The Lorenz Equations and Atmospheric Convection

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Introduction

Atmospheric convection plays a major role in the behavior of Earth's weather systems. The forces created by atmospheric convection result in the motion of Earth's atmosphere and the creation of weather. Meteorologists often attempt to understand the patterns of atmospheric convection in order to make predictions on future weather conditions. However, long-term weather predictions are often prone to inaccuracies and variability. By analyzing the Lorenz model of atmospheric convection, the main variables responsible for the inaccuracies in long-term cases can be determined. In addition, the conditions under which accurate long-terms predictions are possible can also be determined.

The Lorenz Equations

The Lorenz equations were first derived by M.I.T. mathematician and meteorologist Edward Lorenz in 1963 while attempting to model atmospheric convection. This set of nonlinear ordinary differential equations was formed by a simplification on a more complex 12-variable system of equations that was computationally intensive to solve. The Lorenz equations consists of the variables x, y, and z, and the constants s, r, and b with the condition that s, r, and b are greater than 0.

$$\frac{dx}{dt} = s(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

In terms of atmospheric convection, the variable x represents the rate of convective motion, the variable y represents the temperature differences between rising and falling air currents, and the variable z represents the deviation from a linear vertical temperature gradient. The constant s is the Prandtl number, which represents the ratio between fluid viscosity and thermal conductivity. The constant r is the Rayleigh number, which represents the temperature difference between the uniform layers of warm air at the bottom of the system and cool air at the top. The constant b represents the dimensions of the model given as a ratio of width to height. The Lorenz model of thermal atmospheric convection is one of the first known models to exhibit chaos, dramatic shifts in behavior with even the slightest change in initial conditions.

The Lyapunov Exponent

A method of determining the presence of chaos in a system is through its Lyapunov exponent. The value of the Lyapunov exponent λ for a given system indicates that the distance between trajectories with small perturbations in initial conditions is growing increasingly larger if $\lambda > 0$ and not increasing if $\lambda < 0$. To numerically determine an approximation of the Lyapunov exponent for a three-dimensional nonlinear continuous time system we can calculate the "average logarithmic rate of separation" (Sprott). Taking the Lorenz system at s = 10, r = 28, and b = 8/3 and initial conditions of x = 1, y = 1, and z = 1 with a perturbation of $\pm$ 1e-9 we can obtain multiple trajectories of the system (Figure 1). By taking the average of the distances between the trajectories at each point we obtain the change in distance over time. Plotting the natural logarithm of this change in distance over time creates a graph displaying the system's tendency to stretch with a slight shift in initial conditions (Figure 2). A linear regression of this plot can then be calculated to obtain the approximation of the Lyapunov exponent for the system given by the slope of the regression line. By increasing the number of trajectories when calculating average distance, we can improve the accuracy of this approximation; however, a close approximation can be obtained with only three trajectories. The Lyapunov exponent

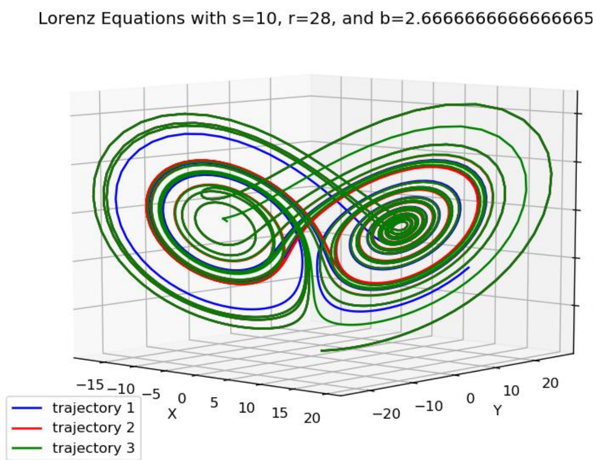

Figure 1: Multiple Trajectories

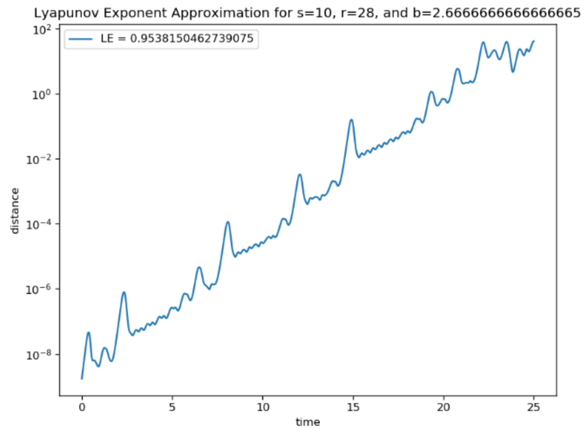

Figure 2: Approximating Lyapunov exponent

approximations can now be determined for varying values of s, r, and b to derive a plot displaying the pattern of chaos in the system in respect to that variable.

The Prandtl Number and Chaos

The Prandtl number consists of the system's fluid viscosity and thermal conductivity. Increases in both the atmospheric viscosity and thermal conductivity can largely be attributed to increases in the temperature of the system. As the value of s increases, the fluid viscosity of the system must be growing at a much more rapid rate than its thermal conductivity. This increase in s can be due to an increase in temperature or pressure of the system (Lasance). To determine the effect that varying s values have on the chaos in this system we can obtain the Lyapunov exponents at each increment of s. It is given that s must always be greater than zero in this model, so we consider the first 100 values of s > 0. A plot of the Lyapunov exponents over s at r = 28 and b = 8/3 shows that there is a region of s values between s = 0 and s = 80 where the system is mostly chaotic with arbitrary flipping between chaotic and non-chaotic at some points (Figure 3). To determine how this behavior changes with r and b, we plot additional lines for increasing s at various r and b values. These additional plots reveal that the values of s at which

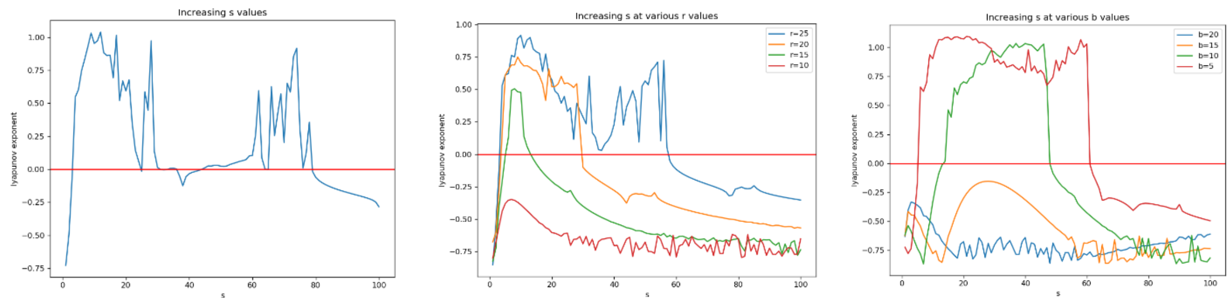

Figure 3: Lyapunov exponents of increasing s at various r and b values

there exists chaos in the system will decrease as the value of r decreases but increase as the value of b decreases. This shift in the pattern of chaos suggests that the values r and b play a large role in its presence in the system.

The Rayleigh Number and Chaos

The Rayleigh number r in this system is the difference in temperature between the layer of air warming the system from the bottom and the layer of air cooling the system from the top. It is assumed that these layers are uniformly distributed along the top and bottom boarders of the system (Danforth). The rising, warm layer of air is created by the heating of the ground by the sun while the falling cool layer is due to the loss of heat at higher altitudes. As the Rayleigh number increases, we expect a larger difference between the top and bottom layer of the system. Realistically, there is a limit to the difference between the two temperatures in a natural setting. It can be assumed that the rising warm air will rarely exceed 55 Celsius in most areas on Earth, thus we will consider the first 50 values of $r > 0$. Plotting the Lyapunov exponents at each increment of r we see the onset of chaos in the system at around $r = 14$ at which the system remains chaotic for all proceeding values of r (Figure 4). To compare this pattern in relation to s and b, we create additional plots displaying the Lyapunov exponents of increasing r at various s and b values. From these plots we notice that increases in s shifts the onset of chaos to the right and causes flips between chaotic and non-chaotic at arbitrary values of r after the onset of chaos

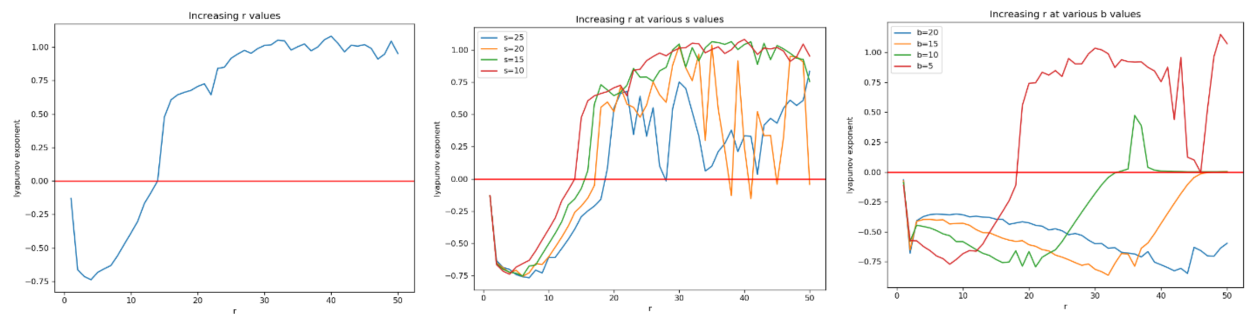

Figure 4: Lyapunov exponents of increasing r at various s and b values

while keeping the overall pattern the same. Increases in b, however, clearly reduces the values of r at which the system is chaotic, indicating that b is a large factor in the presence of chaos in the system.

Dimensions and Chaos

The dimensions of the system, given by b, determine the area in which the convective motion is contained. The top and bottom bounds of the box defined by b consists of the temperature layers that constitute r while the walls of the box determine the range of convective motion that is possible. Increases in b result in a wider model of the system. Since b must be greater than zero, we consider the first 100 values of b when calculating the Lyapunov exponents. A plot of these exponent values over increasing b shows a chaotic system at small values of b until b = 8 where the system shifts to being non-chaotic for all further increases in the dimensions of the system (Figure 5). Plotting these increasing b values in respect to various s and r values yields a shift to the right with increases in s and a shift to the left with decreases in r but minimal change in the overall pattern of chaos in the system.

Main Factors of Chaos

An analysis of the plots created by tracking Lyapunov exponents over varying values of s, r, and b reveal large shifts in the pattern of chaos when changing the difference in temperature

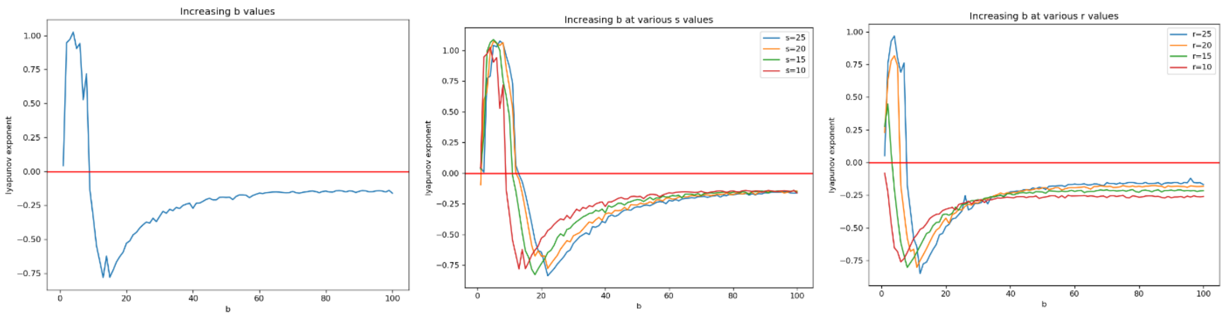

Figure 5: Lyapunov exponents of increasing b at various s and r values

between the top and bottom layers of the system and the dimensions of the system. We observe that at relatively small temperature differences between the horizontal boundaries of the system and relatively large physical spaces, the system will not exhibit any chaotic behavior. Inversely, at large temperature differences and smaller physical spaces the system will have a high tendency to exhibit chaotic behavior. Changes in the ratio of fluid viscosity to thermal conductivity only speed-up or delay the behavior visible on the graph but has minimal change on the overall pattern of chaos. Intuitively, the loss of chaotic behavior at smaller r values can be attributed to the lack of convection when the rising warm air and falling cool air are roughly the same temperature. In terms of weather prediction, a lack of convection results in the loss of much of the motion in the atmosphere, allowing for accurate long-term predictions of weather if these conditions persist.

Equilibrium Points

Understanding where the solutions to the Lorenz model of atmospheric convection will tend towards can aid in the prediction of long-term weather. By analyzing the equilibrium points of the system and their asymptotic behavior of various values of s, r, and b, we can obtain a general idea of the motion of convection over a desired period of time at those values. To obtain the equilibrium points at a given s, r, and b value we can set the system of nonlinear differential

Value Changed	Equilibrium Point 1	Equilibrium Point 2	Equilibrium Point 3
S=1	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
S=2	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
S=3	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
S=4	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
S=5	(-8.48528137423857, -8.48528137423857, 27.0)	(0, 0, 0)	(8.48528137423857, 8.48528137423857, 27.0)
R=1	N/A	(0, 0, 0)	N/A
R=2	(-1.63299316185545, -1.63299316185545, 1.0)	(0, 0, 0)	(1.63299316185545, 1.63299316185545, 1.0)
R=3	(-2.3094010767585, -2.3094010767585, 2.0)	(0, 0, 0)	(2.3094010767585, 2.3094010767585, 2.0)
R=4	(-2.82842712474619, -2.82842712474619, 3.0)	(0, 0, 0)	(2.82842712474619, 2.82842712474619, 3.0)
R=5	(-3.2659863237109, -3.2659863237109, 4.0)	(0, 0, 0)	(3.2659863237109, 3.2659863237109, 4.0)
B=1	(-3*sqrt(3), -3*sqrt(3), 27)	(0, 0, 0)	(3*sqrt(3), 3*sqrt(3), 27)
B=2	(-3*sqrt(6), -3*sqrt(6), 27)	(0, 0, 0)	(3*sqrt(6), 3*sqrt(6), 27)
B=3	(-9, -9, 27)	(0, 0, 0)	(9, 9, 27)
B=4	(-6*sqrt(3), -6*sqrt(3), 27)	(0, 0, 0)	(6*sqrt(3), 6*sqrt(3), 27)
B=5	(-3*sqrt(15), -3*sqrt(15), 27)	(0, 0, 0)	(3*sqrt(15), 3*sqrt(15), 27)

Table 1: Equilibrium points at first five values of each constant:

equations to be equal to zero and then solve the system for the x, y, and z coordinates of the equilibrium points. This can result in one or more equilibrium points of the system; however, to analyze the asymptotic behavior of these points we must derive the Jacobian matrix from the system of nonlinear equations and calculate its eigenvalues. By analyzing the dominant eigenvalue of each equilibrium point and plotting its phase space, we can determine its asymptotic behavior and make predictions on how the system will behave near these points.

Varying Constants and Their Equilibrium Points

To obtain a general idea of how equilibrium points and their behavior change as we vary the constants in the system, we solve the system of nonlinear equations at the first five values of s, r, and b. From the derived equilibrium points (Table 1) we see that the origin is an equilibrium point at all values of the constants; in addition, for every non-origin equilibrium point with positive x and y values, there is an another equilibrium point with same z and the negative of its

Value Changed	Eigenvalue 1	Eigenvalue 2	Eigenvalue 3
S=1	-2. +0.j -1.33333333+8.37987006j -1.33333333-8.37987006j	4.29150262 -6.29150262 -2.66666667	-2. +0.j -1.33333333+8.37987006j -1.33333333-8.37987006j
S=2	-3.93516512+0.j -0.86575077+8.51097709j -0.86575077-8.51097709j	-9. 6. -2.66666667	-3.93516512+0.j -0.86575077+8.51097709j -0.86575077-8.51097709j
S=3	-5.62460018+0.j -0.52103324+8.74837034j -0.52103324-8.74837034j	-11.21954446 7.21954446 -2.66666667	-5.62460018+0.j -0.52103324+8.74837034j -0.52103324-8.74837034j
S=4	-7.0898007 +0.j -0.28843299+9.00889977j -0.28843299-9.00889977j	-13.18877916 8.18877916 -2.66666667	-7.0898007 +0.j -0.28843299+9.00889977j -0.28843299-9.00889977j
S=5	-8.39750712+0.j -0.13457977+9.25859704j -0.13457977-9.25859704j	-15. 9. -2.66666667	-8.39750712+0.j -0.13457977+9.25859704j -0.13457977-9.25859704j
B=1	-12.43601383+0.j 0.21800691+6.58595073j 0.21800691-6.58595073j	-22.82772345 11.82772345 -1.	-12.43601383+0.j 0.21800691+6.58595073j 0.21800691-6.58595073j
B=2	-13.36145276+0.j 0.18072638+8.98870803j 0.18072638-8.98870803j	-22.82772345 11.82772345 -2.	-13.36145276+0.j 0.18072638+8.98870803j 0.18072638-8.98870803j
B=3	-14.07688398 +0.j 0.03844199+10.72757238j 0.03844199-10.72757238j	-22.82772345 11.82772345 -3.	-14.07688398 +0.j 0.03844199+10.72757238j 0.03844199-10.72757238j
B=4	-14.6732966 +0.j -0.1633517+12.13175586j -0.1633517-12.13175586j	-22.82772345 11.82772345 -4.	-14.6732966 +0.j -0.1633517+12.13175586j -0.1633517-12.13175586j
B=5	-15.19200976 +0.j -0.40399512+13.32523142j -0.40399512-13.32523142j	-22.82772345 11.82772345 -5.	-15.19200976 +0.j -0.40399512+13.32523142j -0.40399512-13.32523142j

Table 2: Eigenvalues for s and b values

x and y values. This indicates that the equilibrium points are reflections over y = -x in the plane parallel to its z value, allowing us to derive all the equilibrium points of the system by obtaining only one of them.

The Prandtl Number and Equilibrium Behavior

Looking at the equilibrium points for various s values we see that for each incremental increase in the ratio of fluid viscosity to thermal conductivity, there is no change in the equilibrium points of the system. When calculating the eigenvalues of the three equilibrium points for each s value we see that the origin always contains one positive real eigenvalue and two negative real eigenvalues, making it an unstable saddle point for all s values (Table 2). The two non-origin equilibrium points consists of one real negative eigenvalue and two complex conjugates with negative real parts, indicating that these points are stable spiral focuses for all s

Value Changed	Equilibrium Point 1	Equilibrium Point 2	Equilibrium Point 3	Eigenvalue 1	Eigenvalue 2	Eigenvalue 3
R=0.5	(-1.15470053837925*I, -1.15470053837925*I, -0.5)	(0, 0, 0)	(1.15470053837925*I, 1.15470053837925*I, -0.5)	N/A	[-10.52493781, -0.47506219, -2.66666667]	N/A
R=1	N/A	(0, 0, 0)	N/A	N/A	[-11., 0., -2.66666667]	N/A
R=1.5	(-1.15470053837925, -1.15470053837925, 0.5)	(0, 0, 0)	(1.15470053837925, 1.15470053837925, 0.5)	[-11.1257249 +0.j, -1.27047088+0.88473233j, -1.27047088-0.88473233j]	[-11.43717104, 0.43717104, -2.66666667]	[-11.1257249 +0.j, -1.27047088+0.88473233j, -1.27047088-0.88473233j]

Table 3: Analysis of points around r = 1

values. Thus, it can be assumed that the behavior of the system near equilibrium points does not change greatly for any increase in s.

System Dimensions and Equilibrium Behavior

Changes in the dimensions of the system result in a constant rate of change for the values of x and y in the non-origin equilibrium points. The value of these two equilibrium points is given by the expression $(\pm 3\sqrt{3b}\ , \pm 3\sqrt{3b}, 27)$ where x and y must be the same sign. Calculating the eigenvalues of these equilibrium points at various b values, we see that similar to s, the origin always contains one positive real eigenvalue and two negative real eigenvalues, making it an unstable saddle point for all b values. However, for the two non-origin equilibrium points we notice a shift from an unstable saddle focus to a stable focus for all values of b after b = 3, given by the change in sign in the real parts of the non-dominant eigenvalues. This suggests that two different equilibrium point behaviors for the non-origin points can be expected depending on how large we set the dimensions of the system.

The Rayleigh Number and Critical Point

From increases in the value of r, we see a shift from a single equilibrium point to three equilibrium points, two of which steadily increase in their x, y, and z values as r increases. The shift from a single equilibrium point at the origin to three real equilibrium points from r = 1 to r

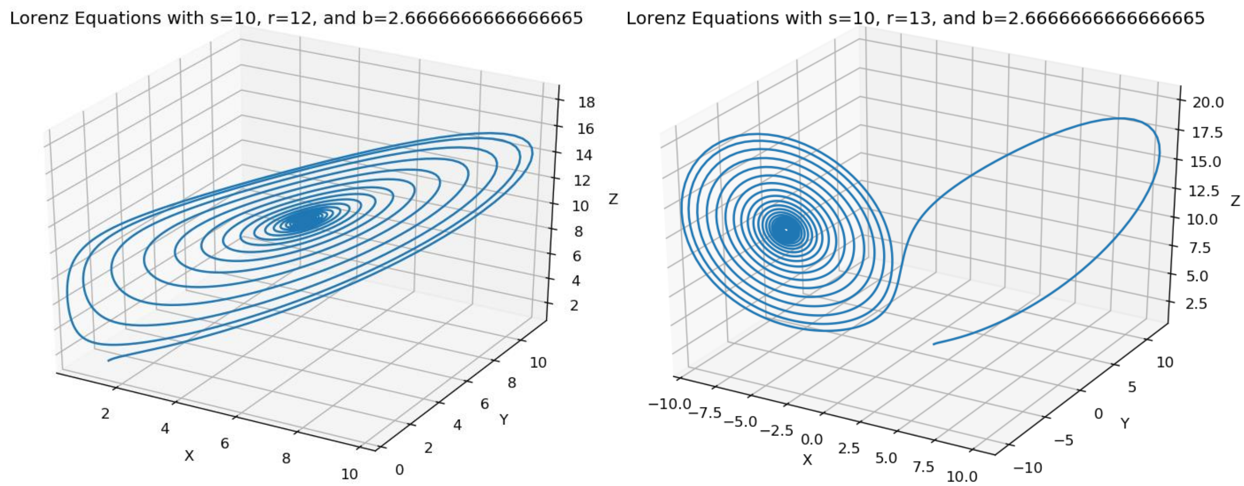

Figure 6: Loss of spiral focus after r = 12

= 2 suggests that there may exist a critical point nearby. Locating and analyzing this critical point can provide insight to the behavior of convective currents at low temperature differences between the horizontal boundaries which can aid in the prediction of weather. Deriving equilibrium points at r = 0.5 and r = 1.5 we see that there is a shift from two imaginary equilibrium points and one real point at the origin at r = 0.5 to only one equilibrium point at r = 1. There is also a shift from one equilibrium point at r = 1 to three real equilibrium points at r = 1.5, indicating that r = 1 is a critical point. Calculating their respective eigenvalues (Table 3), we see that the equilibrium point at r = 1 and the real point at r = 0.5 have negative, real eigenvalues for the non-zero eigenvalues which indicates that they are stable points (Izhikevich). The eigenvalues of the two non-origin points at r = 1.5 consists of a real negative dominant eigenvalue and two complex conjugates with negative real parts which indicates that these points are a stable spiral focus. At the origin for r = 1.5 we notice that while all eigenvalues are real, there exists one positive and two negative values, thus the origin becomes an unstable saddle point after r = 1. This shift from a stable equilibrium point into two stable and one unstable point indicates that there is a supercritical pitchfork bifurcation at r = 1 (Sayama). A plot of these r

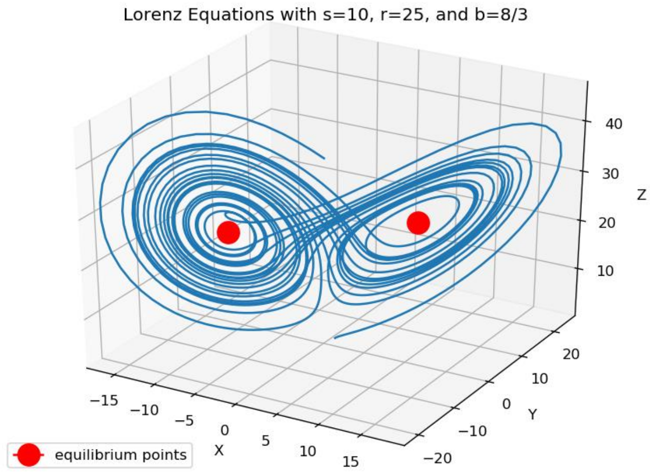

Figure 7: Equilibrium points lie in the strange attractors

values show that a point will converge to the origin for all r values less than or equal to 1, and spiral towards one of the non-origin points for r greater than 1.

Equilibrium Points and Strange Attractors

The spiral focus behavior at the non-origin equilibrium points seen for values of r greater than 1 stay present in the system until around r = 12, after which we see the first jump to the region in which the opposite equilibrium point lies (Figure 6). This value of r was earlier seen to be start of chaotic behavior in the system. For all proceeding r values we see that the solution no longer converges to a fixed point or diverges to infinity, instead it non-periodically circles around the two non-origin equilibrium points with random jumps between the two regions in phase space. This arbitrary and chaotic behavior makes it extremely difficult to make an accurate prediction of where the solution will lie at a given time. However, when plotting the location of the equilibrium points in the phase space of that system, we see that they lie in the set of the strange attractors of that system (Figure 7). This allows us to derive a general idea of the area where atmospheric convection in a system will tend towards, but no information about the specific regions or structure of the system.

Conclusion

The Lorenz model of atmospheric convection suggests that accurate long-term weather predictions are possible when the system does not exhibit chaotic behavior which is true for all values of s, values of r under 13, and large enough values of b. Given that only one variable can change at a time and all other conditions remain the same, we can use the knowledge of asymptotic equilibrium behavior to derive where the solution of the system will tend as time increases. However, conditions in real world weather systems rarely remain constant; as a result, we can expect multiple variables in the Lorenz system of differential equations to be changing at a time. When the increases in the values of horizontal temperature difference and decreases in the dimensions of the system being considered are dramatic enough, we see the onset of chaos in the system and can no longer make accurate long-term weather predictions based on this model.

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Appendix

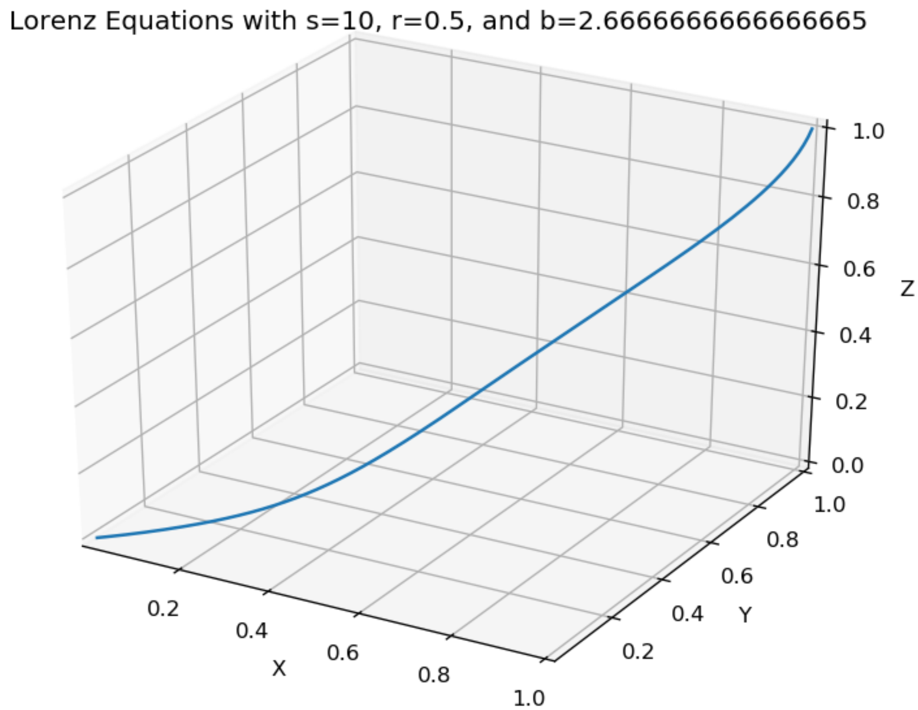

Figure A1: Plot of system at r = 0.5

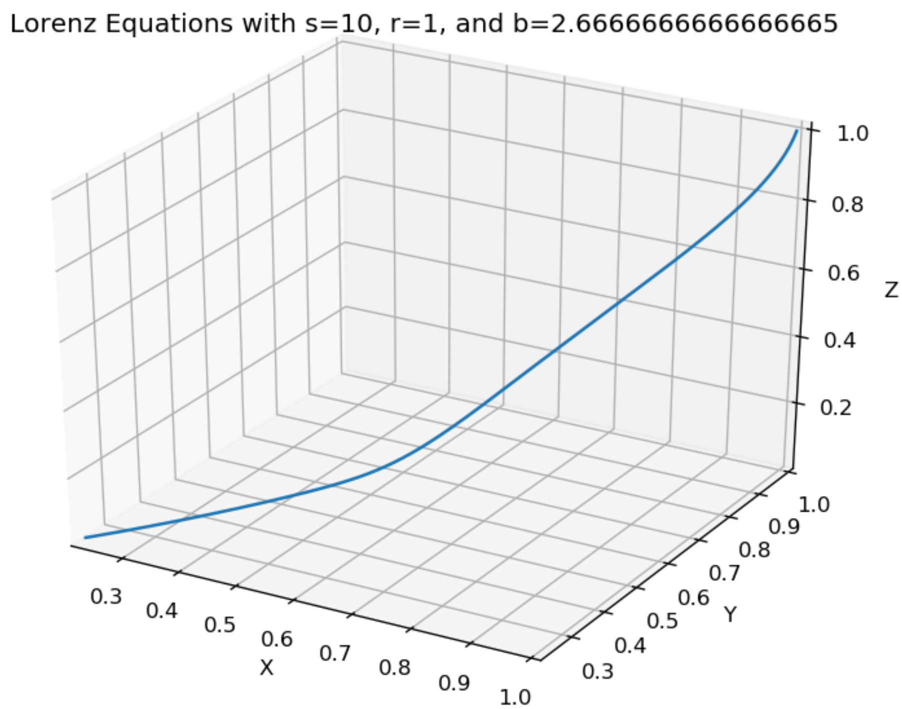

Figure A2: Plot of system at r = 1

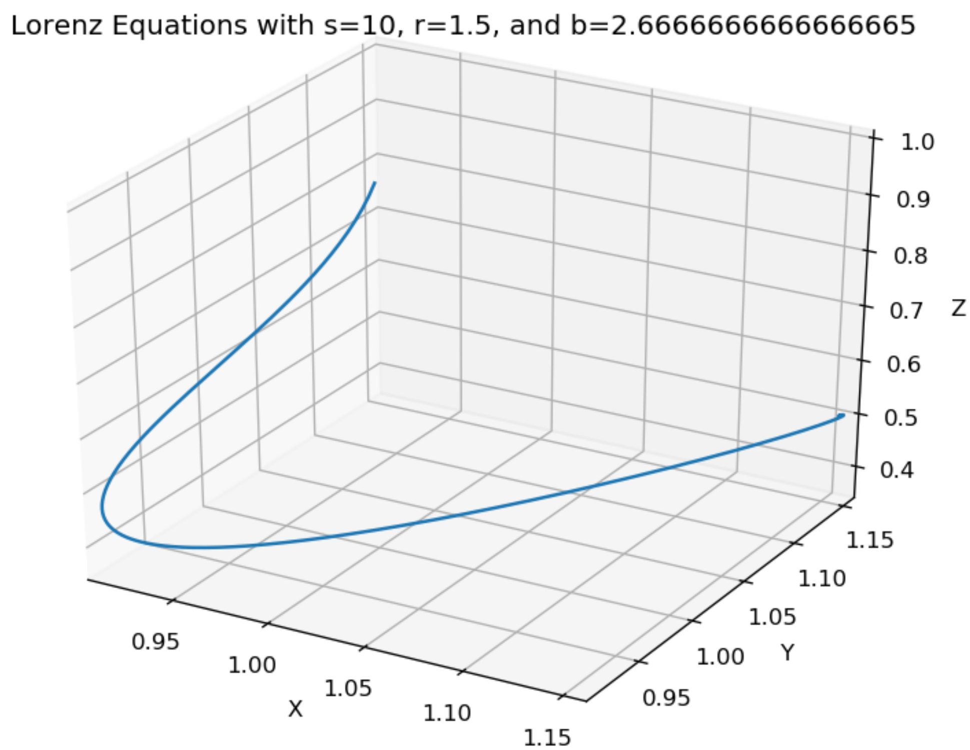

Figure A3: Plot of system at r = 1.5

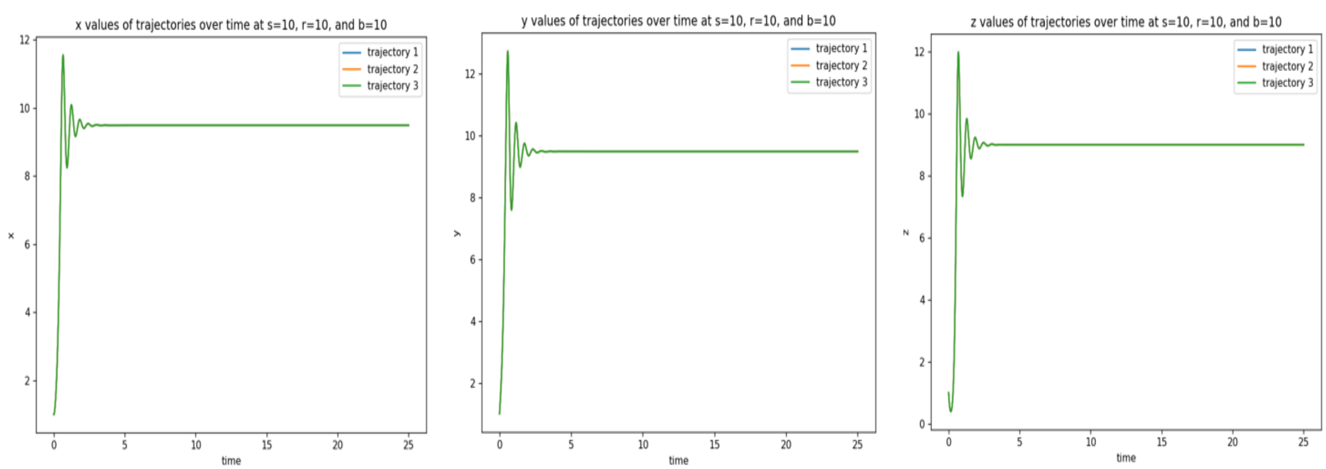

Figure A4: Testing hypothesis of no chaos at small r and large b values

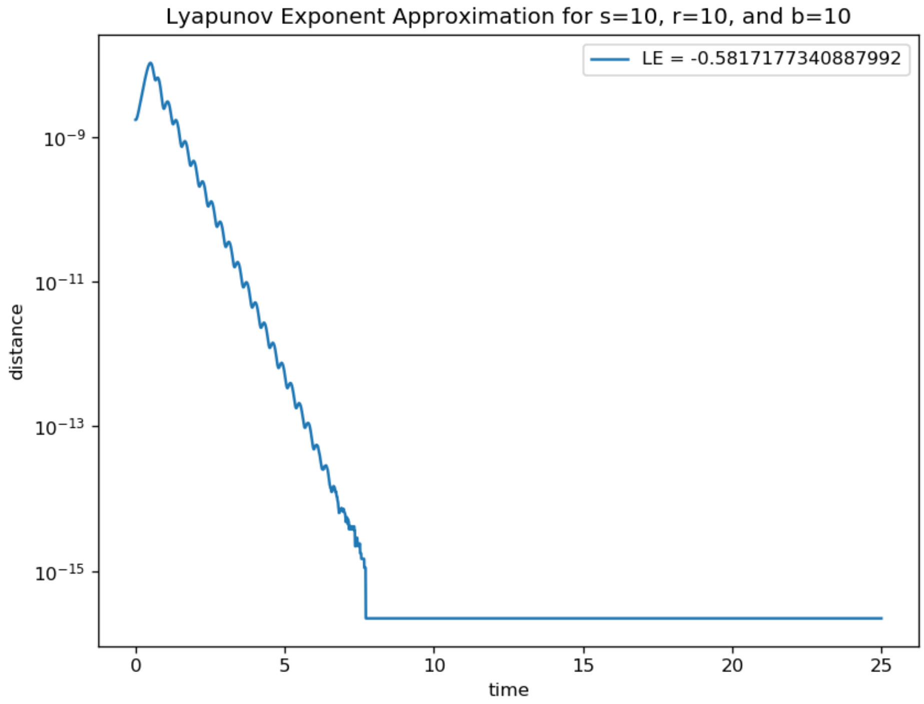

Figure A5: Lyapunov Approximation of hypothesis test

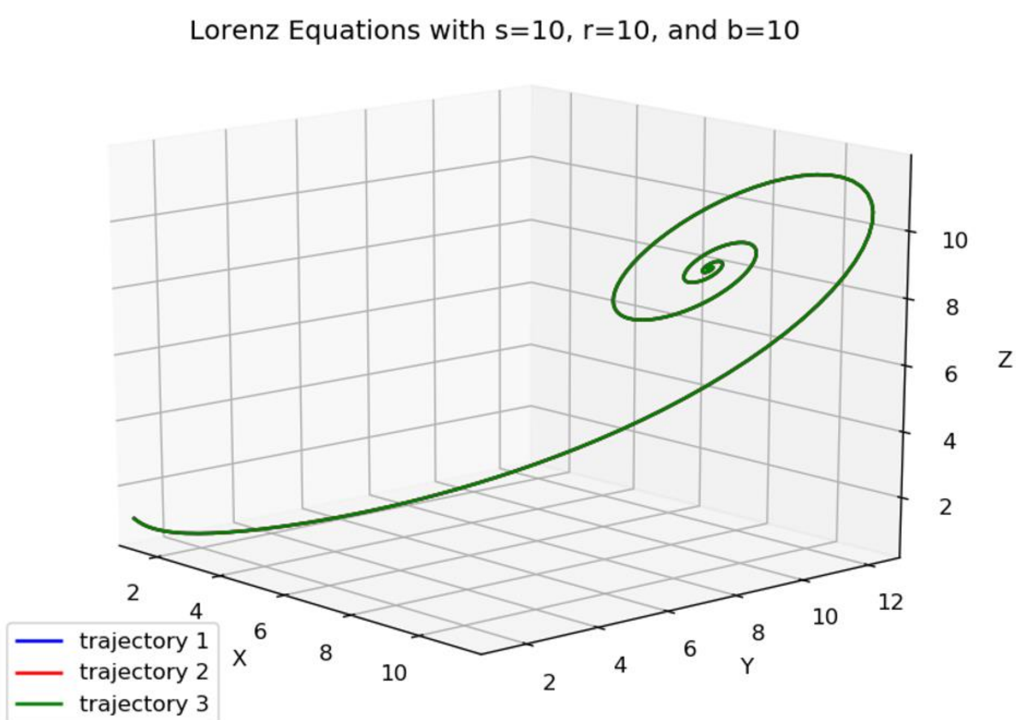

Figure A6: Predicted spiral focus behavior present in phase space