

Série 4

EX3:

$$m \in \mathbb{R}, A_m = \begin{pmatrix} 3 & 4 & 4 \\ m & -m & -1 \\ -m & 2 & 3-m \end{pmatrix}$$

1) P_m le poly caract de A_m .

$$P_m(x) = \det(A_m - x I_3) = \begin{vmatrix} 3-x & 4 & 4 \\ m & -m-x & -1 \\ -m & 2 & 3-m-x \end{vmatrix}$$

$$L_3 \leftarrow L_3 + L_2$$

$$P_m(x) = \begin{vmatrix} 3-x & 4 & 4 \\ m & -m-x & -1 \\ 0 & 2-m-x & 2-m-x \end{vmatrix}$$

$$= (2-m-x) \begin{vmatrix} 3-x & 4 & 4 \\ m & -m-x & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$(C_2 \leftarrow C_2 - C_3)$$

$$P_m(x) = (2-m-x) \begin{vmatrix} 3-x & 0 & 4 \\ m & -m+x & -1 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{dev} \times a^{-1} L_3$$

$$= (2-m-x)(3-x)(-m+x-2)$$

$$\text{Sp}(A_m) = \{2-m, 3, 1-m\}$$

$$2) A_m \text{ est inversible} \Leftrightarrow \det(A_m) \neq 0.$$

$$\Leftrightarrow 0 \notin \text{Sp}(A_m).$$

$$\Leftrightarrow \begin{cases} 2-m \neq 0 \\ \text{et} \\ 1-m \neq 0 \end{cases} \Leftrightarrow \begin{cases} m \neq 2 \\ \text{et} \\ m \neq 1 \end{cases}$$

$$A_m \text{ est inv} \Leftrightarrow m \in \mathbb{R} \setminus \{1, 2\}$$

$$3) \text{ soit } m \in \mathbb{R} \setminus \{1, 2\}$$

M puis A_m est diagonalisable.

$$\text{car: } \text{Sp}(A_m) = \{2-m, 3, 1-m\}.$$

on a:

$$* / 2-m=3 \Leftrightarrow m=-1$$

comme $m \in \mathbb{R} \setminus \{-1, -2\}$ alors $2-m \neq 3$.

$$* / 1-m=3 \Leftrightarrow m=-2$$

comme $m \in \mathbb{R} \setminus \{-1, -2\}$ alors $1-m \neq 3$.

et on a: $1-m \neq 2-m, \forall m$

cl: A_m admet 3 v.p distincts $\Rightarrow A_m$ est diagonalisable

$$A_m \in M_3(\mathbb{R})$$

$$4) \underline{m=0}, A = A_0 = \begin{pmatrix} 3 & 4 & 4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix}$$

$$P_A(x) = (2-x)(3-x)(1-x)$$

$$\text{Sp}(A) = \{1, 2, 3\}$$

(a) Diagonaliser A c'est donner D diagonale, P inversible
tq $A = P \cdot D \cdot P^{-1}$.

Déterminons les sous-espaces propres :

$$E_1(A) = \text{Ker}(A - I_3) = \left\{ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ tq } AX = X \right\}.$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_1(A) \Leftrightarrow AX = X$$

$$\Leftrightarrow \begin{pmatrix} 3 & 4 & 4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{cases} 3x + 4y + 4z = x & (1) \\ -z = y & (2) \\ 2y + 3z = z & (3) \end{cases}$$

$$\Leftrightarrow \begin{cases} (2) : z = -y \\ (3) : z = -y \\ (1) : x = 0 \end{cases}$$

$$\Leftrightarrow X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ -y \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = y V_2.$$

$$E_1(A) = \text{Vect}(V_2)$$

$$V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

$$E_2(A) = \text{Vect} \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \text{Vect}(V_3), V_3 = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

$$E_3(A) = \text{Vect} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \text{Vect}(V_1).$$

$$\underline{d}: A = P \cdot D \cdot P^{-1}$$

$$\text{avec } D = \begin{pmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{pmatrix} \begin{matrix} AV_1 \\ AV_2 \\ AV_3 \\ V_1 \\ V_2 \\ V_3 \end{matrix}$$

$$P = \begin{pmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{pmatrix} \begin{matrix} V_1 \\ V_2 \\ V_3 \\ BC \end{matrix}$$

$$V_1 \in E_3(A) \Rightarrow AV_1 = 3V_1.$$

$$V_2 \in E_{-1}(A) \Rightarrow AV_2 = V_2$$

$$V_3 \in E_2(A) \Rightarrow AV_3 = 2V_3.$$

$$\text{par suite } D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix}.$$

Preuve: le calcul de P^{-1} n'est pas demandé

$$\boxed{4} \mid m=0, A = A_0 = \begin{pmatrix} 3 & 4 & 4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix}.$$

(b) $m=0 \in \mathbb{R} \setminus \{1, 2\}$ d'après 2) A est inv.

$$\text{on a: } A = P \cdot D \cdot P^{-1}.$$

$$\Rightarrow A^{-1} = (P \cdot D \cdot P^{-1})^{-1} = (P^{-1})^{-1} \cdot D^{-1} \cdot P^{-1} \\ = P \cdot D^{-1} \cdot P^{-1}.$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$A^{-1} = P \cdot D^{-1} \cdot P^{-1}$$

$$\text{Sp}(A^{-1}) = \text{Sp}(D^{-1}) = \left\{ \frac{1}{3}, 1, \frac{1}{2} \right\}$$

$$\text{Comme } P = \begin{pmatrix} \boxed{v_1} & \boxed{v_2} & \boxed{v_3} \end{pmatrix}_{B_C}$$

$$\text{Ainsi: } E_{\frac{1}{3}}(A^{-1}) = \text{Vect}(v_1), v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$E_1(A^{-1}) = \text{Vect}(v_2), v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$E_{\frac{1}{2}}(A^{-1}) = \text{Vect}(v_3), v_3 = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

$$D^{-1} = \begin{pmatrix} A^{-1}v_1 & A^{-1}v_2 & A^{-1}v_3 \\ \boxed{1/3} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{1/2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Ecrire: $A^{-1} = P \cdot D^{-1} \cdot P^{-1}$
c'est diagonaliser A^{-1} .

(c)

$$N = 2A + A^{-1} + 3I_3$$

on a:

$$A = P \cdot D \cdot P^{-1}$$

$$A^{-1} = P \cdot D^{-1} \cdot P^{-1}$$

$$I_3 = P \cdot P^{-1}$$

$$N = 2 \cdot P \cdot D \cdot P^{-1} + P \cdot D^{-1} \cdot P^{-1} + 3P \cdot P^{-1}$$

$$= P \underbrace{(2D + D^{-1} + 3I_3)}_{\Delta} P^{-1}$$

$$N = P \cdot \Delta \cdot P^{-1}$$

$$\Delta = 2D + D^{-1} + 3I_3 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} \frac{28}{3} & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & \frac{15}{2} \end{pmatrix}$$

est diagonale.

$$N = P \cdot \Delta \cdot P^{-1}$$

$$P = \begin{pmatrix} v_1 & v_2 & v_3 \\ \boxed{} & \boxed{} & \boxed{} \end{pmatrix}_{BC}$$

$$N \text{ est diagonalisable, } Sp(N) = \left\{ \frac{28}{3}, 6, \frac{15}{2} \right\}$$

$$E_{\frac{28}{3}}(N) = \text{Vect}(v_1)$$

$$E_6(N) = \text{Vect}(v_2)$$

$$E_{\frac{15}{2}}(N) = \text{Vect}(v_3)$$

(d) on a:

$$A = P \cdot D \cdot P^{-1}$$

$$\Rightarrow \forall n \in \mathbb{N},$$

$$A^n = P \cdot D^n \cdot P^{-1}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow D^n = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix}$$

$$P^{-1} = \frac{1}{\det P} \text{ }^t(\text{com } P)$$

$$\text{com } P = \begin{pmatrix} +\Delta_{11} & -\Delta_{12} & +\Delta_{13} \\ -\Delta_{21} & +\Delta_{22} & -\Delta_{23} \\ +\Delta_{31} & -\Delta_{32} & +\Delta_{33} \end{pmatrix}$$

$$\Delta_{11} = \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -1, \quad \Delta_{12} = \begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix} = 0, \quad \Delta_{13} = \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = 0$$

$$\Delta_{21} = \begin{vmatrix} 0 & 4 \\ -1 & -2 \end{vmatrix} = 4, \quad \Delta_{22} = \begin{vmatrix} 1 & 4 \\ 0 & -2 \end{vmatrix} = -2, \quad \Delta_{23} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$

$$\Delta_{31} = \begin{vmatrix} 0 & 4 \\ 1 & 1 \end{vmatrix} = -4, \quad \Delta_{32} = \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = 1, \quad \Delta_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\text{com } P = \begin{pmatrix} -1 & 0 & 0 \\ -4 & -2 & -1 \\ -4 & 1 & 1 \end{pmatrix}$$

$$\det P = \begin{vmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{vmatrix} \stackrel{\text{dev. par } C_1}{=} -1$$

$$P^{-1} = \begin{pmatrix} 1 & 4 & 4 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

$$A^n = P \cdot D^n \cdot P^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 4 & 4 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} 3^n & 4 \cdot 3^n & 4 \cdot 3^n \\ 0 & 2 & 1 \\ 0 & -2^n & -2^n \end{pmatrix}$$

$$A^n = \begin{pmatrix} 3^n & 4 \cdot 3^n - 2^{n+2} & 4 \cdot 3^n - 2^{n+2} \\ 0 & 2 - 2^n & 1 - 2^n \\ 0 & -2 + 2^{n+1} & -1 + 2^{n+1} \end{pmatrix}$$

5) $(x_n)_n, (y_n)_n, (z_n)_n$ des suites réelles

Vérifiant: $x_0 = y_0 = z_0 = 1$ et $\forall n \in \mathbb{N}$

$$\begin{cases} x_{n+1} = 3x_n + 4y_n + 4z_n \\ y_{n+1} = x_n - y_n - z_n \\ z_{n+1} = -x_n + 2y_n + 2z_n \end{cases}$$

on pose $X_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$

$$X_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 & 4 \\ 1 & -1 & -1 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

$$X_{n+1} = A_n \cdot X_n$$

Réque: $X_{n+1} = A \cdot X_n$

par récurrence on P. par

$$X_n = A^n \cdot X_0$$

avec $X_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\begin{cases} x_{n+1} = 3x_n + 4y_n + 4z_n \\ y_{n+1} = -z_n \\ z_{n+1} = 2y_n + 3z_n \end{cases}$$

Soit $X_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$.

$$X_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 & 4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

$$X_{n+1} = A_0 \cdot X_n$$

$$X_{n+1} = A \cdot X_n$$

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = X_n = A^n \cdot X_0 = \begin{pmatrix} 3^n & 4 \cdot 3^n - 2^{n+2} & 4 \cdot 3^n - 2^{n+2} \\ 0 & 2 - 2^n & 1 - 2^n \\ 0 & -2 + 2^{n+1} & -1 + 2^{n+1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathcal{L} : \begin{cases} x_n = 3^n + 4 \cdot 3^n - 2^{n+2} + 4 \cdot 3^n - 2^{n+2} \\ y_n = 2 - 2^n + 1 - 2^n \\ z_n = -2 + 2^{n+1} - 1 + 2^{n+1} \end{cases}$$

$$6] \quad m = -1$$

$$M = A_{-1} = \begin{pmatrix} 3 & 4 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P_M(x) = (3-x)^2(2-x)$$

3 v.p. double

2 v.p. simple.

$$(a) \quad m = -1 \in \mathbb{R} \setminus \{1, 2\}$$

d'après 2) M est inv.

$$\begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix}$$

Théorème de Cayley-Hamilton

$$\begin{aligned} P_M(x) &= (3-x)^2(2-x) \\ &= (x^2 - 6x + 9)(2-x) \\ &= -x^3 + 8x^2 - 21x + 18. \end{aligned}$$

$$\text{C-H: } P_M(M) = 0$$

$$P_M(M) = -M^3 + 8M^2 - 21M + 18I_3$$

$$P_M(M) = 0 \Rightarrow -M^3 + 8M^2 - 21M + 18I_3 = 0$$

$$\Rightarrow \frac{I_3}{3} = \frac{1}{18} [M^3 - 8M^2 + 21M]$$

$$= \frac{1}{18} M [M^2 - 8M + 21I_3]$$

$$M^2 = \frac{1}{18} [M^2 - 8M + 21I_3]$$

(b) Déterminons $E_3(M)$

$$E_3(M) = \ker(M - 3I_3)$$

$$= \left\{ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid M X = 3X \right\}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_3(M) \Leftrightarrow M X = 3X$$

$$\Leftrightarrow \begin{cases} 3x + 4y + 4z = 3x \\ -x + y - z = 3y \\ x + 2y + 4z = 3z \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -z \\ x = -2y - z = -y \\ x = -y \end{cases}$$

$$\Leftrightarrow X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ -y \end{pmatrix} = -y \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$E_3(M) = \text{Vect} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\dim E_3(M) = 1 \neq 2 = m_3$$

cl: M n'est pas diagonalisable.

comme $P_M(z) = (3-z)^2(z-2)$ est scindé sur \mathbb{C} ,
alors M est trigonalisable.