X et Y sont indep

EX 1 :

1/ $f(x) = \frac{2x}{\theta^2} 1_{[0,\theta]}(x)$

$E(X) = \int_{\mathbb{R}} x f(x)\, dx$

$= \int_0^\theta x \frac{2x}{\theta^2}\, dx$

$= \frac{2}{\theta^2} \int_0^\theta x^2\, dx$

$= \frac{2}{\theta^2} \left[\frac{1}{3}x^3\right]_0^\theta$

$= \frac{2}{\theta^2} \cdot \frac{\theta^3}{3} = \frac{2}{3}\theta$.

Exercice 2

$1_A(x) = \begin{cases} 1 & \text{si } x \in A \\ 0 & \text{sinon} \end{cases}$

$E(X^2) = \int_{\mathbb{R}} x^2 f(x)\, dx$

$= \frac{2}{\theta^2} \int_0^\theta x^3\, dx$

$= \frac{2}{\theta^2} \left[\frac{1}{4}x^4\right]_0^\theta$

$= \frac{2}{\theta^2} \cdot \frac{1}{4}\theta^4 = \frac{1}{2}\theta^2$.

$V(X) = E(X^2) - E^2(X)$

$= \frac{\theta^2}{2} - \left(\frac{2}{3}\theta\right)^2$

$= \frac{\theta^2}{2} - \frac{4}{9}\theta^2$

$= \frac{9\theta^2 - 8\theta^2}{18}$

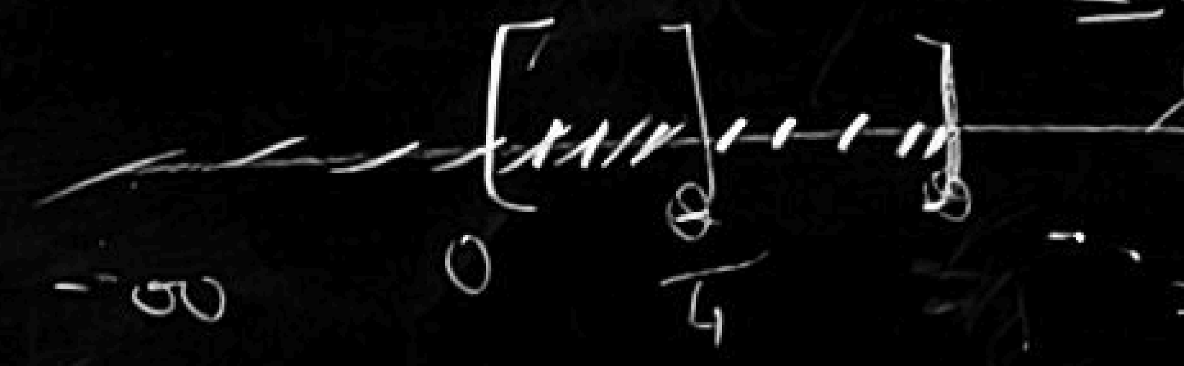
$= \frac{\theta^2}{18}$

$P\left(X < \frac{\theta}{4}\right) = \int_{-\infty}^{\frac{\theta}{4}} f(x)\, dx$

$= \int_{-\infty}^{\frac{\theta}{4}} \frac{2x}{\theta^2} 1_{[0,\theta]}\, dx$

$= \frac{2}{\theta^2} \int_0^{\frac{\theta}{4}} x\, dx = \frac{2}{\theta^2}\left[\frac{1}{2}x^2\right]_0^{\frac{\theta}{4}}$

$= \frac{\cancel{2}}{\theta^2} \cdot \frac{1}{\cancel{2}} \cdot \frac{\theta^2}{16}$



$= \frac{1}{16}$.

Ex 2:

1) $\int_{\mathbb{R}} f(t)\,dt = 1$

$\Leftrightarrow c\int_{\mathbb{R}} t e^{-t} \mathbb{1}_{]0,+\infty[}\,dt = 1$

$\Leftrightarrow c\int_0^{+\infty} t e^{-t}\,dt$

On pose : $u = t \longrightarrow u' = 1$

$v' = e^{-t} \longrightarrow v = -e^{-t}$

$\Rightarrow c\left(\left[-te^{-t}\right]_0^{+\infty} + \int_0^{+\infty} e^{-t}\,dt\right) = 1$

$\Leftrightarrow c\int_0^{+\infty} e^{-t}\,dt = 1$

$\Leftrightarrow c = 1$

2) $E(Y) = \int_{\mathbb{R}} y f(y)\,dy$

$= \int_0^{+\infty} y^2 e^{-y}\,dy$

$$E(y^2) = \underbrace{\left[-y^3 e^{-y}\right]_0^{+\infty}}_{0} + 3\underbrace{\int_0^{+\infty} y^2 e^{-y}\,dy}_{2}$$

$$= 3 \times 2$$

$$= 6$$

$$V(y) = E(y^2) - E^2(y)$$

$$= 6 - (2^2) = 2$$

3) $\mathbb{P}(y < 4 / y > 1)$

$$= \frac{P(y < 4, y > 1)}{P(y > 1)}$$

$$= \frac{\mathbb{P}(1 < y < 4)}{\mathbb{P}(y > 1)}$$

$$P(1 < y < 4) = \int_1^4 y e^{-y} \mathbb{1}_{]0,+\infty[}\,dy$$

$$= \int_1^4 y e^{-y}$$

$$= \left[-y e^{-y}\right]_1^4 + \left[-e^{-y}\right]_1^4$$

$$= -4e^{-4} + e^{-1} - e^{-4} + e^{-1}$$

$$= -5e^{-4} + 2e^{-1}$$

$$= \begin{cases} 0 & \text{si } y \leq 0 \\ \int_0^y t e^{-t} & \text{si } y > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{si } y \leq 0 \\ \left[-te^{-t}\right]_0^y + \left[-e^{-t}\right]_0^y & \text{si } y > 0 \end{cases}$$

$$-ye^{-y} + \underline{0e^{-0}}$$

$$= \begin{cases} 0 & \text{si } y \leq 0 \\ -ye^{-y} - e^{-y} + 1 & \text{si } y > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{si } y \leq 0 \\ 1 - e^{-y}(y+1) & \text{si } y > 0 \end{cases}$$

$$= F_Y(y)$$

$$F_Z(z) = F_Y(z^2)$$

$$= \begin{cases} 1 - e^{-z^2}(z^2+1) & \text{si } z^2 > 0 \\ \cancel{0} & \cancel{\text{si } z^2 \leq 0} \end{cases}$$

$$\Rightarrow f_Z(z) = -\left[-2z e^{-z^2}(z^2+1) + 2z e^{-z^2}\right]$$

$$= -\left[-2z^3 e^{-z^2}\right]$$

$$= 2z^3 e^{-z^2} \underbrace{\mathbb{1}_{\mathbb{R}}}_{=1}$$

$$f_Z(z) = 2z^3 e^{-z^2}$$

exercice 3

Soit $f_X(x) = \frac{1}{2} 1_{]-1,1[}(x)$

Soit $Y = e^{x}$

$F_Y(y) = P(Y \leq y) = P(e^X \leq y)$

$= P(X \leq \ln(y))$

$= F_X(\ln(y))$

$f_Y(y) = (F_X(\ln(y)))'$

$= \ln'(y) \times f_X(\ln(y))$

$= \frac{1}{y} \times \frac{1}{2} 1_{]-1,1[}(\ln y)$

$= \frac{1}{2y} \times 1_{]e^{-1}, e^{1}[}(y)$

$$E(Y) = \int_{\mathbb{R}} y \, f(y)$$

$$= \int_{e^{-1}}^{e^{1}} \frac{1}{2\cancel{y}} \cancel{y} \, dy$$

$$= \int_{e^{-1}}^{e^{1}} \frac{1}{2} \, dy$$

$$= \frac{1}{2}\left(e^{1} - e^{-1}\right)$$

$$E(Y^2) = \int_{e^{-1}}^{e^{1}} \frac{y}{2} \, dy$$

$$= \frac{1}{4}\left[y^2\right]_{e^{-1}}^{e^{1}}$$

$$= \frac{1}{4}\left(e^{2} - e^{-2}\right)$$

$$V(Y) = \frac{1}{4}\left(e^{2} - e^{-2}\right) - \frac{1}{4}\left(e^{1} - e^{-1}\right)^2$$

$$= \frac{1}{4}\left(e^{2} - e^{-2} - e^{2} - e^{-2} + 2e^{0}\right)$$

$$= \frac{1}{4}\left(-2e^{-2} + 2\right)$$

$$= \frac{1}{2} - \frac{e^{-2}}{2}$$

exercice 4 :

$$y = x^2$$

1) $X \sim \mathcal{U}([-1,1])$

$$Y = g(X)$$

$$\Leftrightarrow X = g^{-1}(Y)$$

$$f_Y(y) = f_X(g^{-1}(y)) \times |(g^{-1}(y))'|$$

$$y = x^2 \Rightarrow g(x) = x^2$$

$$g(x) = y$$

$$\Leftrightarrow x^2 = y$$

$$\Leftrightarrow x \begin{cases} \to -\sqrt{y} \\ \to \sqrt{y} \end{cases}$$

$$f_Y(y) = f_X(-\sqrt{y}) \times |(-\sqrt{y})'| + f_X(\sqrt{y}) |(\sqrt{y})'|$$

$$= \frac{1}{2} \mathbb{1}_{[-1,1]}(-\sqrt{y}) \times \left|-\frac{1}{2\sqrt{y}}\right| + \frac{1}{2} \mathbb{1}_{(-1,1)}(\sqrt{y}) \left|\frac{1}{2\sqrt{y}}\right|$$

$$= \frac{1}{4\sqrt{y}} \left(\mathbb{1}_{(-1,1)}(-\sqrt{y}) + \mathbb{1}_{(-1,1)}(\sqrt{y}) \right)$$

$$-1 \le \sqrt{y} \le 1$$

$$\Rightarrow \boxed{0 < y < 1}$$

$$f_Y(y) = \frac{1}{4\sqrt{y}}\left[\underbrace{\mathbb{1}_{[0,1]}(y) + \mathbb{1}_{[0,1]}(y)}_{2}\right]$$

$$= \frac{1}{2\sqrt{y}}\,\mathbb{1}_{(0,1)}(y)$$

2 – $X \sim N(m, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(f_X(-\sqrt{y}) + f_X(\sqrt{y})\right)$$

$$= \frac{1}{2\sqrt{y}} \times \frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{(-\sqrt{y}-m)^2}{2\sigma^2}} + e^{-\frac{(\sqrt{y}-m)^2}{2\sigma^2}}\right]$$

$$= \frac{1}{2\sigma\sqrt{2\pi y}}\left(e^{-\frac{(\sqrt{y}+m)^2}{2\sigma^2}} + e^{-\frac{(\sqrt{y}-m)^2}{2\sigma^2}}\right)$$

Exercice 6 :

X et Y sont indep

$$\begin{cases} X \sim \mathcal{E}(\lambda) \\ Y \sim \mathcal{E}(\mu) \\ f_X(x) = \lambda e^{-\lambda x} \mathbb{1}_{]0,+\infty[}(x) \\ f_Y(y) = \mu e^{-\mu y} \mathbb{1}_{]0,+\infty[}(y) \end{cases}$$

$U = \min(X, Y)$

$$F_U(u) = \mathbb{P}(U < u) = 1 - \mathbb{P}(U \geq u)$$

$$= 1 - \mathbb{P}(\min(X,Y) \geq u)$$

$$= 1 - \mathbb{P}(X \geq u, Y \geq u)$$

$$= 1 - \mathbb{P}(X \geq u) \times \mathbb{P}(Y \geq u)$$

$$= 1 - (1 - F_X(u))(1 - F_Y(u))$$

$$F_X(x) = \mathbb{P}(X \le x)$$

$$= \int_{-\infty}^{x} f_X(u)\,du$$

$$= \int_{-\infty}^{x} \lambda e^{-\lambda t} \mathbb{1}_{]0,+\infty[}(t)\,dt$$

$$= \begin{cases} 0 & \text{si } x \le 0 \\ \int_0^x \lambda e^{-\lambda t}\,dt & \text{si } x > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{si } x \le 0 \\ \left[-e^{-\lambda t}\right]_0^x & \text{si } x > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{si } x \le 0 \\ 1 - e^{-\lambda x} & \text{si } x > 0 \end{cases}$$

$$F_U(u) = 1 - (1 - F_X(u))(1 - F_Y(u))$$

$$= 1 - e^{-\lambda u} \times e^{-\mu u} = 1 - e^{-u(\lambda + \mu)}$$

$$\Rightarrow \quad U \sim \mathcal{E}(\lambda + \mu)$$