Ex6.

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x,y,z) \longmapsto (2x - y,\ -2x + y - 2z,\ x + y + 3z)$$

1/

$$A = \begin{array}{c} \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} \\ \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & -2 \\ 1 & 1 & 3 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}$$

2) $$\det(A) = \begin{vmatrix} 2 & -1 & 0 \\ -2 & 1 & -2 \\ 1 & 1 & 3 \end{vmatrix}$$

$C_1 \leftarrow C_1 + 2C_2$

$$= \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 3 & 1 & 3 \end{vmatrix} = 1 \begin{vmatrix} 0 & -2 \\ 3 & 3 \end{vmatrix}$$

$$= 1(0 - (-2)3)$$

$$= 6 \neq 0 \Rightarrow \text{inversible}$$

$$\cancel{f(x,y,z) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

3) $$\operatorname{com}(A) = \begin{pmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} & - \begin{vmatrix} -2 & -2 \\ 1 & 3 \end{vmatrix} & + \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} \\ - \begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} & + \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} & - \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ + \begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} & - \begin{vmatrix} 2 & 0 \\ -2 & -2 \end{vmatrix} & + \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 & -3 \\ 3 & 6 & -3 \\ 2 & 4 & 0 \end{pmatrix}$$

$${}^{t}\mathrm{com}(A) = \begin{pmatrix} 5 & 3 & 2 \\ 4 & 6 & 4 \\ -3 & -3 & 6 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \cdot {}^{t}\mathrm{com}(A)$$

$$= \frac{1}{6} \begin{pmatrix} 5 & 3 & 2 \\ 4 & 6 & 4 \\ -3 & -3 & 6 \end{pmatrix}$$

$$A^{-1} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{6}x + \frac{1}{2}y + \frac{1}{3}z \\ \frac{2}{3}x + y + \frac{2}{3}z \\ -\frac{1}{2}x - \frac{1}{2}y \end{pmatrix}$$

$$\Rightarrow f^{-1}(x,y,z) = \left(\frac{5}{6}x + \frac{1}{2}y + \frac{1}{3}z,\ \frac{2}{3}x + y + \frac{2}{3}z,\ -\frac{1}{2}x - \frac{1}{2}y\right)$$

Ex6.

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto (2x-y, -2x+y-2z, x+y+3z)$

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc|c} f(e_1) & f(e_2) & f(e_3) & \\ \end{array}$$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & -2 \\ 1 & 1 & 3 \end{pmatrix} \begin{array}{l} e_1 \\ e_2 \\ e_3 \end{array}$$

4)

A est inversible ssi $\mathrm{Com}(A)$ est inversible

$\Rightarrow$ inversible

$$\mathrm{Com}(A)^{-1} = \frac{1}{\det(A)} {}^tA$$

$$= \frac{1}{6}\begin{pmatrix} 2 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 3 \end{pmatrix}$$

5)

$$\det_{\mathcal{B}}(v_1, v_2, v_3) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix}$$

$C_2 \leftarrow C_1 + C_3$

$$= \begin{vmatrix} 2 & 1 & 1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}$$

$= -1 \neq 0 \ldots$

$\Rightarrow B' = (V_1, V_2, V_3)$ forme une base de $\mathbb{R}^3$

6)

b) $P = P_{B \to B'} = \begin{array}{ccc|c} V_1 & V_2 & V_3 & \\ \end{array}$
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}$$

$$P^{-1} = P_{B' \to B} = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} V_1 \\ V_2 \\ V_3 \end{array}$$

$$\begin{cases} V_1 = e_1 + e_2 - e_3 & (1) \\ V_2 = e_1 + e_3 & (2) \\ V_3 = e_1 - e_2 & (3) \end{cases}$$

$\Leftrightarrow$ (1) − (2) $\Rightarrow \boxed{e_2 = V_1 - V_2}$

(3) $\Rightarrow \boxed{e_1} = V_3 + e_2 = \boxed{V_3 + V_1 - V_2}$

$= V_1 - V_2 + V_3$

(2) $\Rightarrow e_3 = e_1 - V_2$

$= V_3 + V_1 - V_2 - V_2$

$\boxed{e_3 = V_3 + V_1 - 2V_2}$

$e_3 = V_1 - 2V_2 + V_3$

Ex6.

f) $D = \text{mat}(f, B')$

$$= \begin{array}{cccc} f(v_1) & f(v_2) & f(v_3) & \\ \multicolumn{3}{c}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}} & \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \end{array}$$

$f(v_1) = f(1,1,-1) = (1,1,-1) = v_1$

$f(v_2) = f(1,0,-1) = (2,0,-2) = 2v_2$

$f(v_3) = 3v_3$

$A = \text{mat}(f, B)$

$D = \text{mat}(f, B')$

$P = P_{B \to B'}$

$D = P^{-1}.A.P$

$A = P.D.P^{-1}$

g) $D^n = \begin{pmatrix} 1^n & & 0 \\ & 2^n & \\ 0 & & 3^n \end{pmatrix}$

car $D = \text{diag}(1,2,3)$

h) pour $n = 0$

$A^0 = P.D^0.P^{-1}$ vrai

pour $n \in \mathbb{N}$

On suppose que $A^n = P D^n . P^{-1}$

et montrons que $A^{n+1} = P.D^{n+1}.P^{-1}$

$A^{n+1} = A^n . A$

$= P.D^n.\underbrace{P^{-1}.P}_{\text{identité}}.D.P^{-1}$

$= P.D^{n+1}.P^{-1}$

V_1

$\forall n \in \mathbb{N} \quad A^n = P \cdot D^n \cdot P^{-1}$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2^n & 3^n \\ 1 & 0 & -3^n \\ -1 & -2^n & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2^n+3^n & 1-2^n & 1-2^{n+1}+3^n \\ 1-3^n & 1 & 1-3^n \\ -1+2^n & -1+2^n & -1+2^{n+1} \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

$U_{n+1} = A U_n$

$U_n = A \cancel{U_{n-1}}$

$\cancel{U_{n-1}} = A \cancel{U_{n-2}}$

$\vdots$

$\cancel{U_1} = A U_0$

$\Rightarrow$ on multiplie membre à membre

$\Rightarrow U_n = A^n U_0, \forall n$

$x_0 = z_0 = 1, \; y_0 = -1$

$\forall n$

$$\begin{cases} x_{n+1} = 2x_n - y_n \\ y_{n+1} = -2x_n + y_n - 2z_n \\ z_{n+1} = x_n + y_n + 3z_n \end{cases}$$

$$U_n = A^n \cdot U_0$$

$$= \begin{pmatrix} 1-2^n+3^n & 1-2^n & 1-2^{n+1}+3^n \\ 1-3^n & 1 & 1-3^n \\ -1+2^n & -1+2^n & -1+2^{n+1} \end{pmatrix} \cdot$$

$$= \begin{pmatrix} 1+2\times 3^n-2^{n+1} \\ 1-2\times 3^n \\ -1+2^{n+1} \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_n = 1+2\times 3^n-2^{n+1} \\ y_n = 1-2\times 3^n \\ z_n = 2^{n+1}-1 \end{cases}$$

$$\begin{pmatrix} 1-2^n+3^n & 1-2^n & 1-2^{m+1}+3^m \\ 1-3^m & 1 & 1-3^m \\ -1+2^m & -1+2^n & -1+2^{m+1} \end{pmatrix}$$