TD1 :

Ex1 :

1) $A = (a_{ij}) \in M_{n,p}(K)$

A de type $(4,2)$ $\Rightarrow A \in M_{4,2}(R) \to \begin{cases} 4 \text{ lignes} \\ 2 \text{ colonnes} \end{cases}$

a_{ij} : i ligne, j colonne

$A \in M_{n,p}(K)$; $A = (a_{ij})$

$1 \le i \le n$

$1 \le j \le p$

a) $a_{ij} = j - 2i$

$$A = \begin{array}{c|c} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} -1 & 0 \\ -3 & -2 \\ -5 & -4 \\ -7 & -6 \end{pmatrix} \end{array}$$

$a_{11} = -1$; $a_{12} = 0$; $a_{21} = -3$ .....

b) A de type $(2,3) \Rightarrow A \in M_{2,3}(C) \begin{cases} 2 \text{ lignes} \\ 3 \text{ colonnes} \end{cases}$

$$A = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ \sqrt{2} & 2 & \sqrt{6} \end{pmatrix} \end{array}$$

c) A de type $(1,5) \Rightarrow A \in M_{1,5}(R) \begin{cases} 1 \text{ ligne} \\ 5 \text{ colonnes} \end{cases}$

$$A = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ 1 & \begin{pmatrix} 3/2 & 2 & 5/2 & 3 & 7/2 \end{pmatrix} \end{array}$$

d) A de type $(n,1)$ et $a_{ij} = 2 - ij = 2 - i$

$$A_{n,1} = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ a_{4,1} \\ \vdots \\ a_{n,1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ -3 \\ \vdots \\ 2-n \end{pmatrix}$$

e) A matrice symétrique $\in M_n(R)$ tq $a_{ij} = \begin{cases} 1 \text{ si } i \neq j \\ i - 2j \text{ sinon} \end{cases}$

$$A = \begin{pmatrix} -1 & 1 & \cdots & & 1 \\ 1 & -2 & & & \vdots \\ \vdots & & -3 & & \\ & & & \ddots & 1 \\ 1 & \cdots & & 1 & -n \end{pmatrix} = \begin{cases} 1 \text{ si } i \neq j \\ -i \text{ si } i = j \end{cases}$$

f) A triangulaire inférieure $\in M_4$

$\{a_{ij} = j^2 - 1$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 8 & 0 \\ 0 & 3 & 8 & 15 \end{pmatrix}$$

g) A est carrée $\in M_n$ tq $a_{ij} = \begin{cases} -3i \text{ si } j \neq i \\ i+1 \text{ sinon} \end{cases}$

$$A = \begin{pmatrix} 2 & -3 & -3 & \cdots & \cdots \\ -6 & 3 & -6 & \cdots & \cdots \\ -9 & -9 & 4 & -9 & \cdots \\ \vdots & \vdots & & 5 & \cdots \\ -3(n-1) & & & & n \\ -3n & \cdots & & & n+1 \end{pmatrix}$$

h) $A \in M_3(K)$; $a_{ij} = 5j - i$; si $i > j$

$$A = \begin{array}{c|c} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & -3 & -2 \\ 3 & 0 & -7 \\ 2 & 7 & 0 \end{pmatrix} \end{array}$$

$\because {}^tA = -A$

$\Rightarrow a_{ii} = -a_{ii}$

$\Rightarrow a_{i,i} = 0$

k) $A \in M_4(R)$ tq $a_{ij} = \begin{cases} 0 \text{ si } i+j \text{ pair} \\ i \text{ sinon} \end{cases}$

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 \\ 4 & 0 & 4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} (1\times 5) + (3\times 6) \\ (2\times 5) + (4\times 6) \end{pmatrix} = \begin{pmatrix} 23 \\ 34 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} (1\times 4) + (2\times 5) + (3\times 7) \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 35 \\ 35 \\ 35 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 2 & 0 \end{pmatrix}}_{\in M_{3;2}(\mathbb{R})} \times \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{pmatrix}}_{M_{2,3}(\mathbb{R})} = \underbrace{\begin{pmatrix} 4 & -1 & 5 \\ 7 & -3 & 10 \\ 2 & 0 & 2 \end{pmatrix}}_{\in M_{(3,3)}(\mathbb{R})}$$

$$\left.\begin{matrix} A \in M_{n,p}(\mathbb{R}) \\ B \in M_{m,q}(\mathbb{R}) \end{matrix}\right| A\times B \text{ existe} \Leftrightarrow p = m$$

$$A \times B \in M_{n,q}(\mathbb{R})$$

Ex: 1) Calculer:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

2) On remarque que $A\times B \neq B \times A$

Ex: calculer:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -5 & -7 & -8 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 10 \\ 4 & 6 & 9 \\ -5 & -7 & -3 \end{pmatrix}$$

Ex: Donner la transposée de chaque matrice:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 5 \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 3 & 0 & 5 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 4 \end{pmatrix} \Rightarrow B^t = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{pmatrix} \Rightarrow C^t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Ex: calculer αA ; $\alpha \in \mathbb{R}$

$5A$

$-3A$ ~~q~~

$$A = \begin{pmatrix} 2 & 0 & 5 \\ 3 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\alpha A = \begin{pmatrix} 2\alpha & 0 & 5\alpha \\ 3\alpha & -\alpha & 4\alpha \\ 0 & 0 & -\alpha \end{pmatrix}$$

$$5A = \begin{pmatrix} 10 & 0 & 25 \\ 15 & -5 & 20 \\ 0 & 0 & -5 \end{pmatrix}$$

$$-3A = \begin{pmatrix} -6 & 0 & -15 \\ -9 & 3 & -12 \\ 0 & 0 & 3 \end{pmatrix}$$

3)

1) $A + B$: impo

2) $B \in M_{3,1}(\mathbb{R})$ avec $B = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$

$A \in M_{1,3}(\mathbb{R})$ avec $A = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$

$B \times A \in M_{3,3}(\mathbb{R})$

$$B \cdot A = \begin{pmatrix} 2 & 0 & -2 \\ 5 & 0 & -5 \\ 7 & 0 & -7 \end{pmatrix}$$

3) $A \times B = M_{1,3}(\mathbb{R}) \times M_{3,1}(\mathbb{R}) = M_{1,1}(\mathbb{R})$

$$A \cdot B = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} = 1 \times 2 + 0 \times 5 + 7(-1) = -5.$$

4) $M_C = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

$$2D = \begin{pmatrix} 2 & 4 \\ -4 & -8 \end{pmatrix}$$

$$M_C - 2D = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ -4 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 \\ 5 & 9 \end{pmatrix}$$

5) $D \cdot C = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 3 \\ -12 & -6 \end{pmatrix}$$

6) $\frac{-2}{3}(A \cdot E') = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \times \boxed{\begin{array}{c} E' \\ \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}}$

$$= -\frac{2}{3} \cdot \begin{pmatrix} 0 & 2 & 2 \end{pmatrix}$$

$$A \cdot E = \begin{pmatrix} 0 & -\frac{4}{3} & -\frac{4}{3} \end{pmatrix}$$

7) $M_E = \frac{-2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$

$$F \cdot T_E = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \times \left(\frac{-2}{3}\right) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= -\frac{2}{3} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -2 & -\frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

8) $A \cdot C$: impo

9) $C \cdot D = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{M_2(\mathbb{R})}$$

$$M_{(CD)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{M_2(\mathbb{R})}$$

10) $tr_C \cdot tr_D = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$

$$= \begin{bmatrix} 6 & -12 \\ 3 & -6 \end{bmatrix}$$

11) $3E = \begin{pmatrix} -2 & -2 & -4 \\ -2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}$

$3E + F = \begin{pmatrix} -3 & -3 & -3 \\ -1 & 0 & -1 \\ -2 & 3 & 0 \end{pmatrix}$

$(3E+F)^2 = \begin{pmatrix} -3 & -3 & -3 \\ -1 & 0 & -1 \\ -2 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} -3 & -3 & -3 \\ -1 & 0 & -1 \\ -2 & 3 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 18 & 0 & 19 \\ 5 & 0 & 3 \\ 3 & 6 & 3 \end{pmatrix}$$

$\boxed{tr(A) = \sum_i a_{ii} = \text{traço}(A)}$

$tr((3E+F)^2) = 18 + 0 + 3 = 21$

12) $H \cdot E = \begin{pmatrix} 4 & -3 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 1 & -1 \end{pmatrix} \frac{2}{3} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

$$= -\frac{2}{3} \begin{pmatrix} 0 & 5 & 5 \\ 0 & -3 & -3 \\ 3 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{10}{3} & -\frac{10}{3} \\ 0 & 2 & \frac{3}{2} \\ -2 & \frac{2}{3} & -\frac{4}{3} \\ \frac{2}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

13) $C \cdot G = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & -2 & 3 \\ -1 & -2 & 3 \end{pmatrix}$$

14) $tr(G \cdot F)$

impo.

15) $E - 3I_3 =$

$-\frac{2}{3} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 7 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

$$= \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & 0 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{11}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & 3 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -3 \end{pmatrix}$$

16)

$$F \cdot {}^tR = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -2 & 1 & -1 \\ -3 & 1 & 0 & 1 \\ -1 & 1 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 & 1 & -1 \\ 3 & -1 & 3 & -2 \\ -3 & 1 & 0 & 1 \end{pmatrix}$$

Ex 3 :

$$M = \begin{pmatrix} 0 & 2 & 4 \\ -1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix}$$

1) $M^2 - 5M + 6I$

$$M^2 = \begin{pmatrix} 0 & 2 & 4 \\ -1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 & 4 \\ -1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} -6 & 10 & 20 \\ -5 & 9 & 10 \\ -5 & 5 & 14 \end{pmatrix}$$

$$5M = \begin{pmatrix} 0 & 10 & 20 \\ -5 & 15 & 10 \\ -5 & 5 & 20 \end{pmatrix}$$

$$6I = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$M^2 - 5M + 6I = \begin{pmatrix} -6 & 10 & 20 \\ -5 & 9 & 10 \\ -5 & 5 & 14 \end{pmatrix} - \begin{pmatrix} 0 & 10 & 20 \\ -5 & 15 & 10 \\ -5 & 5 & 20 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

) On a $M^2 - 5M + 6I = 0$

$\Rightarrow M^2 - 5M = -6I$

$\Rightarrow -\frac{1}{6}(M^2 - 5M) = I$

$\Rightarrow M \times \underbrace{\left[-\frac{1}{6}M + \frac{5}{6}I\right]}_{M^{-1}}$

$= \underbrace{\left[-\frac{1}{6}M + \frac{5}{6}I\right]}_{M^{-1}} \times M = I_3$

donc M est inversible et

$M^{-1} = -\frac{1}{6}M + \frac{5}{6}I$

$$M^{-1} = -\frac{1}{6}\begin{pmatrix} -5 & 2 & 4 \\ -1 & -2 & 2 \\ -1 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{5}{6} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{6} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

2) $A = M - 2I$ et $B = M - 3I$

a) $A^2 = (M - 2I)^2$

$$\begin{pmatrix} 0 & 2 & 4 \\ -1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 & 4 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -2 & 2 & 4 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 2 & 4 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 & 4 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} = A$$

$$B^2 = \begin{pmatrix} -3 & 2 & 4 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 2 & 4 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & -4 \\ 1 & 0 & -2 \\ 1 & -1 & -1 \end{pmatrix} = -B$$

b) ① pour $n = 1$

$A^1 = A$

Sq $A^n = A$ et Mq $A^{n+1} = A$

$A^{n+1} = A^n . A = A . A = A^2 = A$

d'ou d'après le principe de recurrence $A^n = A$.

②

$B^1 = B = (-1)^{1+1} B$

$B^2 = -B = (-1)^{2+1} B$

Sq $B^n = (-1)^{n+1} B$

Mq $B^{n+1} = (-1)^{n+2} B$

$B^{n+1} = B^n . B$

$= (-1)^{n+1} . B . B = (-1)^{n+1} (-1) B$

$= (-1)^{n+2} B$

cl : $B^n = (-1)^{n+1} B$.

3) $A.B = (M - 2I) \times (M - 3I)$

$= M^2 - 3M - 2M + 6I$

$= M^2 - 5M + 6I = O_3$

$B.A = (M - 3I)(M - 2I)$

$= M^2 - 2M + 3M + 6I$

$= M^2 - 5M + 6I = O_3$

$\Rightarrow A \times B = B \times A$.

4) $A = M - 2I \Rightarrow M = A + 2I$ ①

$B = M - 3I \Rightarrow M = B + 3I$ ②

$M = \alpha A + \lambda B$

$3 \times$ ① $- 2 \times$ ② $\Rightarrow 3M - 2M$

$= 3A + \cancel{6I} - 2B - \cancel{6I}$

$\Rightarrow M = 3A - 2B$

$M = 3A + (-2) B$

$$M^n = (3A - 2B)^n = \sum_{k=0}^{n} C_n^k A^{n-k} \cdot (-B)^k$$
$\boxed{AB = BA}$

$$= \sum_{k=0}^{n} 3^{n-k} (-2)^k A \; (-1)^{k+1} B$$

$$= \left(\sum_{k=0}^{n} 3^{n-k} \cdot 2^k\right) \underbrace{AB}_{0_3} = 0_3$$

Ex 4 :

1.

① $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (-x - 6y, 2x + 5y, x - 2y)$

$f(0,0) = (0,0,0)$

soit $u = (x,y)$ et $v = (x',y')$ et $\lambda \in \mathbb{R}$

$\lambda u + v = \lambda(x,y) + (x',y')$

$= (\lambda x + x', \lambda y + y')$

$f(\lambda u + v) = f(\lambda x + x', \lambda y + y')$

$= (-\lambda x - x' - 6\lambda y - 6y', 2\lambda x + 2x' + 5\lambda y + 5y', \lambda x + x' - 2\lambda y - 2y')$

$= \lambda(-x - 6y, 2x + 5y, x - 2y) + (-x' - 6y', 2x' + 5y', x' - 2y')$

$= \lambda f(x,y) + f(x',y') = \lambda f(u) + f(v)$

$\Rightarrow f$ est une app lin de $\mathbb{R}^2$ dans (vers) $\mathbb{R}^3$.

②

B = base canonique de $\mathbb{R}^2 = (e_1, e_2)$ avec $e_1 = (1,0)$ et $e_2 = (0,1)$

B' = b. canonique de $\mathbb{R}^3 = (u_1, u_2, u_3)$

avec $u_1 = (1,0,0)$; $u_2 = (0,1,0)$

$u_3 = (0,0,1)$

$$\text{mat}(f, B, B') = \begin{array}{c} \begin{pmatrix} -1 & -6 \\ 2 & 5 \\ 1 & -2 \end{pmatrix} \end{array} \begin{array}{l} u_1 \\ u_2 \\ u_3 \end{array}$$

- $f(e_1) = (-1, 2, 1) = -u_1 + 2u_2 + u_3$
- $f(e_2) = (-6, 5, -2) = -6u_1 + 5u_2 - 2u_3$

2.

① $f : \mathbb{R}^4 \to \mathbb{R}^2$, $f(x,y,z,t) = (3z + y - 2x, 4x - \frac{2t}{3})$

Soit $u = (x,y,z,t) \in \mathbb{R}^4$; $\alpha \in \mathbb{R}$

$v = (x',y',z',t')$

$f(\alpha u + v) = f(\alpha x + x', \alpha y + y', \alpha z + z', \alpha t + t')$

$= (3\alpha z + 3z' + \alpha y + y' - 2\alpha x - 2x', 4\alpha x + 4x' - \frac{2}{3}\alpha t - \frac{2}{3}t')$

$= \alpha(3z + y - 2x, 4x - \frac{2}{3}t) + (3z' + y' - 2x', 4x' - \frac{2}{3}t')$

$= \alpha f(u) + f(v)$

$\Rightarrow f$ est une app lin de $\mathbb{R}^4$ vers $\mathbb{R}^2$

② B = base canonique de $\mathbb{R}^4$; $B = (e_1, e_2, e_3, e_4)$

$e_1 = (1,0,0,0)$

$e_2 = (0,1,0,0)$

$e_3 = (0,0,1,0)$

$e_4 = (0,0,0,1)$

B' = b.c de $\mathbb{R}^2$, $B' = (u_1, u_2)$

$u_1 = (1,0)$

$u_2 = (0,1)$

$$M = \text{mat}(f, B, B') = \begin{array}{cccc} f(e_1) & f(e_2) & f(e_3) & f(e_4) \\ \end{array}$$
$$\begin{pmatrix} -2 & 1 & 3 & 0 \\ 4 & 0 & 0 & -\frac{2}{3} \end{pmatrix} \begin{array}{l} u_1 \\ u_2 \end{array}$$

3. $f : \mathbb{R}_3[X] \to \mathbb{R}^3$, $f(P) = (P(-3), P(0), P(1))$

①

soit $\lambda \in \mathbb{R}$

soit P et $Q \in \mathbb{R}_3[X]$

$f(\lambda P + Q) = ((\lambda P + Q)(-3) ; (\lambda P + Q)(0) ; (\lambda P + Q)(1))$

$= (\lambda P(-3) + Q(-3), \lambda P(0) + Q(0), \lambda P(1) + Q(1))$

$= \lambda(P(-3), P(0), P(1)) + (Q(-3), Q(0), Q(1))$

$= \lambda f(P) + f(Q)$

$\Rightarrow f$ est une app lin de $\mathbb{R}_3[X]$ dans $\mathbb{R}^3$.

① $B = $ bc de $\mathbb{R}_3[x]$

$B = (1, x, x^2, x^3)$

$B' = $ bc de $\mathbb{R}^3[x] \Rightarrow B' = (e_1, e_2, e_3)$

avec $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$f(1) = (1,1,1) = e_1 + e_2 + e_3$

$f(x) = (-3,0,1) = -3e_1 + e_2$

$f(x^2) = (9,0,1) = 9e_1 + e_3$

$f(x^3) = (-27,0,1) = -27e_1 + e_3$

$$\text{mat}(f, B, B') = \begin{array}{c} \begin{array}{cccc} f(1) & f(x) & f(x^2) & f(x^3) \end{array} \\ \begin{pmatrix} 1 & -3 & 9 & -27 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}$$

4. $f : \mathbb{R}^n \to \mathbb{R}^n$,

$f(x_1, x_2, \dots, x_n) = (x_1, x_2 - x_1, x_3 - x_1, \dots, x_n - x_1)$

① Soit $U = (x_1, x_2, x_3, \dots, x_n)$ et $\lambda \in \mathbb{R}$

soit $V = (y_1, y_2, y_3, \dots, y_n)$

$\lambda U + V = \lambda(x_1, x_2, x_3, \dots, x_n) + (y_1, y_2, y_3, \dots, y_n)$

$= (\lambda x_1 + y_1, \lambda x_2 + y_2, \lambda x_3 + y_3, \dots, \lambda x_n + y_n)$

$f(\lambda U + V) = f(\lambda x_1 + y_1, \lambda x_2 + y_2, \dots, \lambda x_n + y_n)$

$= (\lambda x_1 + y_1, \lambda x_2 + y_2 - \lambda x_1 - y_1, \dots, \lambda x_n + y_n - \lambda x_1 - y_1)$

$= \lambda(x_1, x_2 - x_1, \dots, x_n - x_1) + (y_1, y_2 - y_1, \dots, y_n - y_1)$

$= \lambda f(x_1, x_2, x_3, \dots, x_n) + f(y_1, y_2, y_3, \dots, y_n)$

$= \lambda f(U) + f(V)$

$\Rightarrow f$ est une app lin de $\mathbb{R}^n$ vers $\mathbb{R}^n$

② soit $B = $ bc de $\mathbb{R}^n = (e_1, e_2, \dots, e_n)$

avec $e_1 = (1, 0, 0, \dots, 0)$

$e_2 = (0, 1, 0, \dots)$

$\vdots$

$e_n = (0, 0, \dots, 1)$

$f(e_1) = f(1, 0, \dots) = (1, -1, \dots, -1)$

$f(e_2) = f(0,1,0,\dots,0) = (0,1,0,\dots,0)$

$$\text{mat}(f, B) = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_n \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ -1 & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{pmatrix} \end{array}$$

5. $f : \mathbb{R}_2[x] \to \mathbb{R}_3[x]$, $f(P) = e^{-x^2}(e^{x^2}P)'$

① Soit $\alpha \in \mathbb{R}$, $P, Q \in \mathbb{R}_2[x]$

$f(\alpha P + Q) = e^{-x^2}(e^{x^2}(\alpha P + Q))'$

$= e^{-x^2}(e^{x^2}(\alpha P) + e^{x^2}Q)'$

$= \alpha e^{-x^2}(e^{x^2}P)' + e^{-x^2}(e^{x^2}Q)'$

$= \alpha f(P) + f(Q) \Rightarrow f$ est une app lin de $\mathbb{R}_2[x]$ dans $\mathbb{R}_3[x]$

② B base canonique de $\mathbb{R}_2[x] = (1, x, x^2)$

B' base canonique de $\mathbb{R}_3[x] = (1, x, x^2, x^3)$

$$\text{mat}(f, B, B') = \begin{array}{c} \begin{array}{ccc} f(1) & f(x) & f(x^2) \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array}$$

$f(P) = e^{-x^2}(e^{x^2}P)'$

$= e^{-x^2}(2xe^{x^2}P + P'e^{x^2}) =$

$f(1) = e^{-x^2} \cdot 2x e^{x^2} = 2x$

$f(x) = 2x^2 + 1$

$f(x^2) = 2x^3 + 2x$

6. $f : \mathbb{R}_4[x] \to \mathbb{R}$, $f(P) = \int_{-1}^{1} x P(x)\, dx$

① Soit $\alpha \in \mathbb{R}$, $P, Q \in \mathbb{R}_4[x]$

$f(\alpha P + Q) = \int_{-1}^{1} f(\alpha P + Q)\, dx$

$= \alpha \int_{-1}^{1}$

$\Rightarrow f$ est une app lin de $\mathbb{R}_4[x]$ dans $\mathbb{R}$