Cyber-Physical Systems (CSC.T431)

Hybrid Model / Programming Reactive Systems

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Agenda

- Hybrid Systems
- Programming Reactive Systems

Course Support & Material

- Slides: OCW-i
- Course Web: https://titech-cps.github.io
- Course Slack: titech-cps.slack.com

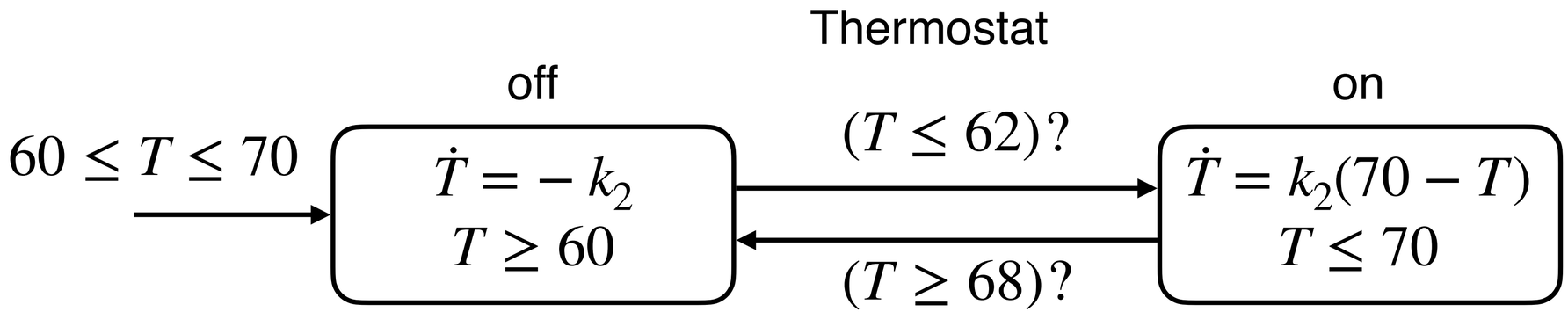
Models of Reactive Computation

- Continuous-Time Model for Dynamical Systems
  - Components run synchronously under continuous time.
  - The execution of a component is described using a system of differential equations.
- Timed Models
  - Processes run asynchronously, with discrete actions
  - Clocks evolve continuously, and constraints on clocks allow synchronous / global coordination
- Hybrid Systems
  - Generalization of Timed Processes
  - During timed transitions, evolution of state/output variables specified differential equations

Hybrid Process

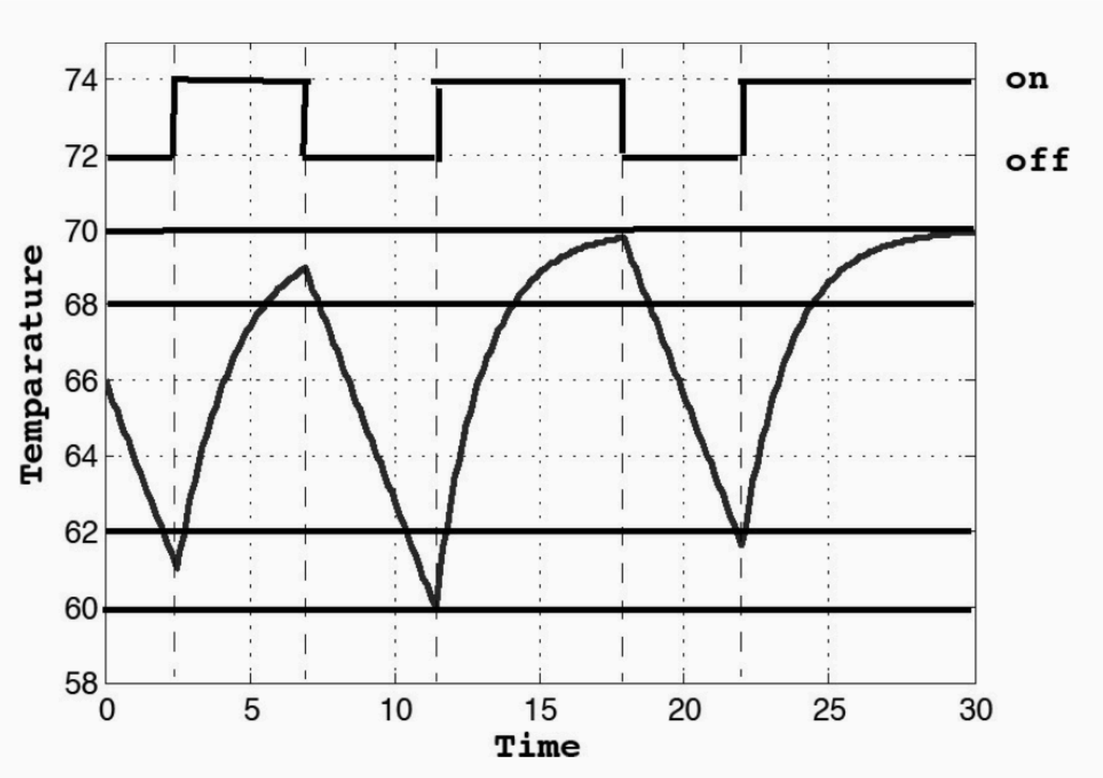
- Similar to timed process
  - Each process has input / output channels and state variables
- **cont** : type of continuously evolving values
  - A variable of type cont takes values from the set of real numbers (or an interval of real numbers) and is updated continuously as time progresses whic a process stays in a mode.
  - Specified by differential / algebraic equations associated with modes.

Ex. Switching Thermostat



- T : continuous-time variable (of type cont) expressing the temperature
- The dynamics of the system is given by differential equations
- The constraint associated with each mode specifies that the process can stay in the mode only as long as the condition holds.
- A mode switch my happen any time while the guard associated with it is satisfied.

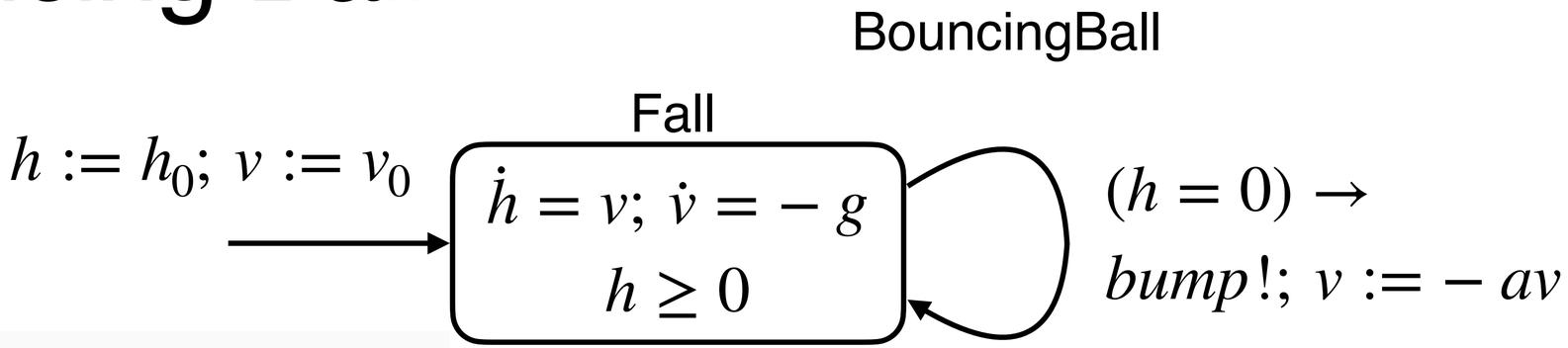
Ex. Switching Thermostat

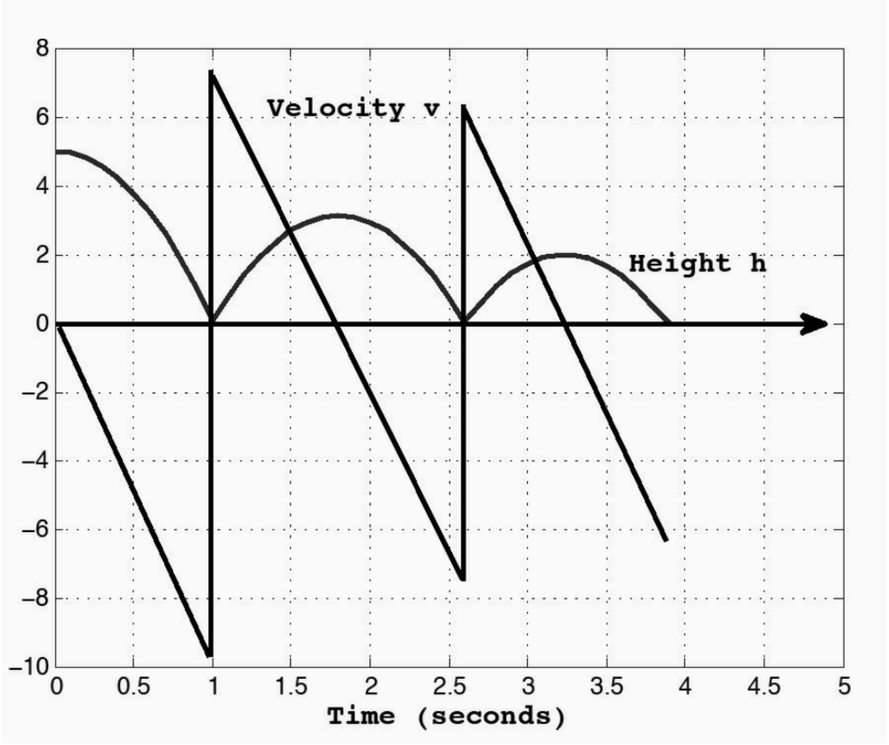


$T_0 = 66, k_1 = 0.6, k_2 = 2$

- If the process switches to off at time t^* with temperature T^*, then until the next mode switch, $T = T^* - k_2(t - t^*)$.
- If the process switches to on at time t^* with temperature T^*, then until the next mode siwtch,
  $T = 70 - (70 - T^*)e^{-k_1(t-t^*)}$.

Ex. Bouncing Ball





- Consider a ball dropped from an initial height h_0 with an initial velocity v_0.
- Differential equations $\dot{h} = v$ and $\dot{v} = -g$ give the dynamics of the falling ball.
- When the ball hits the ground, *i.e.*, when $h = 0$, the velocity discontinuously changes to $-av$ where a is a constant $(0 < a < 1)$.

Hybrid Process

Definition (1/3)

A hybrid process HP consists of:

(1) an asynchronous process P where some of the input, output, and state variables are of type cont,

(2) a continuous-time invariant CI, which is a Boolean expression over the state varibles S,

(3) for every output variable y of type cont, a real-valued expression h_y over the state and input variables of type cont, and

(4) for every state variable x of type cont, a real-valued expression f_x over the state and input variables of type cont.

Hybrid Process

Definition (2/3)

- Inputs, outputs, states, initial states, internal actions, input actions, and output actions of HP are the same as those of the asynchronous process P.
- Given a state s, a real-valued time $\delta > 0$, and an input signal $\bar{u}$ for every input variable u of type cont over the interval $[0, \delta]$, the corresponding timed action of HP is the differentiable state signal $\bar{S}$ over the state variables, and the signal $\bar{y}$ for every output variable y of type cont over $[0, \delta]$ such that:

  (1) for every state variable x, $\bar{x}(0) = s(x)$,

  (2) for every discrete state variable x and time $t \in [0, \delta]$, $\bar{x}(t) = s(x)$,

Hybrid Process

Definition (2/3)

(3) for every output variable y of type cont and time $t \in [0, \delta]$, $\bar{y}(t)$ equals the value of h_y evaluated using the values $\bar{u}(t)$ and $\bar{S}(t)$,

(4) for every state variable x of type cont and time $t \in [0, \delta]$, the time derivative $(d/dt)\bar{x}(t)$ equals the value of f_x evaluated using the values $\bar{u}(t)$ and $\bar{S}(t)$,

(5) for all $t \in [0, \delta]$, the continuous-time invariant CI is satisfied by the values $\bar{S}(t)$ of the state variables at time t.

Ex. Thermostat

Formal Definition (1/2)

- No input variables. A single output variable T of type cont.
- A discrete state variable $mode$ of type $\{\text{off}, \text{on}\}$, and a state variable T of type cont.
  - The initial value of $mode$ is off and the initial value of T is nondeterministically chosen from $[60, 70]$.
- No output tasks. This means that the value of T is not transmitted during the discrete actions
- Two internal tasks corresponding to the mode-switches:
  - $(mode = \text{off} \wedge T \leq 62) \rightarrow mode := \text{on}$
  - $(mode = \text{on} \wedge T \geq 68) \rightarrow mode := \text{off}$

Ex. Thermostat

Formal Definition (2/2)

- The expression defining the value of the output variable T equals to the state variable T
- The expression defining the derivative of the state variable T is given by the conditional expression:

$$\text{if } (mode = \text{off}) \text{ then } -k_2 \text{ else } k_2(70 - T).$$

- The continuous-time invariant CI is given by the expression:

$$\Big[(mode = \text{off}) \rightarrow T \geq 60\Big] \wedge \Big[(mode = \text{on}) \rightarrow T \leq 70\Big].$$

Execution

- At each step, either an internal, input, output, or a timed action is executed.
- Ex. Thermostat
  - $(\text{off}, 66) \xrightarrow{2.5} (\text{off}, 61) \xrightarrow{\varepsilon} (\text{on}, 61) \xrightarrow{3.7} (\text{on}, 69.02) \xrightarrow{\varepsilon} (\text{off}, 69.02) \xrightarrow{4.4} (\text{off}, 60.22) \xrightarrow{\varepsilon}$
  - $(\text{on}, 60.22) \xrightarrow{7.6} (\text{on}, 69.9) \xrightarrow{\varepsilon} (\text{off}, 69.9) \xrightarrow{4.1} (\text{off}, 61.7) \xrightarrow{\varepsilon} (\text{on}, 61.7) \xrightarrow{7.7} (\text{on}, 69.92)$

- The notion of reachability and invariant can be defined in the same way as asynchronous processes.

Process Composition

- Two hybrid processes are compatible and can be composed if
  - their state variables are disjoint,
  - their output variables are disjoint, and
  - there are no cyclic dependencies amon the continuously updated common input / output variables.
- The continuous-time invariant of the composite process is the conjunction of the continuous-time invariants of the component processes.

Execution of the Bouncing Ball

- Assumption: $v_0 = 0$
- Before the 1st bump: $\bar{h}(t) = h_0 - gt^2/2,\ \bar{v}(t) = -gt$
- Time at which the 1st bump occurs: $\delta_1 = \sqrt{2h_0/g}$
- The velocity just before the 1st bump: $v_1 = -\sqrt{2gh_0}$
- The velocity just after the 1st bump: $v_2 = -av_1 = a\sqrt{2gh_0}$
- Between the 1st and 2nd bumps: $\bar{h}(t') = v_2 t' - gt'^2/2,\ \bar{v}(t') = v_2 - gt'$
- Time between 1st and 2nd bumps: $\delta_2 = 2v_2/g$
- The velocity just before the 2nd bump: $-v_2$
- The velocity just after the 2nd bump: $v_3 = -av_2 = a^2 v_1$

$t' = t - \delta_1$

Execution of the Bouncing Ball

- Execution of the bouncing ball
  - $(h_0, 0) \xrightarrow{\delta_1} (0, -v_1) \xrightarrow{bump!} (0, v_2) \xrightarrow{\delta_2} (0, -v_2) \xrightarrow{bump!} (0, v_3) \xrightarrow{\delta_2} \cdots$
- For each i, $v_{i+1} = av_i = a^i v_1$ and $\delta_{i+1} = 2v_{i+1}/g = 2a^i v_1/g$.
- Since $0 < a < 1$, $\lim_{i\to\infty} v_i = 0$ and $\lim_{i\to\infty} \delta_i = 0$.
- We have $\sum_{i=0}^{\infty} \delta_i = v_1(1+a)/(1-a)$

- Zeno behavior: Infinitely many bumps in finite time!

Zeno's Paradox

- Described by Greek philosopher Zeno in context of a race between Achilles and a tortoise
- The tortoise has a head start over Achilles. Suppose the tortoise is ahead of Achilles by d_1 meters at the beginning of the 1st round.
- By the time Achilles has covered this distance d_1, the tortoise has moved a little bit ahead, say d_2 meters, with $d_2 < d_1$.
- In the second round, Achilles has to cover another d_2 meters, but the tortoise has moved farther by d_3 meters, with $d_3 < d_2$.
- By inductive reasoning, for every n, after n rounds, the tortoise is ahead of Achilless by a non-zero distance, and thus Achilles will never be able to catch up with the tortoise!

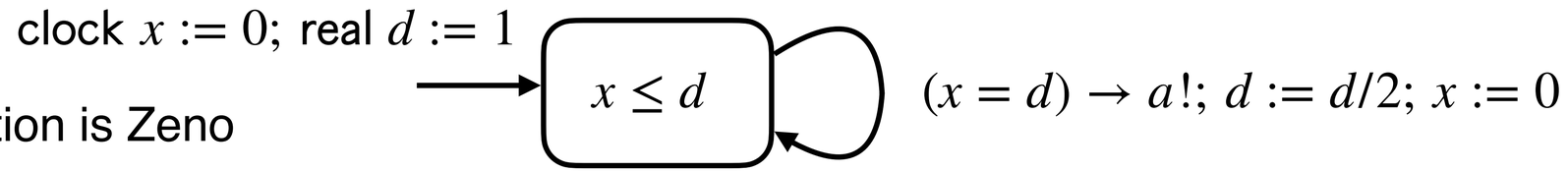
Zeno Execution and Zeno State

- Zeno Execution: An infinite execution of a hybrid process HP is said to be a *Zeno execution* if the sum of the durations of all the timed actions in the execution is bounded by a constant.
- Non-Zeno execution: An execution in which the sum diverges.
- Zeno State: A state s of HP is said to be (1) Zeno if every execution from s is Zeno, or (2) Non-Zeno if there exists some non-Zeno executions from s.
- HP is said to be Zeno process if some reachable state is Zeno.
- Ex. Thermostat: non-Zeno, BouncingBall : Zeno

Zeno Process

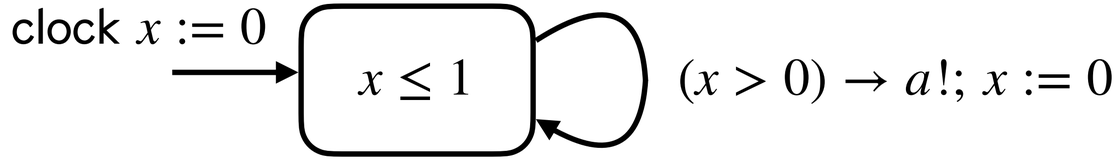
Zeno

Every possible execution is Zeno



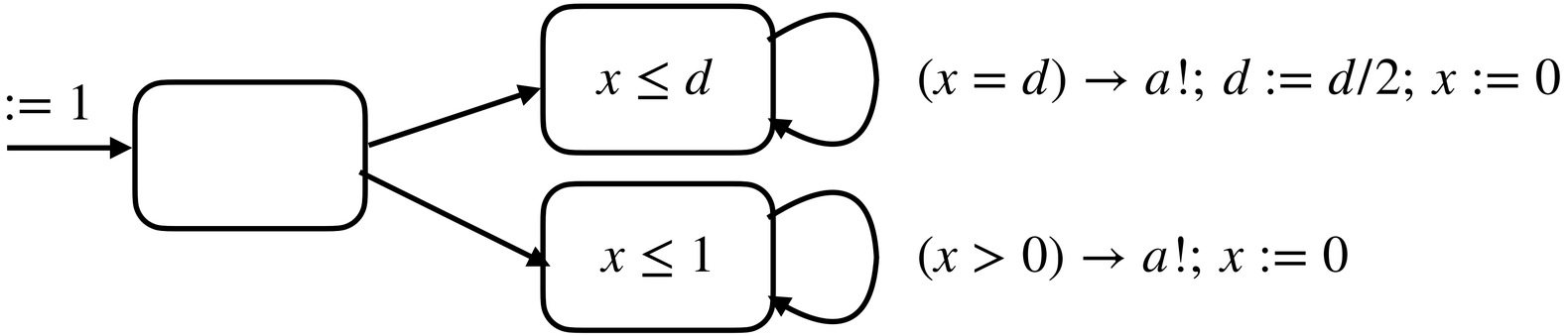
Non-Zeno

Some executions are Zeno
and some are non-Zeno

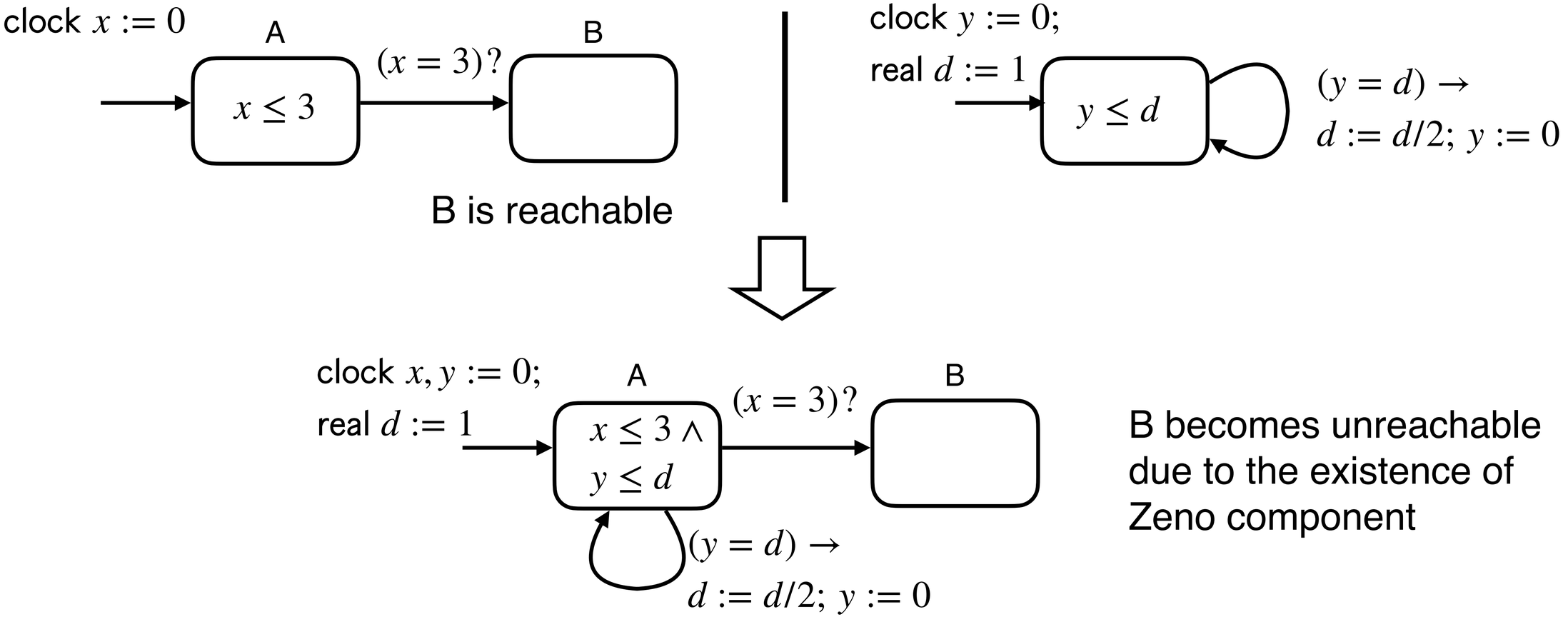


Zeno

Zeno state is reachable

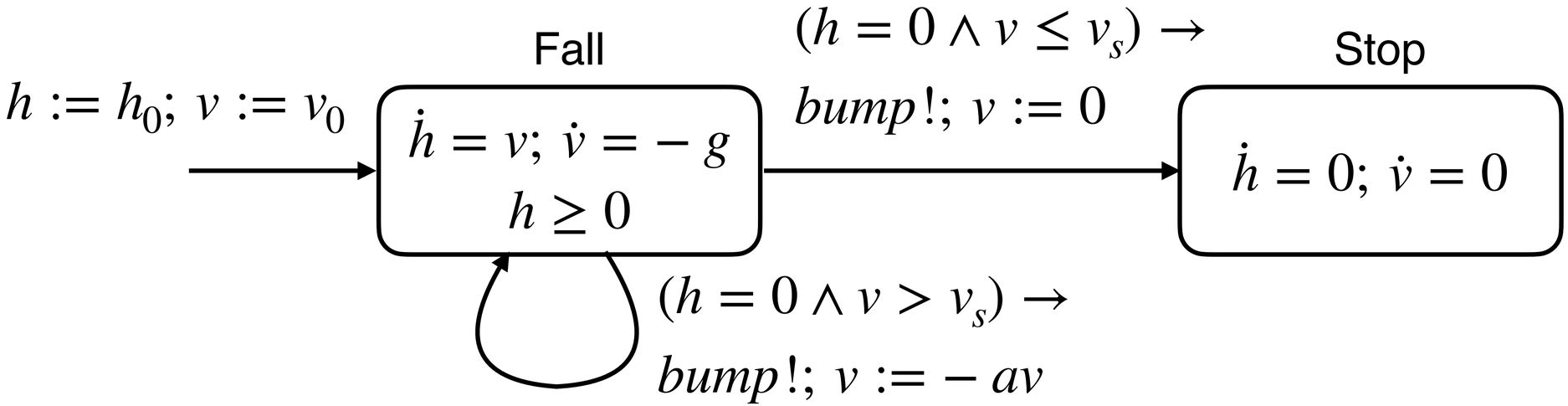


Zeno Process and Reachability



B becomes unreachable due to the existence of Zeno component

A Non-Zeno Model of the Bouncing Ball



- A Zeno process can be converted into a non-Zeno process by modifying the model so that the model does not force mode-switching after a certain short durations.

Stability

A

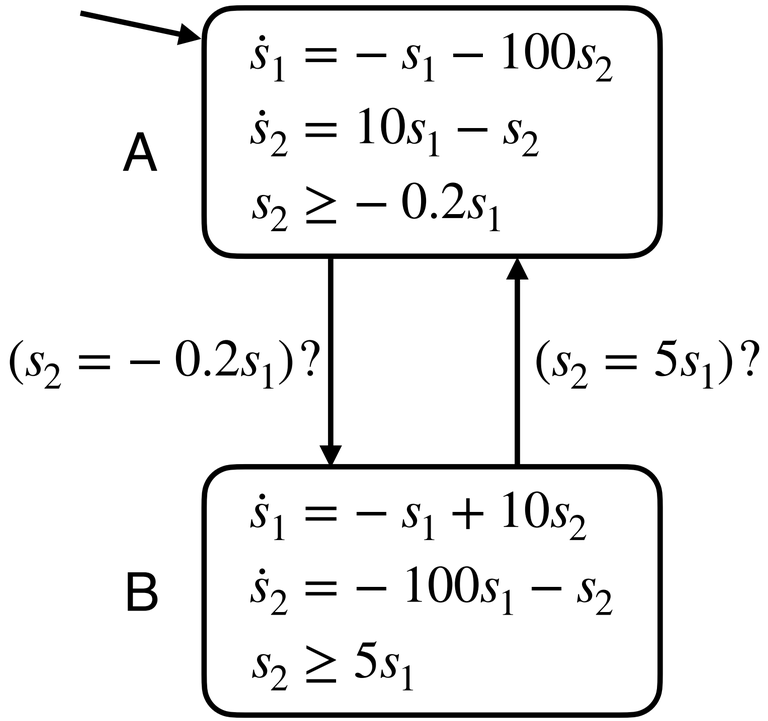
$$\dot{s}_1 = -s_1 - 100s_2$$
$$\dot{s}_2 = 10s_1 - s_2$$
$$s_2 \geq -0.2s_1$$

B

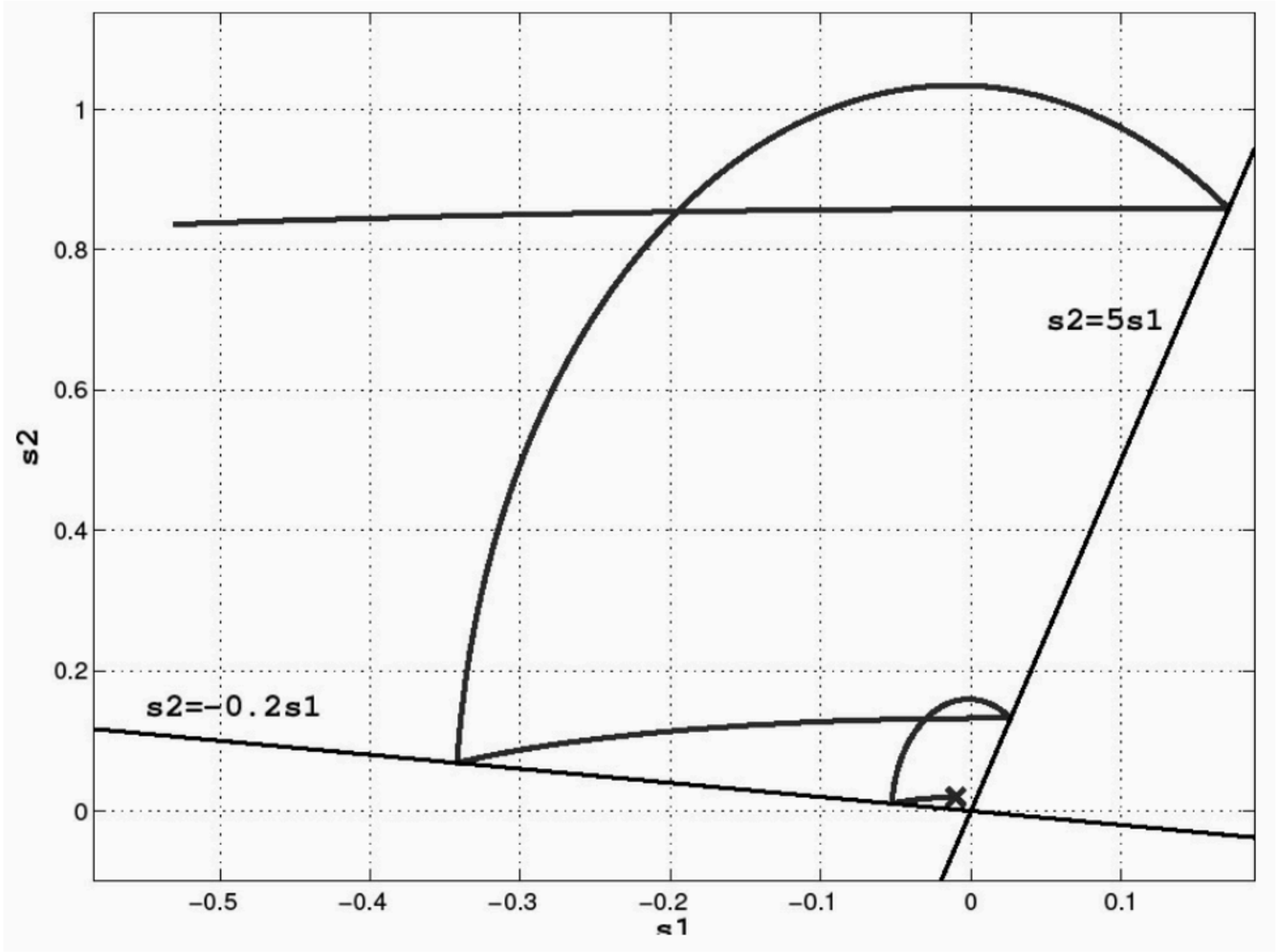
$$\dot{s}_1 = -s_1 + 10s_2$$
$$\dot{s}_2 = -100s_1 - s_2$$
$$s_2 \geq 5s_1$$

- Stability of Dynamical Systems
  - A state s_e of a dynamical system is an equilibrium if the system continues to stay in the state (without external inputs).
  - An equilibrium s_e is stable if we perturb the system state slightly, then the state of the system stays within a bounded distance from s_e.
  - An equilibrium s_e is asymptotically stable if the state of the system converges to s_e.
- Ex. Both A and B are asymptotically stable.

Instability Due to Mode Switching



Mode switching makes the composite system unstable!



Programming Reactive Systems

Programming Reactive Systems

- Reactive Systems: A computational system that reacts with its environment
  - Ex. Embedded Systems, GUI, etc.
- Patterns for Programming Reactive Systems
  - Polling
  - Callback (Interrupts)

Ex. Fan Controller

- Two environmental sensors: Temperature & Humidity
- Controlling policy
  - Turns the fan ON when $DI \geq 75$
  - OFF otherwise

- DI : Discomfort Index $(0.81T + 0.01H(0.99T - 14.3) + 46.3)$
  - T : temperature (Celsius)
  - H : relative humidity (%)
  - Empirically, 50% of people feel uncomfortable if DI reaches 75.

Typical Pattern: Polling (a)

```
#define FAN_PIN 21
DHT12 dht12;

void setup() {
    Wire.begin();
    pinMode(FAN_PIN, OUTPUT);
}
// Main polling loop
void loop () {
    // read sensor values via I2C
    dht12.update();
    float tmp = dht12.temperature();
    float hmd = dht12.humidity();
    // calculate current discomfort index
    float di = 0.81 * tmp + 0.01 * hmd
                * (0.99 * tmp - 14.3) + 46.3;
    // turns the fan on if di >= 75, off otherwise
    digitalWrite(FAN_PIN, di >= 75.0);
}
```

Typical Pattern: Polling (b)

```
#define FAN_PIN 21
TMP11 tmp11;  // temperature sensor
HMD22 hmd22;  // humidity sensor

void setup() {
    Wire.begin();
    pinMode(FAN_PIN, OUTPUT);
}
// Main polling loop
void loop () {
    // read sensor values via I2C
    tmp11.update();    // 3ms
    hmd22.update();    // 15ms
    float tmp = tmp11.read();
    float hmd = hmd22.read();
    // calculate current discomfort index
    float di = 0.81 * tmp + 0.01 * hmd
                * (0.99 * tmp - 14.3) + 46.3;
    // turns the fan on if di >= 75, off otherwise
    digitalWrite(FAN_PIN, di >= 75.0);
}
```

```
TMP11 tmp11;
HMD22 hmd22;

void do_action() {
    float di = 0.81 * tmp + 0.01 * hmd * (0.99 * tmp - 14.3) + 46.3;
    digitalWrite(FAN_PIN, di >= 75.0);
}

void tmp_handler () {
    tmp = tmp11.read();
    do_action();
}

void hmd_handler () {
    hmd = hmd22.read();
    do_action();
}

void setup() {
    Wire.begin();
    tmp11.attach(tmp_handler);
    hmd22.attach(hmd_handler):
}

void loop () { vTaskDelay(portMAX_DELAY); }
```

Typical Pattern: Callbacks (a)

```
TMP11 tmp11;
HMD22 hmd22;
float tmp, hmd;
SemaphoreHandle_t sem;
TaskHandle_t tmp_reader_task, hmd_reader_task;

void tmp_reader_task_fun() {
    for (;;) {
        xTaskNotifyWait(0, 0, NULL, portMAX_DELAY);
        float tmp0 = tmp11.read();
        xSemaphoreTake(sem, portMAX_DELAY);
        tmp = tmp0;
        xSemaphoreGive(sem);
    }
}

void hmd_reader_task_fun() {
    for (;;) {
        xTaskNotifyWait(0, 0, NULL, portMAX_DELAY);
        float hmd0 = hmd22.read();
        xSemaphoreTake(sem, portMAX_DELAY);
        hmd = hmd0;
        xSemaphoreGive(sem);
    }
}
```

Typical Pattern: Callbacks (b)

```
void tmp_handler () { xTaskNotifyFromISR(tmp_reader_task); }

void hmd_handler () { xTaskNotifyFromISR(hmd_reader_task); }

void setup() {
    Wire.begin();
    sem = xSemaphoreCreateBinary();
    xTaskCreatePinnedToCore(tmp_reader_task_fun, "tmp_reader_task",
                            4096, NULL, 1, &tmp_reader_task, 1);
    xTaskCreatePinnedToCore(hmd_reader_task_fun, "hmd_reader_task",
                            4096, NULL, 1, &hmd_reader_task, 1);
    tmp11.attach(tmp_handler);
    hmd22.attach(hmd_handler):
}

void loop () {
    float di;
    xSemaphoreTake(sem, portMAX_DELAY);
    di = 0.81 * tmp + 0.01 * hmd * (0.99 * tmp - 14.3) + 46.3;
    xSemaphoreGive(sem);
    digitalWrite(FAN_PIN, di >= 75.0);
    delay(100);
}
```

Typical Pattern: Callbacks (b)

Polling and Callbacks

- Polling
  - OK for simple system
  - The slowest input govern the input timing
- Callbacks
  - Can handle different timing inputs
  - Split the code into small pieces
  - Callback functions (handlers) may have constraints
- In general, the above patterns are used with threads and/or state machines. This complicates the code and lowers the readability, maintainability, etc.

Functional Reactive Programming (FRP)

- A programming paradigm for reactive systems based on functional (declarative) abstractions to express time-varying values and events.

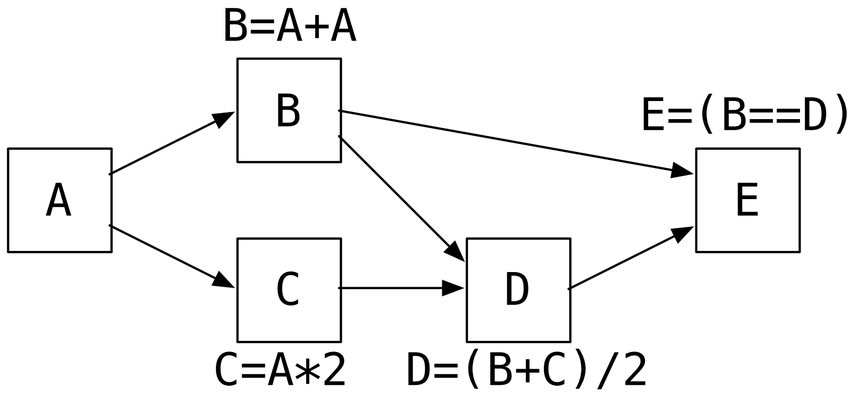
- Time-Varying Values (aka Signals) : Signal α = Time $\rightarrow \alpha$
  - First class entities representing continuously changing data (of type α) over time
  - Ex. temperature :: Signal Float
- Events: Event $\alpha = [(\text{Time}, \alpha)]$
  - List of discrete events of type α
  - Ex. button :: Event Bool

Reactive Systems in FRP

- A reactive system is a function of type Signal $\alpha \rightarrow$ Signal β
  - α : input type, β : output type
- Causality Requirement
  - The value of the output at time t must be determined by input on interval $[0, t]$.
- pure and stateless
  - if output at time t only depends on input at time t
- impure and stateful
  - if output at time t depends on input over the interval $[0, t]$

Glitch Freedom

- A program can be represented as a directed graph without cycles (= DAG) whose nodes and edges correspond to the signals and their dependencies.



- In this example, a *glitch* is the phenomenon that there is an observable difference between B and D due to the delays in updating B, C and D.
- Glitch freedom is a desired property of FRP.

Time-Leak / Space-Leak

- Unrestricted access to a time varying values leads to time/space-leaks

- x :: Signal α = Time $\rightarrow \alpha$
  - The value of x at time t (i.e., $x(t)$) depends on the values of x in the interval $[0,t]$.
- e.g., If $x(t)$ is allowed to be accessed and
  - If $x(t)$ can always be calculated using $\{x(t') \mid t' \in [0,t)\}$, it may take unexpectedly long time (time-leak).
  - If $\{x(t') \mid t' \in [0,t)\}$ is stored in somewhere to calculate $x(t)$, it may require unexpectedly large memory (space-leak)

To Avoid Time-/Space-Leaks

- Restricted representation of Signal α
  - Signal α as a (primitive) type other than Time $\rightarrow \alpha$

- Arrowization
  - Abandon time-varying values themselves and use "functions" on time-varying values
  - *Arrows* are used to construct such "functions"
  - Ex. Yampa (FRP library for Haskell)

FRP Languages / DSLs / Libraries

- Fran [Elliot97]
  - Haskell FRP library for animation
- Yampa [Hudak03][Coutney03]
  - Haskell FRP library based on the notion of Arrows
- FrTime [Cooper04]
  - Extension of Scheme for GUI, simulation
- Flapjax [Meyerovich09], Elm [Czaplicki13]
  - Library/language for client-side web programming
- The majority of the FRP systems proposed so far are based on Haskell or other languages that require rich runtime resource.

Arrowized FRP

- Compose FRP program using *signal functions*
  - Signals are hidden from the code

- Signal Functions
  - SF α $\beta \simeq$ Signal $\alpha \rightarrow$ Signal β
  - SF α β is an *abstract type*
    - Operations on it provide a disciplined way to compose signals
    - The notion of *Arrows* provides the discipline.

Arrow Operations (1)

- Lifting : `arr :: (a -> b) -> SF a b`
- Composition : `(>>>) :: SF a b -> SF b c -> SF a c`
  - Note: `f >>> g = (>>>) f g`

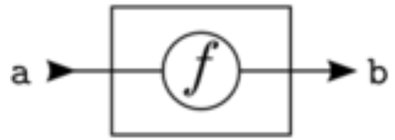
- The following equations should hold.
  - `arr f >>> arr g = arr (g . f)`
    - Note: `(g . f) x = g (f x)`
  - `(f >>> g) >>> h = f >>> (g >>> h)`

Arrow Operations (2)

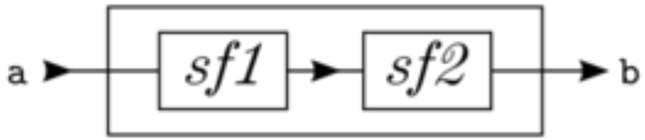
- `(&&&) :: SF a b -> SF a c -> SF a (b, c)`
  - `arr f &&& arr g = arr (f & g)`
    - where `(f & g) x = (f x, g x)`
- `(***) :: SF a b -> SF c d -> SF (a, c) (b, d)`
  - `f *** g = (arr fst >>> f) &&& (arr snd >>> g)`
    - where `fst (x, y) = x, snd (x, y) = y`
- `first :: SF a b -> SF (a, c) (b, c)`
  - `first f = f *** arr id`
    - where `id x = x`
- `second :: SF a b -> SF (c, a) (c, b)`
  - `second f = arr id *** f`

Arrow Operations Visualized

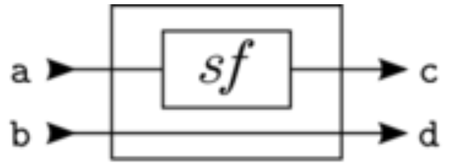
https://wiki.haskell.org/Yampa



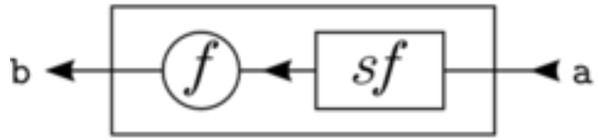
arr f



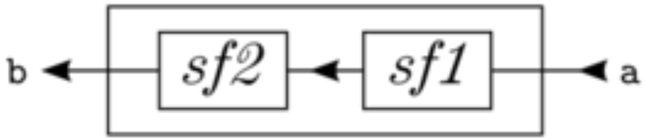
sf1 >>> sf2



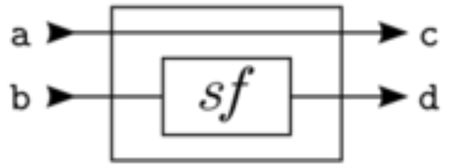
first sf



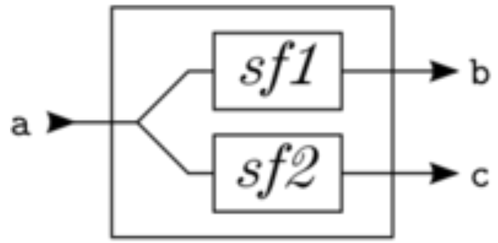
f ^<< sf



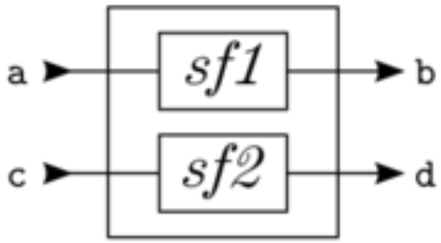
sf2 <<< sf1



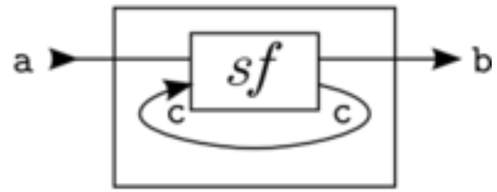
second sf



sf1 &&& sf2



sf1 *** sf2

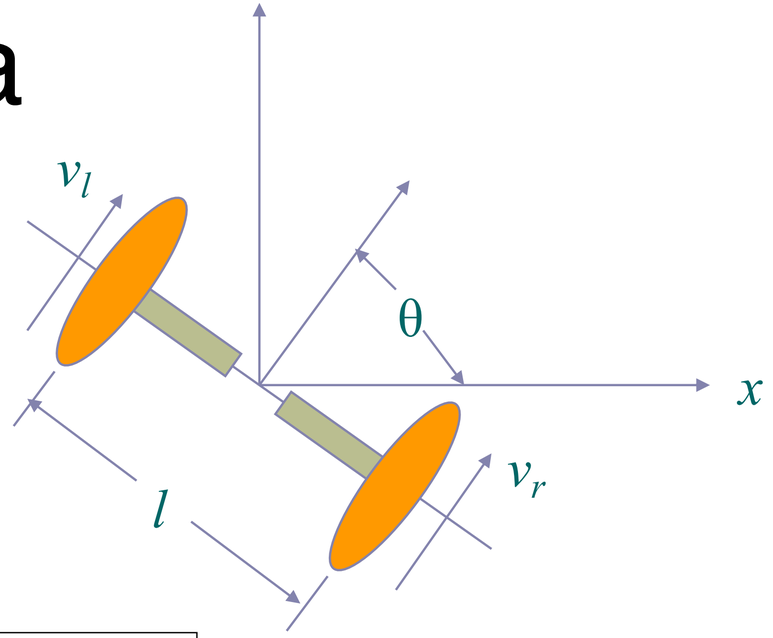


loop sf

Ex. Two-Wheel Robot in Yampa

- Calculate the x position of the robot.

$$x(t) = \int_0^t \frac{vr(t) + vl(t)}{2} \cos\theta(t) dt$$




```
vrSF, vlSF, thetaSF :: SF Input Float

x :: SF Input Float
x = let v = (vrSF &&& vlSF) >>> arr2 (+)
        t = thetaSF >>> arr cos
    in (v &&& t) >>> arr2 (*) >>> integral >>> arr (/ 2)
```

https://wiki.haskell.org/Yampa

```
arr2 :: (a -> b -> c) -> SF (a, b) c
arr2 = arr . uncurry
uncurry :: (a -> b -> c) -> (a, b) -> c
uncurry f (x, y) = f x y
```

Using Arrow Syntax

```
vrSF, vlSF, thetaSF :: SF Input Float

x :: SF Input Float
x = proc input -> do
      vr <- vrSF -< input
      vl <- vlSF -< input
      theta <- thetaSF -< input
      i <- integral -< (vr + vl) * cos theta
      returnA -< (i / 2)
```

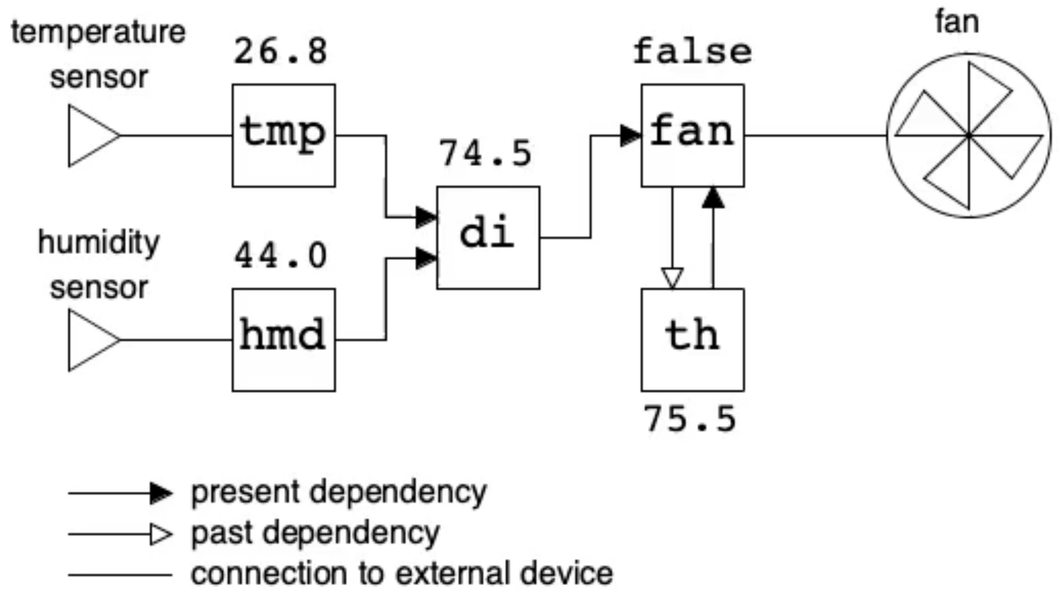
Emfrp [Sawada et al, '16]

A Pure FRP Language for Resource-Constrained Embedded Systems

- Statically-typed pure FRP language
  - Parametric polymorphism, Type inference, Algebraic types & Pattern matching
- Lifting-free, Glitch-free, Space/Time-leaks-free
- Statically determined runtime memory size
- Small memory footprint
  - Can run even in a 8bit microcontrollers such as ATmega32

Emfrp Example

FanController



A program can be seen as a directed graph of time-varying values and their dependencies. Without past dependencies, the graph forms a DAG.

```
module FanController  % module name
in  tmp : Float,      % temperature sensor
    hmd : Float       % humidity sensor
out fan : Bool        % fan switch

% discomfort (temperature-humidity) index
node di = 0.81 * tmp + 0.01 * hmd *
          (0.99 * tmp - 14.3) + 46.3

% fan status
node init[False] fan = di >= th

% threshold
node th = 75.0 +
          if fan@last then -0.5 else 0.5
```

- A fan controller with temperature and humidity sensors.
- Turns ON the fan while the current discomfort index is larger than or equal to the threshold.
- Does a simple hysteresis control to avoid frequent switching of the fan.

Ex. Inverted Pendulum

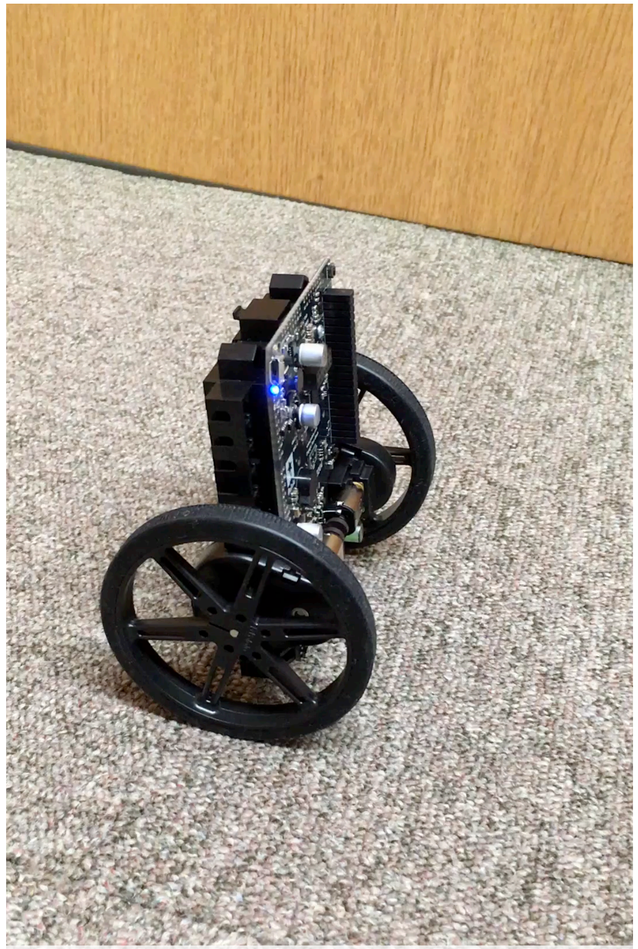
```
module Balancer
in  gyroY  : Int,  # gyroscope (y-axis)
    accX   : Int,  # accelerometer (x-axis)
    encL   : Int,  # left motor encoder
    encR   : Int   # right motor encoder
out motorL : Int,  # left motor
    motorR : Int   # right motor
use Std
...                # definitions of constants

node init[0] angle =
    (angle@last + gyroY * update_time_ms) * 99 / 100

node speedL = encL - encL@last
node speedR = encR - encR@last
node init[0] distL = distL@last + speedL
node init[0] distR = distR@last + speedR

node risingAngleOff  = gyroY * angle_rate_ratio + angle
node init[0] motor =
    motorSpeed(motor@last +
               (angle_resp * risingAngleOff +
                dist_resp * (distL + distR ) +
                speed_resp * (speedL + speedR)) / 100 / gear_ratio)

node diffSpeed = (distL - distR) * dist_diff_resp / 100
node motorL = if accX > 0 then motor + diffSpeed else 0
node motorR = if accX > 0 then motor - diffSpeed else 0
```



Pololu Balboa 32U4
(ATmega 32U4, 32KB Flash, 2.5KB RAM)

Summary

- Hybrid Systems
  - Hybrid Process
  - Zeno Behaviors
  - Stability
- Programming Reactive Systems
  - Functional Reactive Programming