Lecture 24: Theme 24 (Mon 20 Oct 2025 18:58)

Limit when x $\to \infty$

[Example](page 161){this is form $\infty - \infty$} give : $\lim_{x\to-\infty}(x^3-2x) = \lim_{x\to-\infty} x^3\left(1-\frac{2}{x^2}\right) = -\infty$ because $\begin{cases} \lim_{x\to-\infty} x^3 = -\infty \\ \lim_{x\to-\infty} = \left(1-\frac{2}{x^2}\right) = 1 > 0 \end{cases}$

$\implies -\infty \cdot 1 = -\infty$ should $\lim_{x\to-\infty}(x^3-2x) = -\infty$

Example (page 162)form $\frac{\infty}{\infty}$

(1) $\lim_{x\to+\infty}\frac{2x^2+x-1}{x^2-3x+1} = \lim_{x\to+\infty}\frac{\frac{2x^2}{x^2}+\frac{x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{2+\frac{1}{x}-\frac{1}{x^2}}{1-\frac{3}{x}+\frac{1}{x^2}} = \frac{2+0+0}{1-0+0} = 2$

Method 2:

$\frac{2x^2}{x^2} = 2$ this is quick tip if degree of the numerator = degree of the denominator

(2) $\lim_{x\to-\infty}\frac{-x+1}{2x^2+x-1} = \frac{-\frac{x}{x^2}+\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{x}{x^2}-\frac{1}{x^2}} = \frac{\frac{-1}{x}+\frac{1}{x^2}}{2+\frac{1}{x}-\frac{1}{x^2}} = \frac{0+0}{2+0-0} = 0$

(3) $\lim_{x\to+\infty}\frac{x^3+3x^2+2}{2x+1} = \frac{\frac{x^3}{x^3}+\frac{3x^2}{x^3}+\frac{2}{x^3}}{\frac{2x}{x^3}+\frac{1}{x^3}} = \frac{1+\frac{3}{x}+\frac{2}{x^3}}{\frac{2}{x^2}+\frac{1}{x^3}} = \frac{1+0+0}{0+0} = +\infty$

Summary (Quick Tip)

You only need to compare the highest powers (degrees) of the numerator and denominator:

- **If the degree of the numerator equals the degree of the denominator:**
  The limit equals the ratio of the leading coefficients of the numerator and the denominator.
  (As in Example 1)

- **If the degree of the numerator is less than the degree of the denominator:**
  The limit equals 0.
  (As in Example 2)

- **If the degree of the numerator is greater than the degree of the denominator:**
  The limit equals ∞ or $-\infty$.
  (As in Example 3)

Example 1(page 162):

(1) $\lim\limits_{x\to-\infty}\frac{x^3-3x+1}{5-2x}=-\infty;$

(2) $\lim\limits_{x\to-\infty}\frac{1-3x^2-x^3}{4x^2+1}=+\infty;$

(3) $\lim\limits_{x\to+\infty}\frac{3x^2-2x+1}{2x^2-x+5}=\frac{3}{2};$

(4) $\lim\limits_{x\to-\infty}\frac{x+1}{2x^2-x+1}=0;$

Example 2(page 163){Note: $\sqrt{x^2}=|x|$}:

$\lim\limits_{x\to-\infty}\frac{\sqrt{4x^2-x+1}}{x+1}=\frac{\frac{\sqrt{4x^2-x+1}}{x}}{\frac{x+1}{x}}$

$\Longrightarrow \frac{\sqrt{4x^2-x+1}}{x}=\frac{\sqrt{x^2\left(\frac{4x^2}{x^2}-\frac{x}{x^2}+\frac{1}{x^2}\right)}}{x}=\frac{|x|\sqrt{\left(4-\frac{1}{x}+\frac{1}{x^2}\right)}}{x}=\frac{-x\sqrt{\left(4-\frac{1}{x}+\frac{1}{x^2}\right)}}{x}=-\sqrt{\left(4-\frac{1}{x}+\frac{1}{x^2}\right)}=-\sqrt{4}=-2$

$\Longrightarrow \frac{x+1}{x}=1+\frac{1}{x}=1$

$\Longrightarrow \lim\limits_{x\to-\infty}\frac{\sqrt{4x^2-x+1}}{x+1}=\lim\limits_{x\to-\infty}\frac{\frac{\sqrt{4x^2-x+1}}{x}}{\frac{x+1}{x}}=-\frac{2}{1}=-2$

Example 3(page 163):{Note: $\sqrt{x^2}=|x|$}

(1) $\lim\limits_{x\to+\infty}\frac{\sqrt{x^2+1}+x}{x+1}=\lim\limits_{x\to+\infty}\frac{\frac{\sqrt{x^2+1}+\frac{x}{x}}{x}}{\frac{x+1}{x}}$

$\Longrightarrow \frac{\sqrt{x^2+1}+\frac{x}{x}}{x}=\frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}+1}{x}=\frac{|x|\sqrt{\left(1+\frac{1}{x^2}\right)}+1}{x}=\frac{x\sqrt{\left(1+\frac{1}{x^2}\right)}+1}{x}=\sqrt{\left(1+\frac{1}{x^2}\right)}+1=\sqrt{(1+0)}+1=2$

$\Longrightarrow \frac{x+1}{x}=1+\frac{1}{x}=1$

$\Longrightarrow \lim\limits_{x\to+\infty}\frac{\sqrt{x^2+1}+x}{x+1}=\lim\limits_{x\to+\infty}\frac{\frac{\sqrt{x^2+1}+\frac{x}{x}}{x}}{\frac{x+1}{x}}=\frac{2}{1}=2$

(2) $\lim\limits_{x\to-\infty}\frac{\sqrt{x^2-x}-\sqrt{4x^2+1}}{2x+3}=\frac{\frac{\sqrt{x^2-x}}{x}-\frac{\sqrt{4x^2+1}}{x}}{\frac{2x+3}{x}}$

$\Longrightarrow \frac{\sqrt{x^2-x}}{x}=\frac{\sqrt{x^2\left(1-\frac{1}{x}\right)}}{x}=\frac{|x|\sqrt{\left(1-\frac{1}{x}\right)}}{x}=\frac{-x\sqrt{\left(1-\frac{1}{x}\right)}}{x}=-\sqrt{\left(1-\frac{1}{x}\right)}=-\sqrt{1}=-1$

$\Longrightarrow \frac{-\sqrt{4x^2+1}}{x}=\frac{-\sqrt{x^2\left(4+\frac{1}{x^2}\right)}}{x}=\frac{-|x|\sqrt{\left(4+\frac{1}{x^2}\right)}}{x}=\frac{-(-x)\sqrt{\left(4+\frac{1}{x^2}\right)}}{x}=\frac{x\sqrt{\left(4+\frac{1}{x^2}\right)}}{x}=\sqrt{\left(4+\frac{1}{x^2}\right)}=2$

$\frac{2x+3}{x}=2+\frac{3}{x}=2$

$\Longrightarrow \lim\limits_{x\to-\infty}\frac{\sqrt{x^2-x}-\sqrt{4x^2+1}}{2x+3}=\frac{\frac{\sqrt{x^2-x}}{x}-\frac{\sqrt{4x^2+1}}{x}}{\frac{2x+3}{x}}=\frac{-1+2}{2}=\frac{1}{2}$

Example 4(page 163) Give $\lim\limits_{x\to+\infty}\frac{ax^2+bx+3}{cx^3+3x+2}=2$; know $a,b,c\in\mathbb{R}$. Caculate $a+b+c$?

Solve:

Let degree of the numerator = A;

Let degree of the denominator = B;

We know :

Case 1: $A = B \implies$ The limit equals the ratio the leading coefficients of the A and B;

Case 2: $A < B \implies$ The limit = 0;

Case 3: $A > B \implies$ The limit = ∞ or $-\infty$

Given : $\lim\limits_{x\to+\infty} \frac{ax^2+bx+3}{cx^3+3x+2} = 2$ only applies to case 1 (A = B)

$\implies c = 0 \implies \lim\limits_{x\to+\infty} \frac{ax^2+bx+3}{3x+2} = 2$ Not yet consistent with the given condition, so let $a = 0$

$\implies \lim\limits_{x\to+\infty} \frac{bx+3}{3x+2} = 2 \implies$ satisfy given

$\implies \lim\limits_{x\to+\infty} \frac{bx+3}{3x+2} = 2 \implies \frac{b}{3} = 2 \implies b = 6$

$\implies a + b + c = 6$

Additional exercises(page: 163)

Find limit:

(1) $\lim\limits_{x\to-\infty} \frac{x^2+x+1}{2x^3+2x+5} = 0$(Case 2)

(2) $\lim\limits_{x\to+\infty} \frac{2x^2+1}{x^3-3x^2+2} = 0$(Case 2)

(3) $\lim\limits_{x\to+\infty} \frac{\sqrt[3]{x^2-x+1}}{5x^2-1} = 0$(Note: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$)

(4) $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2-x+1}}{2x+1} = \lim\limits_{x\to-\infty} \frac{\frac{\sqrt{x^2-x+1}}{x}}{\frac{2x+1}{x}}$

$\implies \lim\limits_{x\to-\infty} \frac{\sqrt{x^2-x+1}}{x} = \lim\limits_{x\to-\infty} \frac{\sqrt{x^2\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} \frac{|x|\sqrt{\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} \frac{-x\sqrt{\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} -\sqrt{\left(1-\frac{1}{x}+\frac{1}{x^2}\right)} = -1$

$\implies \lim\limits_{x\to-\infty} \frac{2x+1}{x} = \lim\limits_{x\to-\infty} \frac{2x}{x} + \frac{1}{x} = \lim\limits_{x\to-\infty} 2 + \frac{1}{x} = 2$

$\implies \lim\limits_{x\to-\infty} \frac{\sqrt{x^2-x+1}}{2x+1} = \lim\limits_{x\to-\infty} \frac{\frac{\sqrt{x^2-x+1}}{x}}{\frac{2x+1}{x}} = -\frac{1}{2}$

(5) $\lim\limits_{x\to-\infty} \frac{\sqrt[3]{x^3-1}}{\sqrt{2x^2+1}} = \lim\limits_{x\to-\infty} \frac{\frac{\sqrt[3]{x^3-1}}{x}}{\frac{\sqrt{2x^2+1}}{x}}$

$\implies \lim\limits_{x\to-\infty} \frac{\sqrt[3]{x^3-1}}{x} = \lim\limits_{x\to-\infty} \frac{\sqrt[3]{x^3\left(1-\frac{1}{x^3}\right)}}{x} = \lim\limits_{x\to-\infty} \frac{x\sqrt[3]{\left(1-\frac{1}{x^3}\right)}}{x} = \lim\limits_{x\to-\infty} \sqrt[3]{\left(1-\frac{1}{x^3}\right)} = 1$

$\implies \lim\limits_{x\to-\infty} \frac{\sqrt{2x^2+1}}{x} = \lim\limits_{x\to-\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} \frac{|x|\sqrt{\left(2+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} \frac{-x\sqrt{\left(2+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} -\sqrt{\left(2+\frac{1}{x^2}\right)} = -\sqrt{2}$

$\implies \lim\limits_{x\to-\infty} \frac{\sqrt[3]{x^3-1}}{\sqrt{2x^2+1}} = \lim\limits_{x\to-\infty} \frac{\frac{\sqrt[3]{x^3-1}}{x}}{\frac{\sqrt{2x^2+1}}{x}} = \frac{1}{-\sqrt{2}} = \frac{-\sqrt{2}}{2}$

(6) $\lim\limits_{x\to-\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt{2x^2+1}} = \lim\limits_{x\to-\infty} \frac{\frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2}}{\frac{\sqrt{2x^2+1}}{x^2}}$

$\implies \lim\limits_{x\to-\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2} = \lim\limits_{x\to-\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt[3]{x^6}} = \lim\limits_{x\to-\infty} \sqrt[3]{\frac{x^6+x^4+x^2+1}{x^6}} = 1$

$\implies \lim\limits_{x\to-\infty} \frac{\sqrt{2x^2+1}}{x^2} = \lim\limits_{x\to-\infty} \frac{\sqrt{2x^2+1}}{\sqrt{x^4}} = \lim\limits_{x\to-\infty} \sqrt{\frac{2x^2+1}{x^4}} = \lim\limits_{x\to-\infty} \sqrt{\frac{2}{x^2}+\frac{1}{x^4}} =$
0

$\implies \lim\limits_{x\to-\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt{2x^2+1}} = \lim\limits_{x\to-\infty} \frac{\frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2}}{\frac{\sqrt{2x^2+1}}{x^2}} = \frac{1}{0} = +\infty$

Method 2:

We can see numerator the greatest is: $\sqrt[3]{x^6} = x^2$ and all of them are positive. $\implies$ numerator $= +\infty$

We can see denominator is $\sqrt{2x^2+1} \implies$ alway positive

We know : $x^2 > x^1$(The degree of the numerator is greater than the degree of the denominator)

$\implies \frac{positive}{positve} = +\infty$

Method 3:

The greatest degree of numerator is : $\sqrt[3]{x^6} = x^2$

The greatest degree of denominator is : $\sqrt{2x^2} = \sqrt{2} \cdot \sqrt{x^2}$(because $x \to -\infty$ should $|x| = -x$)

$\implies$ The limit after rewriting is : $\lim\limits_{x\to-\infty} \frac{x^2}{\sqrt{2}(-x)} = \frac{x}{-\sqrt{2}} = \frac{negative}{negative} = +\infty$

Medthod 4: Using Casio f(x) 580 :>

(7) $\lim\limits_{x\to-\infty} \frac{x-\sqrt{2x^2+1}}{2x+3\sqrt{x^2+1}} = \lim\limits_{x\to-\infty} \frac{\frac{x-\sqrt{2x^2+1}}{x}}{\frac{2x+3\sqrt{x^2+1}}{x}}$

$\implies \lim\limits_{x\to-\infty} \frac{x-\sqrt{2x^2+1}}{x} = \lim\limits_{x\to-\infty} \frac{-\sqrt{2x^2+1}+x}{x} = \lim\limits_{x\to-\infty} \frac{-\sqrt{x^2\left(2+\frac{1}{x^2}\right)}+x}{x} = \lim\limits_{x\to-\infty} \frac{-|x|\sqrt{\left(2+\frac{1}{x^2}\right)}+x}{x} =$
$\lim\limits_{x\to-\infty} \frac{x\sqrt{\left(2+\frac{1}{x^2}\right)}+x}{x} = \lim\limits_{x\to-\infty} \frac{x\sqrt{\left(2+\frac{1}{x^2}\right)}}{x} + \frac{x}{x} = \lim\limits_{x\to-\infty} \sqrt{\left(2+\frac{1}{x^2}\right)} + 1 = 1+\sqrt{2}$

$\implies \lim\limits_{x\to-\infty} \frac{2x+3\cdot\sqrt{x^2+1}}{x} = \lim\limits_{x\to-\infty} \frac{2x}{x} + \frac{3\cdot\sqrt{x^2+1}}{x} = \lim\limits_{x\to-\infty} 2 + \frac{3\cdot\sqrt{x^2+1}}{x} =$
$\lim\limits_{x\to-\infty} 2 + \frac{3\cdot\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} \frac{3\cdot|x|\sqrt{\left(1+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} \frac{-3x\cdot\sqrt{\left(1+\frac{1}{x^2}\right)}}{x} = \lim\limits_{x\to-\infty} 2 +$
$-3\cdot\sqrt{\left(1+\frac{1}{x^2}\right)} = -1$

$\implies \lim\limits_{x\to-\infty} \frac{x-\sqrt{2x^2+1}}{2x+3\sqrt{x^2+1}} = \lim\limits_{x\to-\infty} \frac{\frac{x-\sqrt{2x^2+1}}{x}}{\frac{2x+3\sqrt{x^2+1}}{x}} = \frac{1+\sqrt{2}}{-1} = -1-\sqrt{2}$

(8) $\lim\limits_{x\to+\infty} \sqrt{\frac{x^3+1}{2x^3+5}} = \lim\limits_{x\to+\infty} \sqrt{\frac{\frac{x^3+1}{x^3}}{\frac{2x^3+5}{x^3}}} = \lim\limits_{x\to+\infty} \sqrt{\frac{1+\frac{1}{x^3}}{2+\frac{5}{x^3}}} = \sqrt{\frac{1+0}{2+0}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

Method 2(Case 1){degree numerator = degree denominator}

$\sqrt{\frac{x^3}{2x^3}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

[Example] (page 164) form $\infty - \infty$

(1) $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+2x}-x\right)=\lim\limits_{x\to+\infty}\frac{\left(\sqrt{x^2+2x}-x\right)\left(\sqrt{x^2+2x}+x\right)}{\left(\sqrt{x^2+2x}+x\right)}=\lim\limits_{x\to+\infty}\frac{x^2+2x-x}{\left(\sqrt{x^2+2x}+x\right)}=$ $\lim\limits_{x\to+\infty}\frac{2x}{\sqrt{x^2+2x}+x}=\lim\limits_{x\to+\infty}\frac{\frac{2x}{x}}{\frac{\sqrt{x^2\left(1+\frac{2}{x}\right)}}{x}+\frac{x}{x}}=\lim\limits_{x\to+\infty}\frac{2}{\frac{|x|\sqrt{\left(1+\frac{2}{x}\right)}}{x}+1}=\lim\limits_{x\to+\infty}\frac{2}{\frac{x\sqrt{\left(1+\frac{2}{x}\right)}}{x}+1}=$ $\lim\limits_{x\to+\infty}\frac{2}{\sqrt{\left(1+\frac{2}{x}\right)}+1}=\frac{2}{\sqrt{1+0}+1}=\frac{2}{2}=1$

(2) $\lim\limits_{x\to+\infty}\left(\sqrt{x+1}-\sqrt{x}\right)=\lim\limits_{x\to+\infty}\frac{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)}=\lim\limits_{x\to+\infty}\frac{1}{\left(\sqrt{x+1}+\sqrt{x}\right)}=$ $\frac{1}{+\infty}=0$

(3) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)=\lim\limits_{x\to-\infty}\frac{\left(\sqrt{x^2+x+1}+x\right)\left(\sqrt{x^2+x+1}-x\right)}{\left(\sqrt{x^2+x+1}-x\right)}=\lim\limits_{x\to-\infty}\frac{x^2+x+1-x^2}{\left(\sqrt{x^2+x+1}-x\right)}=$ $\lim\limits_{x\to-\infty}\frac{x+1}{\left(\sqrt{x^2+x+1}-x\right)}=\lim\limits_{x\to-\infty}\frac{\frac{x+1}{x}}{\frac{\sqrt{x^2+x+1}-x}{x}}=\lim\limits_{x\to-\infty}\frac{1+\frac{1}{x}}{\frac{\sqrt{x^2\left(1+\frac{1}{x}+\frac{1}{x}\right)}-x}{x}}=\lim\limits_{x\to-\infty}\frac{1+\frac{1}{x}}{\frac{|x|\sqrt{\left(1+\frac{2}{x}\right)}-x}{x}}=$ $\lim\limits_{x\to-\infty}\frac{1+\frac{1}{x}}{\frac{-x\left(\sqrt{1+\frac{2}{x}}\right)-x}{x}}=\lim\limits_{x\to-\infty}\frac{1+\frac{1}{x}}{\frac{-x\sqrt{1+\frac{2}{x}}}{x}-\frac{x}{x}}=\lim\limits_{x\to-\infty}\frac{1+\frac{1}{x}}{-\sqrt{1+\frac{2}{x}}-1}=\frac{1+0}{-\sqrt{1+0}-1}=-\frac{1}{2}$

Example 1(page 165):

(1) $\lim\limits_{x\to+\infty}\left(\sqrt{x+3}-\sqrt{x}\right)=\lim\limits_{x\to+\infty}\frac{\left(\sqrt{x+3}-\sqrt{x}\right)\left(\sqrt{x+3}+\sqrt{x}\right)}{\left(\sqrt{x+3}+\sqrt{x}\right)}=\lim\limits_{x\to+\infty}\frac{3}{\left(\sqrt{x+3}+\sqrt{x}\right)}=$ 0

(2) $\lim\limits_{x\to+\infty}\left(\frac{4}{\sqrt{x+2}-\sqrt{x-2}}\right)=\lim\limits_{x\to+\infty}\frac{4\left(\sqrt{x+2}+\sqrt{x-2}\right)}{\left(\sqrt{x+2}-\sqrt{x-2}\right)\left(\sqrt{x+2}+\sqrt{x-2}\right)}=\lim\limits_{x\to+\infty}\frac{4\left(\sqrt{x+2}+\sqrt{x-2}\right)}{(x+2)-(x-2)}=$ $\lim\limits_{x\to+\infty}\left(\sqrt{x+2}+\sqrt{x-2}\right)=+\infty$

(3) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-3x}+x\right)=\lim\limits_{x\to-\infty}\frac{\left(\sqrt{x^2-3x}+x\right)\left(\sqrt{x^2-3x}-x\right)}{\left(\sqrt{x^2-3x}-x\right)}=\lim\limits_{x\to-\infty}\frac{-3x}{\left(\sqrt{x^2-3x}-x\right)}=$ $\lim\limits_{x\to-\infty}\frac{\frac{-3x}{x}}{\frac{\sqrt{x^2-3x}-x}{x}}=\lim\limits_{x\to-\infty}\frac{-3}{\frac{\sqrt{x^2-3x}}{x}-\frac{x}{x}}=\lim\limits_{x\to-\infty}\frac{-3}{\frac{\sqrt{x^2-3x}}{x}-1}=\lim\limits_{x\to-\infty}\frac{-3}{\frac{\sqrt{x^2\left(1-\frac{3}{x}\right)}}{x}-1}=$ $\lim\limits_{x\to-\infty}\frac{-3}{\frac{|x|\sqrt{1-\frac{3}{x}}}{x}-1}=\lim\limits_{x\to-\infty}\frac{-3}{\frac{-x\sqrt{1-\frac{3}{x}}}{x}-1}=\lim\limits_{x\to-\infty}\frac{-3}{-\sqrt{1-\frac{3}{x}}-1}=\frac{-3}{-\sqrt{1-0}-1}=\frac{3}{2}$

(4) $\lim\limits_{x\to+\infty}\left(2x-\sqrt{4x^2+2x-1}\right)=\lim\limits_{x\to+\infty}\frac{\left(2x-\sqrt{4x^2+2x-1}\right)\left(2x+\sqrt{4x^2+2x-1}\right)}{\left(2x+\sqrt{4x^2+2x-1}\right)}=$ $\lim\limits_{x\to+\infty}\frac{6x^2-2x+1}{\left(2x+\sqrt{4x^2+2x-1}\right)}=+\infty$

Example 2(page 165):Value of $A=\lim\limits_{x\to-\infty}\left(\sqrt{x^2+4x+5}+x\right)$ **=?**

$A=\lim\limits_{x\to-\infty}\left(\sqrt{x^2+4x+5}+x\right)=\lim\limits_{x\to-\infty}\frac{\left(\sqrt{x^2+4x+5}+x\right)\left(\sqrt{x^2+4x+5}-x\right)}{\left(\sqrt{x^2+4x+5}-x\right)}=\lim\limits_{x\to-\infty}\frac{4x+5}{\left(\sqrt{x^2+4x+5}-x\right)}=$ $\lim\limits_{x\to-\infty}\frac{\frac{4x+5}{x}}{\frac{\left(\sqrt{x^2+4x+5}-x\right)}{x}}=\lim\limits_{x\to-\infty}\frac{\frac{4x}{x}+\frac{5}{x}}{\frac{\sqrt{x^2+4x+5}}{x}-\frac{x}{x}}=\lim\limits_{x\to-\infty}\frac{4+\frac{5}{x}}{\frac{\sqrt{x^2+4x+5}}{x}-1}=\lim\limits_{x\to-\infty}\frac{4+\frac{5}{x}}{\frac{\sqrt{x^2\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}}{x}-1}=$ $\lim\limits_{x\to-\infty}\frac{4+\frac{5}{x}}{\frac{|x|\sqrt{\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}}{x}-1}=\lim\limits_{x\to-\infty}\frac{4+\frac{5}{x}}{\frac{-x\sqrt{\left(1+\frac{4}{x}+\frac{5}{x^2}\right)}}{x}-1}=\lim\limits_{x\to-\infty}\frac{4+\frac{5}{x}}{-\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}-1}=\frac{4}{-2}=$ -2

$\implies A=-2$

Example 3(page 165):

Give $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+ax}-x\right)=8$. Find $a, a\in\mathbb{R}$

$\lim\limits_{x\to+\infty}\left(\sqrt{x^2+ax}-x\right)=\lim\limits_{x\to+\infty}\frac{\left(\sqrt{x^2+ax}-x\right)\left(\sqrt{x^2+ax}+x\right)}{\left(\sqrt{x^2+ax}+x\right)}=\lim\limits_{x\to+\infty}\frac{ax}{\left(\sqrt{x^2+ax}+x\right)}=$
$\lim\limits_{x\to+\infty}\frac{\frac{ax}{x}}{\frac{\sqrt{x^2+ax}+x}{x}}=\lim\limits_{x\to+\infty}\frac{a}{\frac{\sqrt{x^2+ax}}{x}+\frac{x}{x}}=\lim\limits_{x\to+\infty}\frac{a}{\frac{\sqrt{x^2\left(1+\frac{a}{x}\right)}}{x}+1}=\lim\limits_{x\to+\infty}\frac{a}{\frac{|x|\sqrt{\left(1+\frac{a}{x}\right)}}{x}+1}=$
$\lim\limits_{x\to+\infty}\frac{a}{\frac{x\sqrt{\left(1+\frac{a}{x}\right)}}{x}+1}=\lim\limits_{x\to+\infty}\frac{a}{\sqrt{1+\frac{a}{x}}+1}=\frac{a}{\sqrt{1+0}+1}=\frac{a}{2}=8\implies a=16$

Additional Exercises(page 165):

(1) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+4x+1}+x\right)=\lim\limits_{x\to-\infty}\frac{\left(\sqrt{x^2+4x+1}+x\right)\left(\sqrt{x^2+4x+1}-x\right)}{\left(\sqrt{x^2+4x+1}-x\right)}=\lim\limits_{x\to-\infty}\frac{4x+1}{\sqrt{x^2+4x+1}-x}=$
$\lim\limits_{x\to-\infty}\frac{\frac{4x+1}{x}}{\frac{\sqrt{x^2+4x+1}-x}{x}}=\lim\limits_{x\to-\infty}\frac{4+\frac{1}{x}}{\frac{\sqrt{x^2+4x+1}}{x}-\frac{x}{x}}=\lim\limits_{x\to-\infty}\frac{4+\frac{1}{x}}{\frac{\sqrt{x^2+4x+1}}{x}-1}=\lim\limits_{x\to-\infty}\frac{4+\frac{1}{x}}{\frac{\sqrt{x^2\left(1+\frac{4}{x}+\frac{1}{x^2}\right)}}{x}-1}=$
$\lim\limits_{x\to-\infty}\frac{4+\frac{1}{x}}{\frac{|x|\sqrt{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)}}{x}-1}=\lim\limits_{x\to-\infty}\frac{4+\frac{1}{x}}{\frac{-x\sqrt{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)}}{x}-1}=\lim\limits_{x\to-\infty}\frac{4+\frac{1}{x}}{-\sqrt{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)}-1}=\frac{4+0}{-\sqrt{1+0+0}-1}=$
$\frac{4}{-2}=-2$

(7) $\lim\limits_{x\to+\infty}\{\frac{1}{x+1}\cdot\left(\sqrt{x^2+2x}-x\right)\}$