta$

$|A\times B|^2 = (|A|^2 |B|^2 \sin^2\theta$

$\therefore |A|=|B|=1, \quad (A\times B)^2 = \sin^2\theta$

$\therefore A\cdot B = |A||B|\cos\theta = \cos\theta$

$1-(A\cdot B)^2 = 1-\cos^2\theta = \sin^2\theta$

$\Rightarrow |A\times B|^2 = 1-(A\cdot B)^2$ ▨

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(6) Point $(2, 3, 2)$

plane $3x + 2y - z = 6$

$$\vec{v} = (3, 2, -1)$$

Parametric Equations

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow \quad x = 2 + 3t$$
$$y = 3 + 2t$$
$$z = 4 - t.$$

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b) finding the direction $\overline{PQ}$

$$\overrightarrow{PQ} = \overline{OQ} - \overline{OP} = \begin{pmatrix} -2/3 \\ 2/3 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{-17}{3} \\ \frac{11}{3} \\ 3 \end{pmatrix}$$

Parametric Equations

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -17/3 \\ 11/3 \\ 3 \end{pmatrix} \Rightarrow \quad \begin{aligned} x &= 5 - \tfrac{17}{3}\lambda \\ y &= -3 + \tfrac{11}{3}\lambda \\ z &= -2 + 3\lambda \end{aligned}$$

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Symmetric Equation

$$\frac{x - 5}{-17/3} = \frac{y + 3}{11/3} = \frac{z + 2}{3} = \lambda$$

$$\frac{3(x-5)}{-17} = \frac{3(y+3)}{11} = \frac{z+2}{3}$$

$$\frac{3x - 15}{-17} = \frac{3y + 9}{11} = \frac{z+2}{3}$$

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b) find the equation of the plane passing throug the points $(0,0,0)$, $(1,2,3)$ & $(-2,3,3)$

$$AB = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$AC = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$$

$$AB \times AC = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -2 & 3 & 3 \end{vmatrix} = -3i - (9j) + 7k$$

$$AB \times AC = \begin{pmatrix} -3 \\ 9 \\ 7 \end{pmatrix}$$

Using $\vec{n} \cdot (\vec{r} \cdot \vec{r}) = 0$

$$\begin{pmatrix} -3 \\ -9 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix} = 0$$

$$-3(x-1) - 9(y-2) + 7(z-3) = 0$$

$$\therefore -3x + 3 - 9y + 18 + 7z - 21 = 0$$

$$-3x - 9y + 7z = 0$$

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UNIVERSITY OF ZIMBABWE

DEPARTMENT OF MATHEMATICS AND COMPUTATIONAL SCIENCES

LINEAR ALGEBRA || ASSIGNMENT

Submission date : 01 DECEMBER 2023
Time : ... hours

1

Answer ALL questions carefully numbering them **A1** to **A7**.
Write your name, registration number and your program on each answer script.
Total marks 80.

80

52

A1. (a) Given complex numbers $z_1 = 1+i$ and $z_2 = 1-i$, compute the following expression:

$$\left(\frac{z_1}{z_2}\right)^{16} + \left(\frac{z_2}{z_1}\right)^{8}.$$

[4]

(b) Show that if $z_1, z_2 \in \mathbb{C}$, then $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$. [4]

(*b*) Given that $z = x + iy$, express the equation $2x + y = 5$ in terms of z and $\bar{z}$. [5]

A2. (a) Consider the polynomial

$$p(z) = z^4 - 3z^3 + rz^2 + sz + t,$$

where r, s, and t are real constants. Given that the two roots of $p(z)$ are 2 and $1+2i$, determine the values of r, s and t. [8]

(b) Solve the equation $z^2 + (2i-3)z + (5-i) = 0$. [6]

A3. (a) Find all the cube roots of -8. [6]

(b) Prove that $\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$. [4]

A4. (a) Find a, b and c if, $(a+2b-2c)\mathbf{i} + (b-a)\mathbf{j} + (c-1)\mathbf{k} = 0$. [5]

~~(b) Find a unit vector perpendicular to both $\mathbf{A} = 2\mathbf{i}+\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i}-\mathbf{j}-\mathbf{k}$.~~ [5]

(c) If $\mathbf{A}$ and $\mathbf{B}$ are unit vectors, show that $|\mathbf{A}\times\mathbf{B}|^2 = 1 - (\mathbf{A}\cdot\mathbf{B})^2$. [5]

7a) Equation of a plane passing through the point $(1, -2, 3)$ and parallel to the plane

Equation of a plane pallel.

Point $(1, -2, 3)$

Parallel to plane $\Rightarrow z = 2x + 3y - 4$

$0 = 2x + 3y - z - 4$

$2x + 3y - z - 4 = 0$

$(2, 3, -1)$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Point $(1, -2, 3)$

Equation

$2(x-1) + 3(y+2) - 1(z-3) = 0$

$2x - 2 + 3y + 6 - z + 3 = 0$

$2x + 3y - z + 7 = 0$

$2x + 3y - z = -7$

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