Cyber-Physical Systems (CSC.T431)

Dynamical Systems (1)

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Agenda

- Dynamical Systems (1)

Course Support & Material

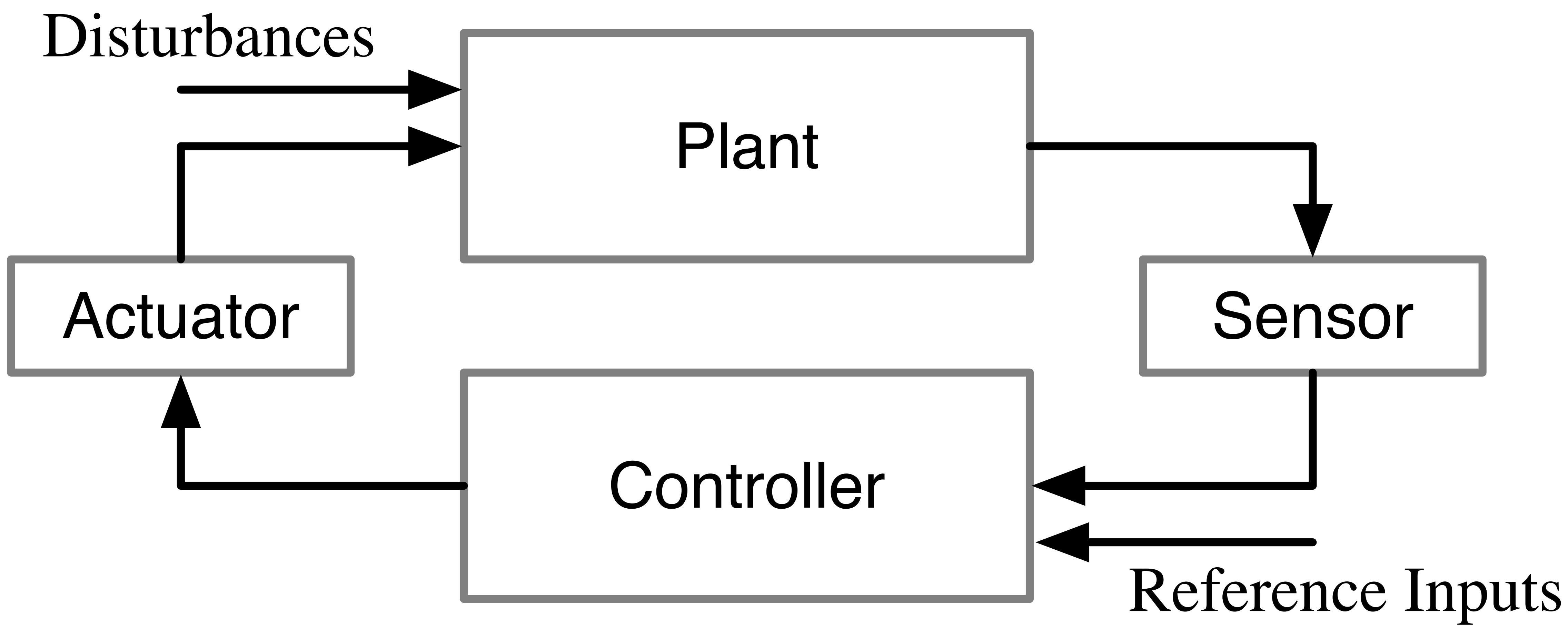
- Slides: OCW-i
- Course Web: https://titech-cps.github.io
- Course Slack: titech-cps.slack.com

Dynamical Systems

- Controllers interacting with physical world via sensors and actuators

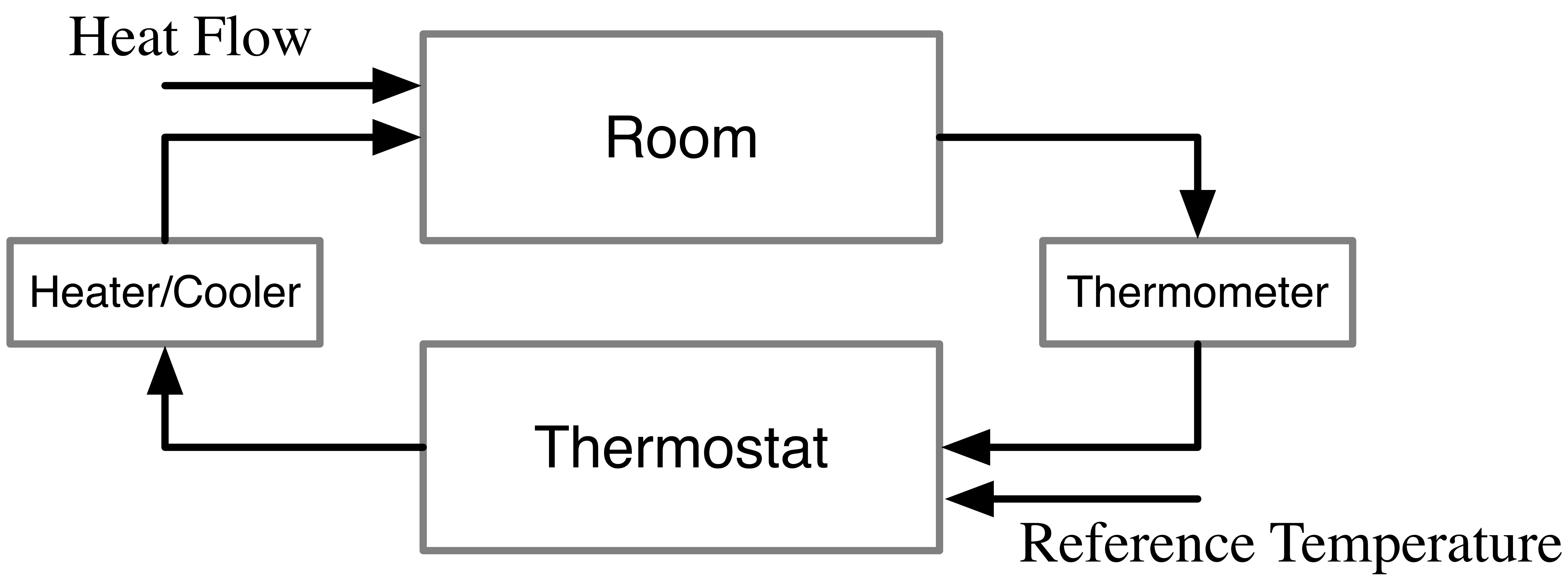
- Physical quantities change continuously over time.
- We use continuous-time models using differential equations.

Control System



- *Plant*: physical world to be controlled
- *Disturbances*: uncontrollable factors from the environment
- *Reference Inputs*: commands/parameters given by the user

Ex. Thermostat



- Goal: to regulate the temperature of the room to be close to the reference temperature

Models of Dynamical Systems

- Like synchronous/asynchronous systems, a dynamical system can be described as a system of connected components with inputs/outputs.
  - While the underlying computation model is synchronous, the values of the variables in a dynamical system change continuously over time.
  - In contrast, in a system consists of SRCs, the values of the variables are updated in a sequence of discrete logical rounds.
- Such components are called continuous-time components.
- We use real to type the variables of continuous-time components.

Signals

- *Signals* are values that change over time, described as functions from the time domain to the types of the values (here, real).

$$\text{time} \to \text{real}$$

- We regard that the time domain consists of non-negative real numbers. Thus time $\cong \mathbb{R}_{\geq 0} = \{v \mid v \in \mathbb{R} \wedge v \geq 0\}$.
- For a variable x in a continuous-time component, $\bar{x}$ denotes the signal over x, i.e., $\bar{x}(t)$ is the value of x at time t.

Signals

- For a finite set of variables V, $\bar{V}$ denotes a signal that gives a valuation over V.

$$\bar{V} : \text{time} \rightarrow V \rightarrow \text{real}$$

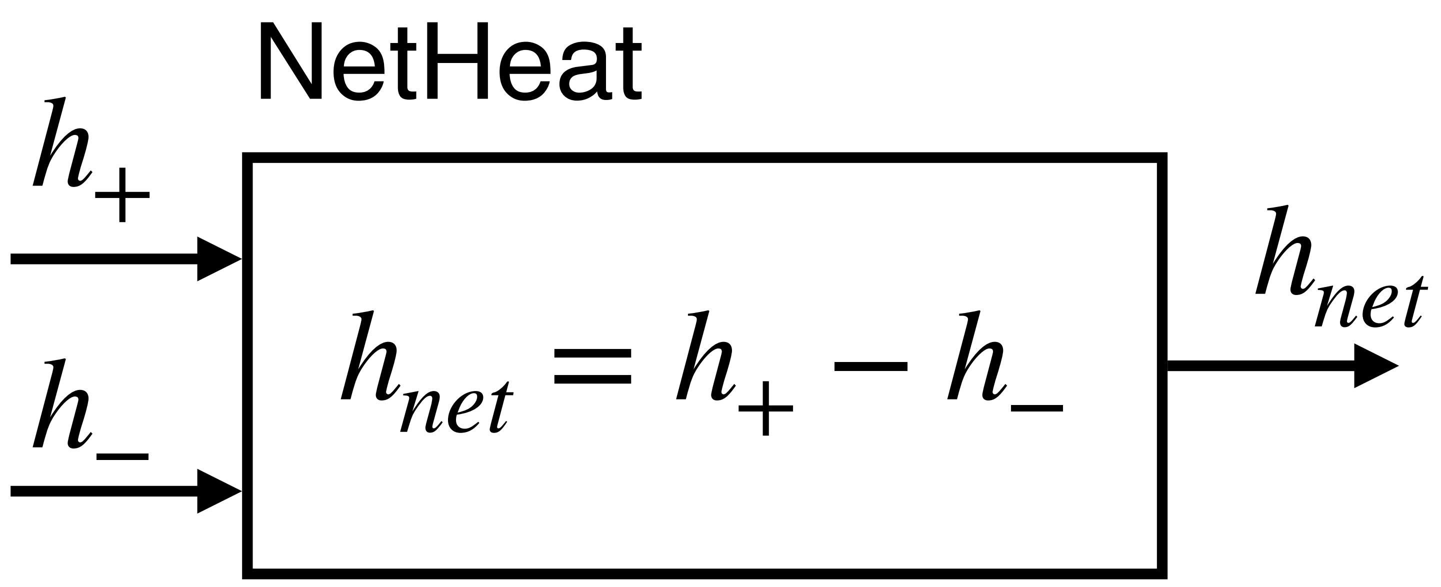
- Thus, for $x \in V$, $\bar{V}(t)(x)$ denotes a value of x at time t.

- Shorthand: Let $V = \{x_1, x_2, \ldots, x_n\}$. We use $\bar{V}$ to denote function from time to a tuple (or a vector) of real values such that $\bar{V}(t) = (\bar{x}_1(t), \bar{x}_1(t),\ \ldots, \bar{x}_n(t))$.
- Thus we can write $\bar{V} : \text{time} \rightarrow \text{real}^n$

Signals

- A signal can be regarded as a function $\mathbb{R}_{\geq 0} \to \mathbb{R}^n$.
- So we can apply some standard mathematical notions such as *continuity* and *differentiability* to signals.
- E.g., Continuity: A signal $\bar{V}$ is continuous if
  - $\forall t \in \text{time}\,.\, \forall \varepsilon > 0.\, \exists \delta > 0.\, \forall t' \in \text{time}\,.\, \Big(\|t - t'\| < \delta \to \|\bar{V}(t) - \bar{V}(t')\| < \varepsilon \Big)$.
  - Note: $\|u - v\|$ denotes the distance between u and v.

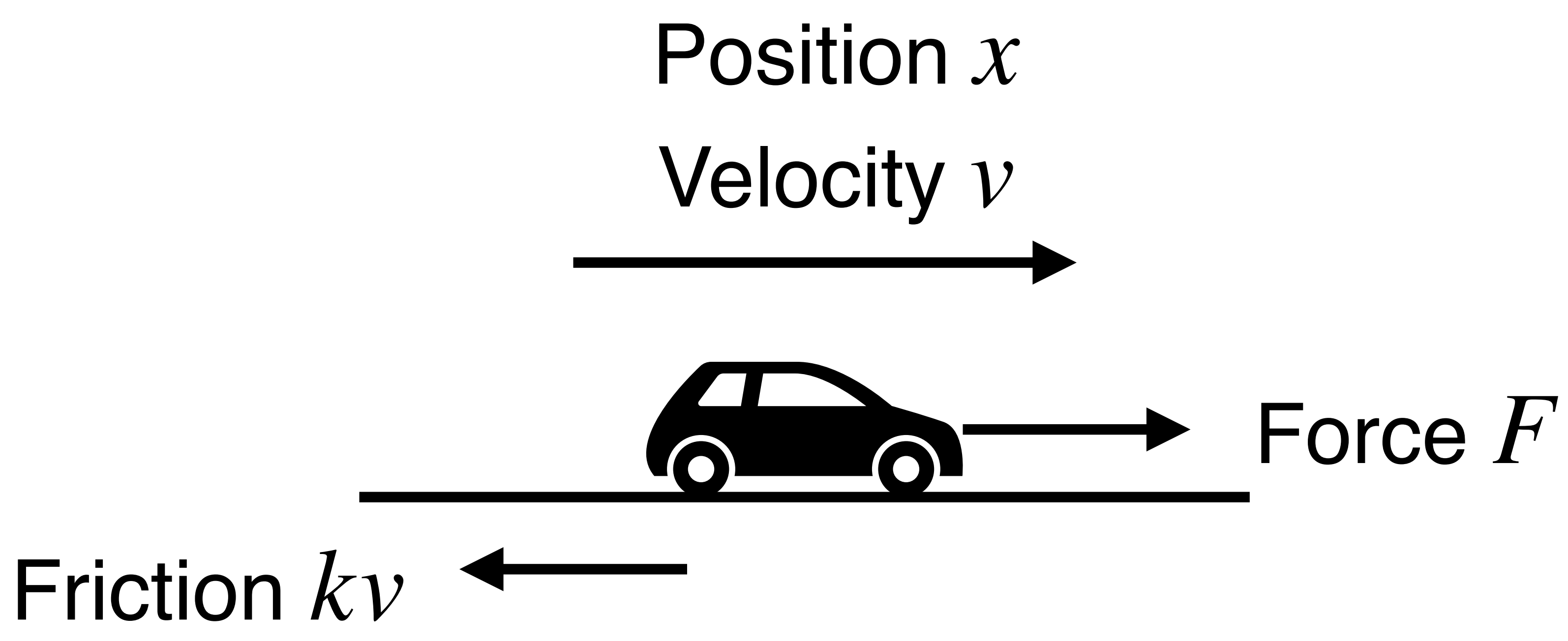
Ex. Heat Flow

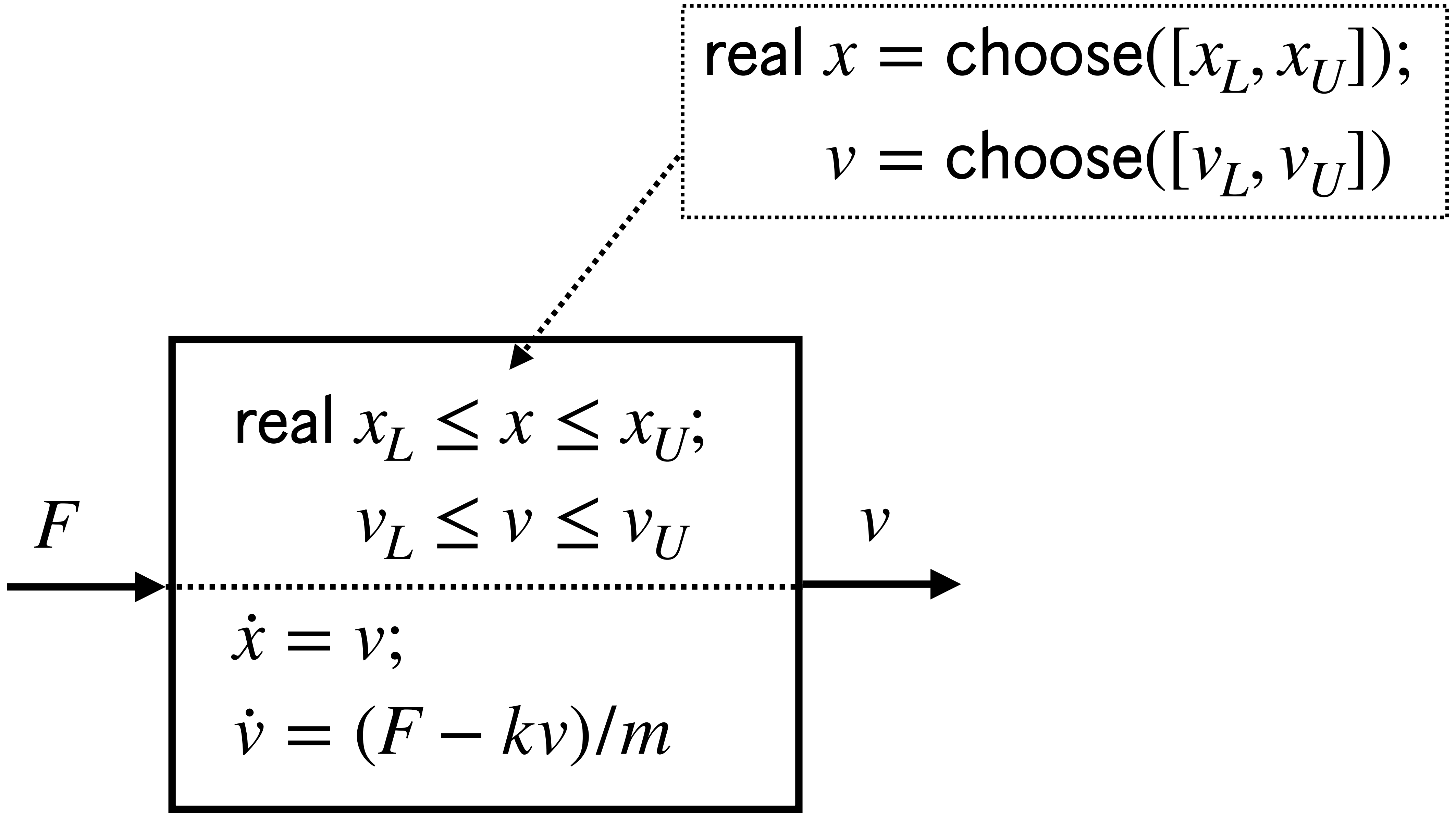


- NetHeat is a continuous-time component with two inputs h_+, h_-, an output h_{net}, and no state variables.
- The behavior is expressed declaratively using an algebraic equation.
- The output signal $\bar{h}_{net}$ can be defined as

$$\bar{h}_{net}(t) = \bar{h}_+(t) - \bar{h}_-(t).$$

Ex. Car Motion





- Using the classical Newtonian mechanics, we have

$$F - k\dot{x} = m\ddot{x}.$$

  - k : the coefficient of the frictional force, m : the mass of the car
  - $\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}$

Ex. Car Motion

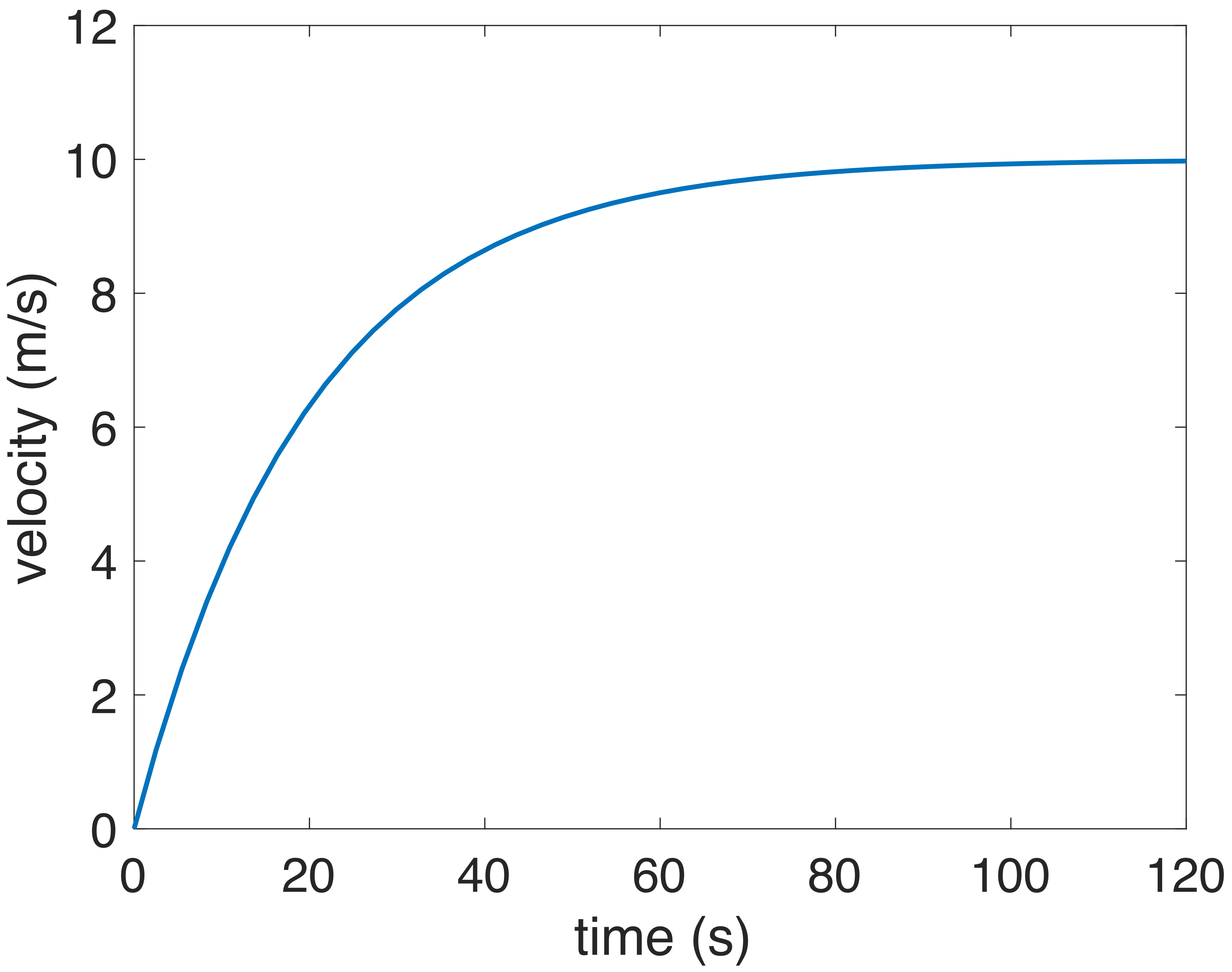
Possible Scenarios

- Case 1: $F = kv_0, \bar{x}(0) = x_0, \bar{v}(0) = v_0$
  - $\dot{x} = v;\ \dot{v} = k(v_0 - v)/m$
  - $\bar{v}(t) = v_0, \bar{x}(t) = x_0 + tv_0$
- Case 2: $F = 0, \bar{x}(0) = 0, \bar{v}(0) = v_0$
  - $\dot{x} = v;\ \dot{v} = -kv/m$
  - $\bar{v}(t) = v_0 e^{-kt/m}, \bar{x}(t) = (mv_0/k)(1 - e^{-kt/m})$
- Case 3: $F = F_0, \bar{x}(0) = 0, \bar{v}(0) = 0$
  - $\dot{x} = v;\ \dot{v} = (F_0 - kv)/m$
  - $\bar{v}(t) = F_0(1 - e^{-kt/m})/k, \bar{x}(t) = F_0 m(e^{-kt/m} - 1 + kt/m)/k^2$

Ex. Case 3

$F_0 = 500, k = 50, m = 1000$

$\bar{v}(t) = 10 - 10e^{-t/20}$



Continuous-Time Component

- $H = (I, O, S, Init, \mathbf{E}_O, \mathbf{E}_S)$
  - I, O, S : finite sets of real-valued input/output/state variables
  - $Init$: initialization constraint, defining $[\![Init]\!]$ as the set of initial states.
    - $\bar{S}(0) \in [\![Init]\!]$
  - $\mathbf{E}_O \in \text{Expr}(I \cup S)^O, \mathbf{E}_S \in \text{Expr}(I \cup S)^S$
    - Expr(V) denotes the set of real-valued expressions over a finite set of real-valued variables V.
    - For each variable in O (S), $\mathbf{E}_O$ ($\mathbf{E}_S$) gives a real-valued expression over $I \cup S$.
    - For $y \in O$ and $t \in \text{time}$, $\bar{y}(t)$ equals the value of $h_y = \mathbf{E}_O(y)$ evaluated using $\bar{I}(t)$ and $\bar{S}(t)$.
    - For $x \in S$ and $t \in \text{time}$, $\frac{d\bar{x}}{dt}$ equals the value of $f_x = \mathbf{E}_S(x)$ evaluated using $\bar{I}(t)$ and $\bar{S}(t)$.

Continuous-Time Component

- $H = (I, O, S, Init, \mathbf{E}_O, \mathbf{E}_S)$
  - $I = \{z_1, \ldots, z_l\}$, $O = \{y_1, \ldots, y_m\}$, $S = \{x_1, \ldots, x_n\}$
    - $\bar{I}(t) = (\bar{z}_1(t), \ldots, \bar{z}_l(t))$, $\bar{O}(t) = (\bar{y}_1(t), \ldots, \bar{z}_m(t))$, $\bar{S}(t) = (\bar{x}_1(t), \ldots, \bar{x}_n(t))$
  - $\bar{S}(0) \in [\![Init]\!] \subseteq Q_S$
  - $\mathbf{E}_O = (h_{y_1}, \ldots, h_{y_m})$, $\quad\quad \mathbf{E}_S = (f_{x_1}, \ldots, f_{x_n})$

$$\begin{cases} \bar{y}_1(t) = h_{y_1}(\bar{I}(t), \bar{S}(t)) \\ \quad\quad\vdots \\ \bar{y}_m(t) = h_{y_m}(\bar{I}(t), \bar{S}(t)) \end{cases} \quad\quad \begin{cases} d\bar{x}_1(t)/dt = f_{x_1}(\bar{I}(t), \bar{S}(t)) \\ \quad\quad\quad\quad\vdots \\ d\bar{x}_n(t)/dt = f_{x_n}(\bar{I}(t), \bar{S}(t)) \end{cases}$$

Existence/Uniqueness of Response Signals

- Problem: Given an input signal, when the system has at least (or exactly) one execution?

- The problem can be seen as an *initial value problem*.
  - An initial value problem is a differential equation $\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t))$ with $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ together with a point $(t_0, \mathbf{x}_0) \in \mathbb{R} \times \mathbb{R}^n$ called initial value.
  - A solution to this initial value problem is a function $\mathbf{x}$ that is a solution of the differential equation and satisfies $\mathbf{x}(t_0) = \mathbf{x}_0$.

Existence

- Let $\mathbf{x} : \text{time} \to \text{real}^n$ be a n-dimentional signal that satisfies $d\mathbf{x}/dt = f(\mathbf{x})$ and $\mathbf{x}(0) = \mathbf{x}_0$.
- There exists at least one solution if f is a continuous function.
- Ex. The RHS of $\dot{v} = (F - kv)/m$ is continuous.

- For a conditional differential equation $\dot{x} = \text{if } x = 0 \text{ then } 1 \text{ else } 0$, there are no differentiable signal $\bar{x}$ that satisfies the equation.

Uniqueness

- Consider the differential equation $\dot{x} = x^{1/3}$ with the initial value $\bar{x}(0) = 0$. This differential equation has two solutions: $\bar{x}(t) = 0$ and $\bar{x}(t) = (2t/3)^{3/2}$.

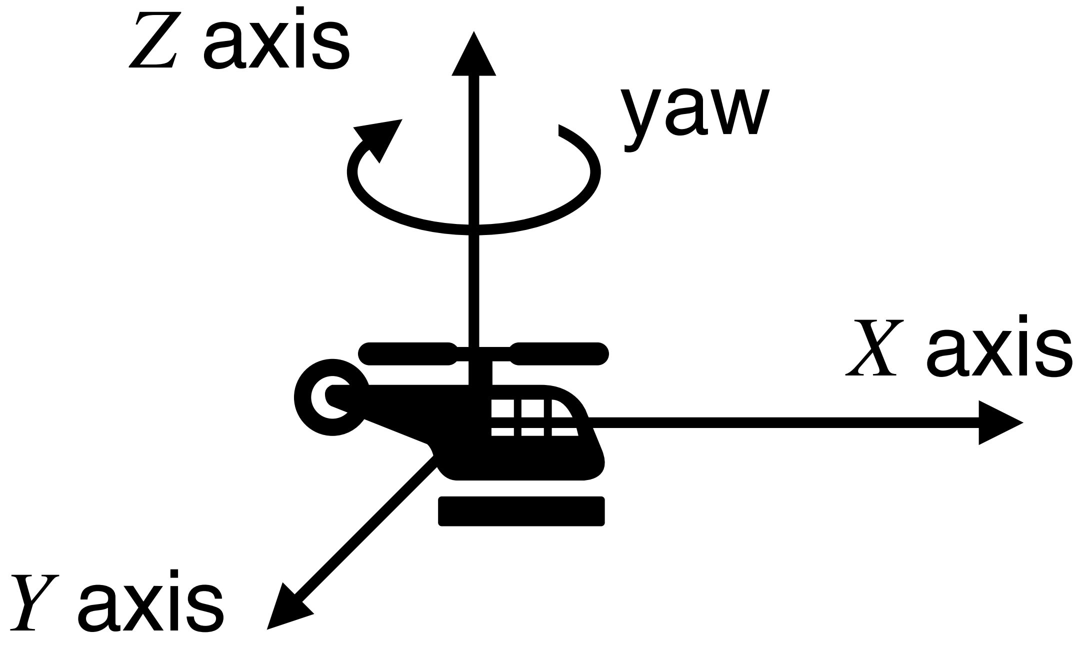
- Let $\mathbf{x} : \text{time} \to \text{real}^n$ be a n-dimentional signal that satisfies $d\mathbf{x}/dt = f(\mathbf{x})$ and $\mathbf{x}(0) = \mathbf{x}_0$.
- There exists exactly one solution if f is *Lipschitz continuous*.
  - A function $f : \text{real}^n \to \text{real}^n$ is said to be Lipschitz continuous if $\exists K \in \mathbb{R} \,.\, \forall u, v \in \mathbb{R}^n \,.\, \|f(u) - f(v)\| \leq K\|u - v\|$.

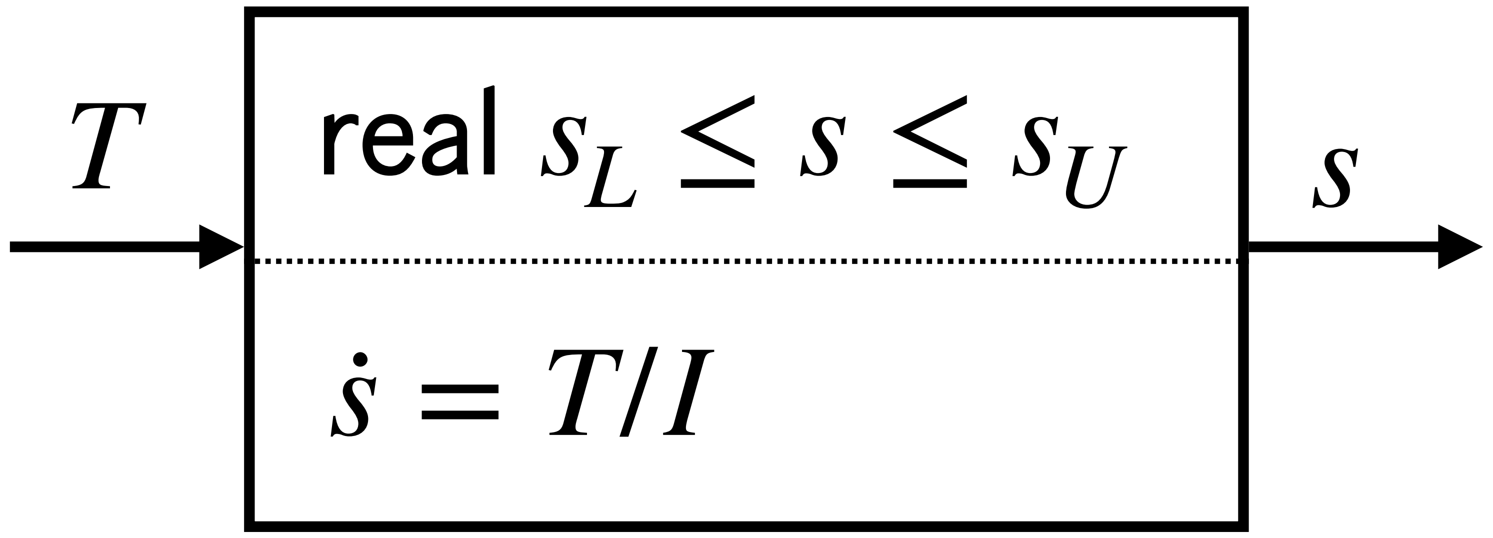
Lipschitz-Continuous Dynamics

- A continuous-time component $H = (I, O, S, Init, \mathbf{E}_O, \mathbf{E}_S)$ is said to have Lipschitz-continuous dynamics if:
  - for each $y \in O$, $h_y = \mathbf{E}_O(y)$ is Lipschitz-continuous, and
  - for each $x \in S, f_x = \mathbf{E}_S(x)$ is Lipschitz-continuous.

- If H has Lipschitz-continuous dynamics, then for a given initial state and a given continuous input signal $\bar{I}$, the corresponding response of the component as a signal over the state and the output variables exists and unique.

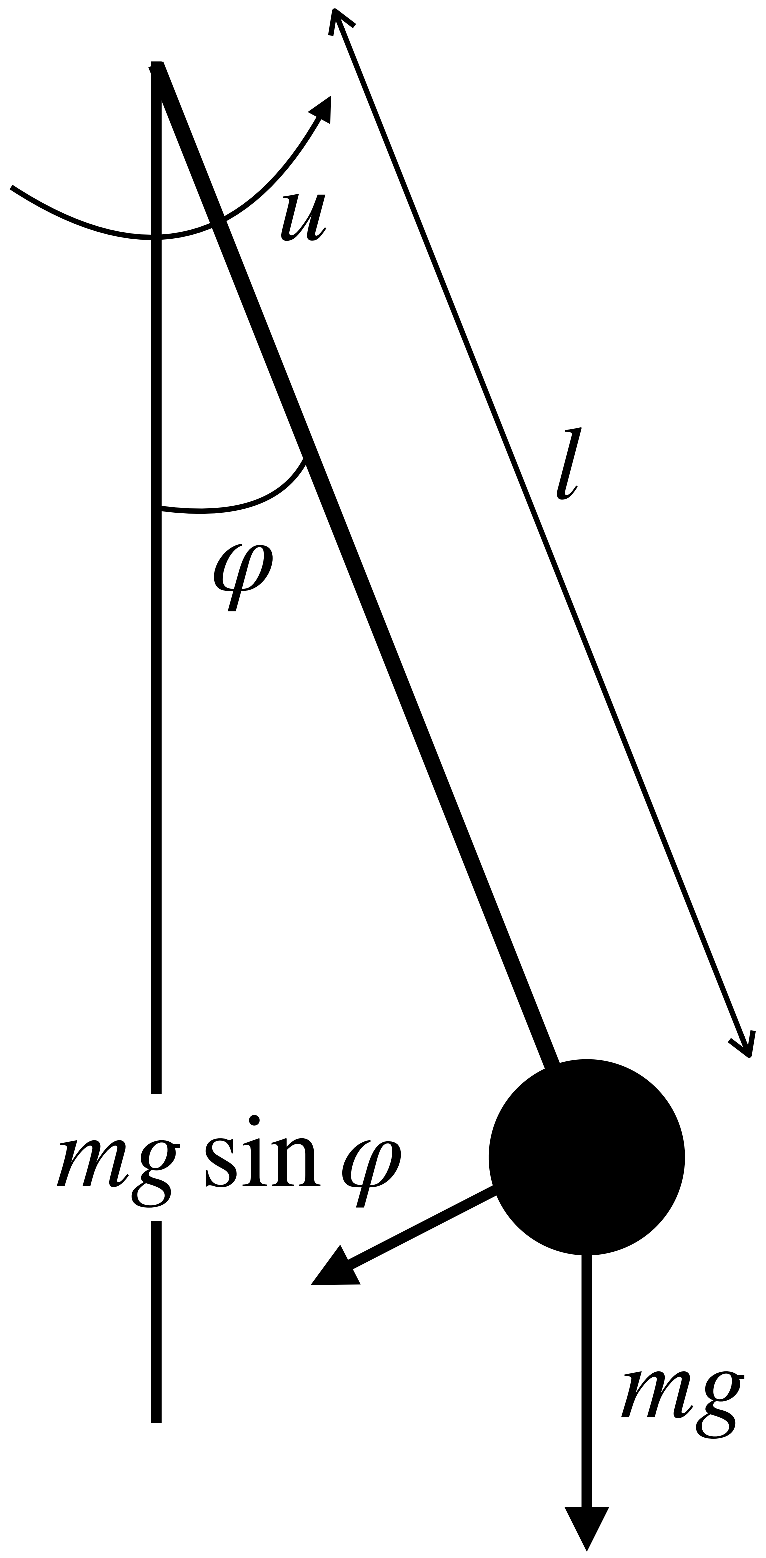
Ex. Helicopter Spin

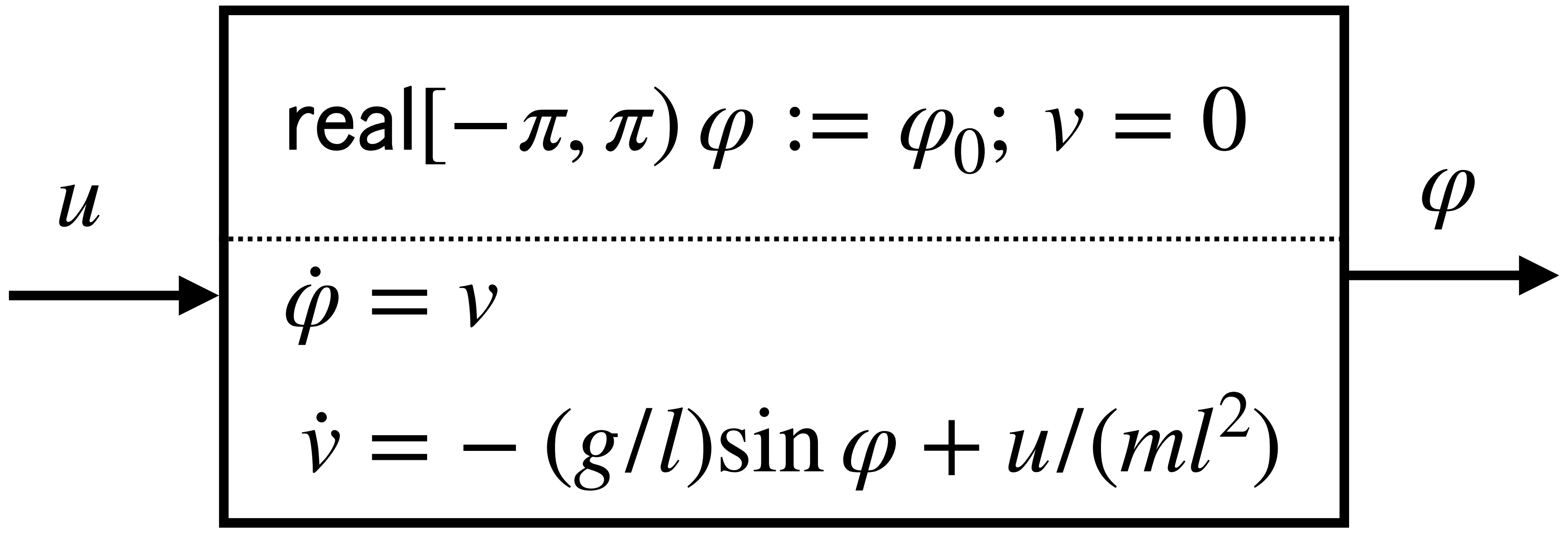




- Goal: keep the helicopter from spinning by controlling the torque applied by the tail rotor.
  - Assumption: the only freedom of motion is the angular rotation around Z-axis.
- The motion is given by $\ddot{\theta} = T/I$
  - where θ : yaw (rotation around Z-axis), T : torque, I : the moment of inertia.
- The goal is represented as the spin $s = \dot{\theta}$ to be zero.

Ex. Simple Pendulum





- The dynamics of the system is described by the second-order nonlinear differential equation

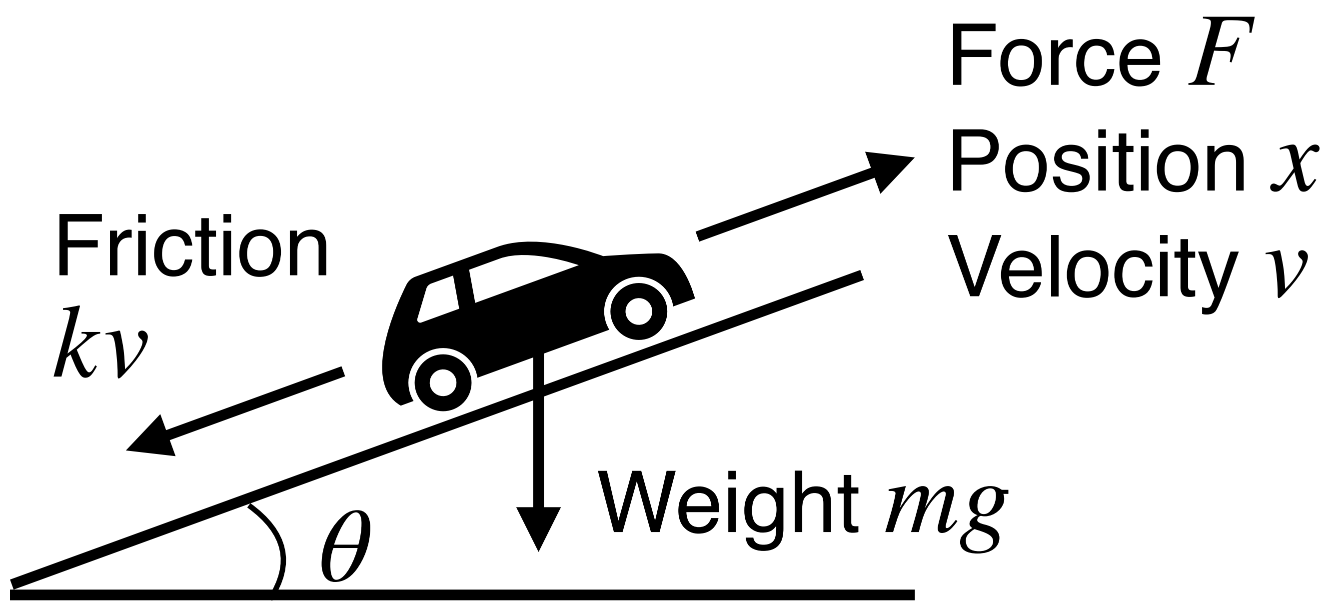
$$ml^2\ddot\varphi = u - mgl\sin\varphi$$

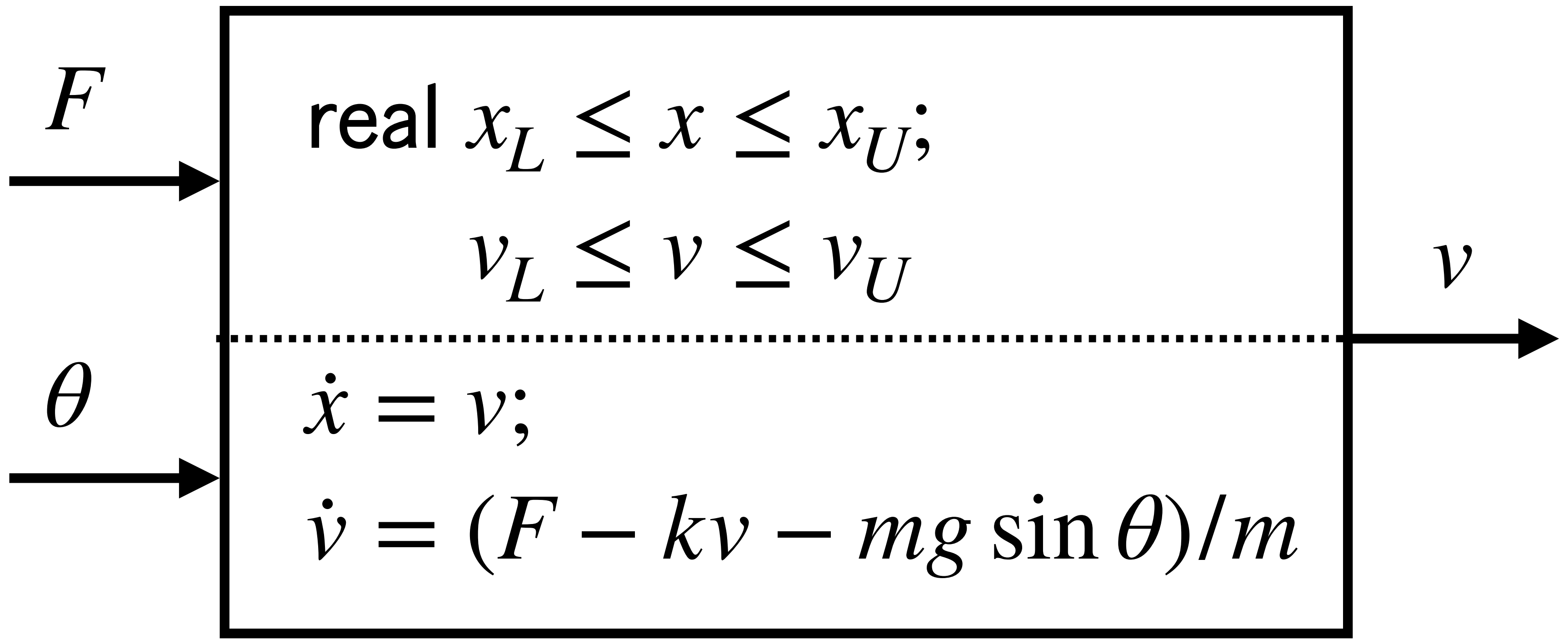
where m is the mass, $g = 9.8m/s^2$ is the gravitational acceleration, l is the length of the rod, φ is the angle of the pendulum, and u is the external torque.

- In the model, the second-order differential equation is replaced with two first-order differential equations.

Models with Disturbance

Ex. Car Motion on a Graded Road





- The grade of the road (θ) models the disturbance (uncontrolled input) of the system.

- Ex. design problem: Design a cruise controller with v as input and F as output such that composed system keeps v within $[v_0 - \Delta, v_0 + \Delta]$ for all continuous input θ in $[-\pi/6, \pi/6]$.

Composing Components

- Continuous-time components can be composed in a way similar to SRCs.
  - variable renaming, parallel composition, output hiding
- Await Dependencies
  - An output variable y awaits an input variable x if the value of y at time t depends on the valu of x at time t.
    - Ex. In NetHeat, h_{net} awaits both h_+ and h_-.
    - Ex. In Car Motion, v awaits neither F nor θ.
  - Suppose that the value an output variable y is described by the expression h_y. The output variable y awaits an input variable w if w occurs in h_y.

Stability

- Correctness requirement for dynamical systems
  - Small perturbations in the input values should not cause disproportionately large changes in the outputs.
- Ex. Cruise Controller
  - Safety: Speed should always be within certain threshold values.
  - Liveness: Actual speed should eventually get close to the desired speed.
  - Stability: If grade of the road changes, speed should change only slowly.
- Formalization
  - Lyapunov stability of equilibria
  - BIBO (Bounded-Input-Bounded-Output) stability of response

Equilibria

- Consider a closed continuous-time component whose state S is an n -dimensional vector, with the dynamics given by the Lipschitz-continuous differential equation $\dot{S} = f(S)$. A state s_e of the system is said to be an *equilibrium* state if $f(s_e) = 0$.
  - closed: no input variables (or input values are fixed)
- If the initial state of the system equals s_e, then the system stays in the state at all times.

Equilibria

Examples

- Car Motion
  - Suppose that $F = 0$. The dynamics of the component is described as:

$$\dot{x} = v;\ \dot{v} = -kv/m$$

  - A state (x_e, v_e) is an equilibrium state iff $v_e = 0$.
- Simple Pendulum
  - Suppose that $u = 0$. The dynamics of the component is described as:

$$\dot{\varphi} = v;\ \dot{v} = -g\sin\varphi/l$$

  - A state (φ_e, v_e) is an equilibrium state iff $(\varphi_e, v_e) = (0, 0)$ or $(\varphi_e, v_e) = (-\pi, 0)$.

Lyapunov Stability

- Consider a closed continuous-time component whose state S is an n -dimensional vector, with the dynamics given by the Lipschitz-continuous differential equation $\dot{S} = f(S)$. Let s_e be an equilibrium state (*i.e.*, $f(s_e) = 0$).
- As we have seen before, if the initial state s_0 equals to s_e then $\bar{S}(t) = s_0$.
- Suppose that the initial state s_0 is perturbed slightly (*i.e.*, $||s_e - s_0||$ is small).

Lyapunov Stability

- Consider a closed continuous-time component whose state S is an n -dimensional vector, with the dynamics given by the Lipschitz-continuous differential equation $\dot{S} = f(S)$. Let s_e be an equilibrium state (*i.e.*, $f(s_e) = 0$).
- As we have seen before, if the initial state s_0 equals to s_e then $\bar{S}(t) = s_0$.
- Suppose that the state s is perturbed slightly (*i.e.*, $||s_e - s||$ equals to a small positive number).
- As time passes, will the system stay close to the equilibrium state? or will the system eventually return to the equilibrium state?

Lyapunov Stability

- Consider a closed continuous-time component H with n state variables S and dynamics given by $\dot{S} = f(S)$ where f is Lipschitz continuous.
- Given an initial state s_0, let $\bar{S}_0$ be the unique response of H from s_0.
  - An equilibrium s_e of H is said to be *stable* if
    $\forall \varepsilon > 0. \exists \delta > 0. \forall s_0 \in [\![Init]\!] . \forall t \in \text{time} . \left(\|s_e - s_0\| < \delta \rightarrow \|\bar{S}_0(t) - s_e\| < \varepsilon \right).$
  - An equilibrium s_e of H is said to be *asymptotically stable* if it is stable and
    $\exists \delta > 0. \forall s_0 \in [\![Init]\!] . \left(\|s_e - s_0\| < \delta \rightarrow \lim_{t \rightarrow \infty} \bar{S}_0(t) = s_e \right).$

Lyapunov Stability

Ex. Car Motion

- Suppose that $F = 0$. The state $(x_e, 0)$ is equilibrium.
- Let us start from the state (x_0, v_0) such that $\|(x_e, 0) - (x_0, v_0)\|$ is small positive number.
- The car will slow down according to $\dot{v} = -kv/m$, with the velocity converging to 0. The position of the car converges to $x_f = x_0 + mv_0/k$.
- Thus, $\|(x_e, 0) - (\bar{x}(t), \bar{v}(t))\|$ is bounded. So $(x_e, 0)$ is stable.
- The state $(x_e, 0)$ is not asymptotically stable because $\bar{x}(t)$ converges to x_f.

Lyapunov Stability
Ex. Simple Pendulum

- Suppose that $u = 0$. The states $(0, 0)$ and $(-\pi, 0)$ are equilibria.
- The state $(-\pi, 0)$ is not stable. If we displace the pendulum slightly from this vertically upward position, the angular velocity v will keep increasing, thereby increasing the displacement φ.
- Clearly $(0, 0)$ is stable.
- $(0, 0)$ is not asymptotically stable (the pendulum may keep swinging forever) if we ignore the effects of dumping (friction).

Summary

- Dynamical Systems (1)
  - Control Systems, Signals
  - Continuous-Time Components
  - Existence/Uniqueness of Responses, Lipschitz-Continuous Dynamics
  - Stability, Equilibria, Lyapunov Stability