4.6 The Gamma Probability Distribution

The continuous *gamma* random variable Y has density

$$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

where the gamma *function* is defined as

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$$

and its expected value (mean), variance and standard deviation are,

$$\mu = E(Y) = \alpha\beta, \quad \sigma^2 = V(Y) = \alpha\beta^2, \quad \sigma = \sqrt{V(Y)}.$$

One important special case of the gamma, is the continuous *chi–square* random variable Y where $\alpha = \frac{\nu}{2}$ and $\beta = 2$; in other words, with density

$$f(y) = \begin{cases} \frac{y^{\frac{\nu-2}{2}}e^{-\frac{y}{2}}}{2^{\nu/2}\Gamma(\frac{\nu}{2})}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

and its expected value (mean), variance and standard deviation are,

$$\mu = E(Y) = \nu, \quad \sigma^2 = V(Y) = 2\nu, \quad \sigma = \sqrt{V(Y)}.$$

Another important special case of the gamma, is the continuous *exponential* random variable Y where $\alpha = 1$; in other words, with density

$$f(y) = \begin{cases} \frac{1}{\beta}e^{-y/\beta}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

and its expected value (mean), variance and standard deviation are,

$$\mu = E(Y) = \beta, \quad \sigma^2 = V(Y) = \beta^2, \quad \sigma = \beta.$$

Exercise 4.6 (The Gamma Probability Distribution)

1. *Gamma distribution.*

   (a) *Gamma function*[8], $\Gamma(\alpha)$.

[8] The gamma *function* is a part of the gamma *density*. There is no closed–form expression for the gamma function except when α is an integer. Consequently, numerical integration is required. We will mostly use the calculator to do this integration.

i. $\Gamma(1.2) = \int_0^\infty y^{1.2-1}e^{-y}\,dy =$
(choose one) (i) **0.92** (ii) **1.12** (iii) **2.34** (iv) **2.67**.
PRGM GAMFUNC ENTER ENTER (again!) 1.2 ENTER

ii. $\Gamma(2.2) \approx$ (choose one) (i) **0.89** (ii) **1.11** (iii) **1.84** (iv) **2.27**.
PRGM GAMFUNC ENTER ENTER (again!) 2.2 ENTER

iii. Notice
$\Gamma(2.2) \approx 1.11 = (2.2-1)\Gamma(2.2-1) = 1.2\Gamma(1.2) \approx 1.2(0.92) \approx 1.11.$
In other words, in general,

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1),\ \alpha > 1$$

(i) **True** (ii) **False**

iv. $\Gamma(1) =$ (choose one) (i) **0** (ii) **0.5** (iii) **0.7** (iv) **1**.
PRGM GAMFUNC ENTER ENTER 1 ENTER

v. $\Gamma(2) = (2-1)\Gamma(2-1) =$ (i) **0** (ii) **0.5** (iii) **0.7** (iv) **1**.

vi. $\Gamma(3) = 2\Gamma(2) =$ (i) **1** (ii) **2!** (iii) **3!** (iv) **4!**.

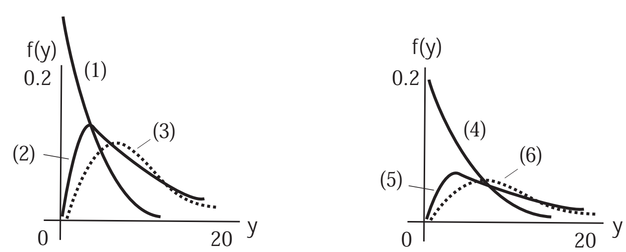
vii. $\Gamma(4) = 3\Gamma(3) =$ (i) **1** (ii) **2!** (iii) **3!** (iv) **4!**.

viii. In general, if n is a positive integer,

$$\Gamma(n) = (n-1)!$$

(i) **True** (ii) **False**

(b) *Graphs of gamma density.* Consider the graphs in Figure 4.9.



(a) increasing a, b = 3 (b) increasing a, b = 5

Figure 4.9: Gamma densities

i. Match gamma density, (α, β), to graph, (1) to (6).

$(\alpha,\beta) =$	$(1,3)$	$(2,3)$	$(3,3)$	$(1,5)$	$(2,5)$	$(3,5)$
graph	(1)					

For $(\alpha, \beta) = (1, 3)$, for example,

PRGM GAMGRPH ENTER ENTER 1 ENTER 3 ENTER 20 ENTER 0.2 ENTER

ii. As α and β grow larger, gamma density becomes (choose one)
(i) **more symmetric.**
(ii) **more skewed.**

iii. As α and β grow larger, "center" (mean) of gamma density
(i) **decreases.**
(ii) **remains the same.**
(iii) **increases.**

iv. As α and β grow larger, "dispersion" (variance) of gamma density
(i) **decreases.**
(ii) **remains the same.**
(iii) **increases.**

(c) *Gamma distribution: area under gamma density.*

i. If $(\alpha, \beta) = (1, 3)$, $P(Y < 1.3) = F(1.3) \approx$ (choose one)
(i) **0.23** (ii) **0.35** (iii) **0.43** (iv) **0.43**.
PRGM GAMDSTR ENTER ENTER 1 ENTER 3 ENTER 1.3 ENTER

ii. If $(\alpha, \beta) = (1, 3)$, $P(Y > 2.7) = 1 - P(Y \leq 2.7) = 1 - F(2.7) \approx$
(i) **0.13** (ii) **0.32** (iii) **0.41** (iv) **0.63**.
First, PRGM GAMDSTR ENTER ENTER 1 ENTER 3 ENTER 2.7 ENTER
then subtract result from 1!

iii. If $(\alpha, \beta) = (2, 5)$, $P(1.4 < Y < 2.7) = P(Y \leq 2.7) - P(Y \leq 1.4) \approx$
(i) **0.07** (ii) **0.11** (iii) **0.21** (iv) **0.33**.
Find PRGM GAMDSTR ENTER ENTER 2 ENTER 5 ENTER 2.7 ENTER
then subtract PRGM GAMDSTR ENTER ENTER 2 ENTER 5 ENTER 1.4 ENTER

(d) *Mean, variance and standard deviation of gamma distribution.*

i. If $(\alpha, \beta) = (2, 5)$, $\mu = E(Y) = \alpha\beta = (2)(5) =$ (choose one)
(i) **10** (ii) **11** (iii) **12** (iv) **13**.

ii. If $(\alpha, \beta) = (1.2, 4.3)$, $\mu = E(Y) =$ (choose one)
(i) **5.16** (ii) **5.34** (iii) **5.44** (iv) **5.66**.

iii. If $(\alpha, \beta) = (2, 5)$, $\sigma^2 = V(Y) = \alpha\beta^2 = (2)(5)^2 =$ (choose one)
(i) **40** (ii) **50** (iii) **60** (iv) **70**.

iv. If $(\alpha, \beta) = (1.2, 4.3)$, $\sigma = \sqrt{\alpha\beta^2} = \sqrt{(1.2)(4.3)^2} \approx$ (choose one)
(i) **3.45** (ii) **3.54** (iii) **4.33** (iv) **4.71**.

2. *Gamma distribution again: time to fix car.* Assume the time, Y, to fix a car is approximately a gamma with mean $\mu = 2$ hours and variance $\sigma^2 = 2$ hours^2.

(a) *What are α and β?* Since

$$\mu = \alpha\beta = 2, \quad \sigma^2 = \alpha\beta^2 = (\alpha\beta)\beta = 2\beta = 2,$$

then $\beta = \frac{2}{2} = 1$ and also $\alpha = \frac{\mu}{\beta} = \frac{2}{1} =$ (choose one)
(i) **2** (ii) **3** (iii) **4** (iv) **5**.

(b) In this case, the gamma density,

$$f(y) = \begin{cases} \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

is given by (choose one)
(i)

$$f(y) = \begin{cases} ye^{-y}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

(ii)

$$f(y) = \begin{cases} \frac{ye^{-y}}{\Gamma(3)}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

(iii)

$$f(y) = \begin{cases} \frac{y^2e^{-y/2}}{2^2\Gamma(1)}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere}. \end{cases}$$

(c) What is the chance of waiting *at most* 4.5 hours?
Since $(\alpha, \beta) = (2, 1)$, $P(Y < 4.5) = F(4.5) \approx$ (choose one)
(i) **0.65** (ii) **0.78** (iii) **0.87** (iv) **0.94**.
PRGM GAMDSTR ENTER ENTER 2 ENTER 1 ENTER 4.5 ENTER

(d) $P(Y > 3.1) = 1 - P(Y \leq 3.1) = 1 - F(3.1) \approx$ (choose one)
(i) **0.18** (ii) **0.28** (iii) **0.32** (iv) **0.41**.
Subtract PRGM GAMDSTR ENTER ENTER 2 ENTER 1 ENTER 3.1 ENTER from one.

(e) What is the 90th percentile waiting time; in other words, what is that time such that 90% of waiting times are less than this time?
If $P(Y < \phi_{0.90}) = 0.90$, then $\phi_{0.90} \approx$ (choose one)
(i) **1.89** (ii) **2.53** (iii) **3.72** (iv) **3.89**.
PRGM GAMINV ENTER ENTER 2 ENTER 1 ENTER 0.9 ENTER

3. *Chi-square distribution: waiting time to order.* At McDonalds in Westville, waiting time to order (in minutes), Y, follows a chi–square distribution.

(a) *Probabilities.* Consider graphs in Figure 4.10.

i. If $\nu = 4$, the probability of waiting less than 3.9 minutes is
$P(Y < 3.9) = F(3.9) \approx$ (choose one)
(i) **0.35** (ii) **0.45** (iii) **0.58** (iv) **0.66**.
2nd DISTR χ^2cdf(0,3.9,4).

ii. If $\nu = 10$, $P(3.6 < Y < 7.0) =$ (choose one)
(i) **0.24** (ii) **0.34** (iii) **0.42** (iv) **0.56**.
2nd DISTR χ^2cdf(3.6,7.0,10).

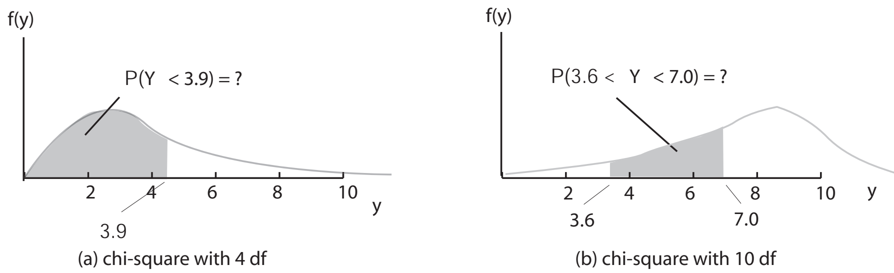


Figure 4.10: Chi–square probabilities

iii. Chance of waiting time *exactly* 3 minutes, say, is *zero*, $P(Y = 3) = 0$.
(i) **True** (ii) **False**

(b) *Percentiles.* Consider graphs in Figure 4.11.

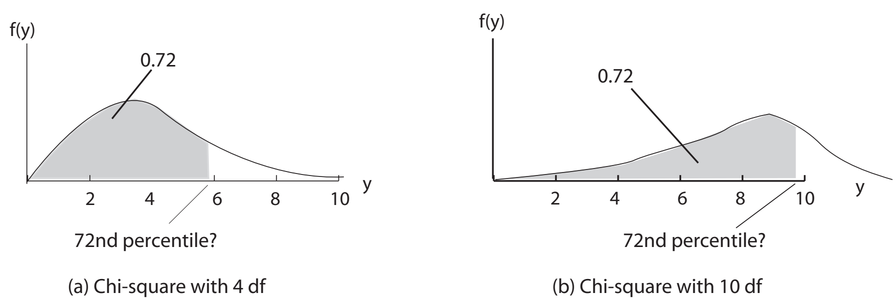


Figure 4.11: Chi–square percentiles

i. If $\nu = 4$ and $P(Y < \phi_{0.72}) = 0.72$, then $\phi_{0.72} \approx$ (choose one)
(i) **3.1** (ii) **5.1** (iii) **8.3** (iv) **9.1**.
PRGM CHI2INV ENTER 4 ENTER 0.72 ENTER

ii. If $\nu = 10$ and $P(Y < \phi_{0.72}) = 0.72$, then $\phi_{0.72} \approx$ (choose one)
(i) **2.5** (ii) **10.5** (iii) **12.1** (iv) **20.4**.
PRGM CHI2INV ENTER 10 ENTER 0.72 ENTER

iii. The 32nd percentile for a chi-square with $\nu = 18$ df, is
(i) **2.5** (ii) **10.5** (iii) **14.7** (iv) **20.4**.
PRGM CHI2INV ENTER 18 ENTER 0.32 ENTER

iv. The 32nd percentile is that waiting time such that 32% of the waiting times are less than this waiting time and 68% are more than this time.
(i) **True** (ii) **False**

4. *Chi–square distribution again.*

(a) If $\nu = 3$, the chi–square density,

$$f(y) = \begin{cases} \frac{y^{\frac{\nu-2}{2}} e^{-\frac{y}{2}}}{2^{\nu/2}\Gamma(\frac{\nu}{2})}, & 0 \le y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

is given by (choose one)

(i)

$$f(y) = \begin{cases} \frac{1}{2}e^{-\frac{y}{2\Gamma(2)}}, & 0 \le y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

(ii)

$$f(y) = \begin{cases} \frac{1}{2\Gamma(2)}e^{-\frac{y}{2}}, & 0 \le y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

(iii)

$$f(y) = \begin{cases} \frac{y^{\frac{1}{2}} e^{-\frac{y}{2}}}{2^{3/2}\Gamma(\frac{3}{2})}, & 0 \le y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

(b) If $\nu = 3$, $P(Y < 3.1) \approx$ (choose one)
(i) **0.62** (ii) **0.67** (iii) **0.72** (iv) **0.81**.
2nd DISTR χ^2cdf(0,3.1,3).

(c) If $\nu = 3$, $P(1.3 < Y < 3.1) \approx$ (choose one)
(i) **0.25** (ii) **0.35** (iii) **0.45** (iv) **0.55**.
2nd DISTR χ^2cdf(1.3,3.1,3).

(d) If $\nu = 3$, $\mu = E(Y) = \nu =$ (choose one)
(i) **3** (ii) **4** (iii) **5** (iv) **6**.

(e) If $\nu = 3$, $\sigma^2 = V(Y) = 2\nu =$ (choose one)
(i) **3** (ii) **4** (iii) **5** (iv) **6**.

(f) If $\nu = 3$ and $P(Y < \phi_{0.90}) = 0.90$, then $\phi_{0.90} \approx$ (choose one)
(i) **3.89** (ii) **4.53** (iii) **5.72** (iv) **6.25**.
PRGM CHI2INV ENTER ENTER 3 ENTER 0.9 ENTER

(g) A chi–square with $\nu = 3$ degrees of freedom is a gamma with parameters $(\alpha, \beta) = (\frac{\nu}{2}, 2) =$ (choose one)
(i) $\left(\mathbf{\frac{0}{2}, 2}\right)$ (ii) $\left(\mathbf{\frac{1}{2}, 2}\right)$ (iii) $\left(\mathbf{\frac{2}{2}, 2}\right)$ (iv) $\left(\mathbf{\frac{3}{2}, 2}\right)$.

5. *Exponential: waiting time for emails.* Assume waiting times for emails follow an exponential distribution,

$$f(y) = \begin{cases} \frac{1}{\beta}e^{-y/\beta}, & 0 \le y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

(a) If $\beta = 2$, the chance of waiting at most 1.1 minutes is[9]

$$P(Y \leq 1.1) = F(1.1) = \int_0^{1.1} \frac{1}{2} e^{-y/2}\, dy = \left[-e^{-y/2}\right]_0^{1.1} = 1 - e^{-\frac{1}{2}(1.1)} =$$

(i) **0.32** (ii) **0.42** (iii) **0.45** (iv) **0.48**.

(b) If $\beta = \frac{1}{3}$,

$$P(Y \leq 1.1) = F(1.1) = \int_0^{1.1} 3e^{-3y}\, dy = \left[-e^{-3y}\right]_0^{1.1} = 1 - e^{-3(1.1)} =$$

(i) **0.32** (ii) **0.42** (iii) **0.75** (iv) **0.96**.

(c) If $\beta = \frac{1}{5}$, $P(Y < 1.1) = F(1.1) = 1 - e^{-5(1.1)} \approx$
(circle one) (i) **0.312** (ii) **0.432** (iii) **0.785** (iv) **0.996**.
It is also possible to use: PRGM EXPDSTR ENTER 1/5 ENTER 1.1 ENTER

(d) If $\beta = \frac{1}{3}$, the chance of waiting at least 0.54 minutes
$P(Y > 0.54) = 1 - F(0.54) = 1 - \left(1 - e^{-(3)(0.54)}\right) = e^{-(3)(0.54)} \approx$
(i) **0.20** (ii) **0.22** (iii) **0.29** (iv) **0.34**.

(e) If $\beta = \frac{1}{3}$,
$P(1.13 < Y < 1.62) = F(1.62) - F(1.13) = e^{-(3)(1.13)} - e^{-(3)(1.62)} \approx$
(i) **0.014** (ii) **0.026** (iii) **0.034** (iv) **0.054**.

(f) If $\beta = \frac{1}{3}$ and $P(Y < \phi_{0.90}) = 0.90$, then $\phi_{0.90} \approx$ (choose one)
(i) **0.45** (ii) **0.65** (iii) **0.77** (iv) **0.89**.
PRGM EXPINV ENTER ENTER 1/3 ENTER 0.9 ENTER

(g) *Expectation and Variance.*
For $\beta = 2$, $\mu = \beta =$ (choose one) (i) **2** (ii) **3** (iii) **4** (iv) **5**.
For $\beta = \frac{1}{3}$, $\mu = \beta =$ (choose one) (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) $\frac{1}{4}$ (iv) $\frac{1}{5}$.
For $\beta = 2$, $\sigma^2 = \beta^2 =$ (choose one) (i) **2** (ii) **3** (iii) **4** (iv) **5**.
For $\beta = \frac{1}{3}$, $\sigma^2 = \beta^2 =$ (choose one) (i) $\frac{1}{3}$ (ii) $\frac{1}{5}$ (iii) $\frac{1}{7}$ (iv) $\frac{1}{9}$.

6. *Exponential again: battery lifetime.* Suppose the distribution of the lifetime of camera flash batteries, Y, is exponential, with mean (average) lifetime of $\mu = \beta = \frac{1}{3}$.

(a) The chance batteries last at least 10 hours is

$$P(Y > 10) = 1 - F(10) = 1 - (1 - e^{-3(10)}) =$$

(choose one) (i) $\boldsymbol{e^{-10}}$ (ii) $\boldsymbol{e^{-20}}$ (iii) $\boldsymbol{e^{-30}}$ (iv) $\boldsymbol{e^{-40}}$.

[9] Unlike the gamma and chi-square distributions, it is fairly easy to integrate exponential densities.

(b) The chance batteries last at least 15 hours, given that they have already lasted at least 5 hours is

$$P(Y > 15|Y > 5) = \frac{P(Y > 15, Y > 5)}{P(Y > 5)} = \frac{P(Y > 15)}{P(Y > 5)} = \frac{1 - (1 - e^{-3(15)})}{1 - (1 - e^{-3(5)})} =$$

(choose one) (i) $\boldsymbol{e^{-10}}$ (ii) $\boldsymbol{e^{-20}}$ (iii) $\boldsymbol{e^{-30}}$ (iv) $\boldsymbol{e^{-40}}$.

(c) In other words[10],

$$P(Y > 15|Y > 5) = P(Y > 10).$$

This is an example of the "memoryless" property of the exponential.
(i) **True** (ii) **False**

(d) *Implication of memoryless property of exponential: independence.* If

$$P(Y > s + t|Y > t) = P(Y > s);\; s, t \geq 0$$

and also

$$P(Y > s + t|Y > t) = \frac{P(Y > s + t, Y > t)}{P(Y > t)} = \frac{P(Y > s + t)}{P(Y > t)}$$

then, combining the last two equations,

$$\frac{P(Y > s + t)}{P(Y > t)} = P(Y > s)$$

or

$$P(Y > s + t) = P(Y > t)P(Y > s)$$

But $P(Y > a) = e^{-\frac{a}{\beta}}$, and so

$$e^{-\frac{s+t}{\beta}} = e^{-\frac{s}{\beta}} e^{-\frac{t}{\beta}}$$

(i) **True** (ii) **False**

4.7 The Beta Probability Distribution

The *beta* random variable Y, with parameters $\alpha > 0$ and $\beta > 0$, has density

$$f(y) = \begin{cases} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere,} \end{cases}$$

[10]The chance a battery lasts at least 10 hours or more, is the same as the chance a battery lasts at least 15 hours, given that it has already lasted 5 hours or more. This is kind of surprising, because it seems to imply the battery's life starts "fresh" after 5 hours, as though the battery "forgot" about the first five hours of its life.

and distribution function, called an *incomplete beta function*,

$$F(y) = \int_0^y \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)}\, dt = I_y(\alpha,\beta),$$

where

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\, dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and its expected value (mean), variance and standard deviation are,

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta}, \quad \sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}, \quad \sigma = \sqrt{V(Y)}.$$

Exercise 4.7 (The Beta Probability Distribution)

1. *Beta Distribution.*

   (a) *Graphs of beta density.* Consider the graphs in Figure 4.12.

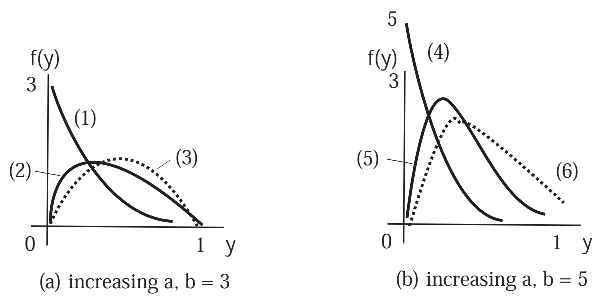


Figure 4.12: Beta densities

i. Match beta density, (α, β), to graph (1) to (6).

$(a,b) =$	$(1,3)$	$(2,3)$	$(3,3)$	$(1,5)$	$(2,5)$	$(3,5)$
graph	(1)					

For $(\alpha,\beta) = (1,3)$, for example,

PRGM BETAGRPH ENTER ENTER 1 ENTER 3 ENTER 3 ENTER

ii. If α and β become more equal, beta density becomes (choose one)
(i) **more symmetric.**
(ii) **more skewed.**

iii. If β decreases, "center" (mean) of beta density
(i) **decreases.**
(ii) **remains the same.**
(iii) **increases.**

iv. All beta densities are defined on (choose one)
(i) $\mathbf{(0,1)}$ (ii) $\mathbf{(-1,1)}$ (iii) $\mathbf{(0,\infty)}$ (iv) $\mathbf{(-\infty,0)}$

v. If $\alpha = 1$ and $\beta = 1$ beta is the (choose one)
(i) **uniform.** (ii) **normal.** (iii) **gamma.** (iv) **exponential.**
PRGM BETAGRPH ENTER ENTER 1 ENTER 1 ENTER 2 ENTER

(b) *Beta distribution: area under beta density.*

i. For $(\alpha, \beta) = (1, 5)$, $P(Y < 0.2) = F(0.2) \approx$ (choose one)
(i) **0.35** (ii) **0.41** (iii) **0.67** (iv) **0.77**.
PRGM BETADSTR ENTER ENTER 1 ENTER 5 ENTER 0.2 ENTER

ii. For $(\alpha, \beta) = (2, 5)$, $P(Y < 0.2) = F(0.2) \approx$ (choose one)
(i) **0.34** (ii) **0.41** (iii) **0.67** (iv) **0.77**.
PRGM BETADSTR ENTER ENTER 2 ENTER 5 ENTER 0.2 ENTER

iii. For $(\alpha, \beta) = (2, 5)$, $P(0.1 \leq Y < 0.7) = F(0.7) - F(0.1) \approx$
(i) **0.65** (ii) **0.71** (iii) **0.87** (iv) **0.96**.
PRGM BETADSTR ENTER ENTER 2 ENTER 5 ENTER 0.7 ENTER subtract PRGM BETADSTR ENTER ENTER 2 ENTER 5 ENTER 0.1 ENTER

(c) *Mean, variance and standard deviation of beta.*

i. If $(\alpha, \beta) = (1, 5)$, $\mu = E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{1}{1+5} =$ (choose one)
(i) $\mathbf{\frac{1}{5}}$ (ii) $\mathbf{\frac{1}{6}}$ (iii) $\mathbf{\frac{1}{7}}$ (iv) $\mathbf{\frac{1}{8}}$.

ii. If $(\alpha, \beta) = (1, 5)$, $\sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} =$ (choose one)
(i) $\mathbf{\frac{4}{252}}$ (ii) $\mathbf{\frac{5}{252}}$ (iii) $\mathbf{\frac{6}{252}}$ (iv) $\mathbf{\frac{7}{252}}$.

2. *Beta distribution again: proportion of time car broken.* Assume the *proportion* of time, Y, a car is broken is approximately a beta with the following density,

$$f(y) = \begin{cases} 5(1-y)^4, & 0 < y < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

(a) *What are α and β?* Since the general density is given by,

$$f(y) = \begin{cases} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere,} \end{cases}$$

which, when compared to the density in this example[11],

$$\alpha - 1 = 0, \quad \beta - 1 = 4,$$

and so $\alpha = 1$ and $\beta =$ (choose one)
(i) **2** (ii) **3** (iii) **4** (iv) **5**.

[11] Also notice $B(1,5) = \frac{\Gamma(1)\Gamma(5)}{\Gamma(1+5)} = \frac{(1)4!}{5!} = \frac{1}{5}$, so $\frac{1}{B(1,5)} = 5$.

(b) What is the chance the car is broken *at most* 65% of the time?
Since $(\alpha, \beta) = (1, 5)$, $P(Y < 0.65) = F(0.65) \approx$ (choose one)
(i) **0.65** (ii) **0.78** (iii) **0.87** (iv) **0.99**.
PRGM BETADSTR ENTER ENTER 1 ENTER 5 ENTER 0.65 ENTER

(c) $P(0.15 \leq Y \leq 0.45) = F(0.45) - F(0.15) \approx$ (choose one)
(i) **0.34** (ii) **0.39** (iii) **0.45** (iv) **0.56**.
PRGM BETADSTR ENTER ENTER 1 ENTER 5 ENTER 0.45 ENTER
subtract PRGM BETADSTR ENTER ENTER 1 ENTER 5 ENTER 0.15 ENTER.

(d) What is the 80th percentile proportion broken time?
If $P(Y < \phi_{0.80}) = 0.80$, then $\phi_{0.80} \approx$ (choose one)
(i) **0.13** (ii) **0.28** (iii) **0.35** (iv) **0.54**.
PRGM BETAINV ENTER ENTER 1 ENTER 5 ENTER 0.8 ENTER

4.8 Some General Comments

All of the probability distributions discussed could be used to model “real” data. A good probability model is one that closely matches the actual data.

4.9 Other Expected Values

Just as for the discrete case, *moment–generating functions*, $m(t)$, are useful in calculating the *moments* of the distribution of any[12] *continuous* random variable Y. Furthermore, $m(t)$ uniquely identifies any probability distribution. Consequently, it is possible to use either the probability distribution or its associated (and possibly easier to mathematically manipulate) moment–generating function when working with probability distributions.

- The *moment of random variable Y taken about the origin* is defined by,

$$\mu'_k = E\left(Y^k\right).$$

- The *moment of random variable Y taken about its mean* (or *kth central moment of Y*) is defined by,

$$\mu_k = E((Y - \mu)^k).$$

[12] This is true as long as $m(t)$ exists. The function $m(t)$ exists as long as there is a constant b such that $m(t) < \infty$ for $|t| \leq b$.

- In the continuous case, the moment–generating function of Y is defined by,

$$\begin{aligned}
m(t) \;=\; E\left(e^{tY}\right) &= \int_{-\infty}^{\infty} e^{ty} f(y)\,dy \\
&= \int_{-\infty}^{\infty} \left[1 + ty + \frac{(ty)^2}{2!} + \cdots\right] f(y)\,dy \\
&= \int_{-\infty}^{\infty} f(y) + t\int_{-\infty}^{\infty} yf(y) + \frac{t^2}{2!}\int_{-\infty}^{\infty} y^2 f(y) + \cdots \\
&= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots
\end{aligned}$$

- Function $m(t)$ "generates" moments of a distribution by successively differentiating $m(t)$ and evaluating the results at $t = 0$,

$$\left.\frac{d^k m(t)}{dt^k}\right]_{t=0} = m^{(k)}(0) = \mu_k', \quad k = 1, 2, \ldots$$

Exercise 4.8 (Other Expected Values)

1. *Exponential.* Let Y have an exponential density given by

$$f(y) = \begin{cases} \frac{1}{\beta}e^{-y/\beta}, & 0 \leq y < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) *Determine* $m(t)$.

$$m(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f(y)\,dy = \int_0^{\infty} e^{ty}\frac{1}{\beta}e^{-y/\beta}\,dy = \frac{1}{\beta}\int_0^{\infty} e^{-y/\beta^*}\,dy$$

   where $\beta^* = \frac{\beta}{1-\beta t}$. However, since $\frac{1}{\beta^*}\int_0^{\infty} e^{-y/\beta^*}\,dy = 1$, then

$$m(t) = \frac{1}{\beta} \times \beta^* \left[\frac{1}{\beta^*}\int_0^{\infty} e^{-y/\beta^*}\,dy\right] = \frac{1}{\beta} \times \beta^* = \frac{1}{\beta} \times \frac{\beta}{1-\beta t} =$$

   (i) $\boldsymbol{\frac{1}{1-\beta t}}$ (ii) $\boldsymbol{\frac{2}{1-\beta t}}$ (iii) $\boldsymbol{\frac{3}{1-\beta t}}$ (iv) $\boldsymbol{\frac{4}{1-\beta t}}$.

   (b) *Determine* $E(Y)$*; that is, show* $E(Y) = \beta$ *using* $m(t)$. First notice

$$E(Y) = E\left(Y^1\right) = \mu_1'.$$

   So

$$\mu_1' = m^{(1)}(0) = \left.\frac{d^1 m(t)}{dt^1}\right]_{t=0} = \left.\frac{d\left\{(1-\beta t)^{-1}\right\}}{dt}\right]_{t=0} = \left[\beta(1-\beta t)^{-2}\right]_{t=0} =$$

   (i) $\boldsymbol{\beta}$ (ii) $\boldsymbol{2\beta}$ (iii) $\boldsymbol{3\beta}$ (iv) $\boldsymbol{\beta + 1}$.

(c) *Determine* $E\left(Y^2\right)$. First notice

$$E\left(Y^2\right) = \mu_2'.$$

So

$$\mu_2' = m^{(2)}(0) = \left.\frac{d^2 m(t)}{dt^2}\right]_{t=0} = \left.\frac{d^2\left\{(1-\beta t)^{-1}\right\}}{dt^2}\right]_{t=0} = \left[2\beta^2(1-\beta t)^{-3}\right]_{t=0}$$

which equals (choose one)
(i) $\boldsymbol{2\beta^2}$ (ii) $\boldsymbol{\beta^2+\beta}$ (iii) $\boldsymbol{\beta+3\beta}$ (iv) $\boldsymbol{\beta^3+\beta}$.

(d) *Determine* $V(Y)$*; that is, show* $V(Y) = \beta^2$.
$V(Y) = E[Y^2] - (E[Y])^2 = 2\beta^2 - \beta^2 =$
(i) $\boldsymbol{\beta^2}$ (ii) $\boldsymbol{2\beta^2+\beta}$ (iii) $\boldsymbol{\beta^2+3\beta}$ (iv) $\boldsymbol{\beta^3+\beta}$.

2. *Gamma. Assume* moment–generating function for gamma Y is

$$m(t) = \frac{1}{(1-\beta t)^{\alpha}}.$$

(a) Moment–generating function $m(t) = \frac{1}{(1-1.5t)^{2.5}}$ is gamma where
$(\alpha, \beta) =$ (i) $\mathbf{(1.5, 2.5)}$ (ii) $\mathbf{(1.5, 2.0)}$ (iii) $\mathbf{(2.5, 1.0)}$ (iv) $\mathbf{(2.5, 1.5)}$,
and $\mu = \alpha\beta = (2.5)(1.5) =$ (i) **1.00** (ii) **2.75** (iii) **3.00** (iv) **3.75**,
and $\sigma^2 = \alpha\beta^2 =$ (circle one)
(i) **5.083** (ii) **5.095** (iii) **5.123** (iv) **5.625**,
and so $P(0 \leq Y \leq 3) \approx$ (i) **0.35** (ii) **0.40** (iii) **0.45** (iv) **0.58**.
PRGM GAMDSTR ENTER ENTER 2.5 ENTER 1.5 ENTER 3 ENTER.

(b) *Moment–generating function for* $W = -2Y - 9$.
In general, $m_Y(t) = E\left(e^{tY}\right)$ and

$$m_W(t) = E\left(e^{tW}\right) = E\left(e^{t(-2Y-9)}\right) = E\left(e^{(-2t)Y-9t}\right) = e^{-9t}E\left(e^{(-2t)Y}\right) =$$

(i) $\boldsymbol{5e^{-9t}m_Y(t)}$ (ii) $\boldsymbol{e^{-9t}m_Y(-2t)}$
(iii) $\boldsymbol{e^{-9t}m_W(-2t)}$ (iv) $\boldsymbol{e^{-9t}m_Y(-2t^2)}$.
For gamma moment–generating function, $m_Y(t) = \frac{1}{(1-1.5t)^{2.5}}$,

$$m_W(t) = e^{-9t}m_Y(-2t) =$$

(i) $\boldsymbol{e^{-2t}\left(\frac{1}{(1-1.5t)^{2.5}}\right)}$ (ii) $\boldsymbol{e^{-9t}\left(\frac{1}{(1+3t)^{2.5}}\right)}$
(iii) $\boldsymbol{e^{-9}\left(\frac{1}{(1+1.5t)^{2.5}}\right)}$ (iv) $\boldsymbol{e^{-9}\left(\frac{1}{(1-1.5t)^{2.5}}\right)}$.

3. *Normal. Assume* moment–generating function for normal Y is

$$m(t) = e^{\mu t + (1/2)t^2\sigma^2}.$$

(a) Moment–generating function $m(t) = e^{2t+(1/2)t^2(2)}$ is normal where
$\mu =$ (circle one) (i) **1** (ii) **2** (iii) **3** (iv) **4**,
and $\sigma = \sqrt{\sigma^2} \approx$ (circle one) (i) **1.41** (ii) **2.41** (iii) **3.41** (iv) **4.41**,
and so $P(0 \leq Y \leq 3) \approx$
(circle one) (i) **0.41** (ii) **0.68** (iii) **0.91** (iv) **0.98**.
2nd DISTR normalcdf(0,3,2,$\sqrt{2}$)

(b) Moment–generating function $m(t) = e^{2t+(1/2)t^2}$ is normal where
$\mu =$ (circle one) (i) **1** (ii) **2** (iii) **3** (iv) **4**,
and $\sigma = \sqrt{\sigma^2} \approx$ (circle one) (i) **1** (ii) **1.41** (iii) **1.51** (iv) **2**,
and so $P(0 \leq Y \leq 3) \approx$
(circle one) (i) **0.41** (ii) **0.68** (iii) **0.82** (iv) **0.98**.
2nd DISTR normalcdf(0,3,2,1)

4. *Uniform*[13]. *Assume* moment–generating function for uniform Y is

$$m(t) = \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}.$$

(a) Moment–generating function $m(t) = \frac{e^{2t}-e^t}{t}$ is uniform which is defined on
$(\theta_1, \theta_2) =$ (i) $\mathbf{(0,1)}$ (ii) $\mathbf{(1,2)}$ (iii) $\mathbf{(-1,1)}$ (iv) $\mathbf{(1,3)}$,
and $\mu = \frac{\theta_1+\theta_2}{2} =$ (circle one) (i) **1.0** (ii) **1.5** (iii) **2.0** (iv) **2.5**,
and $\sigma = \frac{\theta_2-\theta_1}{\sqrt{12}} \approx$ (circle one)
(i) **0.289** (ii) **0.295** (iii) **0.323** (iv) **0.433**,
and $P(0 \leq Y \leq 3) \approx$ (circle one) (i) **0.25** (ii) **0.50** (iii) **0.75** (iv) **1**.
Since uniform defined on (1,2), probability must be 1.

(b) *Moment–generating function for* $W = 5Y + 3$.
In general, $m_Y(t) = E\left(e^{tY}\right)$ and

$$m_W(t) = E\left(e^{tW}\right) = E\left(e^{t(5Y+3)}\right) = E\left(e^{(5t)Y+3t}\right) = e^{3t}E\left(e^{(5t)Y}\right) =$$

(i) $\boldsymbol{5e^{3t}m_Y(t)}$ (ii) $\boldsymbol{e^{3t}m_Y(5t)}$ (iii) $\boldsymbol{e^{3t}m_W(5t)}$ (iv) $\boldsymbol{e^{3t}m_Y(5t^2)}$.
For uniform moment–generating function, $m_Y(t) = \frac{e^{2t}-e^t}{t}$,

$$m_W(t) = e^{3t}m_Y(5t) =$$

(i) $\boldsymbol{e^{3t}\left(\frac{e^{10t}-e^{5t}}{5t}\right)}$ (ii) $\boldsymbol{e^{5t}\left(\frac{e^{10t}-e^{5t}}{5t}\right)}$ (iii) $\boldsymbol{e^{3}\left(\frac{e^{10t}-e^{5t}}{5t}\right)}$ (iv) $\boldsymbol{e^{5}\left(\frac{e^{10t}-e^{5t}}{5t}\right)}$.

[13] Although the beta distribution does not have a closed–form moment–generating function, a special case of the beta distribution, the uniform distribution, does have a closed–form moment–generating function.

4.10 Tchebysheff's Theorem

As was true in the discrete case, *Tchebysheff's theorem* states, for *continuous* random variable Y with finite μ and σ^2 and for $k > 0$,

$$P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} \quad \text{or} \quad P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

These two (equivalent) inequalities allow us to specify (very loose) lower bounds on probabilities when the distribution is not known.

Exercise 4.10 (Tchebysheff's Theorem)

1. *Tchebysheff's theorem and exponential: battery lifetime.* Suppose the distribution of the lifetime of camera flash batteries, Y, is exponential, with parameter $\beta = \frac{1}{3}$.

   (a) $\mu = \beta =$ (choose one) (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{4}}$ (iii) $\mathbf{\frac{1}{5}}$ (iv) $\mathbf{\frac{1}{6}}$.

   (b) $\sigma = \sqrt{\beta^2} \approx$ (choose one) (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{4}}$ (iii) $\mathbf{\frac{1}{5}}$ (iv) $\mathbf{\frac{1}{6}}$.

   (c) According to Tchebysheff's theorem, the probability battery lifetime is *within* $k = 2$ standard deviations of mean battery lifetime is *at least*

   $$P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{2^2} =$$

   (i) **0.75** (ii) **0.85** (iii) **0.95** (iv) **0.98**.
   In fact, since $\mu = \frac{1}{3}$ and $\sigma = \frac{1}{3}$,

   $$\begin{aligned} P(|Y-\mu| < k\sigma) &= P(\mu - k\sigma < Y < \mu + k\sigma) \\ &= P\left(\frac{1}{3} - 2\left(\frac{1}{3}\right) < Y < \frac{1}{3} + 2\left(\frac{1}{3}\right)\right) \\ &= P\left(-\frac{1}{3} < Y < \frac{3}{3}\right) \\ &= P(0 \leq Y \leq 1) \\ &= F(1) \ = \ 1 - e^{-3(1)} \ \approx \end{aligned}$$

   (i) **0.56** (ii) **0.76** (iii) **0.88** (iv) **0.95**.
   or: PRGM EXPDSTR ENTER ENTER 1/3 ENTER 1
   Tchebysheff's approximation, 0.75, is a (very) low bound on the actual probability, 0.95.

   (d) According to Tchebysheff's theorem, the probability battery lifetime is *within* $k = 1.5$ standard deviations of mean battery lifetime is *at least*

   $$P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{1.5^2} \approx$$

(i) **0.56** (ii) **0.74** (iii) **0.85** (iv) **0.88**.
In actual fact, since $\mu = \frac{1}{3}$ and $\sigma = \frac{1}{3}$,

$$\begin{aligned}
P(|Y - \mu| < k\sigma) &= P(\mu - k\sigma < Y < \mu + k\sigma) \\
&= P\left(\frac{1}{3} - 1.5\left(\frac{1}{3}\right) < Y < \frac{1}{3} + 1.5\left(\frac{1}{3}\right)\right) \\
&= P\left(-\frac{1}{6} < Y < \frac{5}{6}\right) \\
&= P\left(0 \leq Y \leq \frac{5}{6}\right) \\
&= F\left(\frac{5}{6}\right) = 1 - e^{-3\left(\frac{5}{6}\right)} \approx
\end{aligned}$$

(i) **0.56** (ii) **0.76** (iii) **0.88** (iv) **0.92**.
Tchebysheff's approximation, 0.56, is a (very) low bound on the actual probability, 0.92.

(e) According to Tchebysheff's theorem, the probability battery lifetime is *beyond* $k = 1.5$ standard deviations from mean battery lifetime is *at most*

$$P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2} = \frac{1}{1.5^2} =$$

(i) **0.32** (ii) **0.36** (iii) **0.35** (iv) **0.44**.
In actual fact, since $\mu = 4$ and $\sigma \approx 1.55$,

$$\begin{aligned}
P(|Y - \mu| \geq k\sigma) &= P(Y \leq \mu - k\sigma) + P(Y \geq \mu + k\sigma) \\
&\approx P\left(Y \leq \frac{1}{3} - 1.5\left(\frac{1}{3}\right)\right) + P\left(Y \geq \frac{1}{3} + 1.5\left(\frac{1}{3}\right)\right) \\
&= P\left(Y \geq \frac{5}{6}\right) \\
&= 1 - F\left(\frac{5}{6}\right) =
\end{aligned}$$

(i) **0.05** (ii) **0.06** (iii) **0.07** (iv) **0.08**.
Tchebysheff's approximation, 0.44, is a (very) high bound on the actual probability, 0.08.

2. *Tchebysheff's theorem and gamma: time to fix a car.* Suppose the distribution of the time to fix a car, Y, is gamma, with parameters $\alpha = 2$, $\beta = 4$. Let repair costs equal $C = 25Y + 250$.

   (a) $E(Y) = \mu_Y = \alpha\beta =$ (choose one) (i) **6** (ii) **7** (iii) **8** (iv) **9**.

(b) $E(C) = \mu_C = E(25Y + 250) = 25E(Y) + 250 = 25(8) + 250 =$
(choose one) (i) **350** (ii) **400** (iii) **450** (iv) **500**.

(c) $\sigma_Y = \sqrt{\alpha\beta^2} \approx$ (choose one) (i) **5.34** (ii) **5.66** (iii) **6.54** (iv) **7.23**.

(d) $\sigma_C = \sqrt{V(25Y + 250)} = \sqrt{25^2 V(Y)} = 25\sqrt{\sigma^2} = 25\sigma \approx$
(choose one) (i) **138.22** (ii) **139.98** (iii) **141.42** (iv) **144.33**.

(e) According to Tchebysheff's theorem, the probability repair cost is *within* $k = 2$ standard deviations of mean repair cost is *at least*

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{2^2} =$$

(i) **0.75** (ii) **0.85** (iii) **0.95** (iv) **0.98**.
In fact, since $\mu_C = 450$ and $\sigma_C \approx 141.42$ (equivalently, $\sigma_C^2 = 20000$), so

$$\mu = \alpha\beta = 450, \quad \sigma^2 = \alpha\beta^2 = (\alpha\beta)\beta = (450)\beta = 20000$$

and so $\beta_C = \frac{20000}{450} = \frac{400}{9}$ and $\alpha_C = \frac{450}{\frac{400}{9}} =$ (choose one)
(i) $\mathbf{\frac{81}{8}}$ (ii) $\mathbf{\frac{82}{8}}$ (iii) $\mathbf{\frac{83}{8}}$ (iv) $\mathbf{\frac{84}{8}}$,
then

$$\begin{aligned} P(|C - \mu| < k\sigma) &= P(\mu - k\sigma < C < \mu + k\sigma) \\ &= P(450 - 2(141.42) < C < 450 + 2(141.42)) \\ &= P(167.16 < C < 732.84) \approx \end{aligned}$$

(i) **0.56** (ii) **0.76** (iii) **0.88** (iv) **0.96**.
PRGM GAMDSTR ENTER ENTER 81/8 ENTER 400/9 ENTER 732.84 ENTER
subtract PRGM GAMDSTR ENTER ENTER 81/8 ENTER 400/9 ENTER 167.16 ENTER
Tchebysheff's approximation, 0.75, is a (very) low bound on the actual probability, 0.96.

4.11 Expectations of Discontinuous Functions and Mixed Probability Distributions

Not covered.

4.12 Summary

CONTINUOUS	$f(y)$	$m(t)$	μ	σ^2
Beta	$\frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Uniform	$1/(\theta_2-\theta_1)$	$\frac{e^{t\theta_2}-e^{t\theta_1}}{t(\theta_2-\theta_1)}$	$(\theta_1+\theta_2)/2$	$(\theta_2-\theta_1)^2/12$
Exponential	$\frac{1}{\beta}e^{-\frac{1}{\beta}y}$	$\frac{1}{1-\beta t}$	β	β^2
Gamma	$\frac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}$	$\frac{1}{(1-\beta t)^\alpha}$	$\alpha\beta$	$\alpha\beta^2$
Normal	$\frac{1}{\sqrt{2\pi}\sigma}e^{-(y-\mu)^2/2\sigma^2}$	$e^{\mu t+\sigma^2t^2/2}$	μ	σ^2