Cyber-Physical Systems (CSC.T431)

Dynamical Systems (2)

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Agenda

- Dynamical Systems (2)

Course Support & Material

- Slides: OCW-i
- Course Web: https://titech-cps.github.io
- Course Slack: titech-cps.slack.com

Linear Components

Definition

- A continuous-time component H with input variables I, output variables O, and state variables S is said to be a *linear component* if
  1. for every $y \in O$, the expression h_y is a linear expression over $I \cup S$, and
  2. for every $x \in S$, the expression f_x is a linear expression over $I \cup S$.
- Note
  - $h_y = \mathbf{E}_O(y)$ and $f_x = \mathbf{E}_S(x)$
  - A *linear expression* over a set of variables V is an expression of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ where $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n$ are constants.

Linear Components
Examples

- Linear
  - NetHeat: $I = \{h+, h_-\}$, $O = h_{out}$, $h_{out} = h_+ - h_-$
  - Car: $I = \{F\}$, $S = \{x, v\}$, $\dot{x} = v$; $\dot{v} = (F - kv)/m$
  - Helicopter: $I = \{T\}$, $S = \{s\}$, $\dot{s} = T/I$
- Nonlinear
  - Car on a graded road: $I = \{F, \theta\}$, $S = \{x, v\}$, $\dot{x} = v$; $\dot{v} = (F - kv - mg\sin\theta)/m$
    - If you use $d = \sin\theta$ as an input instead of θ, the component can be treated as linear.
  - Simple Pendulum: $I = \{u\}$, $S = \{\varphi, v\}$, $\dot{\varphi} = v$; $\dot{v} = -(g/l)\sin\varphi + u/(ml^2)$

Affine Expressions

- Expressions of the form $a_1x_1 + a_2x_2 + \cdots a_nx_n + a_{n+1}$ with $a_{n+1} \neq 0$ are called *affine expressions*.
    - An affine expression is not a linear expression.
- If some of h_y or f_x in H are affine expressions, H is not a linear component.
- Ex. State variable t that models the time: $\dot{t} = 1$

Matrix-Based Representation

- Let H be a linear component with input, output, and state variables
  - $I = \{u_1, \ldots, u_m\}$, $O = \{y_1, \ldots, y_k\}$, and $S = \{x_1, \ldots, x_n\}$.
- We regard I, O, and S as m-, k- and n-dimensional column vectors
  - $I = (u_1 \ \cdots \ u_m)^T$, $O = (y_1 \ \cdots \ y_m)^T$, and $S = (x_1 \ \ldots \ x_n)^T$.
- The dynamics of H can be represented using matrices A (dimension $n \times n$), B (dimension $n \times m$), C (dimension $k \times n$), and D (dimension $k \times m$) as:
  - $\dot{S} = AS + BI \quad$ ($\dot{x}_i = A_{i,1}x_1 + \cdots + A_{i,n}x_n + B_{i,1}u_1 + \cdots + B_{i,m}u_m$ for $i \in \{1,\ldots,n\}$)
  - $O = CS + DI \quad$ ($y_j = C_{j,1}x_1 + \cdots + C_{j,n}x_n + D_{j,1}u_1 + \cdots + D_{j,m}u_m$ for $j \in \{1,\ldots,k\}$)

Matrix-Based Representation

Ex. Car

- $I = (F), O = (v), S = (x\ v)^T$
- $A = \begin{pmatrix} 0 & 1 \\ 0 & -k/m \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}, C = (0 \quad 1), D = (0)$

- $\dot{S} = AS + BI \quad \begin{cases} \dot{x} = 0x + 1v + 0F \\ \dot{v} = 0x + (-k/m)v + (1/m)F \end{cases}$
- $O = CS + DI \quad \{ v = 0x + 1v + 0F$

Linear Response

Linearity of Input-Output Transformation

- Let H be a continuous-time component with Lipschitz-continuous dynamics with $I = \{x\}$ and $O = \{y\}$.
- Given an input signal $\bar{x} : \mathsf{time} \to \mathsf{real}$, there is a unique output signal $\bar{y} : \mathsf{time} \to \mathsf{real}$ corresponding to the execution of H on $\bar{x}$.
- Thus we can regard H as a transformer (function) on signals: $\bar{x} \mapsto \bar{y}$.
- The transformation is *linear* if the following hold.
  - If $\bar{x}_1 \mapsto \bar{y}_1$ and $\bar{x}_2 \mapsto \bar{y}_2$ then $\alpha\bar{x}_1 + \beta\bar{x}_2 \mapsto \alpha\bar{y}_1 + \beta\bar{y}_2$ $(\alpha, \beta \in \mathsf{real})$.

Linear Response

Linearity of Input-Output Transformation

- Theorem: Let H be a linear component with input variables I and output variables O. For all input signals $\bar{I}_1$ and $\bar{I}_2$ and constants $\alpha, \beta \in$ real, if the output signals generated by H from the <u>initial state 0</u> in response to $\bar{I}_1$ and $\bar{I}_2$ are $\bar{O}_1$ and $\bar{O}_2$ respectively, then the output signal generated by H from the initial state 0 in response to the input signal $\alpha\bar{I}_1 + \beta\bar{I}_2$ is $\alpha\bar{O}_1 + \beta\bar{O}_2$.

- Proof: Suppose that the dynamics of H is given by $\dot{S} = AS + BI$ and $O = CS + DI$ where S is the set of state variables and A, B, C, D are constant matrices. Suppose that the initial state s_0 is 0.

Linear Response

Linearity of Input-Output Transformation

- Proof (cont'd): By solving $\dot{S} = AS + BI$ with $s_0 = 0$, we have $\bar{S}(t) = \int_0^t e^{A(t-\tau)} B\bar{I}(\tau)d\tau$ for any I. (See a few pages later)
- Thus, if $\bar{S}_1$ and $\bar{S}_2$ are state response signals corresponding to $\bar{I}_1$ and $\bar{I}_2$ respectively, then the state response signal corresponding to $\alpha\bar{I}_1 + \beta\bar{I}_2$ is $\alpha\bar{S}_1 + \beta\bar{S}_2$.
- Since $O = CS + DI$, the output signal corresponding to $\alpha\bar{I}_1 + \beta\bar{I}_2$ is clearly $\alpha\bar{O}_1 + \beta\bar{O}_2$.

Solving Linear Differential Equations

$\dot{S} = AS$ with $\bar{S}(0) = s_0$

- Let's solve $\dot{S} = AS$ with $\bar{S}(0) = s_0$.
- We define a sequance of signals $\bar{S}_0, \bar{S}_1, \ldots$ inductively as: $\bar{S}_0(t) = s_0$ and $\bar{S}_{n+1}(t) = s_0 + \int_0^t A\bar{S}_n(\tau)d\tau$. The sequence <u>approximate</u> the solution.
- Let us calculate first several signals.
  - $\bar{S}_1(t) = s_0 + \int_0^t A\bar{S}_0(\tau)d\tau = s_0 + \int_0^t As_0 d\tau = s_0 + Ats_0 = (I + At)s_0$
  - $\bar{S}_2(t) = s_0 + \int_0^t A(I + A\tau)s_0 d\tau = (I + At + \frac{A^2t^2}{2})s_0$
  - $\bar{S}_3(t) = s_0 + \int_0^t A(I + A\tau + \frac{A^2\tau^2}{2})s_0 d\tau = (I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{2 \cdot 3})s_0$

$$\boxed{\frac{d}{dt}\bar{S}_{n+1}(t) = A\bar{S}_n(t)}$$

Solving Linear Differential Equations

$\dot{S} = AS$ with $\bar{S}(0) = s_0$

- By mathematical induction, we have $\bar{S}_n(t) = (\sum_{j=0}^{n} \frac{(At)^j}{j!})s_0$.
- The sequence converges to the unique solution:
  $\bar{S}(t) = \lim_{n\to\infty} \bar{S}_n(t) = (\sum_{j=0}^{\infty} \frac{(At)^j}{j!})s_0$.
- From $e^a = \sum_{j=0}^{\infty} \frac{a^j}{j!}$ for a real number a, we define the *matrix exponential* e^A for a matrix A as $\sum_{j=0}^{\infty} \frac{A^j}{j!}$.
- Thus, the solution of $\dot{S} = AS$ with $\bar{S}(0) = s_0$ is $\bar{S}(t) = e^{At}s_0$.

Solving Linear Differential Equations

$\dot{S} = AS + BI$ with $\bar{S}(0) = s_0$

- By solving $\dot{S} = AS + BI$ with $\bar{S}(0) = s_0$, we have
  $$\bar{S}(t) = e^{At}s_0 + \int_0^t e^{A(t-\tau)}B\bar{I}(\tau)d\tau.$$

- The next slide shows the proof for single-dimensional case.

Solving Linear Differential Equations

Single-Dimensional Case: $\dot{s} = as + bu$ with $\bar{s}(0) = s_0$

u : input variable, s : state variable

$$\frac{d\bar{s}(t)}{dt} = a\bar{s}(t) + b\bar{u}(t)$$

$$\Leftrightarrow \frac{d\bar{s}(t)}{dt} - a\bar{s}(t) = b\bar{u}(t)$$

$$\Leftrightarrow e^{-at}\frac{d\bar{s}(t)}{dt} - e^{-at}a\bar{s}(t) = e^{-at}b\bar{u}(t)$$

$$\Leftrightarrow e^{-at}\frac{d\bar{s}(t)}{dt} + \frac{de^{-at}}{dt}\bar{s}(t) = e^{-at}b\bar{u}(t)$$

$$\Leftrightarrow \frac{de^{-at}\bar{s}(t)}{dt} = e^{-at}b\bar{u}(t)$$

$$\Leftrightarrow e^{-at}\bar{s}(t) = \int_0^t e^{-a\tau}b\bar{u}(\tau)d\tau + C$$

$$\Leftrightarrow \bar{s}(t) = \int_0^t e^{a(t-\tau)}b\bar{u}(\tau)d\tau + e^{at}C$$

$$\therefore \bar{s}(t) = e^{at}s_0 + \int_0^t e^{a(t-\tau)}b\bar{u}(\tau)d\tau$$

Matrix Exponential e^A

Examples

- Diagonal matrix $\mathbf{D}(a_1, a_2, \ldots, a_n)$
  - $$= \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$
- $e^{\mathbf{D}(a_1, a_2, \ldots, a_n)} = \mathbf{D}(e^{a_1}, e^{a_2}, \ldots, e^{a_n})$

- Two dimensional matrix:
  $$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$
- $$e^A = I + A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

Matrix Exponential e^A

Properties

- $e^0 = I$
- $e^{aA}e^{bA} = e^{(a+b)A}$
- $e^A e^{-A} = I$
- $AB = BA \rightarrow e^A e^B = e^B e^A = e^{A+B}$ (← does not holds)
- If P is invertible, $e^{PAP^{-1}} = Pe^A P^{-1}$
- $e^{A^T} = (e^A)^T$

Eigenvalues and Eigenvectors

Let's think back to your first year of undergrad.

- Let A be an $n \times n$-matrix. If $Ax = \lambda x$ holds for a scalar λ and a non-zero vector x of dimension n, then λ is called an *eigenvalue* of A and x is called an *eigenvector* of A corresponding to λ.
- A has at most n distinct eigenvalues. They correspond to the solutions of the characteristic equation $\det(A - \lambda I) = 0$ where $\det$ gives the determinant of a matrix.
- If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all distinct, the corresponding eigenvectors $x_1, x_2, \ldots, x_n$ are linearly independent.

Eigenvalues and Eigenvectors

Example

- Let $A = \begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix}$.
  - Since $\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 6 = (\lambda - 6)(\lambda - 1)$, the eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = 1$.
  - By solving $\begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 6 \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$ and $\begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$, we obtain the eigenvectors $x_1 = (x_{11}\ x_{12})^T = (3\ 1)^T$ and $x_2 = (x_{21}\ x_{22})^T = (2\ \ -1)^T$.

Similarity Transformations

How to calculate matrix exponentials

- Consider the dynamical system H given by $\dot{S} = AS$ with $\bar{S}(0) = s_0$ where S is the n-dimensional state vector, A is an $n \times n$-matrix, and s_0 is the initial state.
- Suppose P is an invertible $n \times n$-real matrix and P^{-1} is its inverse. Consider $S' = P^{-1}S$ and $J = P^{-1}AP$. Such matrices A and J are said to be similar.
- Similarity transformation: Define a dynamical system H' with $\dot{S}' = \frac{d}{dt}(P^{-1}S) = P^{-1}\dot{S} = P^{-1}AS = P^{-1}APS' = JS'$ and $\bar{S}'(0) = P^{-1}\bar{S}(0) = P^{-1}s_0$.
- By solving this, we obtain $\bar{S}'(t) = e^{Jt}\bar{S}'(0)$. Thus $\bar{S}(t) = Pe^{Jt}P^{-1}s_0$.

Similarity Transformations

How to calculate matrix exponentials

- Suppose that A has n linearly independent real eigenvectors $x_1, x_2, \ldots, x_n$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- If $P = (x_1 \; x_2 \; \cdots \; x_n)$ is an $n \times n$-matrix whose columns are $x_1, x_2, \ldots, x_n$, then $J = P^{-1}AP = \mathbf{D}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.
- Thus $e^{Jt} = \mathbf{D}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})$

- The state response given by $\dot{S} = AS$ with $\bar{S}(0) = s_0$ is $\bar{S}(t) = P\mathbf{D}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})P^{-1}s_0$.

Similarity Transformations

Example

- Consider a dynamical system whose state response is given by $\dot{s}_1 = 4s_1 + 6s_2;\ \dot{s}_2 = s_1 + 3s_2.$
- The eigenvectors of the 2×2-matrix A used to express the response are $x_1 = (3\ 1)^T$ and $x_2 = (2\ \ -1)^T$ with corresponding eigenvalues 6 and 1.
- So if $P = (x_1\ x_2)$, then we obtain
$$P^{-1}AP = \begin{pmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}.$$
- Thus $\bar{s}_1(t) = ((3e^{6t} + 2e^{t})s_{01} + 6(e^{6t} - e^{t})s_{02})/5$ and $\bar{s}_2(t) = ((e^{6t} - e^{t})s_{01} + (2e^{6t} + 3e^{t})s_{02})/5.$

Stability

- Consider an n-dimensional linear system H given by $\dot{S} = AS$ with $\bar{S}(0) = s_0$.
- The response of the system: $\bar{S}(t) = e^{At}s_0$
- A state s_e is an equilibrium of H if $As_e = 0$. So we can obtain s_e by solving the system of n linear equations.
- Clearly the state 0 is an equilibrium. If A is invertible (i.e., $\mathrm{rank}(A) = n$), 0 is the unique solution of $As_e = 0$ and is the only equilibrium.

Stability

- Consider that there exists a nonzero equilibrium s_e of H. Define H' with the state vector $S' = S - s_e$.
- The dynamics of H' is given by $\dot{S}' = AS'$ and $\bar{S}'(t) = \bar{S}(t) - s_e$ holds for each time t. The state 0 is an equilibrium of H'.
- This means that if we know how to analyze whether the equilibrium 0 is stable, the same technique can be used to analyze whether an arbitrary equilibrium is stable.
- Abbreviation:
  - "H is (asymptotically) stable" = "the state 0 of H is a (asymptotically) stable equilibrium"

Lyapunov Stability

- A linear system H is *stable* if
  $\forall \varepsilon > 0. \exists \delta > 0. \forall s_0 \in [\![Init]\!] . \forall t \in \text{time} . (\|s_0\| < \delta \rightarrow \|e^{At} s_0\| < \varepsilon)$.
- H is *asymptotically stable* if it is stable and
  $\exists \delta > 0. \forall s_0 \in [\![Init]\!] . (\|s_0\| < \delta \rightarrow \lim_{t \to \infty} e^{At} s_0 = 0)$.
  - The value of δ that makes the condition true is called the *region of attraction*.

Stability of Single-Dimensional System

- Consider a single-dimensional linear system whose dynamics is given by $\dot{x} = ax$ with $\bar{x}(0) = x_0$. For each time t, $\bar{x}(t) = e^{at}x_0$.
- If $a < 0$, the system is stable and asymptotically stable.
- if $a = 0$, the system is stable but not asymptotically stable.
- if $a > 0$, the system is not stable.

Diagonal State Dynamics Matrix

- Consider an n-dimensional linear system given by $\dot{S} = AS$ with $\bar{S}(0) = s_0$ where $A = \mathbf{D}(a_1, a_2, \ldots, a_n)$ and $s_0 = (s_{01}\ s_{02}\ \cdots\ s_{0n})^T$.
- Since A is diagonal, $\bar{S}(t) = (e^{a_1 t} s_{01}\ e^{a_2 t} s_{02}\ \cdots\ e^{a_n t} s_{0n})^T$.

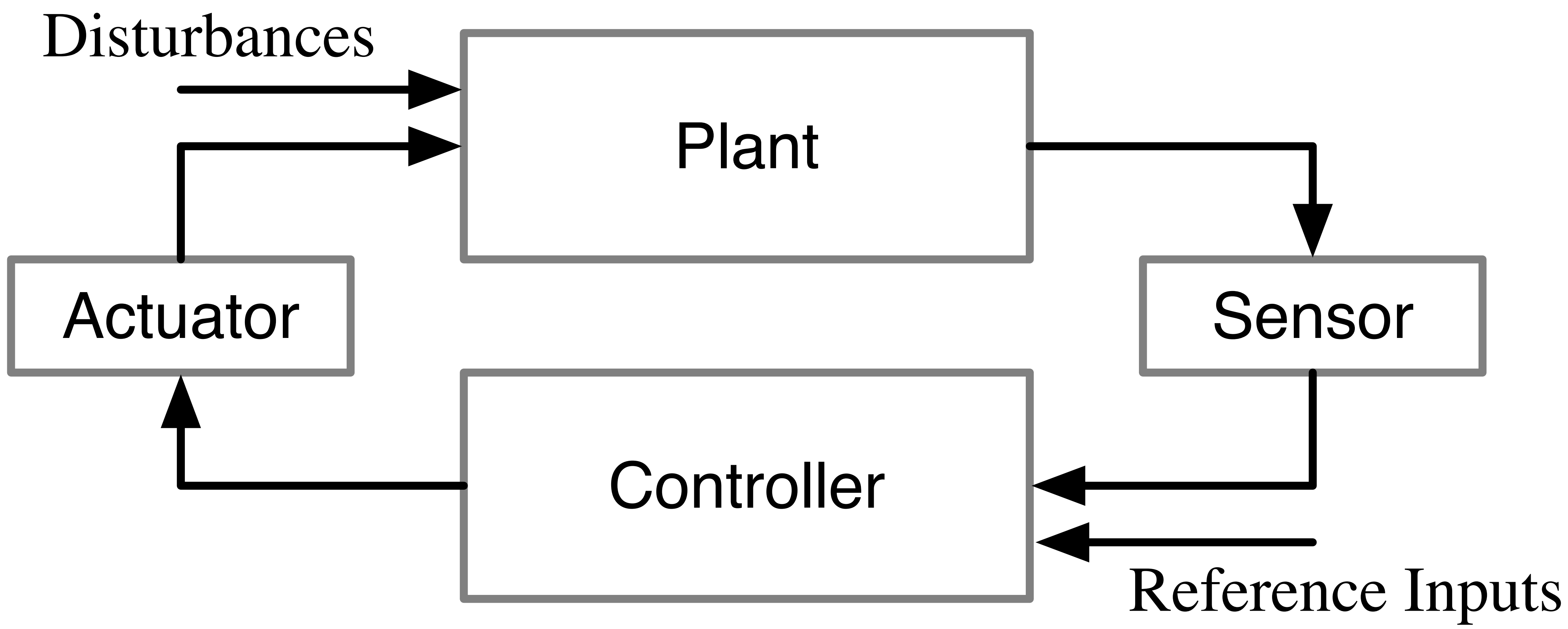
- If none of a_i is positive, then the system is stable
- If all of a_i is negative, then the system is asymptotically stable.
- If $a_i > 0$ for some i, the system is not stable.

Diagonalizable State Dynamics Matrix

- Given an n-dimensional linear system H with $\dot{S} = AS$, we choose an invertible matrix P and define H' with $\dot{S}' = JS'$ where $J = P^{-1}AP$ and $S' = P^{-1}S$.
- Proposition: H is (asymptotically) stable iff H' is (asymptotically) stable.
- If A has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we can choose P so that $J = \mathbf{D}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Thus, we can analyze the stability of H using these eigenvalues.

- Theorem: If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are complex numbers, then the system is asymptotically stable if each λ_i has a negative real part.

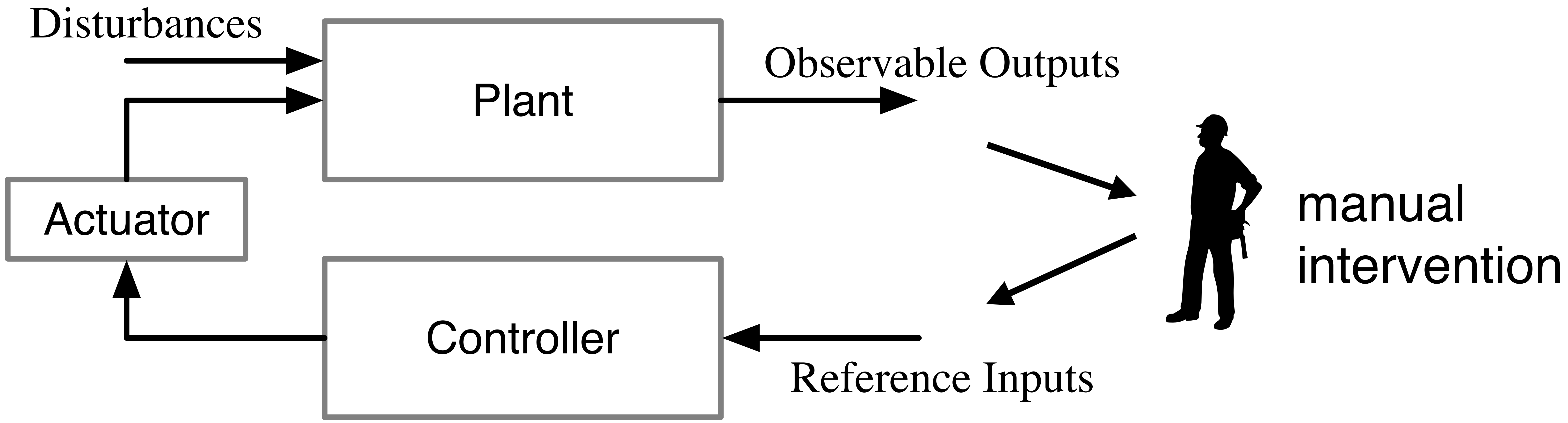
Designing Controllers



- *Plant*: physical world to be controlled
- *Disturbances*: uncontrollable factors from the environment
- *Reference Inputs*: commands/parameters given by the user

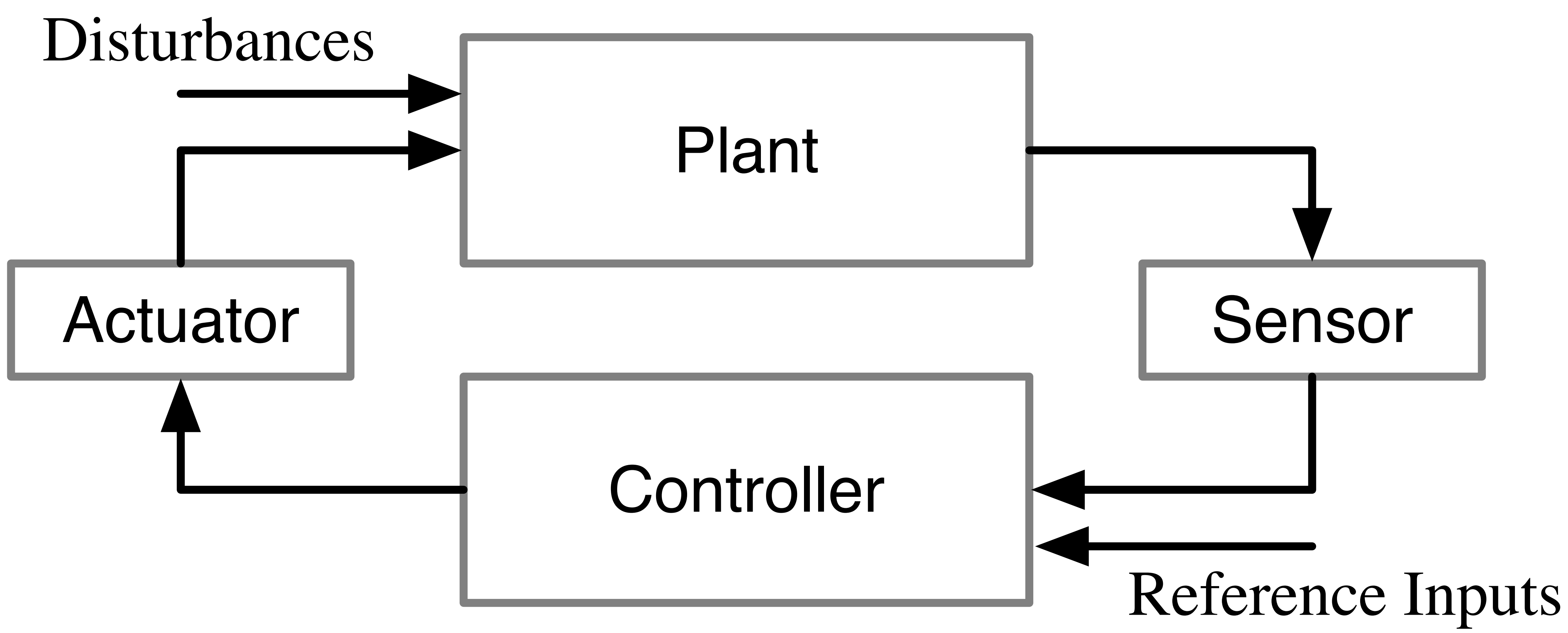
- Problem: Design a controller so that the composed system is stable.

Open-Loop Controller



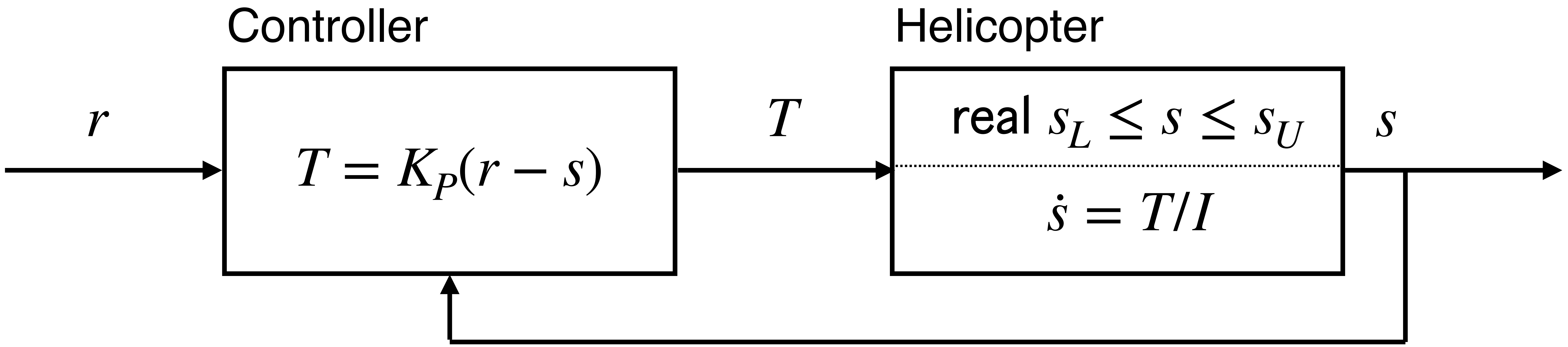
- An open-loop controller makes its decision <u>without sensing</u> the plant.
- It relies on the model of the plant.
- It works under the assumption that the behavior of the plant is predictable and accurately captured by the model.

Feedback Controller



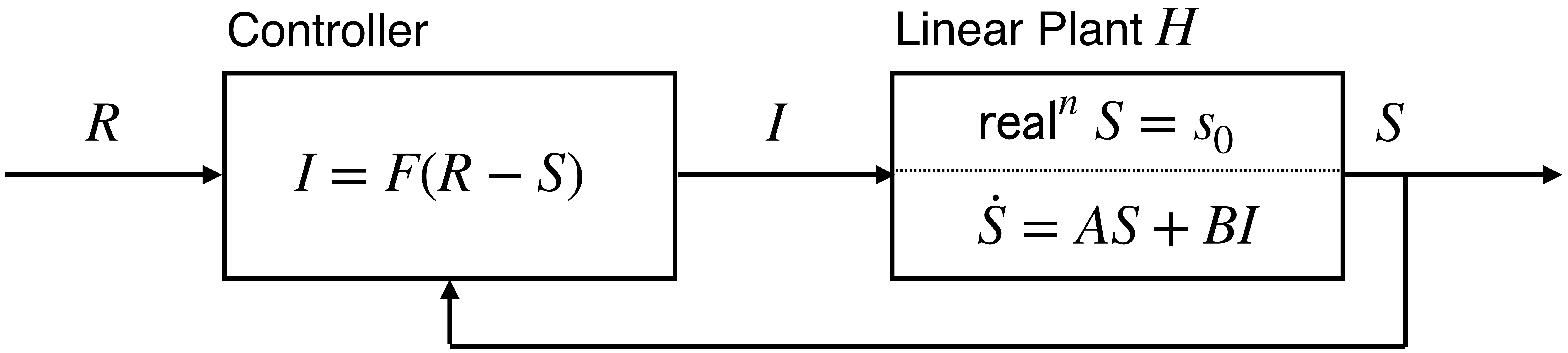
- A feedback controller uses sensors to obtain the information of the plant and maintains the model of the plant to adjust the controlling outputs.

Stabilizing Helicopter Model



- Controller: r : reference input, $K_P > 0$
  - The output T is proportional to the error signal $r - s$. (called a *proportional controller*)
- The dynamics of the composed system: $\dot{s} = K_P(r - s)/I$
  - One-dimensional linear system with the negative coefficient ($-K_P/I$) of the state s.
  - Thus the composed system is asymptotically stable.

Linear State Feedback



- State S : n-dim, Input I : m-dim, Reference R : n-dim vectors
- Gain matrix F : $m \times n$-dim
- The dynamics of the composed system: $\dot{S} = (A - BF)S + BFR$
- Problem: Choose the gain matrix F

Design of the Gain Matrix

- Problem: Given a linear plant H with $\dot{S} = AS + BI$. Choose F so that the composed system with $\dot{S} = (A - BF)S + BFR$ is (asymptotically) stable.
- Suppose $R = 0$.
- Choose F such that every eigenvalue of $A - BF$ has a negative real part.

- Example: A linear plant with two state variables s_1, s_2 and ont input variable u.
  - $\dot{s}_1 = 4s_1 + 6s_2 + 2u$
  - $\dot{s}_2 = s_1 + 3s_2 + u$

Design of the Gain Matrix

- $A - BF$
  - $\begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} (f_1 \; f_2) = \begin{pmatrix} 4 - 2f_1 & 6 - 2f_2 \\ 1 - f_1 & 3 - f_2 \end{pmatrix}$
- The characteristic polynomial is $\lambda^2 + (2f_1 + f_2 - 7)\lambda + (6 - 2f_2)$.
- The eigenvalues λ_1, λ_2 satisfy $2f_1 + f_2 - 7 = -\lambda_1 - \lambda_2$ and $6 - 2f_2 = \lambda_1 \lambda_2$.
- If we choose $\lambda_1 = -1$ and $\lambda_2 = -2$, we need to solve $2f_1 + f_2 - 7 = 3$ and $6 - 2f_2 = 2$. Thus we have $f_1 = 4$ and $f_2 = 2$.
- The controller provides $u = 4(r_1 - s_1) + 2(r_2 - s_2)$.
- The composed system is asymptotically stable because $\lambda_1, \lambda_2 < 0$.

Determining Controllability

- In general, to obtain the gain matrix, we need to solve the equation $\det(A - BF - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$ with mn unknowns.
- When is the equation guaranteed to have solutions?
- Does the existence of a solution depend on the choice of eigenvalues?
- If (A, B) satisfies a certain property, for every choice of the eigenvalues, it is possible to choose the entries of F so as to satisfy the above equation.
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Determining Controllability

- Given an $n \times n$-matrix A and $n \times m$-matrix B. Define the $n \times nm$ matrix $\mathbf{C}(A, B)$ as $(B \;\; AB \;\; A^2B \quad \cdots \quad A^{n-1}B)$. (called the *controllability matrix*)
- (A, B) is said to be *controllable* if the rank of $\mathbf{C}(A, B)$ is n.
    - All the rows of $\mathbf{C}(A, B)$ is linearly independent.
- Theorem: The following two statements are equivalent.
    - The rank of $\mathbf{C}(A, B)$ is n.
    - For every choice of complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that a complex number appears in this list exactly when its conjugate also appears, there exists a $m \times n$-matrix F such that the eivenvalues of $A - BF$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Summary

- Dynamical Systems (2)
  - Linear Components
    - Matrix-Based Representations for Linear Components
    - Linear Response
    - Solving Linear Differential Equations
    - Stability of Linear Components
  - Designing Controllers
    - Open-Loop Controller, Feedback Controller
    - Gain Matrix