PDE NOTES

TOLIBJON ISMOILOV

Contents

E-mail address: `tolibjon.iismoilov@gmail.com`.
Date: February 14, 2023.

1. THEORY OF DISTRIBUTIONS

Taqsimotlar nazariyasi (yohud umumlashgan funksiyalar nazariyasi)

1.1. **Space of test functions. Test funksiyalar fazolari**

Ushbu bo‘limda biz ixtiyoriy $\Omega \subseteq \mathbb{R}^d$ ochiq to‘plam uchun bir nechta funksiyanal fazolarni qaraymiz. $\Omega = \mathbb{R}^d$ bo‘lgan hollarda biz ayrim belgilashlarda shunchaki $\mathbb{R}^d$ tushurib ketamiz.

Ta’rif 1.1. $f : \Omega \to \mathbb{C}$ funksiyaning (supporti) *dastagi* deb

$$\{x \in \Omega : f(x) = 0\} \tag{1.1}$$

to‘plamning $\mathbb{R}^d$ fazodagi yopilmasiga aytiladi va $\operatorname{supp}(f)$ bilan belgilanadi.

Agar $\operatorname{supp}(f)$ kompakt to‘plam bo‘lsa, f funkisya *kompakt dastakli* deb ataladi.

Istalgan $n \geq 0$ butun son uchun $\Omega \subseteq \mathbb{R}^d$ to‘plamda quyidagi funksiyalar sinflarini qaraymiz.

(*i*) $\mathscr{E}^n(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \in C^n(\Omega)\}$ ya’ni k marta uzliksiz differensiallanuvchi funksiyalar to‘plami. Agar $n = 0$ bo‘lsa, $\mathscr{E}^0(\Omega)$ bilan Ω to‘plamda aniqlangan uzliksiz funkiyalar to‘plamini belgilab olamiz.

(*ii*) $\mathscr{E}(\Omega)$ bilan Ω to‘plamda aniqlangan *silliq* funksiyalar sinfini belgilab olamiz, ya’ni:

$$\mathscr{E}(\Omega) = \bigcap_{n=0}^{\infty} \mathscr{E}^n(\Omega) \tag{1.2}$$

(*iii*) Aylaylik $K \subset \Omega \subseteq \mathbb{R}^d$ kompakt to‘plam, $\mathscr{D}_K^n(\Omega)$ va $\mathscr{D}_K(\Omega)$ sinflarni quyidagicha aniqlaymiz

$$\mathscr{D}_K^n(\Omega) = \left\{f \in \mathscr{E}^k(\Omega) \mid \operatorname{supp}(f) \subseteq K\right\} \tag{1.3}$$

$$\mathscr{D}_K(\Omega) = \{f \in \mathscr{E}(\Omega) \mid \operatorname{supp}(f) \subseteq K\} = \bigcap_{n=0}^{\infty} \mathscr{D}_K^n(\Omega) \tag{1.4}$$

(*iv*) $\mathscr{K}(\Omega)$ esa Ω to‘plamning barcha kompakt qism-to‘plamlari sinfini belgilasin. $\mathscr{D}(\Omega)$ bilan Ω to‘plamda aniqlangan kompakt dastakli silliq funsiyalar sinfini belgilaymiz.

$$\mathscr{D}(\Omega) = \bigcup_{K \in \mathscr{K}(\Omega)} \mathscr{D}_K(\Omega) \tag{1.5}$$

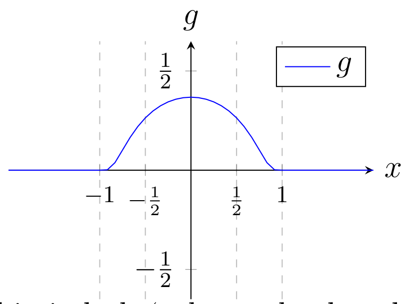
Ha dastlab bu funksiyalar sinflari juda abstrakt va juda kichik sinflar bo‘lib tuyilishi mumkin. Lekin aslida bu sinflar biz bilgan $L^p(\Omega)$ fazolarda zich va bunday funksiyalar yordamida biz juda ko‘p hossalarni isbotlashimiz mumkin.

Misol 1.1. (*i*) $d = 1$ va $\Omega = \mathbb{R}$ bo‘lgan holatni qaraylik. Quyidagicha aniqlangan $g : \mathbb{R} \to \mathbb{R}$ funksiyani qaraylik:

$$g(x) = \begin{cases} \exp\left(-\dfrac{1}{1-|x|^2}\right), & \text{agar } |x| < 1 \\ 0, & \text{agar } |x| \geq 1. \end{cases} \tag{1.6}$$

To‘g‘ri birinchi bor bunday funksiyani tasavvur qilish, va uni nima sababdan biz bu funksiyani qarashimiz kerakligini anglab olish qiyin lekin ozgina fikr yuritib ushbu funksiya qanday hossalarga ega ekanligini ko‘rish mumkin. Keling shu yerda ushbu funksiya grafigini qaraylik.

O‘ng tomondagi grafikdan shuni ko‘rishimiz mumkinki g kompakt dastakli funksiya va $\mathrm{supp}(g) = [-1, 1]$. Hosila ta’rifidan foydalanib g funksiya -1 va 1 nuqtalarda ham cheksiz ko‘p marotaba differensiallanuvchi bo‘lishini ko‘rish mumkin. Bundan ko‘rinadiki



$$g \in \mathscr{D}(\mathbb{R}) \subset \mathscr{E}(\mathbb{R})$$

Yuqorida aniqlangan g funksiya yordamida biz juda ko‘p kompakt dastakli silliq funksiyalar hosil qila olamiz. Masalan, $f(x) = \sin(x) \cdot g(x) \in \mathscr{D}(\mathbb{R})$. Yoki istalgan silliq funksiyani $g(x)$ ga ko‘paytirish orqali biz $\mathscr{D}(\mathbb{R})$ sinfga tegishli funksiya hosil qilamiz.

(ii) Aytaylik $d \geq 1$ ixtiyoriy butun son bo‘lsin. Yuqoridagi g funksiyaning $\mathbb{R}^d$ fazoda ham analogi mavjud va uni soddagina

$$\mathbb{R}^d \ni x \longmapsto g\left(\|x\|\right)$$

kabi aniqlashimiz mumkin. Bu funksiyaning dastagi ese $\overline{B_1(0)}$—birlik yopiq shar bo‘ladi.

(iii) $(Gaussian)$ $g_\alpha : \mathbb{R}^d \mapsto \mathbb{R}$ $(d \geq 1)$ quyidagicha aniqlangan bo‘lsin

$$g_\alpha(x) = e^{-\alpha|x|^2}, \quad (\alpha > 0).$$

Albatta bu funksiya kompakt dastakli emas, lekin $g_\alpha \in \mathscr{E}(\mathbb{R}^d)$. Keyinchalik biz bu funksiyalar Furye transform bilan bog‘liq ajoyib hossalarga ega ekanligini ko‘rishimiz mumkin. Aytish lozim bo‘lgan yana bir jihati shunda-ki, g_α va uning barcha hosilalari $|x| \to \infty$ bo‘lganda, ixtiyoriy ratsional funksiyaga nisbatan kuchliroq 0 ga yaqinlashadi. Buni quyidagicha ifodalash mumkin,

$$\sup_{x \in \mathbb{R}^d} |x^\beta \partial^\gamma g_\alpha(x)| \leq C_{\beta,\gamma} \text{[1]}$$

ixtiyoriy $\beta, \gamma \in \mathbb{N}_0^d$, multiindekslar uchun.

[1] $\beta = (\beta_1, \beta_2, ..., \beta_d) \in \mathbb{N}_0^d$ bo‘lsa $x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$ va $\partial^\beta = \frac{\partial^{\beta_1+\beta_2+\ldots+\beta_d}}{\partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \cdots \partial_{x_d}^{\beta_d}}$

2. Dirichlet problem for Poisson equation

3. Evolution equations

3.1. Unbounded operators in Banach spaces.

3.1.1. *Dissipative operators.*

3.1.2. *Resolvent and spectrum.*

3.2. Operator-valued functions.

3.2.1. *Measurability and continuity.*

3.3. Semigroup theory.

3.4. Hille-Yosida and related theorems.

References

[1] L. C. Evans. *Partial Differential Equations: Second Edition*. Graduate Studies in Mathematics. AMS, 2 edition, 2010.