Lecture 7: Lecture 7

Max-Min (part 3)

I. The problem has a model

Example 1:

Solve:

Consider the $V(r) = k(R-r)r^2 \implies V(r) = kRr^2 - kr^3$

$V'(r) = 2kRr - 3kr^2 = kr(2R-3r)$

$V'(r) = 0 \iff \left[\begin{array}{l} r = 0 \\ r = \frac{2R}{3} \end{array}\right.$

Variation table:

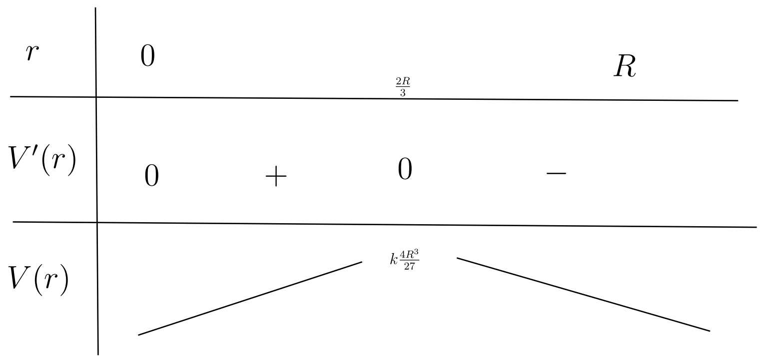

Figure 1: Variation table R

$\implies \max\limits_{[0;R)} V(r) = k\frac{4R^3}{27}$ at $r = \frac{2R}{3}$

Example 2:

Solve:

Consider the $f(v)$ on the domain $(0;+\infty)$

$f(v) = \frac{209{,}4v}{0{,}36v^2+13{,}2v+264}$

$f'(v) = -\frac{-75{,}384v^2+55281{,}6}{(0{,}36v^2+13{,}2v+264)^2}$

$f'(v) = 0 \iff v = \frac{10\sqrt{66}}{3}$ (beacause $v > 0$)

Variation table:

v	0		$\frac{10\sqrt{66}}{3}$		$+\infty$
$f'(v)$		$+$	0	$-$	
$f(v)$		$\nearrow$	$6,4041$	$\searrow$	

Figure 2: example 2 xeco

$\Longrightarrow \max\limits_{(0;+\infty)} f(v) \approx 6,4041$ at $v \approx 27,09(km)$

Example 3: Be careful with the convertion units

Summary:

Sell : 60.000/item $\Longrightarrow x$ item $= 60000 \cdot x$

Publitaion cost : $C(x) = 0,0001x^2 - 0,2x + 10000$ (ten thousand dong)

Release cost: 20000/item $\Longrightarrow x$ item $= 20000x$

Earn 90 million dong from ads

$T(x)$ as total cost = publication cost + release cost

Solve:

a)True :

convert dong to ten thousand dong : $\frac{20000x}{10} = 2x$

$T(x) = 0,0001x^2 - 0,2x + 10000 + 2x = 0,0001x^2 + 1,8x + 10000$

b) True

Convert dong to ten thousand dong : $\frac{60000x}{10000} = 6x$

Revenue x books = sell (ten thousand dong) + Earn from ads = $6x + 9000$

Profit = Revenue - cost

Let $P(x)$ be profit function

$\Longrightarrow P(x) = 6x + 9000 - (0,0001x^2 + 1,8x + 10000)$

$= -0,0001x^2 - 4,2x - 1000$

$P(1000) = 3100$ (ten thousand dong)

$\Longrightarrow 3100 \cdot 10000 = 31000000$ (thousand dong)

c) True

$T(x) = 0,0001x^2 + 1,8x + 10000$

$\Longrightarrow M(x) = \frac{T(x)}{x} = 0,0001x + 1,8 + \frac{1000}{x}$

$M'(x) = 0,0001 - \frac{1000}{x^2}$

$M'(x) = 0 \iff x = 10000(x > 0)$

Variation table :

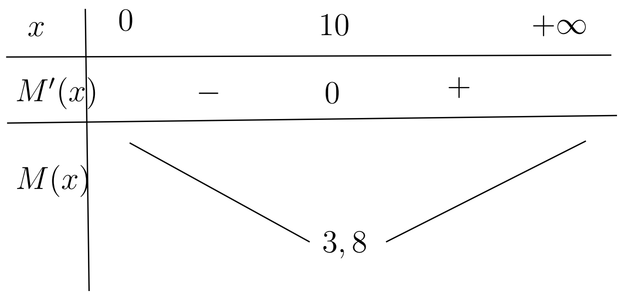

Figure 3: VND

$\implies M(10000) = 3,8$ (ten thousand dong)
$\implies 3,8 \cdot 10000 = 38000$ (thousand dong)
d)True
$P(x) = -0,0001x^2 - 4,2x - 1000$
$P'(x) = -0,0002x - 4,2$
$P'(x) = 0 \iff x = 21000$

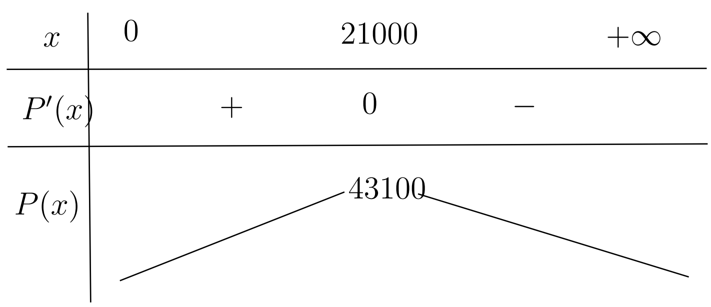

Figure 4: Max Profit

$\implies 43100 \cdot 10000 = 431000000$ (million dong)
II.The problem needs to build a model

Example 4:

Solve:

A square aluminum sheet $\implies$ All four sides are equal:

$h = x$

$l = 100 - 2x$

$w = 100 - 2x$

$100 - 2x > 0 \implies 0 < x < 50 \implies (0; 50)$

$\implies V = l \cdot w \cdot h$

$= (100 - 2x)^2 \cdot x$

$= 4x^3 - 400x^2 + 10000x$

Consider $V(x)$ on the domain $(0; 50)$

$V'(x) = 12x^2 - 800x + 10000$

$V'(x) = 0 \iff \left[\begin{array}{l} x = 50 \\ x = \frac{50}{3} \end{array}\right.$

Variation table:

x	0		$\frac{50}{3}$		50
$V'(x)$		+	0	−	0
$V(x)$			74074, 074		

Figure 5: tam nhom cua bac 5

$\implies \max\limits_{(0;50)} V(x) = 74074,074$ at $x = \frac{50}{3}$

Example 5:

Idea: $B\hat{E}C = C\hat{E}H - B\hat{E}H$

Solve:

Draw $AB \perp CD (H \in AB) \implies HA = 1,6, HB = 6,4$

We have:

in $\triangle BHE : tanB\hat{E}H = \frac{6,4}{x}$ **Note:** $tan = \frac{opposite}{adjacent}$

in $\triangle CHE : tanC\hat{E}H = \frac{8,1}{x}$

$\implies tanB\hat{E}C = tan(CEH) \hat{-} B\hat{E}H = \frac{tanC\hat{E}H - tanB\hat{E}H}{1 + tanC\hat{E}H \cdot tanB\hat{E}H} = \frac{1,7x}{x^2 + 51,84}$ **Note:** $\frac{tan\alpha - tan\beta}{1 + tan\alpha \cdot tan\beta}$

Consider $f(x) = \frac{1,7x}{x^2 + 51,84}(0; +\infty)$

$f'(x) = \frac{-1,7x^2 + 88,128}{(x^2 + 51,84)^2}$

$f'(x) = 0 \iff x = \frac{36}{5}(x > 0)$
Variation table:

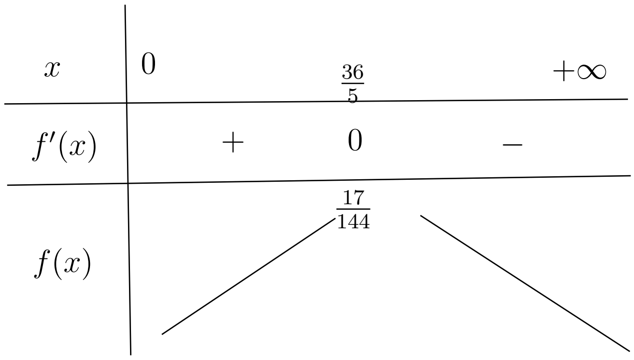

Figure 6: Find BEC

$\Longrightarrow \max\limits_{(0;+\infty)} f(x) = \frac{17}{144}$ at $x = \frac{36}{5}$
III.HomeWork
Sentence 1: Be careful Procducts sold
Idea:
Products sold $= x$
Price $= P$
Total cost $= Q$
Sales = price $\cdot$ Products sold $= P \cdot x = (600 - 2x) \cdot x = -2x^2 + 600x$
Profit = Sales - Total cost $= -2x^2 + 600x - 0,2x^2 - 28x - 200 = -2,2x^2 + 572x - 200$
Consider $f(x) = -2,2x^2 + 572x - 200(0; +\infty)$
$f'(x) = -4,4x + 572$
$f'(x) = 0 \iff x = 130$
Variation table:

x	0		130		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$		$\nearrow$	36980	$\searrow$	

Figure 7: Be careful with Products sold

$\implies \max\limits_{(0;+\infty)} f(x) = 36980$ at $x = 130$

Sentence 2:

Skill: main skill "Unfold(trải phẳng)"

Calculator tool :

- Similar triangles/Thales' + Unfold theorem or deratives + pytago theorem

Solve:

Before:

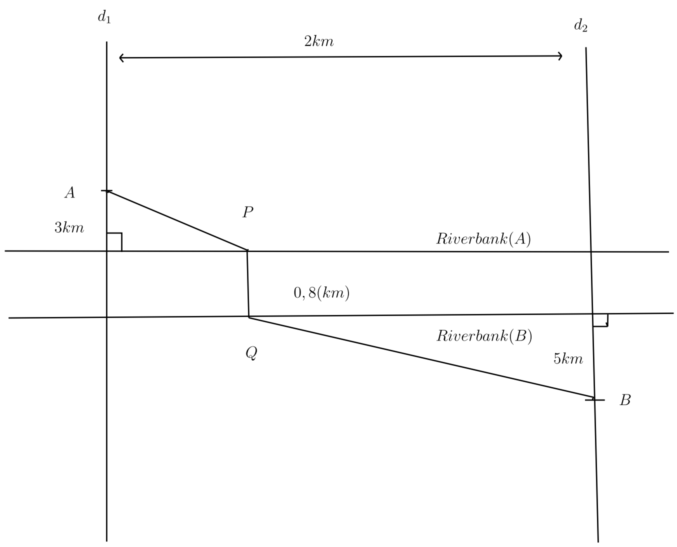

Figure 8: Bridge-DongAnh

Medthod 1:Using deratives + pytago

We can see : AB = AP + P + Q + QB

+) PQ = width of river $\implies$ PQ is constant = 0,8 $\implies$ should we need to find only min of sum AP + QB

Let side $AP = x \implies QB = 2 - x$

Apply theoreo pytago we have :

$AP = \sqrt{3^2 + x^2} = \sqrt{9 + x^2}$

$QB = \sqrt{25 + (2 - x)^2}$

Consider $f(x) = \sqrt{9 + x^2} + \sqrt{25 + (2 - x)^2}$

$f'(x) = \frac{x}{\sqrt{9+x^2}} + \frac{2-x}{\sqrt{25+(2-x)^2}}$

$f'(x) = 0 \iff \frac{x}{\sqrt{9+x^2}} = \frac{x-2}{\sqrt{25+(2-x)^2}}$

$\iff x \cdot \sqrt{25 + (2 - x)^2} = (x - 2) \cdot \sqrt{9 + x^2}$

Squaring both sides, we get:

$\iff x^2 \cdot [25 + (2 - x)^2] = (x - 2)^2 \cdot (9 + x^2)$

$\iff (x^2 \cdot 25) + x^2 \cdot (2 - x)^2 = ((x - 2)^2 \cdot 9) + (x - 2)^2 \cdot x^2$

$\iff 25x^2 + x^2(2 - x)^2 - 9(x - 2)^2 - x^2(x - 2)^2 = 0$

Because $x^2(x - 2)^2$ and $x^2(2 - x)^2$ cancel each other out should we have :

$\iff 25x^2 - 9(x - 2)^2 = 0$

Apply the identity $(A - B)^2 = 2AB + B^2$ with $(x - 2)^2$ we have :

$\iff 25x^2 - 9(4 - 4x + x^2)$

$\iff 25x^2 - 36 + 36x - 9x^2 = 0$

$\iff 16x^2 + 36x - 36 = 0 \iff x = \frac{3}{4}(0 < x < 2)$

Variation table:

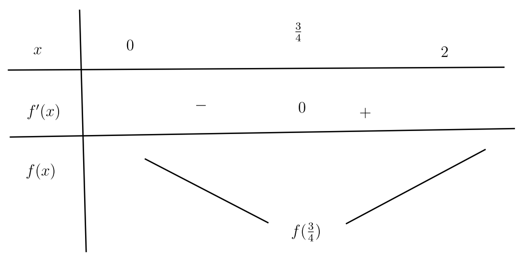

Figure 9: min do lech

$x = d_1 = \frac{3}{4} = 0,75 \implies$ answer A

Method 2: Using Similar triatangles/Thales' + Unfold theorem

Apply technique Unfold(trai hinh) we have :

After using technique Unfold remove Riverbank(2) we have figure :

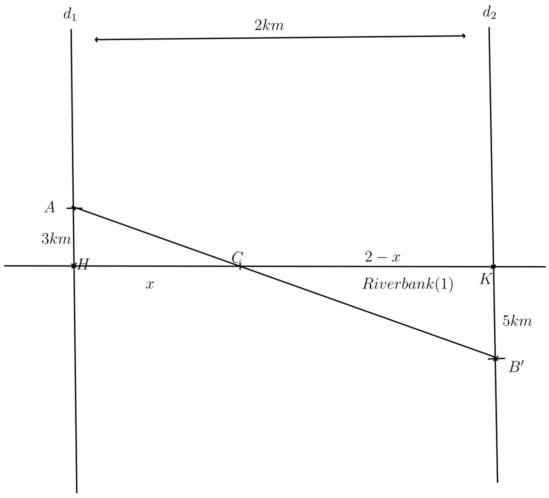

Figure 10: Remove riverbank(2)

Solve:

1.Suppose :

+) Riverbank(1) is $y = 0 \implies$ Riverbank(2) is $y = 0,8$

+) d_1 through $A = x = 0, d_2$ through $B = x = 2$ (you can assign coodrinates freely, as long as the distance between d_1 and d_2 is $2km$

Based on the above, we have :

+)Point A is $3km$ from Riverbank(1) $\rightarrow y = 0 + -3 \implies A(0, -3)$

+)Point B is $5km$ from Riverbank(1) $\rightarrow y = 0,8 + 5 \implies B(2; 5,8)$

2.Unfold

Remove Riverbank(2) by translate point B forward by a distance equal to the rive's width(0,8km) introduce an auxiliary point B'

$\implies B(2; 5,8) \rightarrow B'(2; 5,8 - 0,8) = (2; 5)$

3.Now problem become:

- find shortest path $A(0; -3) \rightarrow B'(2; 5)$ and cut Reverbank(1) at 1 point is C(x,0)
- We know shortest path both two point is a line

+) The location :

- before draw a line $A \rightarrow B'$ we can see line AB' cut Riverbank(1) at $C(x, 0)$ we need to find x
- before draw a line $A \rightarrow B'$ we get 2 right triangle that is $\triangle AHC$ and $\triangle B'KC$

+) A, C, B' collinear create to 2 vertical angles $A\hat{C}H$ and $B'\hat{C}H$(Verical angles are equal)

We have :

$\hat{H} = \hat{K} = 90°$

$A\hat{C}H = B'\hat{C}H$ (Vertical angles)

$\implies \triangle AHC \sim \triangle B'KC(g.g)$

$\implies$ ratio : $\frac{AH}{B'K} = \frac{HC}{KC}$

$\iff \frac{3}{5} = \frac{x}{2-x}$

$\iff x = 0,75$

Sentence 3:
Key idea: Time = workload / productivity
Summarize:
15 machine $\implies 1 \leq x \leq 15$
C = productivity : 30(products)/h $\implies 30 \cdot x_{machine}$<productivity>
A = Maintenance: 48000/machine $\implies 48000 \cdot x_{machine}$ <cost>
B = Monitoring: 24000/h $\implies 24000 \cdot x_h$ <cost>
Time $= x_h = \frac{workload}{productivity} = \frac{6000}{30x} = \frac{200}{x}$
Total cost = A + B
Consider $C(x) = 48000x + \frac{4800000}{x}$
$C'(x) = 48000 - \frac{4800000}{x^2}$
$C'(x) = 0 \iff x = 10 (1 \leq x \leq 15)$
Variation table

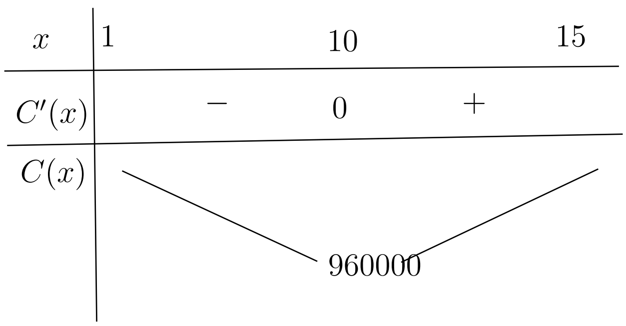

Figure 11: Be careful x hour

$\implies \min\limits_{[1;15]} C(x) = 960000$ at $x = 10$
Sentence 4:

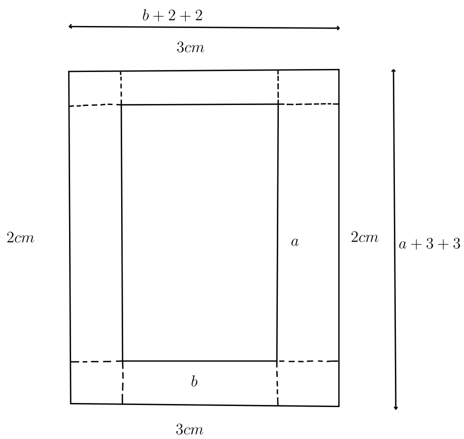

Figure 12: nhin hinh la ra

a) $S_{rectangle} = a \cdot b \implies$ proposition true.
b) $l = a + 3 + 3 = a + 6$
$w = b + 2 + 2 = b + 4$
$\implies$ proposition false. c)
we know : $ab = 384$
$\implies b = \frac{384}{a}$ (Factor out "a" to make "a" the common variable)
$\implies S_{magazine} = (a + 6)(b + 4)$
$= (a + 6)(\frac{384}{a} + 4)$
$= 384 + 4a + \frac{2304}{a} + 24$
$= 4a + \frac{2304}{a} + 408$
Using AM-GM we have :
$4a + \frac{2304}{a} \geq 2 \cdot \sqrt{4a \cdot \frac{2304}{a}} + 408 = 600 \implies$ proposition true.
The "=" occur when $4a = \frac{2304}{a} \implies a = 24 \implies b = 16$
d) $P = 2 * (a + 6 + b + 4) = 100 \implies$ proposition false.
Sentence 5
a) false
$S_{base} = \pi \cdot r^2$
$V = S_{base} \cdot h = \pi \cdot r^2 \cdot h$
$\implies V = \pi \cdot R^2 \cdot h$
b) true
$S_{total} = 2\pi R(R + h) = 10,08\pi$
C) false
if $V = 1dm^3 \implies V = \pi r^2 h \implies h = \frac{V}{\pi r^2} \implies h = \frac{1}{\pi r^2}$
$\implies S_{total} = 2\pi R^2 + \frac{2}{R}$
d)
Method 1: Using the deratives
Consider$S(v) = 2\pi R^2 + \frac{2}{R}(0; +\infty)$
$S'(v) = 4\pi R - \frac{2}{R^2}$

$S'(v) = 0 \iff R = \frac{1}{\sqrt[3]{2\pi}}$
Variation table:

v	0		$\frac{1}{\sqrt[3]{2\pi}}$		$+\infty$
$S'(v)$		$-$	0	$+$	
$S(v)$		$\searrow$	$V(\frac{1}{\sqrt[3]{2\pi}})$	$\nearrow$	

Figure 13: happy

Method 2: Using the AM-GM inequality
We have : $2\pi R^2 + \frac{2}{R} = 2\pi R^2 + \frac{1}{R} + \frac{1}{R} \geq 3 \cdot \sqrt[3]{2\pi R^2 \cdot \frac{1}{R} \cdot \frac{1}{R}} = 3 \cdot \sqrt[3]{2\pi}$
The "=" sign in used when $2\pi R^2 = \frac{1}{R} \implies R = \frac{1}{\sqrt[3]{\frac{1}{2\pi}}}$ **Note:** $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} (a, b > 0)$
Sentence 6:

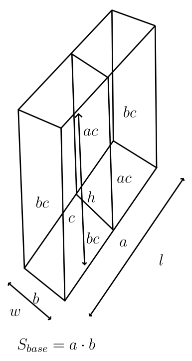

Figure 14: rectangle

a) $V = l \cdot w \cdot h$

$= a \cdot b \cdot c = 1,296$
$\implies$ proposition true.
b) $S_{base} = l \cdot w$
$= a \cdot b$
$\implies$ proposition true.
c) See Figure 13
$\implies$ proposition true.
d) We know $V = abc = 1,296$ (product)
$S_{rectangularPrism} = ab + 2ac + 3bc$ (sum)
$\implies$ Find S_{min} with condition $V = abc = 1,296$
$\implies$ Using AM-GM inequality
Simplify the expresstion : $\frac{S}{V} = \frac{ab+2ac+3bc}{abc} = \frac{1}{c} + \frac{2}{b} + \frac{3}{a}$
Using AM-GM we have:
$\frac{1}{c} + \frac{2}{b} + \frac{3}{a} \geq 3\sqrt[3]{\frac{1}{c} \cdot \frac{2}{b} \cdot \frac{3}{a}} = 3\sqrt[3]{\frac{6}{1,296}} = 5$
The "=" occur when $\frac{1}{c} = \frac{2}{b} = \frac{3}{a}$,we know $abc = 1,296$
$b = 2c$
$a = 3c$
$\implies abc = 2c \cdot 3c \cdot c = 1,296$
$6c^3 = 1,296$
$\implies c = 0,6$
$\implies b = 1,2; a = 1,8$
$\implies$ proposition true.
Sentence 7:
Summarize:
Basis price : 30000/kg
Selling price : 50000/kg
$\implies$ sold : 25kg
Expected price decrease : 4000/kg $\implies$ sales up : 50kg
Solve:
a) sale 4000/kg = sales up 50kg $\implies$ Not sure extra sales from discount
$\implies$ Price dropped is : Basic price - Price dropped $\implies$ $50000 - x$ $\implies$ proposition a true.
b) decrease 4000 $\implies$ increase 50kg
$\frac{50}{4000}$ $\implies$ ratio (Selling price : 50000)
sold / increase price = ratio $\implies$ sold = increase price $\cdot$ ratio = $(50000 - x) \cdot 0,0125 = 625 - 0,0125x$
$\implies$ proposition b true.
d)
Profit = sales - cost
sales = selling price * sold
cost = Basic price * sold
Let profit function = $P(x)$
Consider $P(x) = [(650x - 0,0125x^2)] - (19500000 - 375x)$ on the interval $(0; +\infty)$
$\iff P(x) = -0,0125x^2 + 1025x - 19500000$
$P'(x) = -0,025x + 1025$
$\iff P'(x) = 0 \iff x = 41000$
Variation table:

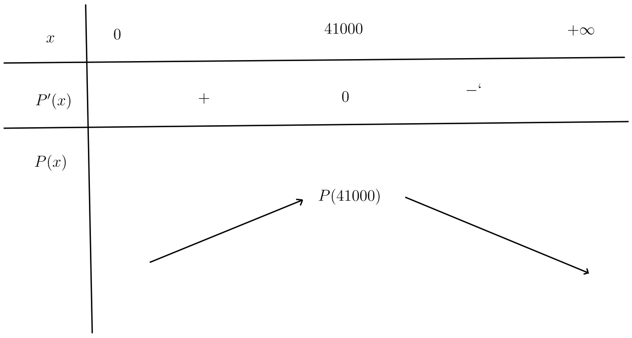

Figure 15: sentence b so hard

Sentence 8:

a)true

$B = A_{base} = l \cdot w$

$\implies V = B \cdot h$

b)false

$w = \frac{3}{2}$

$V = 18$

$l = w \cdot 3 = \frac{3}{2} \cdot 3 = \frac{9}{2}$

$V = l \cdot w \cdot h$

$\implies h = \frac{V}{l \cdot w}$

$\implies h = \frac{18}{\frac{9}{2} \cdot \frac{3}{2}} = \frac{8}{3}$

c) true

$V = 18$

$w = x$

$l = 3x$

$h = \frac{V}{l \cdot w} = \frac{18}{3x \cdot x} = 6x^2$

Surface area (without lid) :

$S_{open} = S_{lateral} + S_{base}$

$= 2 \cdot \frac{6}{x^2}(3x + x) + 3x \cdot x$

$= 3x^2 + \frac{48}{x}$

d) true

Method 1 : using AM-GM

Let$w = x$

$h = \frac{V}{l \cdot w} \implies h = \frac{18}{3x^2}$

$S_{openMin} = 3x^2 + \frac{48}{x} = 3x^2 + \frac{48}{2x} + \frac{48}{2x} = 3x^2 + \frac{24}{x} + \frac{24}{x} \geq 3 \cdot \sqrt[3]{3x^2 \cdot \frac{24}{x} \cdot \frac{24}{x}} = 3 \cdot \sqrt[3]{1778} = 36$ (Minium)

The "=" occurs when $3x^2 = \frac{24}{x} \iff x = 2$

$h(2) = \frac{3}{2}$

Method 2: using deratives
$f(x) = 3x^2 + \frac{48}{x}$
$f'(x) = -48x^2 + 6x$
$f'(x) = 0 \iff x = 2$
Substituting 1 billion into $f(x)$, we find that $f(x) > 0$. Therefore, the right side of the number line starts with a "+" sign.
Variation consider sign :

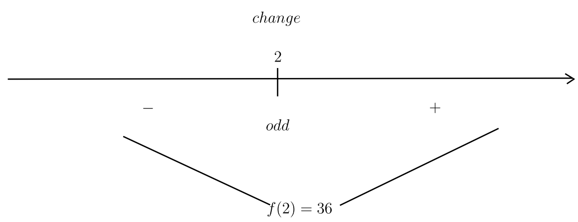

Figure 16: consider sign

$\implies h = \frac{3}{2}$
Sentence 9:
a) Sales = price * sold = 10 * 60 = 600 $\implies$ proposition true
b) decrease 1000 $\implies$ sold +30
decrease 2000 $\implies$ 60
$\implies$ Sales $= 8 \cdot (60 + 60)$
$= 960$
$\implies$ proposition false
c) Sales = unit price * sum sold
unit price $= x$
sum sold = (initial sales quantity + the quantity sold has dereased)
initial quantity sold = 60
Original price : 10
new price : x
$\implies$ discount amount : $10 - x$
Sales volume has decreased $= 30(10 - x) = 300 - 30x$
Sum sold $= 60 + 300 - 30x$
$= 360 - 30x$
Sales $= x(360 - 30x)$
$= -30x^2 + 360x$
$\implies$ proposition false

d) Profit = sales - cost
sales = price a product * sum product sold
sum product sold = 360 - 30x
price a product unknow $\Longleftarrow$ Let $= x$
Cost = original price a product * sum sold original price = 6 , sum sold = 360 - 30x
Consider $P(x) = x(360 - 30x) - 6(360 - 30x)$ on the interval $(0; +\infty)$
$= -30x^2 + 360x + 180x - 2160$
$P'(x) = -60x + 360 + 180$
$P'(x) = 0 \iff x = 9$
Variation table:

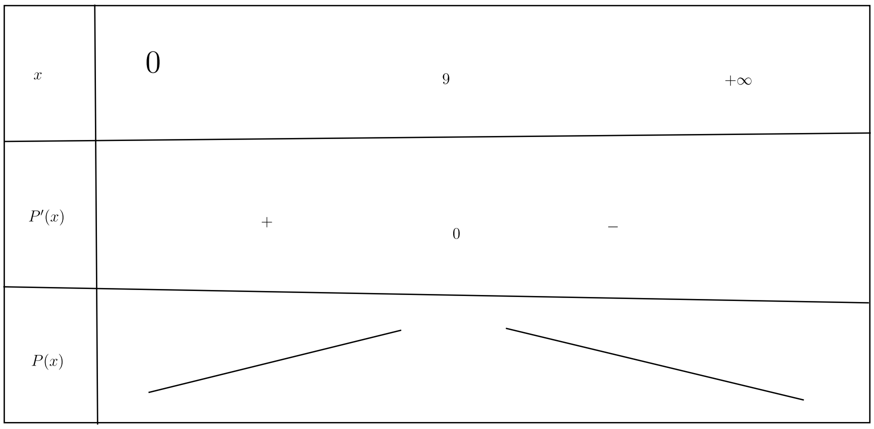

Figure 17: see tit

$\Longrightarrow$ Amount to be reduced $10 - 9 = 1 \Longrightarrow$ proposition true.
Setence 10:

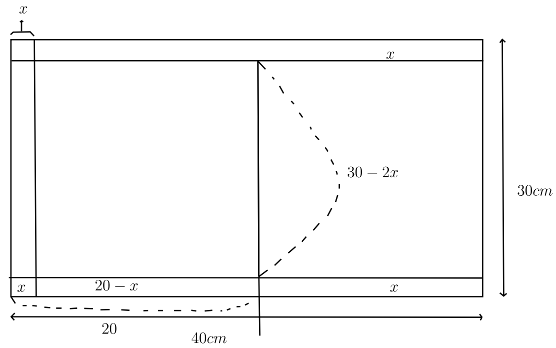

Figure 18: situation

a) true
$40 \cdot 30 = 1200(cm^2)$
b) false
look at Figure 15: situation
c) true
$V = (30 - 2 \cdot 5) \cdot (20 - 5) \cdot 5 = 1500$
d)

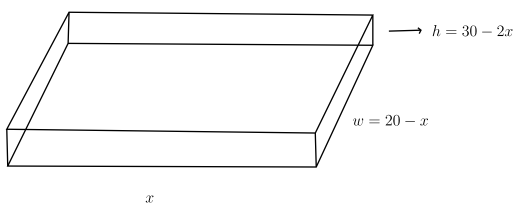

Figure 19: Let see pic

Consider $V(x) = (30 - 2x)(20 - x) \cdot x$ on the interval $[0; 15]$ because $x \geq 0, 30 - 2x \geq \iff x \leq 15, 20 - x \geq 0 \iff x \leq 20 \implies and, x < 15 \implies x = 15 \implies 0 \leq x <= 15$

$= 600x - 70x^2 + 2x^3$
$V'(x) = 6x^2 - 140x + 600$
$V'(x) = 0 \iff x = \frac{35-5\sqrt{13}}{3}(x \in [0;15])$
Variation table:

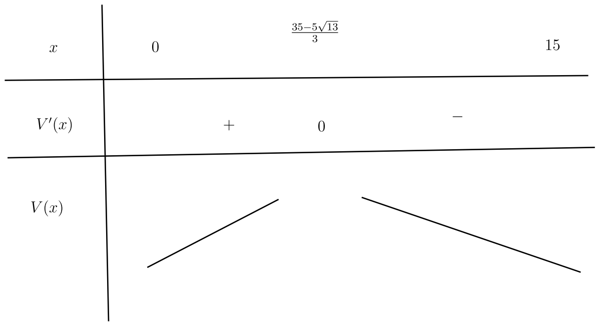

Figure 20: table

Sentence 11:
a) $50000 \cdot 25 = 1250000 \implies$ proposition true
b) $1250000 - (30000 \cdot 25) = 500000 \implies$ proposition true
c)Original ratio : $\frac{50}{4000} = \frac{1}{80} = 0,0125$
Price decrease : $50000 - x$
Increase in quantity sold : $(50000 - x) \cdot 0,0125$
$= 625 - 0,0125x$
Total "vải" sold $= 25 + 625 - 0,0125x$
$= 650 - 0,0125x \implies$ proposition false
d) Profit = sales - cost
$\implies P(x) = x \cdot (650 - 0,0125x) - 30000(650 - 0,0125x)$
$= -0,0125x^2 + 1025x - 1950000$
$P'(x) = -0,025x + 1025$
$P'(x) = 0 \iff x = 41000$
Variation table:

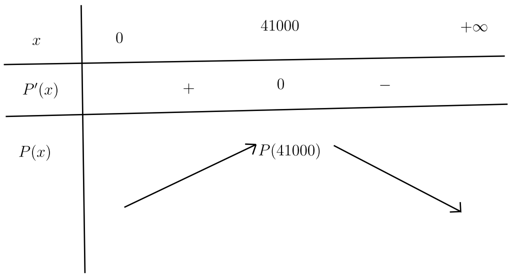

Figure 21: figxxx