SCALER

Combinatorics Basics

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Question

Given 10 Girls and & 7 Boys. How many different pairs?

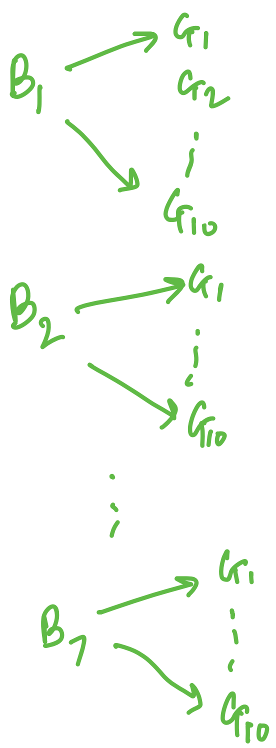
Pair → 1 Girl, 1 Boy

(B G)

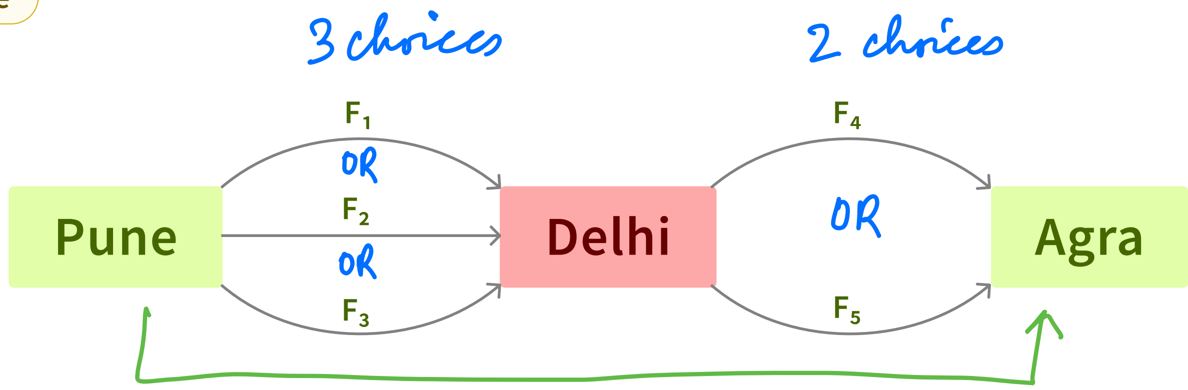
7 AND 10

$7 * 10$ possibilities

Boys	Girls
B_1	G_1
B_2	G_2
B_3	G_3
B_4	G_4
B_5	G_5
B_6	G_6
B_7	G_7
	G_8
	G_9
	G_{10}



Example



Number of ways to reach Agra from Pune via Delhi

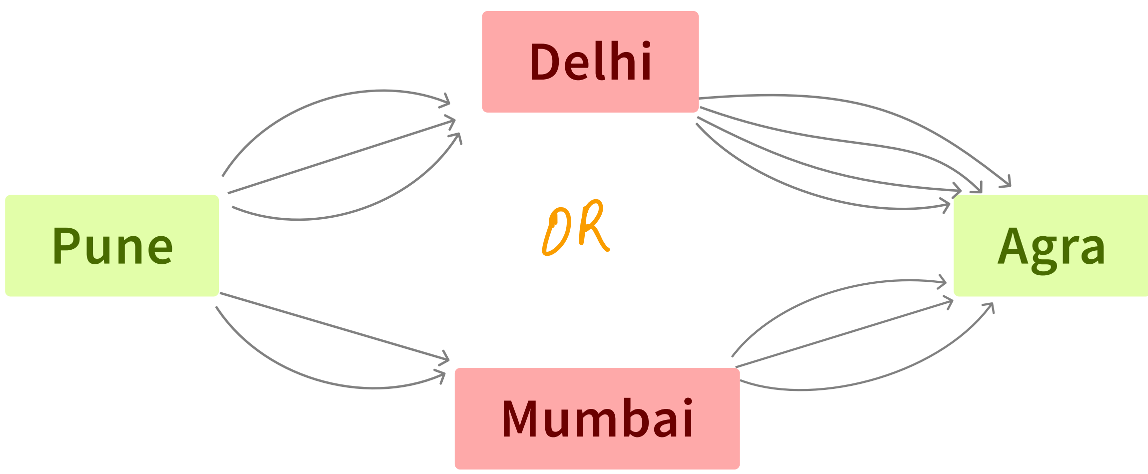
Pune $\rightarrow$ Delhi AND Delhi $\rightarrow$ Agra

3 ways * 2 ways

$F_1\ F_4$
$F_1\ F_5$
$F_2\ F_4$
$F_2\ F_5$
$F_3\ F_4$
$F_3\ F_5$

Example

Number of ways of reaching Agra from Pune ?



$\underbrace{\text{Pune} \xrightarrow{3} \text{Delhi} \xrightarrow{4} \text{Agra}}_{(3*4)}$ OR $\underbrace{\text{Pune} \xrightarrow{2} \text{Mumbai} \xrightarrow{3} \text{Agra}}_{2*3)}$

$(3*4) + 2*3) = 18.$

Permutations (Arrangements)

Question

Given 3 distinct characters. In how many ways, we can arrange them?

S = “ a b c ”

$\underline{3} * \underline{2} * \underline{1} = 6$ ways.

$3! = 6$ ways

$a \to b \to c$

$a \to c \to b$

$b \to a \to c$

$b \to c \to a$

$c \to a \to b$

$c \to b \to c$

Question

In how many ways, you can arrange 4 distinct characters?

$\underline{4} * \underline{3} * \underline{2} * \underline{1} = 4! = 24$ ways.

Question

In how many ways n distinct characters can be arranged?

$\underline{n} * \underline{n-1} * \underline{n-2} * \underline{n-3} \ldots \underline{2} * \underline{1}$

$n!$ ways.

Question

Given 5 distinct characters, in how many ways can we arrange 2 characters?

abcde

$\underline{5} * \underline{4} = 5*4 = 20.$

Question

3 distinct characters?

abcde

$\underline{5} * \underline{4} * \underline{3} = 60$ ways.

$\underline{3} * \underline{4} * \underline{5} = 60$ ways

Question

4 characters out of N distinct characters?

$\underline{N} * \underline{(N-1)} * \underline{(N-2)} * \underline{(N-3)}$

$N(N-1)(N-2)(N-3)$ ways.

$$\frac{n!}{(n-4)!}$$

Question

Given N distinct characters, in how many ways can we arrange r characters?

$$\underset{1}{\underline{n}} * \underset{2}{\underline{n-1}} * \underset{3}{\underline{n-2}} * \underset{4}{\underline{n-3}} * \underset{5}{\underline{n-4}} * \underset{6}{\underline{n-5}} \ldots * \underset{r-1}{\underline{n-(r-2)}} * \underset{r}{\underline{n-(r-1)}}$$

$$n-(r-1) \equiv n-r+1.$$

$$= n * (n-1) * (n-2) * \ldots * (n-r+1)$$

$$= \frac{n * (n-1) * (n-2) * \ldots * (n-r+1) * (n-r) * (n-r-1) * (n-r-2) \ldots * 1}{(n-r) * (n-r-1) * (n-r-2) \ldots * 1}$$

$$= \frac{n!}{(n-r)!}$$

$$\boxed{{}^{n}P_{r} = \frac{n!}{(n-r)!}}$$

Combinations (Selections)

Question

In how many ways can we select 3 players from a pool of 4 players?

[P_1 P_2 P_3 P_4]

$(P_1 P_2 P_3)$ $(P_1 P_2 P_4)$ $(P_1 P_3 P_4)$ $(P_2 P_3 P_4)$

4 ways

6: $P_1 P_2 P_3$, $P_1 P_3 P_2$, $P_2 P_1 P_3$, $P_2 P_3 P_1$, $P_3 P_1 P_2$, $P_3 P_2 P_1$

6: $P_1 P_2 P_4$, $P_1 P_4 P_2$, $P_2 P_1 P_4$, $P_2 P_4 P_1$, $P_4 P_1 P_2$, $P_4 P_2 P_1$

6: $P_1 P_3 P_4$, …

6: $P_2 P_3 P_4$, …

(3)

Question

Number of ways to arrange the players in 3 slots

Given 4 players → [P_1 P_2 P_3 P_4]

6: $P_1 P_2 P_3$, $P_1 P_3 P_2$, $P_2 P_1 P_3$, $P_2 P_3 P_1$, $P_3 P_1 P_2$, $P_3 P_2 P_1$

6: $P_1 P_2 P_4$, $P_1 P_4 P_2$, $P_2 P_1 P_4$, $P_2 P_4 P_1$, $P_4 P_1 P_2$, $P_4 P_2 P_1$

6: $P_1 P_3 P_4$, …

6: $P_2 P_3 P_4$, …

Nr. of arrangements = 24. $\left({}^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24\right)$

Nr. of arrangements = Nr. of arrangements for one selection * Nr. of selections.

⇒ Nr. of selections = $\dfrac{\text{Nr. of arrangements}}{\text{Nr. of arrangements for one selection}} = \dfrac{24}{6} = 4$.

$$\text{No. of selections } ({}^nC_r) = \frac{\text{No. of arrangements } ({}^nP_r)}{\text{No. of arrangements for one selection}}$$

No. of arrangements for one selection $\longrightarrow r!$

${}^nC_r$ $\downarrow$ No. of ways to select r items out of n.

$$\boxed{{}^nC_r = \frac{{}^nP_r}{r!}}$$

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\boxed{{}^nC_r = \frac{n!}{(n-r)!\cdot(r!)}}$$

Properties

$$^nC_r = \frac{n!}{(n-r)!\cdot r!}$$

1. $^NC_0 = \frac{n!}{(n-0)!\cdot 0!} = \frac{n!}{n!\cdot 1} = 1$

2. $^NC_N = \frac{n!}{(n-n)!\cdot n!} = \frac{n!}{0!\cdot n!} = \frac{n!}{n!} = 1$

3. $^NC_{N-r} = \frac{n!}{(n-(n-r))!\cdot (n-r)!} = \frac{n!}{r!\cdot (n-r)!} = {^nC_r}$

$$\boxed{\begin{aligned} {^nC_0} &= 1 \\ {^nC_n} &= 1 \\ {^nC_{n-r}} &= {^nC_r} \end{aligned}}$$

Question

Given N distinct elements, select r distinct elements ^nC_r

n^{th} element

select OR not select

$[1, \ldots, n-1]$ (n)

$\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \ldots\ \underline{r-1}\ \underline{r}$

$^{n-1}C_{r-1}$

$[1, n-1]$

$\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \ldots\ \underline{r}$

$^{n-1}C_r$

Total no. of ways $= \left(^{n-1}C_{r-1} + {}^{n-1}C_r \right)$

$$\Rightarrow \boxed{^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r} \ *$$

$25 \longrightarrow 15$

VK

VK ✗

$24 \longrightarrow 14$ $\qquad$ $24 \longrightarrow 15$

$$^{25}C_{15} = \boxed{^{24}C_{14} + {}^{24}C_{15}}$$

VK✓ $\qquad$ VK✗

$$^5C_2 = {}^4C_1 + {}^4C_2$$

$$10 \qquad 4 + 6$$

$$RHS = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

$${}^nC_r = \frac{n!}{(n-r)!\cdot r!}$$

$$= \frac{(n-1)!}{(n-r)!\cdot(r-1)!} + \frac{(n-1)!}{(n-1-r)!\cdot r!}$$

$$= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right)$$

$$= \frac{(n-1)!}{(r-1)!\cdot(n-r-1)!} * \frac{\cancel{r} + (n-\cancel{r})}{(n-r)\cdot r}$$

$$= \frac{n!}{r!\,(n-r)!}$$

$$= {}^nC_r = LHS.$$

Proved.

[Break till 10:39 PM]

Pascal Traingle

Generate the Pascal's traingle for given N

0 →	${}^0C_0$						1				
1 →	${}^1C_0$	${}^1C_1$					1	1			
2 →	${}^2C_0$	${}^2C_1$	${}^2C_2$			➡	1	2	1		
3 →	${}^3C_0$	${}^3C_1$	${}^3C_2$	${}^3C_3$			1	3	3	1	
4 →	${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$		1	4	6	4	1
i →	${}^iC_0$	${}^iC_1$	${}^iC_2$	- - - - - - -	${}^iC_i$						

row i col j of Pascal's Δ → ${}^{row}C_{col} = {}^{i}C_{j}$

Individually, calc. ${}^nC_r$ will take $O(n)$ time.

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Top cell

Top left cell

n \ r	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$\boxed{nCr[n][r] = {}^nC_r[n-1][r-1] + {}^nC_r[n-1][r]}$$

Code

Find all ${}^nC_r$ values upto a certain n

$\longrightarrow O(n^2)$ T.C

```
fn PascalsTriangle(n){
    nCr[n+1][n+1] = {0}
    for(i → 0 to n){
        nCr[i][0] = 1
        nCr[i][i] = 1
        for(j → 1 to i-1){
            nCr[i][j] = nCr[i-1][j-1] + nCr[i-1][j]
            // If mod m is needed, do the foll.
            // nCr[i][j] = (nCr[i-1][j-1] + nCr[i-1][j]) % m)
        }
    }
    return nCr
}
```

$\boxed{O(n^2)\ TC}$

${}^nC_r \% m$

$$= \left(\frac{n!}{(n-r)! \cdot r!}\right) \% m = (n! \% m) * \text{modInverse}((n-r)!, m) * \text{modInverse}((r!), m)$$

$$\binom{10}{5} \% 7 = 2$$

$$\left(\frac{10 \% 7}{5}\right) \% 7 = \left(\frac{3}{5}\right) \% 7 =$$

N^{th} Column Title

Find the N^{th} column title

N =	1	2	3		26	27	28		50	51	52	53	54	- -	
	A	B	C	- -	Z	AA	AB	- - - - - -	AX	AY	AZ	BA	BB	- — -	BZ

$n = 3 \rightarrow C$

$n = 30 \rightarrow AD$

$n = 50 \rightarrow AX$

$1 \rightarrow$ 'A'

$3 \rightarrow (3-1) +$ 'A'

$5 \rightarrow (5-1) +$ 'A' = 'E'

$26 \rightarrow (26-1) +$ 'A' = 'Z'

$$
\begin{array}{r|c|l}
26 & 27-1=26 & \\ \hline
26 & 1-1=0 & 0 \xrightarrow{+\text{'A'}} \text{'A'} \\ \hline
 & 0 & 0 \xrightarrow{+\text{'A'}} \text{'A} \\
\end{array}
$$

"AA"

$$
\begin{array}{r|c|l}
26 & 50-1=49 & \\ \hline
26 & 1-1=0 & 23 \xrightarrow{+\text{'A'}} \text{'X'} \\ \hline
 & 0 & 0 \xrightarrow{+\text{'A'}} \text{'A'} \\
\end{array}
$$

"AX"

BX

$$
\begin{array}{r|c|l}
26 & 76-1=75 & \\ \hline
26 & 2-1=1 & \longrightarrow 23 \longrightarrow \text{'X'} \\ \hline
 & 0 & \longrightarrow 1 \xrightarrow{+\text{'A'}} \text{'B'} \\
\end{array}
$$

```
fn colTitle (n) {
    ans = ""
    while (n > 0) {
        n = n-1
        ans = (char)(n % 26 + 'A') + ans
        n = n/26
    }
    return ans
}
```

T.C. → $O(\log(n))$

S.C. → $O(1)$

Code

$$a^{\phi(m)} \% m = 1$$ → Euler Totient fn,

If m is Prime, $\phi(m) = m-1$

$$\boxed{a^{m-1} \% m = 1}$$

$$\boxed{a^{-1} \% m = a^{m-2} \% m}$$

$$\left(\frac{x}{a}\right) \% m$$

$$= (x * a^{-1}) \% m$$

$$= (x * \underbrace{a^{m-2}}_{\text{mod Inverse}(a, m)}) \% m$$

mod Inverse(a, m)

⇓

$a^{-1} \% m$ ~ $a^{m-2} \% m$ if m is Prime.

$a^{\phi(m)-1} \% m$

$$a^{m-1} \% m$$

$$= (a^{m-2} * a) \% m$$

$$\Rightarrow ((a^{m-1} \% m) * (a^{-1} \% m)) \% m$$

$$= a^{m-2} \% m$$

$$\Rightarrow a^{-1} \equiv a^{m-2} \% m$$