PDE NOTES

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Contents

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1. THEORY OF DISTRIBUTIONS

Taqsimotlar nazariyasi (yohud umumlashgan funksiyalar nazariyasi)

1.1. Space of test functions. Test funksiyalar fazolari

Ushbu bo'limda biz ixtiyoriy $\Omega \subseteq \mathbb{R}^d$ ochiq to'plam uchun bir nechta funksiyanal fazolarni qaraymiz. $\Omega = \mathbb{R}^d$ bo'lgan hollarda biz ayrim belgilashlarda shunchaki $\mathbb{R}^d$ tushurib ketamiz.

Ta'rif 1.1. $f : \Omega \to \mathbb{C}$ funksiyaning (supporti) *dastagi* deb

$$\{x \in \Omega : f(x) = 0\} \tag{1.1}$$

to'plamning $\mathbb{R}^d$ fazodagi yopilmasiga aytiladi va $\mathrm{supp}(f)$ bilan belgilanadi.

Agar $\mathrm{supp}(f)$ kompakt to'plam bo'lsa, f funkisya *kompakt dastakli* deb ataladi.

Istalgan $n \geq 0$ butun son uchun $\Omega \subseteq \mathbb{R}^d$ to'plamda quyidagi funksiyalar sinflarini qaraymiz.

(i) $\mathscr{E}^n(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \in C^n(\Omega)\}$ ya'ni k marta uzliksiz differensiallanuvchi funksiyalar to'plami. Agar $n = 0$ bo'lsa, $\mathscr{E}^0(\Omega)$ bilan Ω to'plamda aniqlangan uzliksiz funkiyalar to'plamini belgilab olamiz.

(ii) $\mathscr{E}(\Omega)$ bilan Ω to'plamda aniqlangan *silliq* funksiyalar sinfini belgilab olamiz, ya'ni:

$$\mathscr{E}(\Omega) = \bigcap_{n=0}^{\infty} \mathscr{E}^n(\Omega) \tag{1.2}$$

(iii) Aylaylik $K \subset \Omega \subseteq \mathbb{R}^d$ kompakt to'plam, $\mathscr{D}^n_K(\Omega)$ va $\mathscr{D}_K(\Omega)$ sinflarni quyidagicha aniqlaymiz

$$\mathscr{D}^n_K(\Omega) = \left\{f \in \mathscr{E}^k(\Omega) \mid \mathrm{supp}(f) \subseteq K\right\} \tag{1.3}$$

$$\mathscr{D}_K(\Omega) = \{f \in \mathscr{E}(\Omega) \mid \mathrm{supp}(f) \subseteq K\} = \bigcap_{n=0}^{\infty} \mathscr{D}^n_K(\Omega) \tag{1.4}$$

(iv) $\mathscr{K}(\Omega)$ esa Ω to'plamning barcha kompakt qism-to'plamlari sinfini belgilasin. $\mathscr{D}(\Omega)$ bilan Ω to'plamda aniqlangan kompakt dastakli silliq funsiyalar sinfini belgilaymiz.

$$\mathscr{D}(\Omega) = \bigcup_{K \in \mathscr{K}(\Omega)} \mathscr{D}_K(\Omega) \tag{1.5}$$

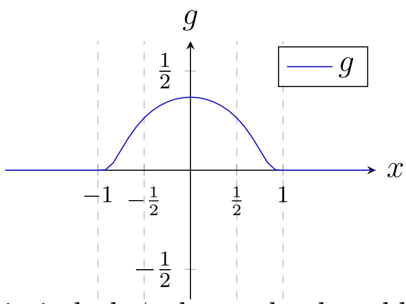
Ha dastlab bu funksiyalar sinflari juda abstrakt va juda kichik sinflar bo'lib tuyilishi mumkin. Lekin aslida bu sinflar biz bilgan $L^p(\Omega)$ fazolarda zich va bunday funksiyalar yordamida biz juda ko'p hossalarni isbotlashimiz mumkin.

Misol 1.1. (i) $d = 1$ va $\Omega = \mathbb{R}$ bo'lgan holatni qaraylik. Quyidagicha aniqlangan $g : \mathbb{R} \to \mathbb{R}$ funksiyani qaraylik:

$$g(x) = \begin{cases} \exp\left(-\dfrac{1}{1-|x|^2}\right), & \text{agar } |x| < 1 \\ 0, & \text{agar } |x| \geq 1. \end{cases} \tag{1.6}$$

To'g'ri birinchi bor bunday funksiyani tasavvur qilish, va uni nima sababdan biz bu funksiyani qarashimiz kerakligini anglab olish qiyin lekin ozgina fikr yuritib ushbu funksiya qanday hossalarga ega ekanligini ko'rish mumkin. Keling shu yerda ushbu funksiya grafigini qaraylik.

O‘ng tomondagi grafikdan shuni ko‘rishimiz mumkinki g kompakt dastakli funksiya va $\text{supp}(g) = [-1, 1]$. Hosila ta’rifidan foydalanib g funksiya -1 va 1 nuqtalarda ham cheksiz ko‘p marotaba differensiallanuvchi bo‘lishini ko‘rish mumkin. Bundan ko‘rinadiki



$$g \in \mathscr{D}(\mathbb{R}) \subset \mathscr{E}(\mathbb{R})$$

Yuqorida aniqlangan g funksiya yordamida biz juda ko‘p kompakt dastakli silliq funksiyalar hosil qila olamiz. Masalan, $f(x) = \sin(x) \cdot g(x) \in \mathscr{D}(\mathbb{R})$. Yoki istalgan silliq funksiyani $g(x)$ ga ko‘paytirish orqali biz $\mathscr{D}(\mathbb{R})$ sinfga tegishli funksiya hosil qilamiz.

(ii) Aytaylik $d \geq 1$ ixtiyoriy butun son bo‘lsin. Yuqoridagi g funksiyaning $\mathbb{R}^d$ fazoda ham analogi mavjud va uni soddagina

$$\mathbb{R}^d \ni x \longmapsto g(\|x\|)$$

kabi aniqlashimiz mumkin. Bu funksiyaning dastagi ese $\overline{B_1(0)}$—birlik yopiq shar bo‘ladi.

(iii) $(Gaussian)$ $g_\alpha : \mathbb{R}^d \mapsto \mathbb{R}$ $(d \geq 1)$ quyidagicha aniqlangan bo‘lsin

$$g_\alpha(x) = e^{-\alpha|x|^2}, \quad (\alpha > 0).$$

Albatta bu funksiya kompakt dastakli emas, lekin $g_\alpha \in \mathscr{E}(\mathbb{R}^d)$. Keyinchalik biz bu funksiyalar Furye transform bilan bog‘liq ajoyib hossalarga ega ekanligini ko‘rishimiz mumkin. Aytish lozim bo‘lgan yana bir jihati shunda-ki, g_α va uning barcha hosilalari $|x| \to \infty$ bo‘lganda, ixtiyoriy ratsional funksiyaga nisbatan kuchliroq 0 ga yaqinlashadi. Buni quyidagicha ifodalash mumkin,

$$\sup_{x \in \mathbb{R}^d} |x^\beta \partial^\gamma g_\alpha(x)| \leq C_{\beta,\gamma}$$ [1]

ixtiyoriy $\beta, \gamma \in \mathbb{N}_0^d$, multiindekslar uchun.

[1] $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{N}_0^d$ bo‘lsa $x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$ va $\partial^\beta = \frac{\partial^{\beta_1+\beta_2+\ldots+\beta_d}}{\partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \cdots \partial_{x_d}^{\beta_d}}$

2. Dirichlet problem for Poisson equation

3. Evolution equations

3.1. Unbounded operators in Banach spaces.

3.1.1. *Dissipative operators.*

3.1.2. *Resolvent and spectrum.*

3.2. Operator-valued functions.

3.2.1. *Measurability and continuity.*

3.3. Semigroup theory.

3.4. Hille-Yosida and related theorems.

3.5. The abstract problems.

3.5.1. *The homogeneous abstract problem.* Aytaylik X Banah fazosi bo'lib $A : D(A) \mapsto X$ zich aniqlangan m-dissipative operator bo'lsin. Berilgan $x \in X$ uchun quyidagi shartlarni qanoatlantiruvchi $u \in C([0, +\infty), X)$ funksiyalarni qidiramiz:

$$\begin{cases} u \in C\left((0,+\infty), D(A)\right) \cap C^1\left((0,+\infty), X\right); & (3.1)\\ u'(t) = Au(t), \quad \forall t > 0; & (3.2)\\ u(0) = x. & (3.3)\end{cases}$$

Albatta ixtiyoriy $x \in X$ uchun bunday funksiya topish murakkab masalaga aylanishi mumkin, lekin ayrim x lar uchun yuqoridagi masalani soddagina yechishimiz mumkin. Buning uchun Lumer-Phillips teoremasining natijasi o'laroq A operator hosil qilgan $\{T(t)\}_{t\geq 0}$ siquvchi yarim-gruppadan foydalanishimiz mumkin.

Teorema 3.1 (Yechimning mavjudligi va yagonaligi)**.** *Istalgan $x \in D(A)$ uchun*

$$\begin{cases} u \in C\left([0,+\infty), D(A)\right) \cap C^1\left([0,+\infty), X\right); & (3.4)\\ u'(t) = Au(t), \quad \forall t \geq 0; & (3.5)\\ u(0) = x. & (3.6)\end{cases}$$

abstrakt Koshi masalasining yagona yechimi mavjud bo'lib, ushbu yechim

$$u(t) = T(t)x, \quad t \geq 0 \tag{3.7}$$

kabi aniqlangan funksiya bo'ladi.

Isbot. ∎

Gilbert fazolarida yuqoridagi natijani yanada yaxshilasa bo'ladi. Buni quyidagi teoremada ko'rishimiz mumkin.

Teorema 3.2. *Aytaylik X haqiqiy sonlar ustidagi Gilbert fazo va $A\colon D(A) \mapsto X$ zich aniqlangan operator. Aytaylik A o'z-o'ziga qo'shma va manfiy aniqlangan operator. Ixtiyoriy $x \in X$ uchun $u(t) = T(t)x$, $t \geq 0$ kabi aniqlangan funksiya bo'lsin. U holda $u \in C([0, +\infty), X)$ va u funksiya (3.1)–(3.3) Koshi masalasining yagona yechimi bo'ladi. Bundan tashqari,*

$$\|Au(t)\| \leq \frac{1}{t\sqrt{2}}\|x\|, \tag{3.8}$$

$$-\langle Au(t), u(t)\rangle \leq \frac{1}{2t}\|x\|^2, \tag{3.9}$$

va agarda $x \in D(A)$

$$\|Au(t)\|^2 \leq -\frac{1}{2t}\langle Ax, x\rangle. \tag{3.10}$$

Isbot. ■

3.5.2. *The non-homogeneous abstract problem.* Ushbu paragraf ichida X Banah fazosi, $A\colon D(A) \mapsto X$ zich aniqlangan m-dissipative operator bo'lsin. $\{T(t)\}_{t\geq 0}$ bilan esa A operator hosil qiladigan siquvchi yarim-gruppani belgilab olamiz.
$T > 0$ son va $f\colon [0, T] \mapsto X$ funksiya berilgan. Ixtiyoriy $x \in X$ uchun quyidagi abstract Koshi masalasini qaraylik:

$$\begin{cases} u \in C([0, T], D(A)) \cap C^1([0, T], X); & (3.11)\\ u'(t) = Au(t) + f(t), \quad \forall t \in [0, T]; & (3.12)\\ u(0) = x. & (3.13)\end{cases}$$

Eslatma. Teorema 3.1 dan ko'rinadiki bunday yechim yagona bo'lishga majbur.

Oddiy differensial tenglamalar kabi bu masalada ham biz Duhamel formulasini keltirishimiz mumkin.

Lemma 3.3 (Duhamel's formula)**.** *Aytaylik $x \in D(A)$ va $f \in C([0, T], X)$. Agar $u \in C([0, T], D(A)) \cap C^1([0, T], X)$ funksiya (3.11)–(3.13) Koshi masalasining yechimi bo'lsa, u holda*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\, ds, \quad \forall t \in [0, T]. \tag{3.14}$$

Isbot. ■

Teorema 3.4. *Aytaylik $x \in D(A)$ va $f \in C([0, T], X)$. Agar*

(i) $f \in L^1((0, T), D(A))$;
(ii) $f \in W^{1,1}((0, T), X)$;

shartlarning birortasi o'rinli bo'lsa, (3.14) bilan aniqlangan u funksiya (3.11)–(3.13) Koshi masalasining yagona yechimi bo'ladi.

Isbot. ■

Natija 3.5. *Berilgan $x \in D(A)$ va $f \in C([0, T], X)$ lar uchun $u\colon [0, T] \mapsto X$ funksiya (3.14) bilan aniqlangan bo'lsin. Aytaylik quyidagilarning birortasi o'rinli:*

(i) $u \in C((0, T), D(A))$;
(ii) $u \in C^1((0, T), X)$.

U holda u funksiya (3.11)–(3.13) Koshi masalasining yagona yechimi bo'ladi.

Lemma 3.6 (Gronwall's lemma)**.** *Biror $T > 0$ son bo'lsin, aytaylik $\theta \in L^1(0,T)$ va $\theta \geq 0$ d.b. Agar biror $\varphi \in L^1(0,T), \varphi \geq 0$ funksiya uchun $\theta\varphi \in L^1(0,T)$ va*

$$\varphi(t) \leq C_1 + C_2 \int_0^t \theta(s)\varphi(s)\, ds, \quad d.b. t \in (0,T)$$

o'rinli bo'lsa, quyidagi tengsizlik ham deyarli barcha $t \in (0,T)$ lar uchun o'rinli

$$\varphi(t) \leq C_1 e^{C_2 \int_0^t \theta(s)\, ds}$$

Isbot. ■

3.5.3. *Semilinear abstract problems.* Ushbu paragrafda ham 3.5.2 paragrafdagi barcha belgilashlardan foydalanamiz.

Ta'rif 3.1. $F\colon X \mapsto X$ funksiya X fazoda lokal Lipshitz uzliksiz deb aytiladi agar ixtiyoriy $M > 0$ son uchun shunday $L(M)$ soni mavjud bo'lib

$$\|F(x) - F(y)\| \leq L(M)\|x - y\|, \quad \forall x, y \in B_M(0)^2.$$

Eslatma. $L(M)$ soni M soniga nisbatan funksiya deb qaralsa o'suvchi (non-decreasing) funksiya bo'ladi.

$F\colon X \mapsto X$ lokal Lipshitz uzliksiz funksiya bo'lsin. Berilgan $x \in X$ uchun biz shunday $T > 0$ va quyidagi Koshi masalasini qanoatlantiradigan $u\colon [0,T] \mapsto X$ funksiyalarni qidiramiz:

$$\begin{cases} u \in C\left([0,T], D(A)\right) \cap C^1\left([0,T], X\right); & (3.15)\\ u'(t) = Au(t) + F\left(u(t)\right), \quad \forall t \in [0,T]; & (3.16)\\ u(0) = x. & (3.17) \end{cases}$$

Eslatma. Agar $u \in C\left([0,T], X\right)$ funksiya (3.15)—(3.17) masalaning yechimi bo'lsa, Lemma 3.3 ga ko'ra u quyidagi integral tenglamaning ham yechimi bo'ladi:

$$u(t) = T(t)x + \int_0^t T(t-s)F(u(s))\, ds, \quad \forall t \in [0,T]. \tag{3.18}$$

Bu faktdan va Lemma 3.6 dan foydalanib (3.15)—(3.17) Koshi masalasining yechimi ko'pi bilan bitta ekanligini ko'rishimiz mumkin.

Dastlab biz (3.18) masalaning yechimi ayrim holatlarda mavjud ekanligini keltiramiz va ushbu holatlarda yechimning qay darajada regulyar ekanligini o'rganamiz.

Teorema 3.7. *X refleksiv Banah fazosi bo'lsin. $T > 0$ va $x \in X$ bo'lib $u \in C\left([0,T], X\right)$ funksiya (3.18) masalaning yechimi bo'lsin. Agar $x \in D(A)$ bo'lsa, u funksiya (3.15)—(3.17) Koshi masalasining yagona yechimi bo'ladi.*

[2] $B_M(0) = \{x \in X\colon \|x\| < M\}$

4. THE HEAT EQUATION

Aytaylik $\Omega \subset \mathbb{R}^d$ chegaralangan ochiq to'plam bo'lsin va $\partial\Omega$ Lipshitz uzliksiz. $C_0(\Omega)$ bilan $\overline{\Omega}$ da uzliksiz va $\partial\Omega$ to'plamda aynan nolga teng bo'lgan funksiyalar sinfini belgilab olamiz. Albatta $C_0(\Omega) \subset L^2(\Omega)$ bo'ladi. Ushbu sinfni sup-norma bilan birgalikda qarasak Banah fazosiga aylanadi va bunda $C_0(\Omega) \hookrightarrow L^2(\Omega)$ uzliksiz akslantirish.

4.1. The Laplacian on different domains.

$$\begin{cases} D(B) = \left\{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\right\} \\ \quad Bu = \Delta u, \quad \forall u \in D(B) \end{cases} \tag{4.1}$$

$$\begin{cases} D(A) = \left\{u \in H_0^1(\Omega) \cap C_0(\Omega) : \Delta u \in C_0(\Omega)\right\} \\ \quad Au = \Delta u, \quad \forall u \in D(A) \end{cases} \tag{4.2}$$

Tasdiq 4.1. *(4.1) va (4.2) da aniqlangan A va B operatorlar quyidagi hossalarga ega.*

(i) *$D(B)$ to'plam $L^2(\Omega)$ fazoda zich;*
(ii) *$B : D(B) \mapsto L^2(\Omega)$ o'z-o'ziga qo'shma operator;*
(iii) *$B \leq 0$ ya'ni B dissipative operator;*
(iv) *$D(A)$ to'plam $C_0(\Omega)$ fazoda zich;*
(v) *$A : D(A) \mapsto C_0(\Omega)$ m-dissipative operator;*

Isbot. ■

Lumer-Phillips teoremasiga ko'ra A va B operatorlarning har biri siquvchi yarim-gruppalar generatori bo'ladi. Biz bu yarim-grupplarni mos ravishda $\{T(t)\}_{t\geq 0}$ va $\{S(t)\}_{t\geq 0}$ bilan belgilab olamiz.

Tasdiq 4.2. *Ixtiyriy $\varphi \in C_0(\Omega)$ va $t \geq 0$ uchun $T(t)\varphi = S(t)\varphi$ tenglik o'rinli bo'ladi.*

4.2. The Semilinear Heat Equation.

Aytaylik $F \in C(\mathbb{R}, \mathbb{R})$ lokal Lipshitz uzliksiz funksiya va $F(0) = 0$ bo'lsin. Biz $F\colon C_0(\Omega) \mapsto C_0(\Omega)$ akslantirishni quyidagicha aniqlaymiz:

$$F(u)(x) = F(u(x)), \quad \forall u \in C_0(\Omega),\ \forall x \in \Omega.$$

Bundan kelib chiqadiki $F\colon C_0(\Omega) \mapsto C_0(\Omega)$ ham lokal Lipshitz uzliksiz akslantirish bo'ladi.

Berilgan $\varphi \in C_0(\Omega)$ funksiyalar uchun, biz shunday $T > 0$ va quyidagi masalani qanoatlantiradigan $u\colon [0,T] \mapsto C_0(\Omega)$ funksiyani qidiramiz:

$$\begin{cases} u \in C\left([0,T], C_0(\Omega)\right) \cap C\left((0,T], H_0^1(\Omega)\right) \cap C^1\left((0,T], L^2(\Omega)\right); & (4.3) \\ \Delta u \in C\left((0,T], L^2(\Omega)\right); & (4.4) \\ u_t - \Delta u = F(u), \quad \forall t \in [0,T]; & (4.5) \\ u(0) = \varphi. & (4.6) \end{cases}$$

Quyidagi teoremada biz (4.3)—(4.6) masalaning yechimi va unga mos integral tenglamaning yechimi aynan bir xil bo'lishini ko'ramiz.

Teorema 4.3. *Aytaylik $\varphi \in C_0(\Omega)$, $T > 0$ va $u \in C([0,T], C_0(\Omega))$ bo'lsin. Bu holatda $u\colon [0,T] \mapsto C_0(\Omega)$ funksiya (4.3)—(4.6) masalaning yechimi bo'lishi uchun*

$$u(t) = T(t)\varphi + \int_0^t T(t-s)F(u(s))\, ds, \quad \forall t \in [0,T] \tag{4.7}$$

tenglik o'rinli bo'lishi zarur va yetarlidir.

Isbot. ■

Biz bilamizki (4.7) tenglama lokal yechimga ega va agar ushbu yechimning normasi chegaralangan bo‘lsa bu yechim global bo‘lishga majbur. Shu faktdan foydalangan holda biz quyida bir nechta holatlarni o‘rganamiz.

Avvalo biz quyidagi maksimum prinsipini isbotalab olamiz.

Teorema 4.4 (Maksimum prinsipi). *$T > 0$, $f \in C\left([0,T], C_0(\Omega)\right)$ va $\varphi \in C_0(\Omega)$ bo‘lsin. Aytaylik $u \in C\left([0,T], C_0(\Omega)\right) \cap C\left((0,T), H_0^1(\Omega)\right) \cap C^1\left((0,T), L^2(\Omega)\right)$ va $\Delta u \in C\left((0,T), L^2(\Omega)\right)$ fuksiya uchun*

$$\begin{cases} u'(t) - \Delta u(t) = f(t), \quad \forall t \in (0,T); & (4.8) \\ u(0) = \varphi. & (4.9) \end{cases}$$

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[2] L. C. Evans. *Partial Differential Equations: Second Edition*. Graduate Studies in Mathematics. AMS, 2 edition, 2010.