Exercice n 5 :

1) $A = \mathrm{mat}(f, B_1, B_2)$

avec $B_1 = (e_1, e_2, e_3) = \big((1,0,0), (0,1,0), (0,0,1)\big)$

$B_2 = (u_1, u_2) = \big((1,0), (0,1)\big)$

$$A = \begin{array}{c c} \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} & \\ \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 1 \end{pmatrix} & \begin{array}{c} u_1 \\ u_2 \end{array} \end{array}$$

$f(e_1) = f(1,0,0) = (1,0) = 1u_1$

$f(e_2) = f(0,1,0) = (-1,-2) = -u_1 - 2u_2$

$f(e_3) = f(0,0,1) = (2,1) = 2u_1 + u_2$

2-

* $A + 5I_2 \longrightarrow$ impossible

* $\underset{(3,2)}{{}^tA} \cdot \underset{(2,3)}{A} = \begin{pmatrix} 1 & 0 \\ -1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 5 \end{pmatrix}$$

* $A\,({}^tA - 2I_3) \longrightarrow$ impossible

3) $S = A \cdot {}^tA - 6I_2$

$\underset{(2,3)}{A} \cdot \underset{(3,2)}{{}^tA} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -2 \\ 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 4 \\ 4 & 5 \end{pmatrix}$$

$S = \begin{pmatrix} 6 & 4 \\ 4 & 5 \end{pmatrix} - 6I_2$

$= \begin{pmatrix} 0 & 4 \\ 4 & -1 \end{pmatrix}$

$$\cdot\ S^2 = \begin{pmatrix} 0 & 4 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & -4 \\ -4 & 17 \end{pmatrix}$$

$$S^2 + S - 10 I_2 = \begin{pmatrix} 16 & -4 \\ -4 & 17 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = 6 I_2$$

b) $S^2 + S - 10 I_2 = 6 I_2$

$S^2 + S = 16 I_2$

$\Rightarrow S(S + I_2) = 16 I_2$

$\Rightarrow S\left(\frac{1}{16}(S + I_2)\right) = I_2$

donc S est invert $S^{-1} = \frac{1}{16}(S + I_2)$.

4) a) $N = {}^tA\,A - 5 I_3$

$$= \begin{pmatrix} -4 & -1 & 2 \\ -1 & 0 & -4 \\ 2 & -4 & 0 \end{pmatrix}$$

Soit $a = (x, y, z) \in \mathbb{R}_3$

On pose $X = \underset{B_1}{mat}(a) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$Y = \underset{B_1}{mat}(g(a)) = NX = \begin{pmatrix} -4x - y + 2z \\ -x - 4z \\ 2x - 4y \end{pmatrix}$$

$\Rightarrow g(a) = g(x, y, z)$

$= (-4x - y + 2z,\ -x - 4z,\ 2x - 4y)$

d) $M = mat(g, B_1')$

$$= \begin{array}{cc} & \begin{array}{ccc} g(e_1') & g(e_2') & g(e_3') \end{array} \\ & \begin{pmatrix} -4 & -4 & -1 \\ 0 & 4 & 3 \\ 1 & 2 & -4 \end{pmatrix} \begin{array}{c} e_1' \\ e_2' \\ e_3' \end{array} \end{array}$$

$g(e_1') = g(0,1,1) = (1,-4,-4) = e_3' - 4e_1'$

$g(e_2') = g(0,0,1) = (2,-4,0)$
$= 2e_1 - 4e_2$
$= 2e_3' - 4(e_1' - e_2')$
$= 2e_3' - 4e_1' + 4e_2'$

$g(e_3') = g(e_1) = g(1,0,0)$
$= (-4,-1,2)$
$= -4e_1 - e_2 + 2e_3$
$= -4e_3' - e_1' + e_2' + 2e_3'$
$= -e_1' + 3e_2' - 4e_3'$

c) $M = mat(g, B_1')$

$N = mat(g, B_1)$

$P = Pass(B_1, B_1')$

$$\boxed{M = P^{-1} N P}$$

$$\left(N = P M P^{-1}\right)$$

$e_1' = e_2 + e_3$; $e_2' = e_3$; $e_3' = e_1$

b) $B_1' = (e_1', e_2', e_3')$

. Soit $\alpha, \beta, \gamma \in \mathbb{R}$

tq $\alpha e_1' + \beta e_2' + \gamma e_3' = 0$

$\alpha(e_2 + e_3) + \beta\, e_3 + \gamma\, e_1 = 0$

$\Rightarrow \gamma e_1 + \alpha e_2 + (\alpha + \beta) e_3 = 0$

or (e_1, e_2, e_3) est la Bc de $\mathbb{R}_3$ est libre $\Rightarrow \begin{cases} \gamma = 0 \\ \alpha = 0 \\ \alpha + \beta = 0 \end{cases}$

$\Rightarrow \boxed{\alpha = \beta = \gamma = 0}$

alors B_1' est une famille libre

et $\mathrm{card}(B_1') = 3 = \dim \mathbb{R}^3$

$\underline{\underline{cl}}$ B_1' est une base de $\mathbb{R}^3$.

c) $P = \mathrm{Pass}(B_1, B_1')$

$$P = \begin{array}{c} \begin{array}{ccc} e_1' & e_2' & e_3' \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}$$

$\begin{cases} e_1' = e_2 + e_3 \\ e_2' = e_3 \\ e_3' = e_1 \end{cases} \Rightarrow \begin{cases} e_2 = e_1' - e_3 \\ \quad = e_1' - e_2' \\ e_3 = e_2' \\ e_1 = e_3' \end{cases}$

Ex n°6

1) $u(P) = -2P + (X+1)P'$

Soient A et $B \in \mathbb{R}_n[X]$

$\alpha \in \mathbb{R}$

$u(\alpha A + B) = -2(\alpha A + B) + (X+1)(\alpha A + B)'$

$= -2\alpha A - 2B + (X+1)\alpha A' + (X+1)B'$

$= \alpha(-2A + (X+1)A') - 2B + (X+1)B'$

$= \alpha u(A) + u(B)$

$\Rightarrow u$ est une app linéaire

Soit $P \in \mathbb{R}_n[X] \Rightarrow \deg P \leq n$

$\deg(P') \leq n-1$

$\deg((X+1)P') \leq n-1+1$
$\leq n$

$\Rightarrow \deg(u(P)) \leq n$.

donc $u(P) \in \mathbb{R}_n[X]$

d'où u est un endomorphisme de $\mathbb{R}_n[X]$.

2) $A = mat(u, BC) =$

$$\begin{array}{c} \begin{array}{cccc} u(1) & u(X) \;\cdots & u(X^k) \;\cdots & u(X^n) \end{array} \\ \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -1 & \vdots & \vdots \\ \vdots & 0 & 0 & \vdots \\ \vdots & \vdots & k & \vdots \\ \vdots & \vdots & k-2 & \vdots \\ \vdots & \vdots & 0 & n \\ \vdots & 0 & \vdots & n-2 \\ 0 & & 0 & \end{pmatrix} \begin{array}{c} 1 \\ X \\ \vdots \\ \\ \\ \\ \\ X^n \end{array} \end{array}$$

$u(1) = -2$

$u(X) = -2X + (X+1)$

$= -X+1$

$u(X^k) = -2X^k + (X+1)kX^{k-1}$

$= -2X^k + kX^k + kX^{k-1}$

$= (k-2)X^k + kX^{k-1}$

$u(X^n) = (n-2)X^n + nX^{n-1}$

3) a) $P_k = X^{n-k}$, $0 \leq k \leq n$

$\mathcal{B} = (P_k ; 0 \leq k \leq n) = (X^n ; X^{n-1}, \dots, X ; 1)$

$P_0 = X^n$

$P_1 = X^{n-1}$ $\dots\dots$ $P_{n-1} = X$; $P_n = 1$.

Or $(1, X, \dots, X^n)$ est la BC de $\mathbb{R}_n[X]$.

alors cette famille est libre.

Soient $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ / $\sum_{i=0}^{n} \alpha_i P_i = 0$

$\iff \alpha_0 P_0 + \alpha_1 P_1 + \dots + \alpha_n P_n = 0$

$\iff \alpha_0 X^n + \alpha_1 X^{n-1} + \dots + \alpha_{n-1} X + \alpha_n = 0$

$\iff \alpha_i = 0 \ \forall i = 0, \dots, n.$

$\implies \mathcal{B}$ est une famille libre

et $\text{card}(\mathcal{B}) = n+1 = \dim(\mathbb{R}_n[X])$

Donc $\mathcal{B}$ est une base de $\mathbb{R}_n[X]$.

b) $S = \text{Pass}(\mathcal{B}_C, \mathcal{B})$

$$
\begin{array}{c}
\begin{array}{ccccc} X^n & X^{n-1} & \cdots & X & 1 \end{array} \\
= \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & & 1 & 0 \\
\vdots & \vdots & & 0 & \vdots \\
0 & 1 & & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\begin{array}{c} 1 \\ X \\ \vdots \\ X^{n-1} \\ X^n \end{array}
\end{array}
$$

Correction exercice 6. Serie Calculs matriciels

1. $u(P) = -2P + (X+1)P'$

Soit A et $B \in \mathbb{R}_n[X]$

$\alpha \in \mathbb{R}$

$u(\alpha A + B) = -2(\alpha A + B) + (X+1)(\alpha A + B)'$

$= -2\alpha A + (X-1)\alpha A' - 2B + (X-1)B'$

$= \alpha(-2A + (X-1)A' - 2B + (X-1)B')$

$= \alpha\, u(A) + u(B)$

$\rightarrow$ u est une application linéaire

Soit $P \in \mathbb{R}_n[X] \Rightarrow \deg P \leq n$

$\deg(P') \leq n-1$

$\deg((X+1)P') \leq n-1+1$

$\leq n$

$\Rightarrow \deg(u(P)) \leq n$

donc $u(P) \in \mathbb{R}_n[X]$

d'où u est un endomorphisme de $\mathbb{R}_n[X]$

2) $A = \text{mat}(u, BC) =$

$$\begin{array}{c c} \begin{array}{cccccc} u(1) & u(X) & \cdots & u(X^k) & \cdots & u(X^n) \end{array} & \\ \begin{pmatrix} -2 & 1 & \cdots & 0 & \cdots & 0 \\ 0 & -1 & \cdots & \vdots & & \vdots \\ \vdots & \vdots & & k & & \vdots \\ 0 & 0 & & k-2 & & n \\ 0 & 0 & & 0 & & n-2 \end{pmatrix} & \begin{array}{c} 1 \\ X \\ \vdots \\ X^k \\ \vdots \\ X^n \end{array} \end{array}$$

. $u(1) = -2$

. $u(X) = -2X + (X+1) = -X+1$

$u(X^k) = -2X^k + kX^k + kX^{k-1} = (k-2)X^k + kX^{k-1}$

$u(X^n) = (n-2)X^n + n(X)^{n-1}$

3) a) $P_k = X^{n-k}$, $0 \leq k \leq n$

$B = (P_k \; ; \; 0 \leq k \leq n)$

$$\left.\begin{array}{l} P_0 = X^n \\ P_1 = X^{n-1} \\ \vdots \\ P_{n-1} = X \\ P_n = 1 \end{array}\right\}$$ base canonique de $\mathbb{R}_n[X]$ inversée

on $(1, X, \dots, X^n)$ est la Base de $\mathbb{R}_n[X]$

alors cette famille est libre

Soient $\alpha_0 \dots \alpha_n \in \mathbb{R} \,/\, \sum_{i=0}^{n} \alpha_i P_i = 0$

$\iff \alpha_0 P_0 + \alpha_1 P_1 + \dots \alpha_n P_n = 0$

$\iff \alpha_0 X^n + \alpha_1 X^{n-1} + \dots + \alpha_{n-1} X + \alpha_n = 0$

$\iff \alpha_i = 0 \quad \forall i \in \{0 ; \dots n\}$

$\implies B$ est une famille libre et $\mathrm{card}(B) = n+1 = \dim \mathbb{R}_n[X]$

Donc B est une Base de $\mathbb{R}_n[X]$

b) $S = \mathrm{Pass}(B_c, B)$

$$= \begin{array}{c} \begin{array}{ccccc} X^n & X^{n-1} & \dots & X & 1 \end{array} \\ \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & & 1 & 0 \\ \vdots & \vdots & & 0 & \vdots \\ \vdots & 0 & & \vdots & \vdots \\ 0 & 1 & & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} 1 \\ X \\ \vdots \\ \vdots \\ X^{n-1} \\ X^n \end{array}$$

c) $M = mat_B(u)$

$$= \begin{array}{c} \begin{array}{cccc} u(x^n) & u(x^{n-1}) & \dots & u(x^n) \end{array} \\ \left(\begin{array}{cccc} & 0 & & 0 \\ n-2 & n-3 & & \\ n & n-1 & & \\ 0 & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & & 0 \\ & & & -2 \end{array} \right) \end{array} \begin{array}{c} X^n \\ X^{n-1} \\ \vdots \end{array}$$

$u(x^n) = -2X^n + (X+1) n X^{n-1}$

$= (n-2) X^n + n (X^{n-1})$

$= (n-2) P_0 + n P_1$

$u(x^{n-1}) = (n-3) P_1 + (n-1) P_2$

$u(x) = -P_{n-1} + P_n$

$u(1) = -2P_n$

d) $\left.\begin{array}{l} M = mat(u, B) \\ A = mat(u, B_c) \\ S = Pas(B_c, B \end{array}\right\} M = S^{-1} A . S$