Lorenz Equations and Atmospheric Convection

By Tohidul Islam

Purpose

The goal of this project is to study the Lorenz system and how it can be used to determine the reliability of long-term weather predictions. Understanding the patterns of atmospheric convection can allow meteorologists to make accurate predictions in the motion of the Earth’s atmosphere; however, these predictions often become less accurate as we stray further from the present. An analysis of the Lorenz System can help to indicate the conditions that make long-term weather prediction so difficult.

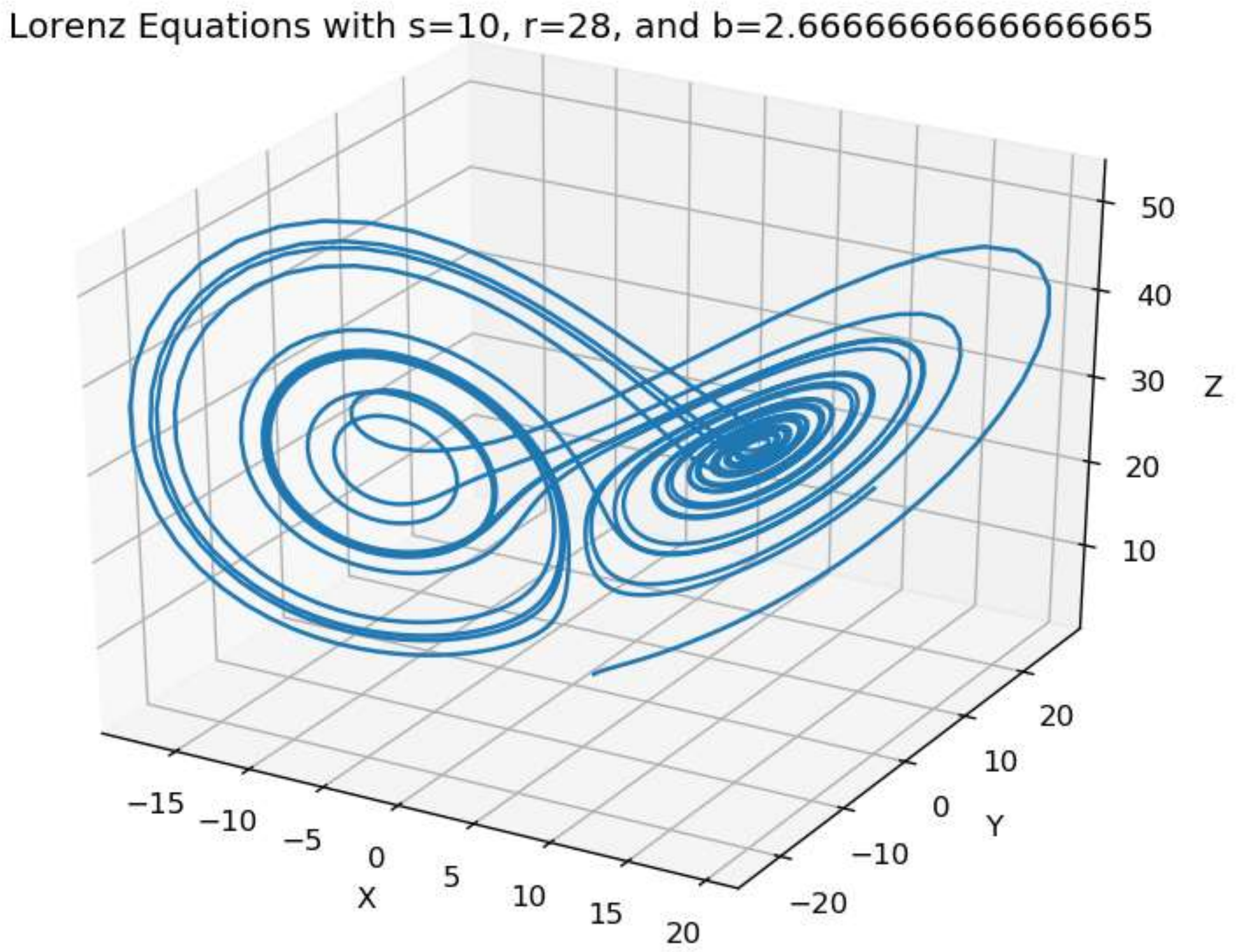
Background

The Lorenz Equations were first derived by M.I.T. meteorologist Edward Lorenz while trying to mathematically model weather systems for long-term predictions. Lorenz simplified his initial model of twelve variables to what is now known as the Lorenz System where:

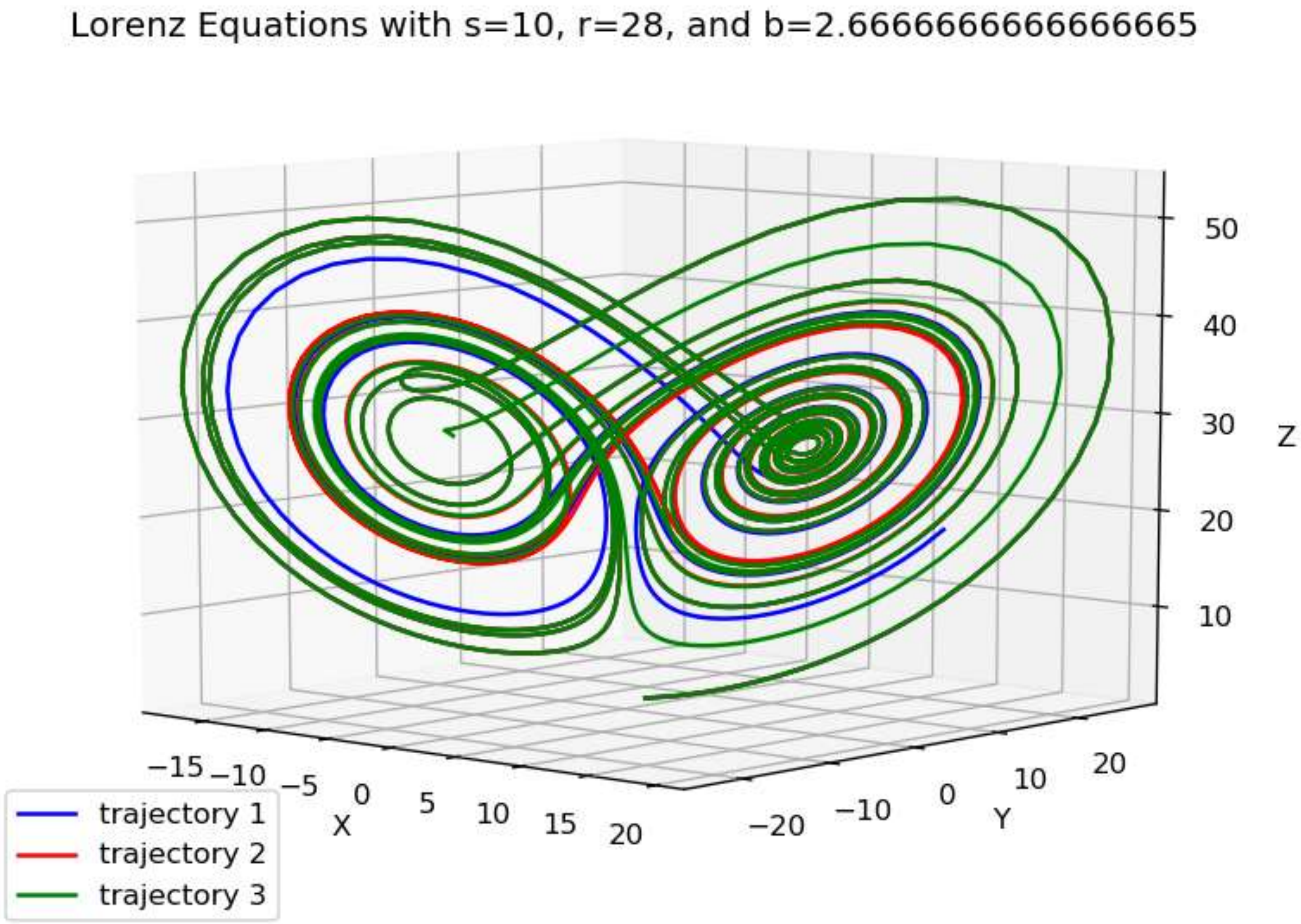
- X represents the rate of convective motion
- Y represents the temperature differences between rising and falling air currents
- Z represents the deviation from a linear vertical temperature gradient
- S is the Prandtl number, the ratio between fluid viscosity and thermal conductivity
- R is the temperature difference between the top and bottom of the system
- B is the ratio between the dimensions of the model's box

This model was one of the first known to exhibit chaotic behavior.

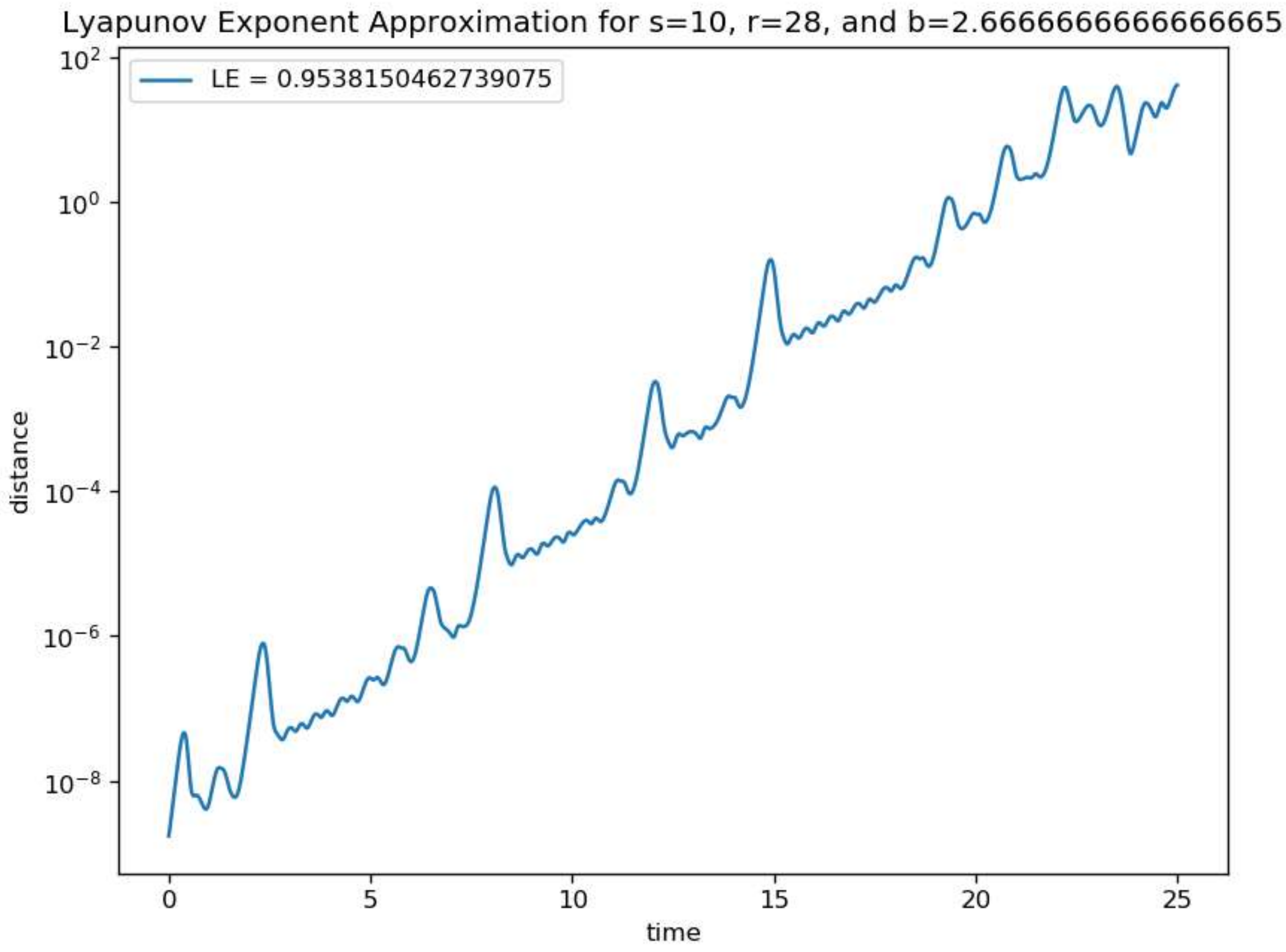
The Lorenz System

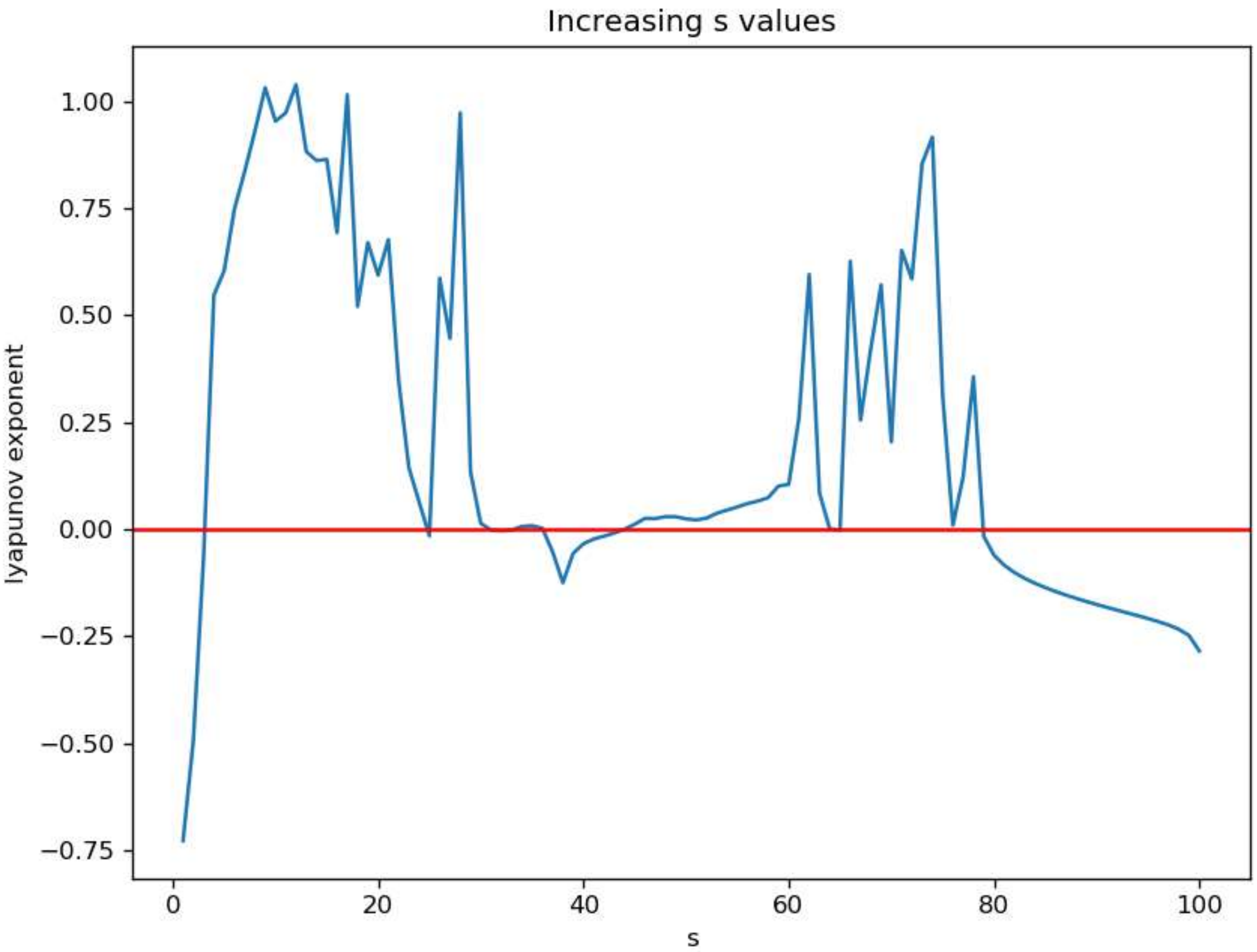


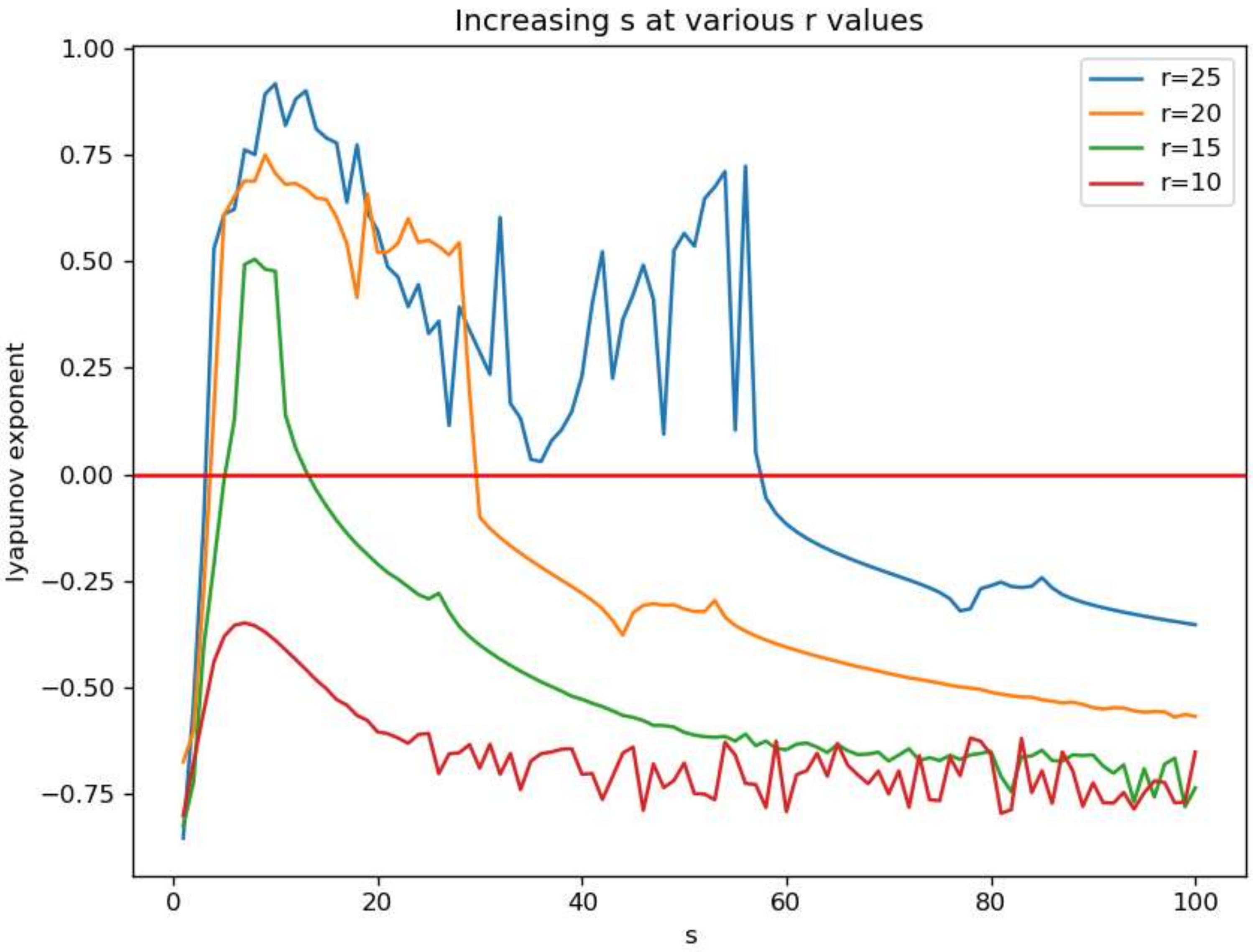
Lorenz System with Perturbed Initial Conditions

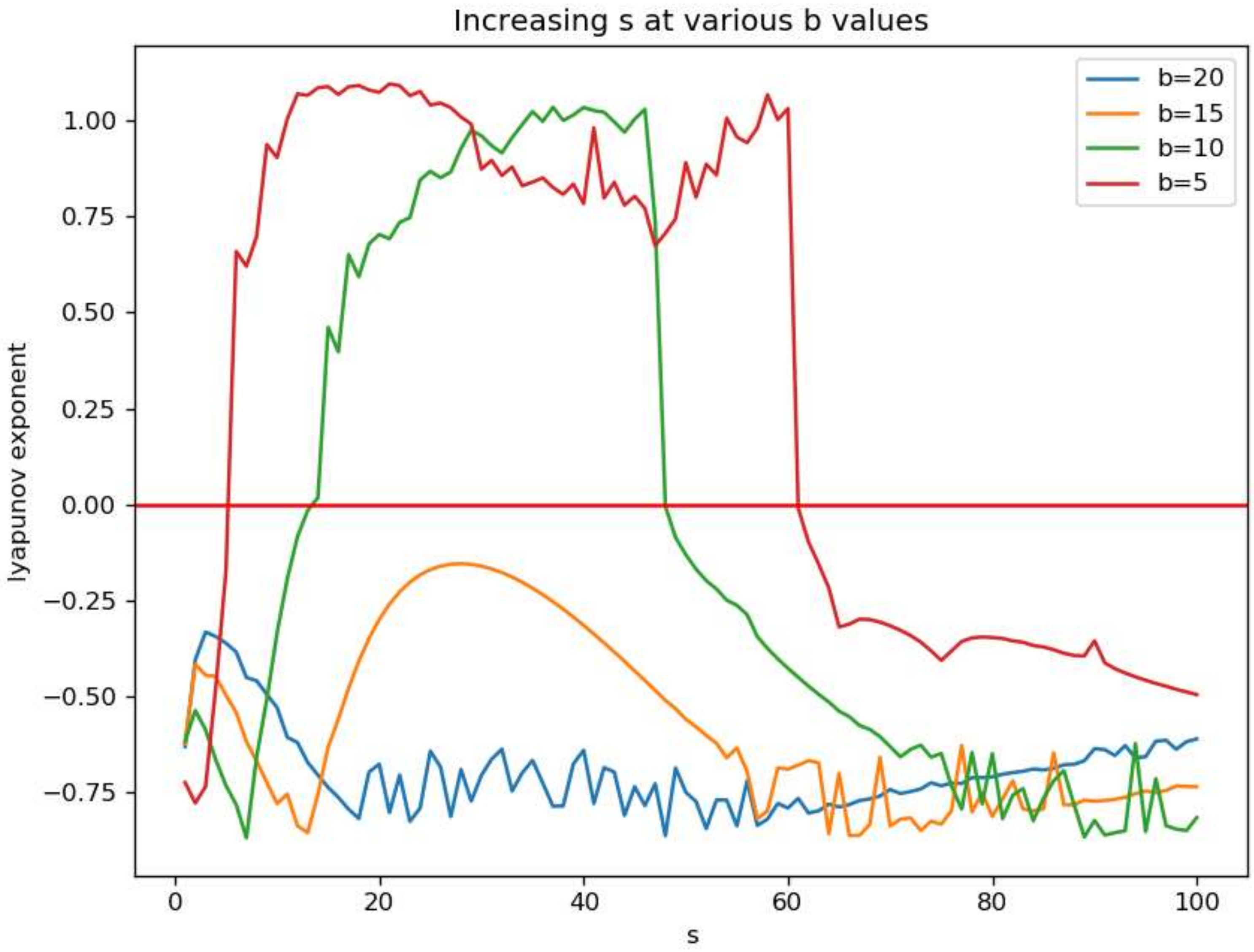


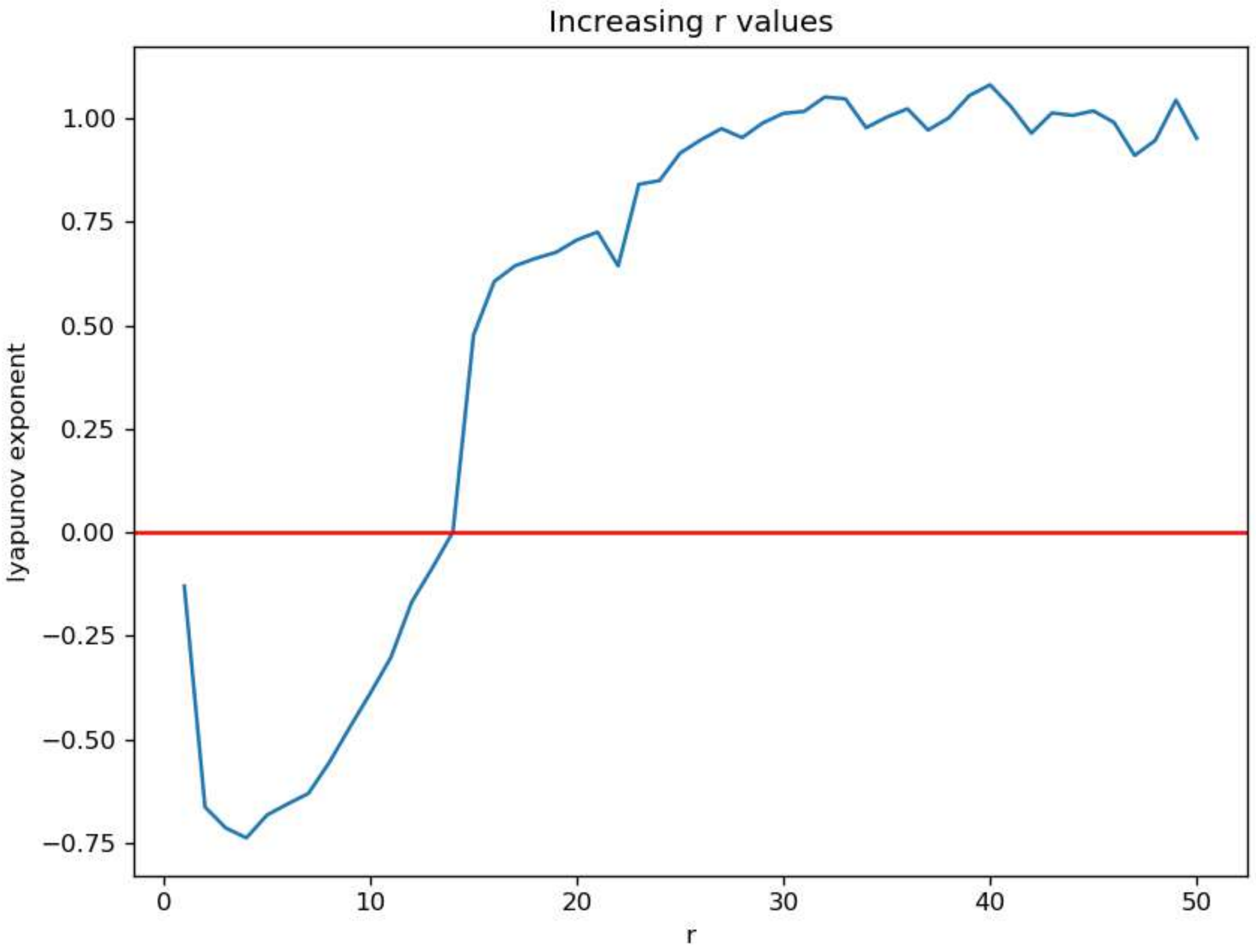
Lyapunov Exponent Approximation

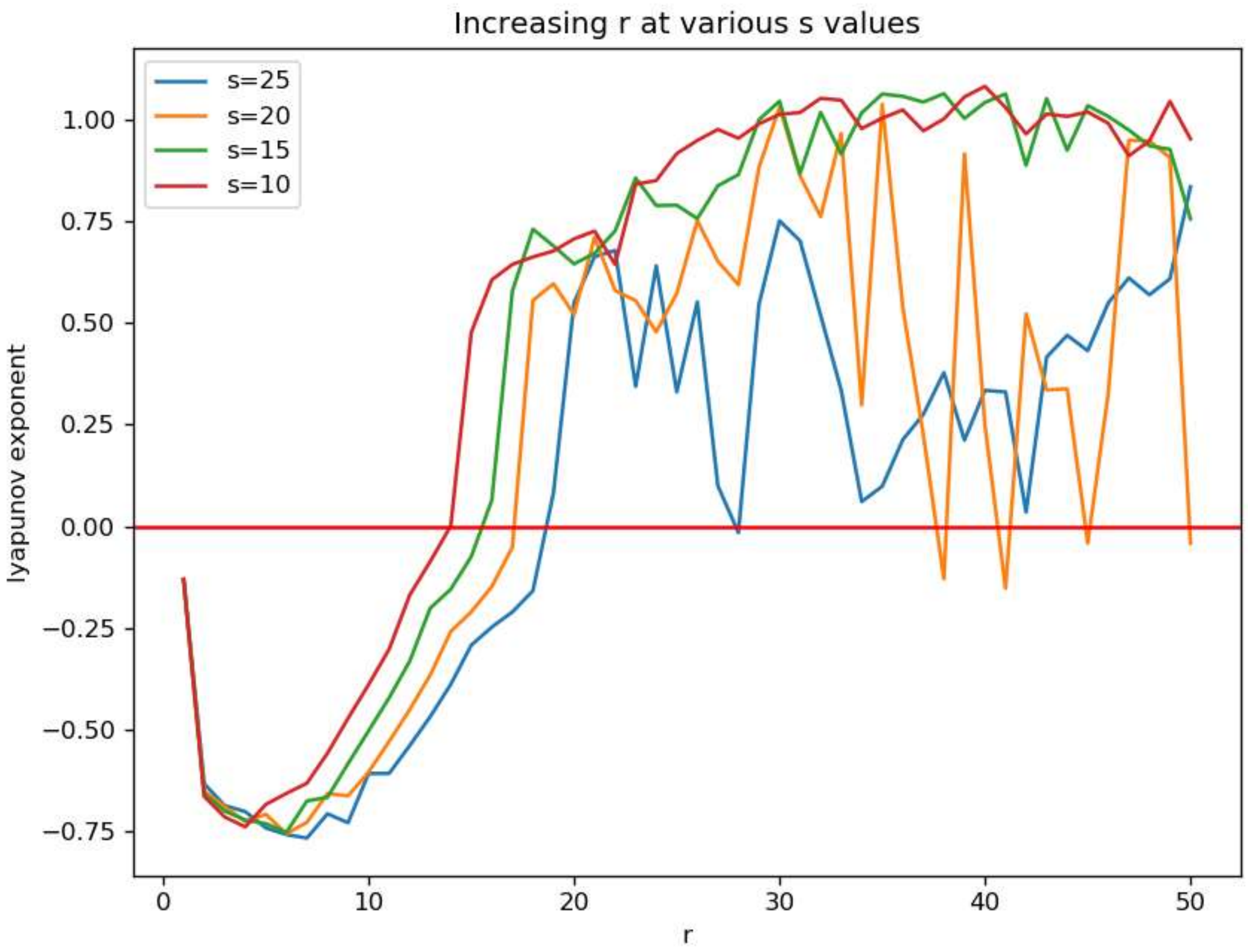


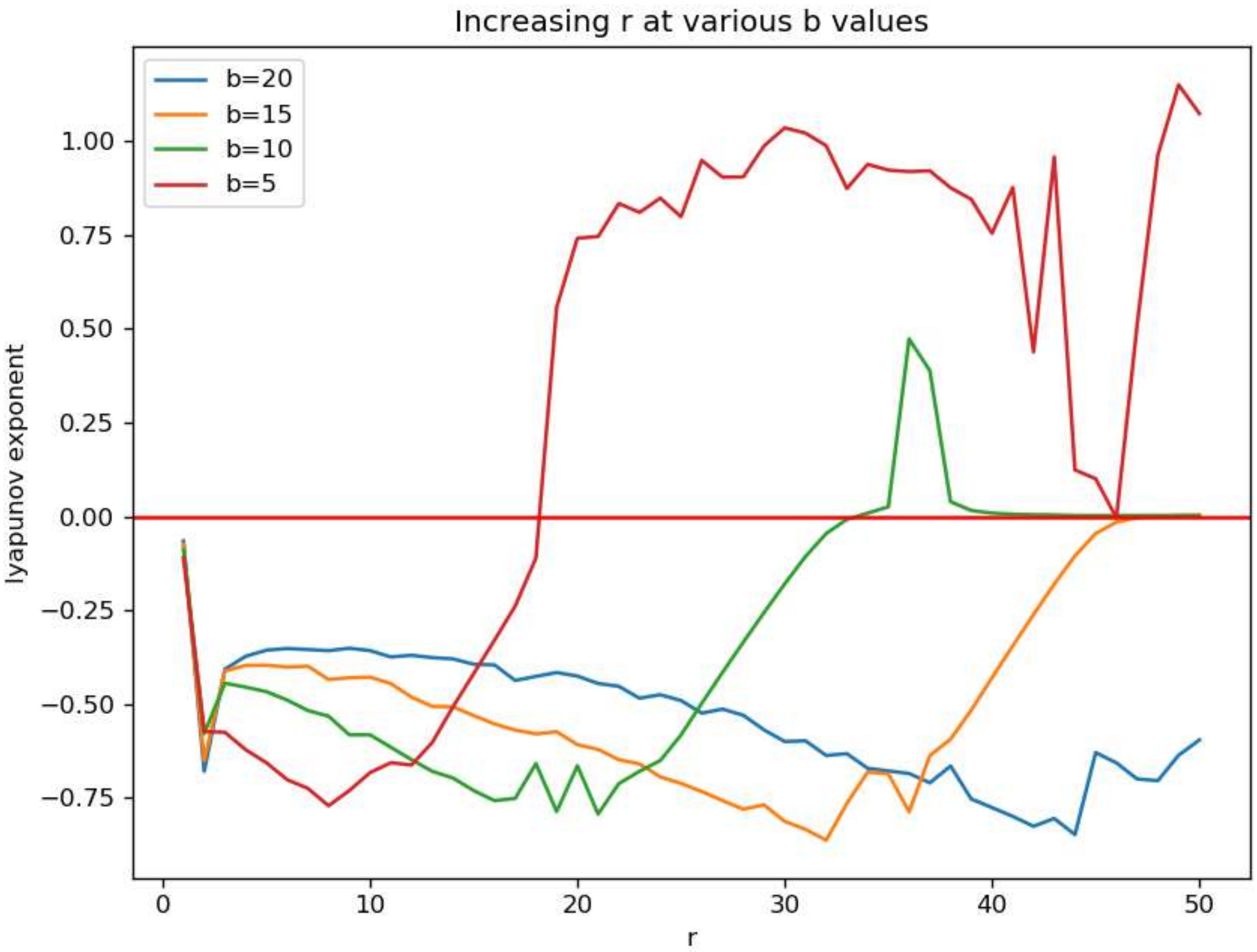

Increasing s values
lyapunov exponent
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0
20
40
60
80
100
s


Increasing s at various r values
r=25
r=20
r=15
r=10
lyapunov exponent
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0
20
40
60
80
100
s

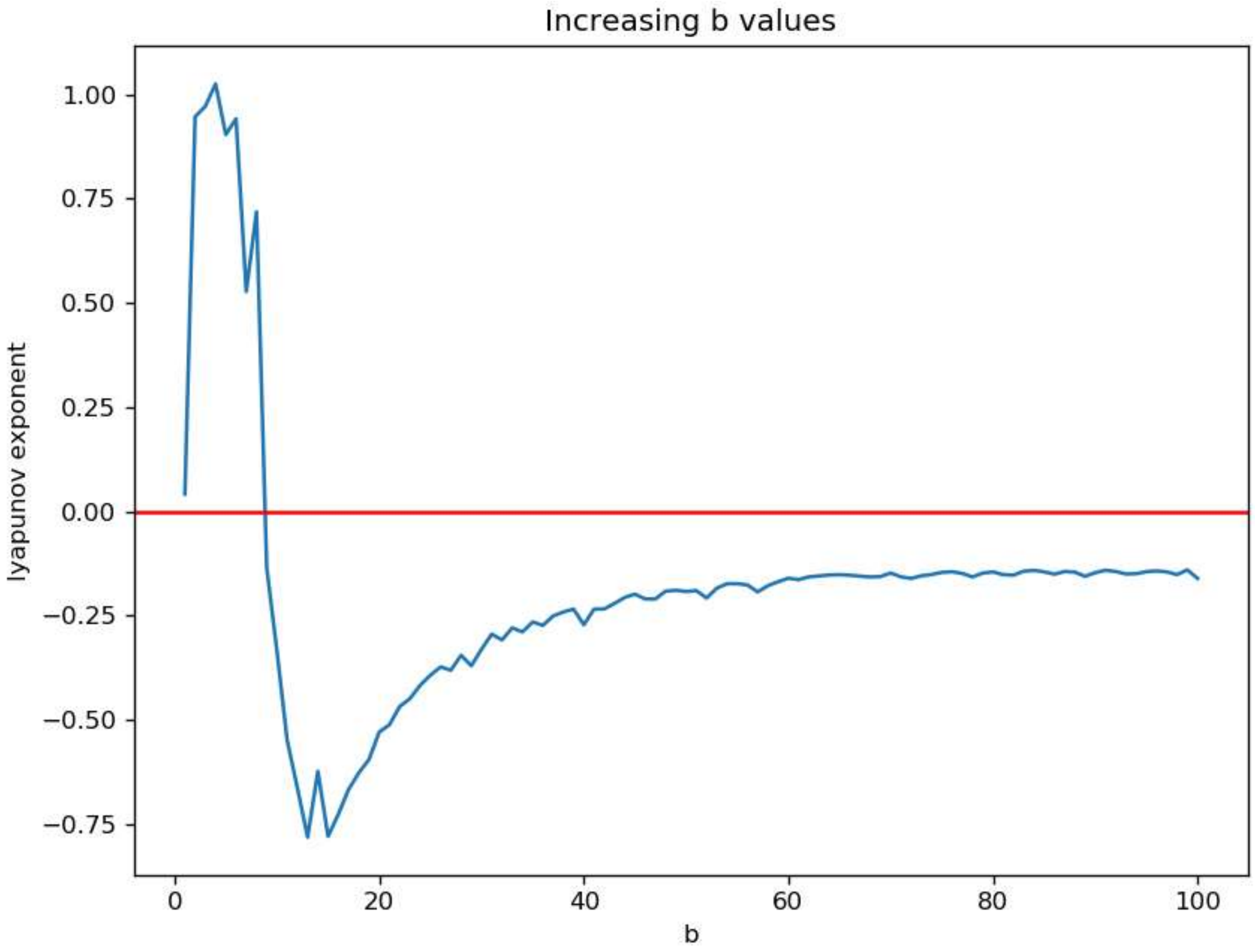

Increasing s at various b values
b=20
b=15
b=10
b=5
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
lyapunov exponent
0
20
40
60
80
100
s

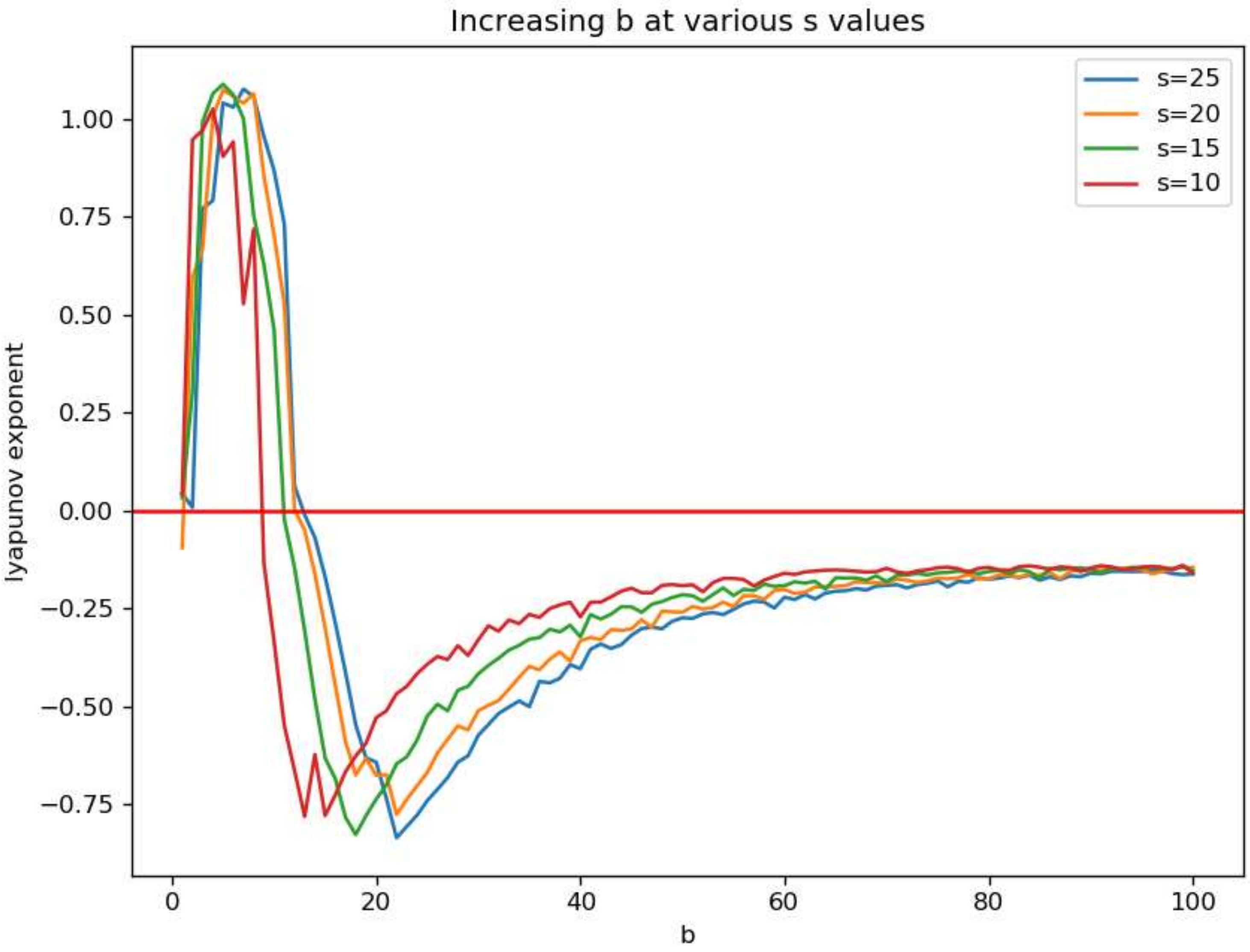

Increasing r values
lyapunov exponent
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0
10
20
30
40
50
r

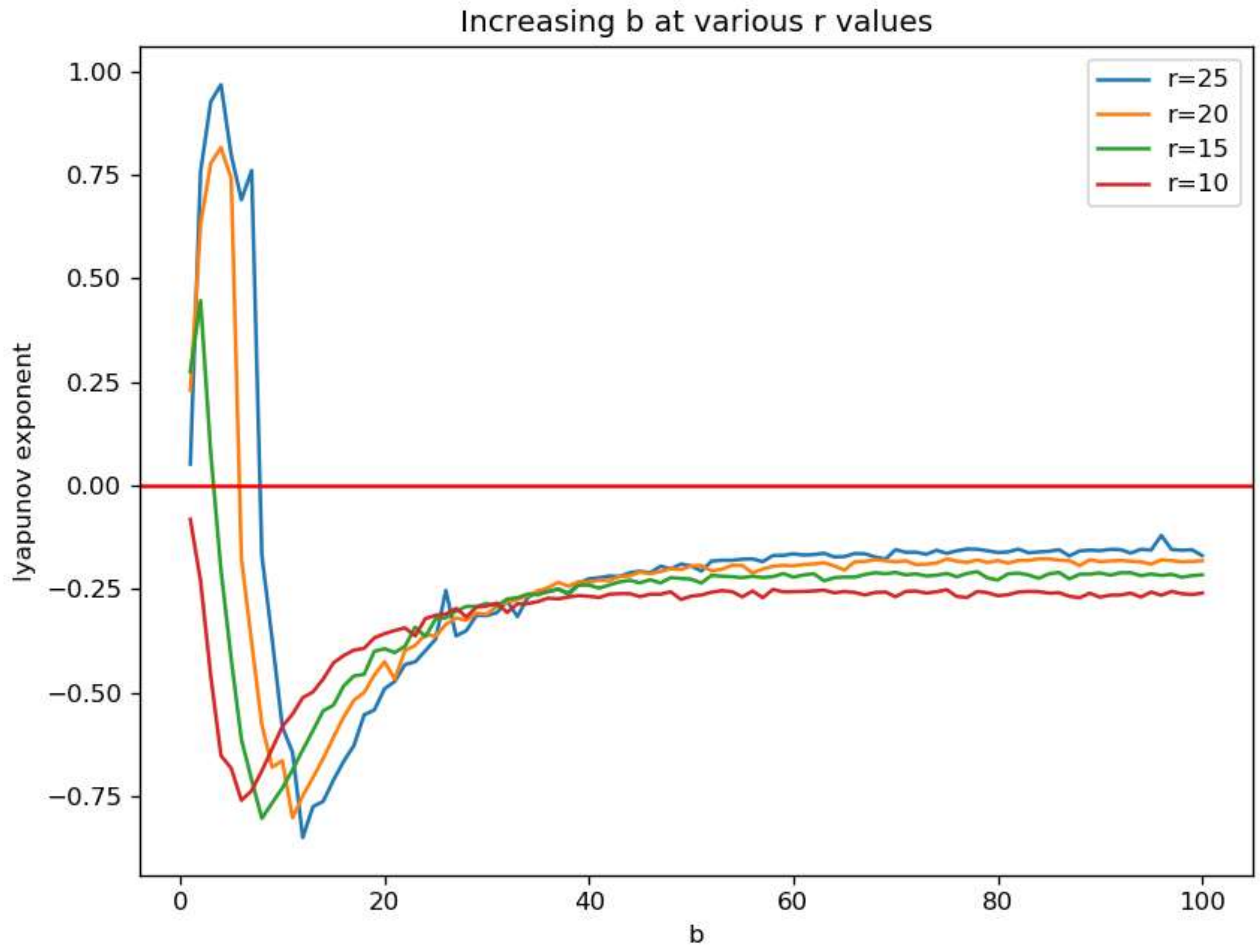

Increasing r at various s values
s=25
s=20
s=15
s=10
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
lyapunov exponent
0
10
20
30
40
50
r



Increasing r at various b values
b=20
b=15
b=10
b=5
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
lyapunov exponent
0
10
20
30
40
50
r


Increasing b values
lyapunov exponent
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0
20
40
60
80
100
b

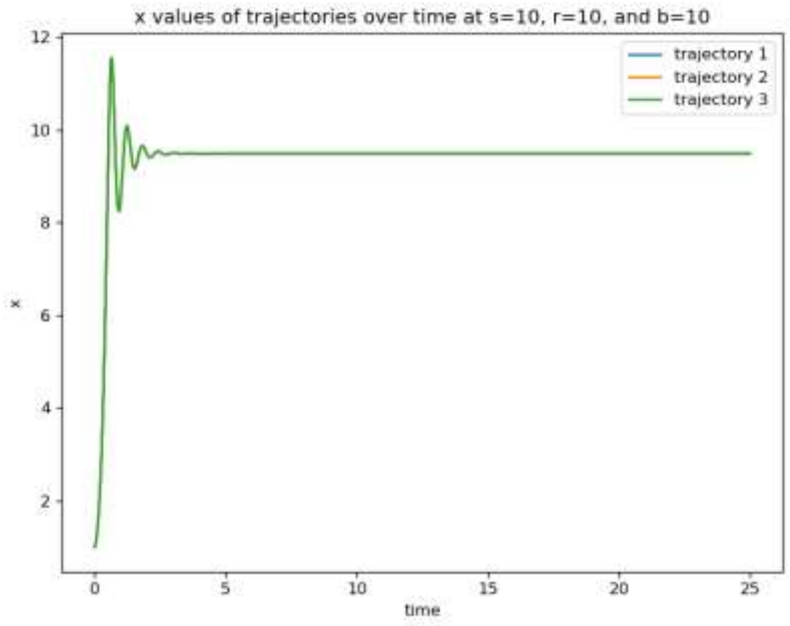

Increasing b at various s values
s=25
s=20
s=15
s=10
lyapunov exponent
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0
20
40
60
80
100
b

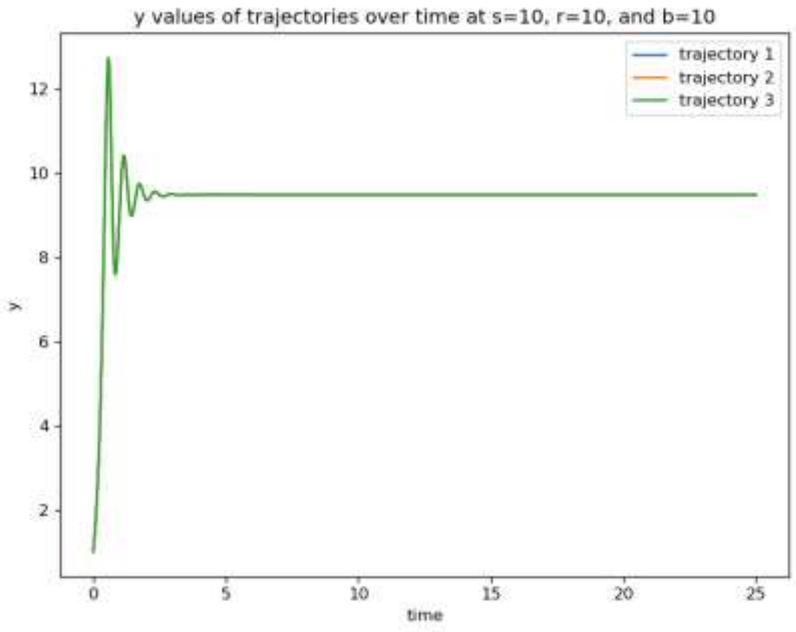

Increasing b at various r values
r=25
r=20
r=15
r=10
lyapunov exponent
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0
20
40
60
80
100
b

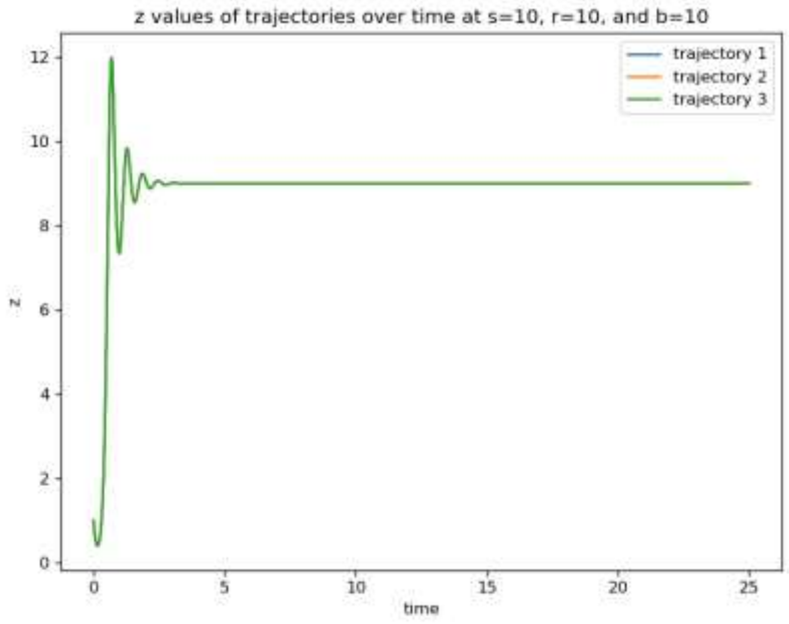
Observations

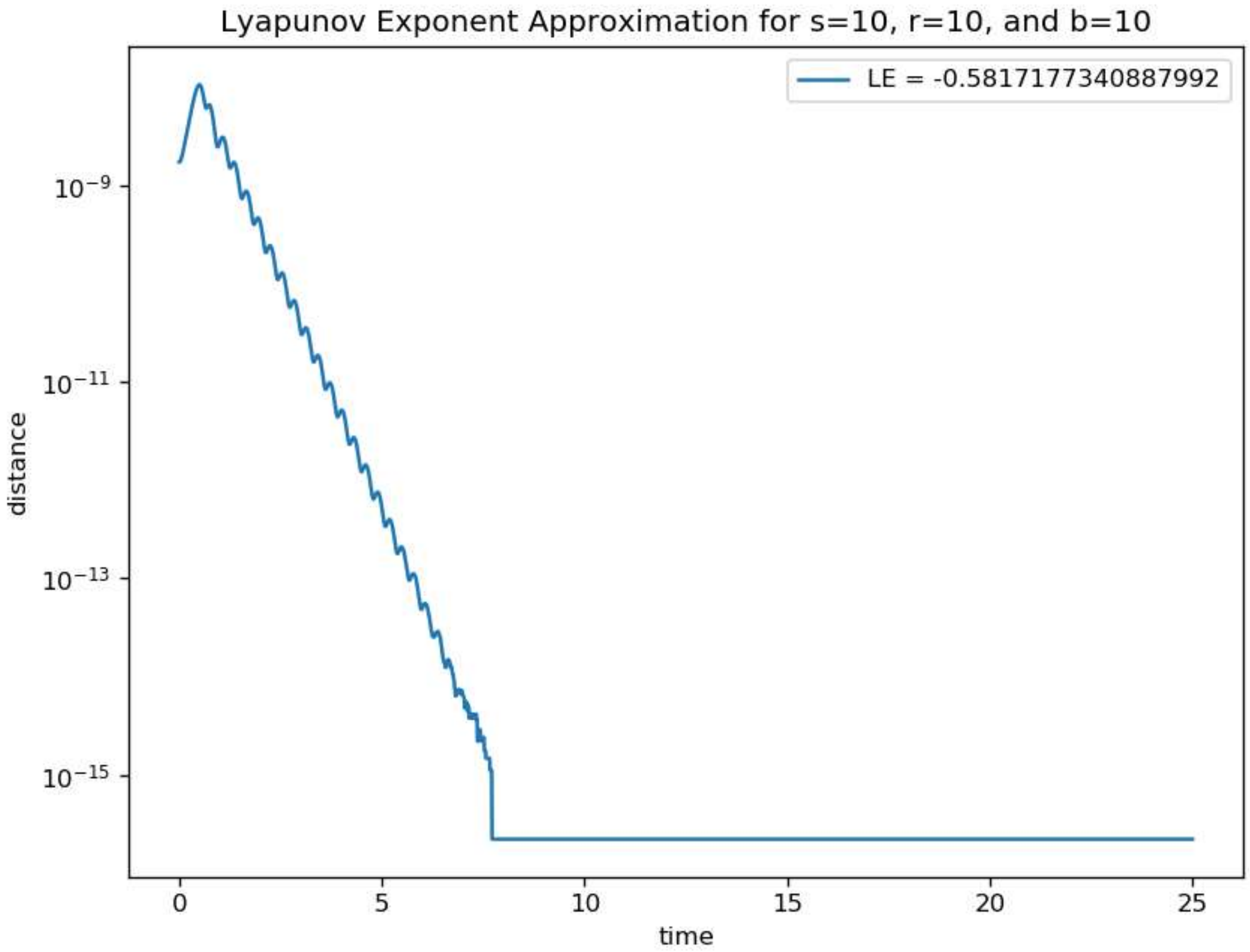
The system will tend to be not chaotic at relatively small values of r and large values of b. The value of s does not seem to significantly affect the presence of chaos in the system if these conditions are met. In terms of weather prediction this indicates that the temperature differences between the top and bottom of the system, as well as the size of the model, are the main factors contributing to chaos in the system.

Picking Values of s=r=b=10

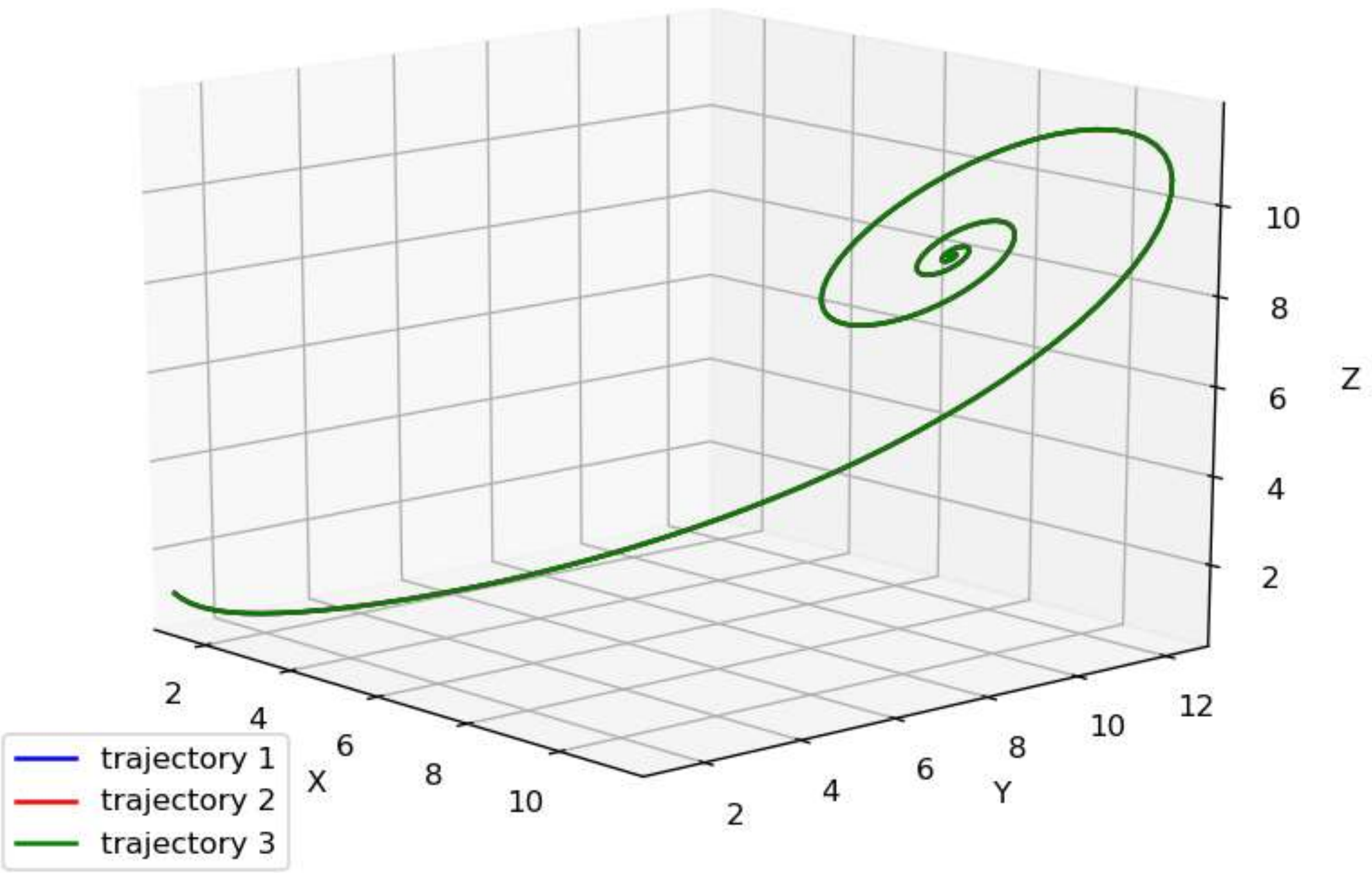







Lyapunov Exponent Approximation for s=10, r=10, and b=10
LE = -0.5817177340887992
distance
10^{-9}
10^{-11}
10^{-13}
10^{-15}
0
5
10
15
20
25
time

Lorenz Equations with s=10, r=10, and b=10



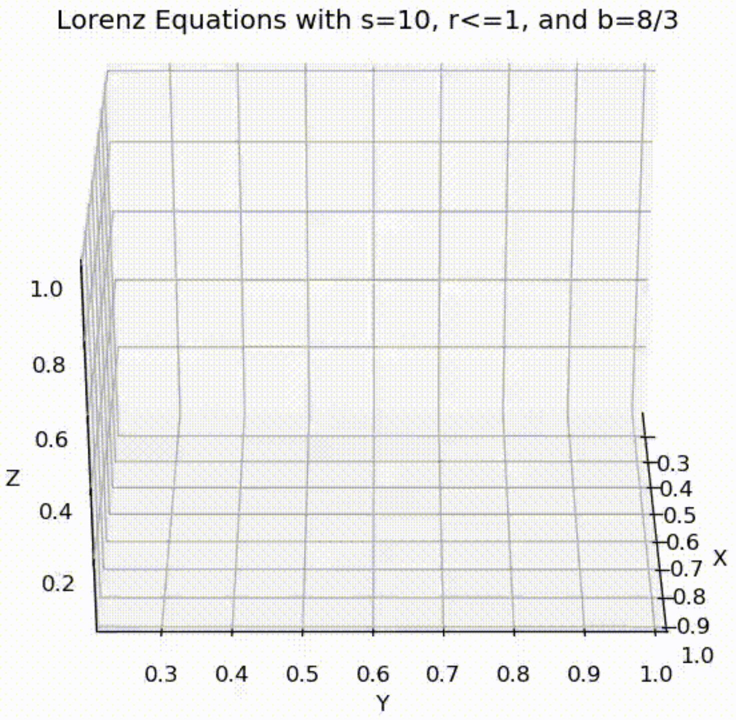
Value Changed	Equilibrium Point 1	Equilibrium Point 2	Equilibrium Point 3
S=1	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
S=2	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
S=3	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
S=4	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
S=5	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
R=1	N/A	(0, 0, 0)	N/A
R=2	(-1.63299316185545, -1.63299316185545, 1.0)	(0, 0, 0)	(1.63299316185545, 1.63299316185545, 1.0)
R=3	(-2.3094010767585, -2.3094010767585, 2.0)	(0, 0, 0)	(2.3094010767585, 2.3094010767585, 2.0)
R=4	(-2.82842712474619, -2.82842712474619, 3.0)	(0, 0, 0)	(2.82842712474619, 2.82842712474619, 3.0)
R=5	(-3.2659863237109, -3.2659863237109, 4.0)	(0, 0, 0)	(3.2659863237109, 3.2659863237109, 4.0)
B=1	(-3*sqrt(3), -3*sqrt(3), 27)	(0, 0, 0)	(3*sqrt(3), 3*sqrt(3), 27)
B=2	(-3*sqrt(6), -3*sqrt(6), 27)	(0, 0, 0)	(3*sqrt(6), 3*sqrt(6), 27)
B=3	(-9, -9, 27)	(0, 0, 0)	(9, 9, 27)
B=4	(-6*sqrt(3), -6*sqrt(3), 27)	(0, 0, 0)	(6*sqrt(3), 6*sqrt(3), 27)
B=5	(-3*sqrt(15), -3*sqrt(15), 27)	(0, 0, 0)	(3*sqrt(15), 3*sqrt(15), 27)

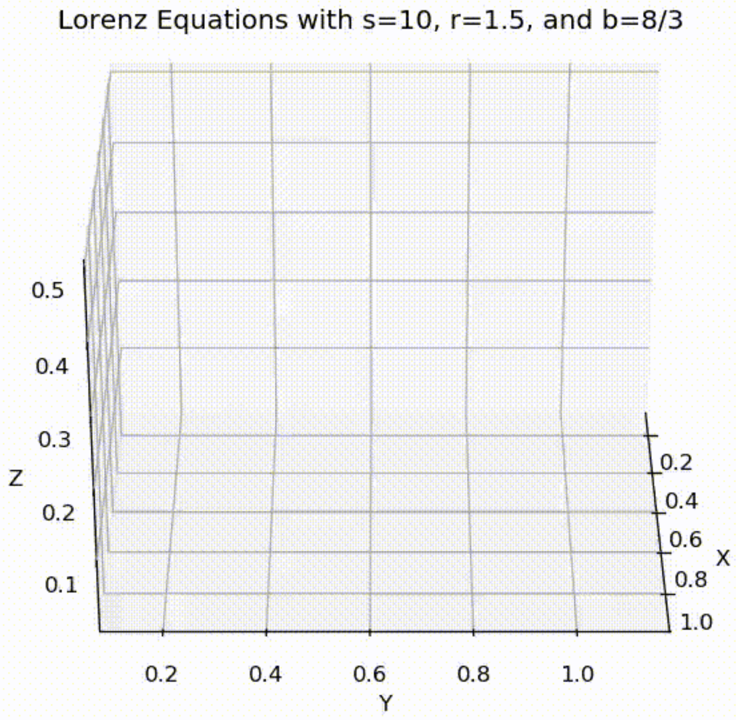
Value Changed	Eigenvalue 1	Eigenvalue 2	Eigenvalue 3
S=1	-2. +0.j -1.33333333+8.37987006j -1.33333333-8.37987006j	4.29150262 -6.29150262 -2.66666667	-2. +0.j -1.33333333+8.37987006j -1.33333333-8.37987006j
S=2	-3.93516512+0.j -0.86575077+8.51097709j -0.86575077-8.51097709j	-9. 6. -2.66666667	-3.93516512+0.j -0.86575077+8.51097709j -0.86575077-8.51097709j
S=3	-5.62460018+0.j -0.52103324+8.74837034j -0.52103324-8.74837034j	-11.21954446 7.21954446 -2.66666667	-5.62460018+0.j -0.52103324+8.74837034j -0.52103324-8.74837034j
S=4	-7.0898007 +0.j -0.28843299+9.00889977j -0.28843299-9.00889977j	-13.18877916 8.18877916 -2.66666667	-7.0898007 +0.j -0.28843299+9.00889977j -0.28843299-9.00889977j
S=5	-8.39750712+0.j -0.13457977+9.25859704j -0.13457977-9.25859704j	-15. 9. -2.66666667	-8.39750712+0.j -0.13457977+9.25859704j -0.13457977-9.25859704j
B=1	-12.43601383+0.j 0.21800691+6.58595073j 0.21800691-6.58595073j	-22.82772345 11.82772345 -1.	-12.43601383+0.j 0.21800691+6.58595073j 0.21800691-6.58595073j
B=2	-13.36145276+0.j 0.18072638+8.98870803j 0.18072638-8.98870803j	-22.82772345 11.82772345 -2.	-13.36145276+0.j 0.18072638+8.98870803j 0.18072638-8.98870803j
B=3	-14.07688398 +0.j 0.03844199+10.72757238j 0.03844199-10.72757238j	-22.82772345 11.82772345 -3.	-14.07688398 +0.j 0.03844199+10.72757238j 0.03844199-10.72757238j
B=4	-14.6732966 +0.j -0.1633517+12.13175586j -0.1633517-12.13175586j	-22.82772345 11.82772345 -4.	-14.6732966 +0.j -0.1633517+12.13175586j -0.1633517-12.13175586j
B=5	-15.19200976 +0.j -0.40399512+13.32523142j -0.40399512-13.32523142j	-22.82772345 11.82772345 -5.	-15.19200976 +0.j -0.40399512+13.32523142j -0.40399512-13.32523142j

Equilibrium Analysis at r = 0.5, 1, and 1.5

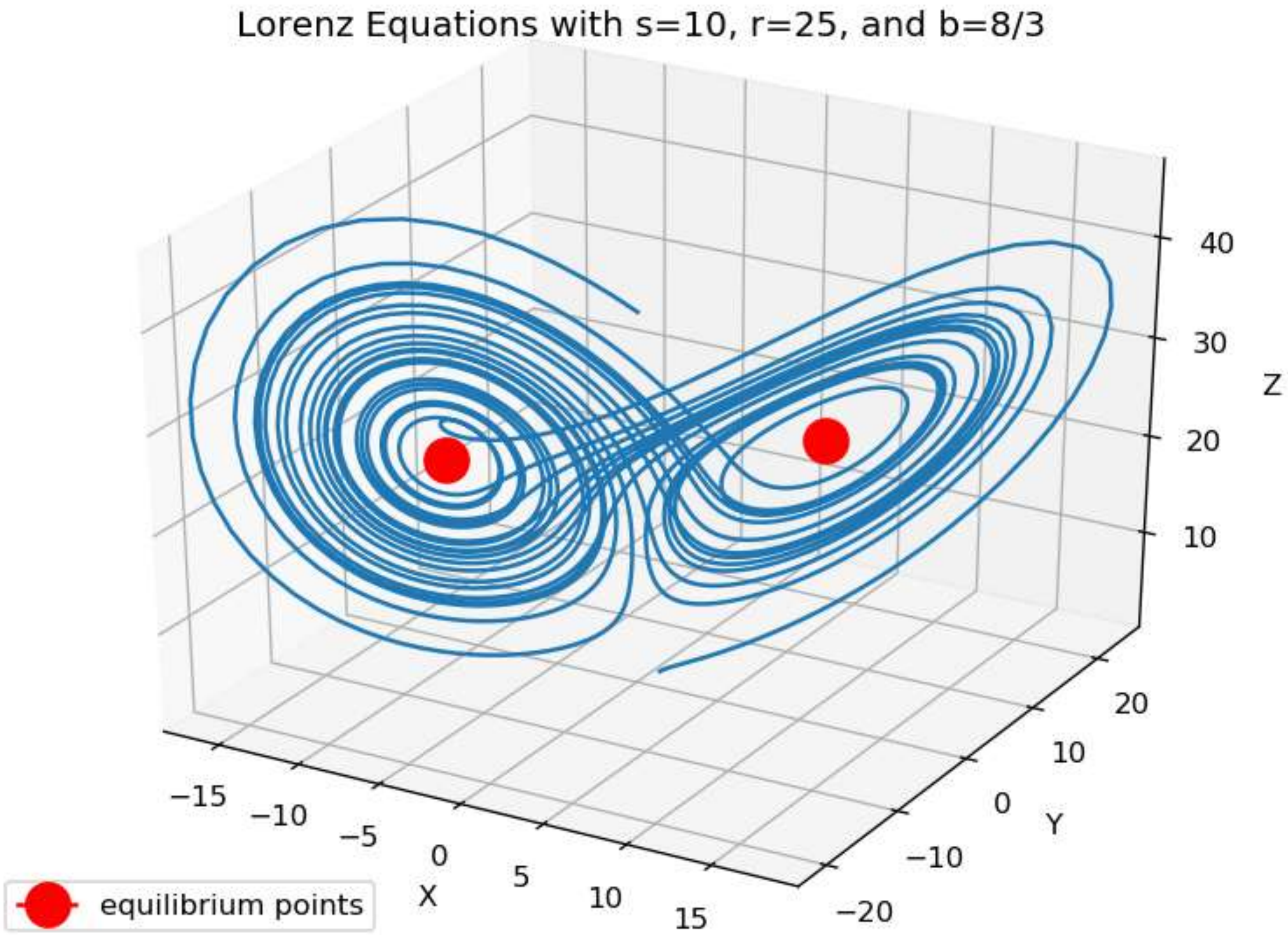
Value Changed	Equilibrium Point 1	Equilibrium Point 2	Equilibrium Point 3	Eigenvalue 1	Eigenvalue 2	Eigenvalue 3
R=0.5	(-1.15470053837925*I, -1.15470053837925*I, -0.5)	(0, 0, 0)	(1.15470053837925*I, 1.15470053837925*I, -0.5)	N/A	[-10.52493781, -0.47506219, -2.66666667]	N/A
R=1	N/A	(0, 0, 0)	N/A	N/A	[-11., 0., -2.66666667]	N/A
R=1.5	(-1.15470053837925, -1.15470053837925, 0.5)	(0, 0, 0)	(1.15470053837925, 1.15470053837925, 0.5)	[-11.1257249 +0.j, -1.27047088+0.88473233j, -1.27047088-0.88473233j]	[-11.43717104, 0.43717104, -2.66666667]	[-11.1257249 +0.j, -1.27047088+0.88473233j, -1.27047088-0.88473233j]

Animations of Lorenz System at r<=1 and r=1.5



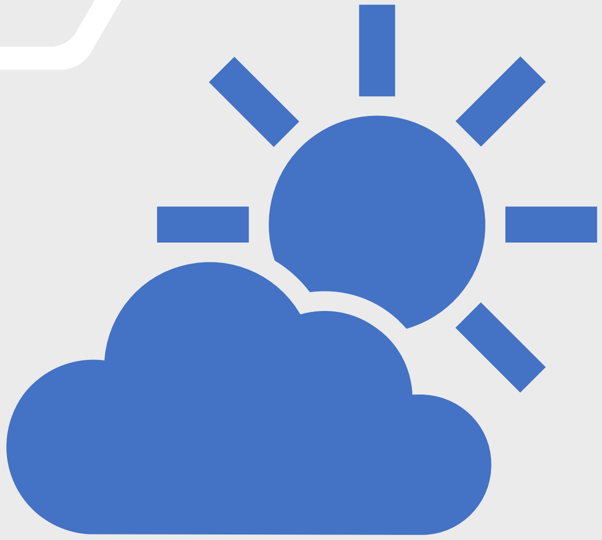


Location of Stable Equilibrium Points



Conclusion

With enough of a difference in the horizontal temperatures or dimensions, the model will experience the onset of chaos which makes accurate long-term predictions impossible. However, at smaller temperature differences we can derive an accurate prediction of atmospheric behavior due to predictable equilibrium point behavior and the lack of chaos.



References

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Thank You