

Lecture 24: Theme 24 (Mon 20 Oct 2025 18:58)

Limit when $x \rightarrow \infty$

[Example](page 161){this is form $\infty - \infty$ } give : $\lim_{x \rightarrow -\infty} (x^3 - 2x) =$

$$\lim_{x \rightarrow -\infty} x^3 \left(1 - \frac{2}{x^2}\right) = -\infty \text{ because } \begin{cases} \lim_{x \rightarrow -\infty} x^3 = -\infty \\ \lim_{x \rightarrow -\infty} \left(1 - \frac{2}{x^2}\right) = 1 > 0 \end{cases}$$

$$\Rightarrow -\infty \cdot 1 = -\infty \text{ should } \lim_{x \rightarrow -\infty} (x^3 - 2x) = -\infty$$

Example (page 162) form $\frac{\infty}{\infty}$

$$(1) \lim_{x \rightarrow +\infty} \frac{2x^2 + x - 1}{x^2 - 3x + 1} = \lim_{x \rightarrow +\infty} \frac{\frac{2x^2}{x^2} + \frac{x}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{3x}{x^2} + \frac{1}{x^2}} = \lim_{x \rightarrow +\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{3}{x} + \frac{1}{x^2}} = \frac{2+0+0}{1-0+0} = 2$$

Method 2:

$\frac{2x^2}{x^2} = 2$ this is quick tip if degree of the numerator = degree of the denominator

$$(2) \lim_{x \rightarrow -\infty} \frac{-x+1}{2x^2+x-1} = \frac{\frac{-x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{x}{x^2} - \frac{1}{x^2}} = \frac{\frac{-1}{x} + \frac{1}{x^2}}{2 + \frac{1}{x} - \frac{1}{x^2}} = \frac{0+0}{2+0-0} = 0$$

$$(3) \lim_{x \rightarrow +\infty} \frac{x^3+3x^2+2}{2x+1} = \frac{\frac{x^3}{x^3} + \frac{3x^2}{x^3} + \frac{2}{x^3}}{\frac{2x}{x^3} + \frac{1}{x^3}} = \frac{1 + \frac{3}{x} + \frac{2}{x^3}}{\frac{2}{x^2} + \frac{1}{x^3}} = \frac{1+0+0}{0+0} = +\infty$$

Summary (Quick Tip)

You only need to compare the highest powers (degrees) of the numerator and denominator:

- **If the degree of the numerator equals the degree of the denominator:**

The limit equals the ratio of the leading coefficients of the numerator and the denominator.

(As in Example 1)

- **If the degree of the numerator is less than the degree of the denominator:**

The limit equals 0.

(As in Example 2)

- **If the degree of the numerator is greater than the degree of the denominator:**

The limit equals ∞ or $-\infty$.

(As in Example 3)

Example 1(page 162):

$$(1) \lim_{x \rightarrow -\infty} \frac{x^3-3x+1}{5-2x} = -\infty;$$

$$(2) \lim_{x \rightarrow -\infty} \frac{1-3x^2-x^3}{4x^2+1} = +\infty;$$

$$(3) \lim_{x \rightarrow +\infty} \frac{3x^2-2x+1}{2x^2-x+5} = \frac{3}{2};$$

$$(4) \lim_{x \rightarrow -\infty} \frac{x+1}{2x^2-x+1} = 0;$$

Example 2(page 163){Note: $\sqrt{x^2} = |x|$ }

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2-x+1}}{x+1} &= \frac{\frac{\sqrt{4x^2-x+1}}{x}}{\frac{x+1}{x}} \\ &\Rightarrow \frac{\frac{\sqrt{4x^2-x+1}}{x}}{\frac{x+1}{x}} = \frac{\sqrt{x^2\left(\frac{4x^2}{x^2}-\frac{x}{x^2}+\frac{1}{x^2}\right)}}{x} = \frac{|x|\sqrt{\left(4-\frac{1}{x}+\frac{1}{x^2}\right)}}{x} = \frac{-x\sqrt{\left(4-\frac{1}{x}+\frac{1}{x^2}\right)}}{x} = \\ &= -\sqrt{\left(4-\frac{1}{x}+\frac{1}{x^2}\right)} = -\sqrt{4} = -2 \\ &\Rightarrow \frac{x+1}{x} = 1 + \frac{1}{x} = 1 \\ &\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2-x+1}}{x+1} = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt{4x^2-x+1}}{x}}{\frac{x+1}{x}} = -\frac{2}{1} = -2 \end{aligned}$$

Example 3(page 163):{Note: $\sqrt{x^2} = |x|$ }

$$\begin{aligned} (1) \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+1}+x}{x+1} &= \lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{x^2+1}+x}{x}}{\frac{x+1}{x}} \\ &\Rightarrow \frac{\frac{\sqrt{x^2+1}+x}{x}}{\frac{x+1}{x}} = \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}+1}{x} = \frac{|x|\sqrt{\left(1+\frac{1}{x^2}\right)}+1}{x} = \frac{x\sqrt{\left(1+\frac{1}{x^2}\right)}+1}{x} = \sqrt{\left(1+\frac{1}{x^2}\right)}+ \\ &= \sqrt{(1+0)}+1 = 2 \\ &\Rightarrow \frac{x+1}{x} = 1 + \frac{1}{x} = 1 \\ &\Rightarrow \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+1}+x}{x+1} = \lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{x^2+1}+x}{x}}{\frac{x+1}{x}} = \frac{2}{1} = 2 \\ (2) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x}-\sqrt{4x^2+1}}{2x+3} &= \frac{\frac{\sqrt{x^2-x}-\sqrt{4x^2+1}}{x}}{\frac{2x+3}{x}} \\ &\Rightarrow \frac{\frac{\sqrt{x^2-x}-\sqrt{4x^2+1}}{x}}{\frac{2x+3}{x}} = \frac{\sqrt{x^2\left(1-\frac{1}{x}\right)}-\sqrt{4x^2\left(1+\frac{1}{x^2}\right)}}{2x+3} = \frac{|x|\sqrt{\left(1-\frac{1}{x}\right)}-\sqrt{4\left(1+\frac{1}{x^2}\right)}}{2x+3} = \frac{-x\sqrt{\left(1-\frac{1}{x}\right)}-\sqrt{4\left(1+\frac{1}{x^2}\right)}}{2x+3} = \\ &= -\sqrt{1} = -1 \\ &\Rightarrow \frac{-\sqrt{4x^2+1}}{x} = \frac{-\sqrt{x^2\left(4+\frac{1}{x^2}\right)}}{x} = \frac{-|x|\sqrt{\left(4+\frac{1}{x^2}\right)}}{x} = \frac{-(-x)\sqrt{\left(4+\frac{1}{x^2}\right)}}{x} = \frac{x\sqrt{\left(4+\frac{1}{x^2}\right)}}{x} = \\ &= \sqrt{\left(4+\frac{1}{x^2}\right)} = 2 \\ &\frac{2x+3}{x} = 2 + \frac{3}{x} = 2 \\ &\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x}-\sqrt{4x^2+1}}{2x+3} = \frac{\frac{\sqrt{x^2-x}-\sqrt{4x^2+1}}{x}}{\frac{2x+3}{x}} = \frac{-1+2}{2} = \frac{1}{2} \end{aligned}$$

Example 4(page 163) Give $\lim_{x \rightarrow +\infty} \frac{ax^2+bx+3}{cx^3+3x+2} = 2$; know $a, b, c \in \mathbb{R}$. Calculate $a + b + c$?

Solve:

Let degree of the numerator = A;

Let degree of the denominator = B;

We know :

Case 1: $A = B \implies$ The limit equals the ratio the leading coefficients of the A and B;

Case 2: $A < B \implies$ The limit = 0;

Case 3: $A > B \implies$ The limit = ∞ or $-\infty$

Given : $\lim_{x \rightarrow +\infty} \frac{ax^2+bx+3}{cx^3+3x+2} = 2$ only applies to case 1 ($A = B$)

$\implies c = 0 \implies \lim_{x \rightarrow +\infty} \frac{ax^2+bx+3}{3x+2} = 2$ Not yet consistent with the given condition, so let $a = 0$

$$\implies \lim_{x \rightarrow +\infty} \frac{bx+3}{3x+2} = 2 \implies \text{satisfy given}$$

$$\implies \lim_{x \rightarrow +\infty} \frac{bx+3}{3x+2} = 2 \implies \frac{b}{3} = 2 \implies b = 6$$

$$\implies a + b + c = 6$$

Additional exercises(page: 163)

Find limit:

$$(1) \lim_{x \rightarrow -\infty} \frac{x^2+x+1}{2x^3+2x+5} = 0(\text{Case 2})$$

$$(2) \lim_{x \rightarrow +\infty} \frac{2x^2+1}{x^3-3x^2+2} = 0(\text{Case 2})$$

$$(3) \lim_{x \rightarrow +\infty} \frac{\sqrt[3]{x^2-x+1}}{5x^2-1} = 0(\text{Note: } \sqrt[n]{x^m} = x^{\frac{m}{n}})$$

$$(4) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x+1}}{2x+1} = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt{x^2-x+1}}{x}}{\frac{2x+1}{x}}$$

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x+1}}{x} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(1-\frac{1}{x}+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{|x|\sqrt{(1-\frac{1}{x}+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{-x\sqrt{(1-\frac{1}{x}+\frac{1}{x^2})}}{x} =$$

$$\lim_{x \rightarrow -\infty} -\sqrt{(1-\frac{1}{x}+\frac{1}{x^2})} = -1$$

$$\implies \lim_{x \rightarrow -\infty} \frac{2x+1}{x} = \lim_{x \rightarrow -\infty} \frac{2x}{x} + \frac{1}{x} = \lim_{x \rightarrow -\infty} 2 + \frac{1}{x} = 2$$

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x+1}}{2x+1} = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt{x^2-x+1}}{x}}{\frac{2x+1}{x}} = -\frac{1}{2}$$

$$(5) \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3-1}}{\sqrt{2x^2+1}} = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt[3]{x^3-1}}{x}}{\frac{\sqrt{2x^2+1}}{x}}$$

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3-1}}{x} = \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3(1-\frac{1}{x^3})}}{x} = \lim_{x \rightarrow -\infty} \frac{x\sqrt[3]{(1-\frac{1}{x^3})}}{x} = \lim_{x \rightarrow -\infty} \sqrt[3]{(1-\frac{1}{x^3})} =$$

1

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2+1}}{x} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(2+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{|x|\sqrt{(2+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{-x\sqrt{(2+\frac{1}{x^2})}}{x} =$$

$$\lim_{x \rightarrow -\infty} -\sqrt{(2+\frac{1}{x^2})} = -\sqrt{2}$$

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3-1}}{\sqrt{2x^2+1}} = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt[3]{x^3-1}}{x}}{\frac{\sqrt{2x^2+1}}{x}} = \frac{1}{-\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

$$\begin{aligned}
(6) \quad \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt{2x^2+1}} &= \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2}}{\frac{\sqrt{2x^2+1}}{x^2}} \\
\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2} &= \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt[3]{x^6}} = \lim_{x \rightarrow -\infty} \sqrt[3]{\frac{x^6+x^4+x^2+1}{x^6}} = 1 \\
\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2+1}}{x^2} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2+1}}{\sqrt{x^4}} = \lim_{x \rightarrow -\infty} \sqrt{\frac{2x^2+1}{x^4}} = \lim_{x \rightarrow -\infty} \sqrt{\frac{2}{x^2} + \frac{1}{x^4}} = \\
0 \quad \Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt{2x^2+1}} &= \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2}}{\frac{\sqrt{2x^2+1}}{x^2}} = \frac{1}{0} = +\infty
\end{aligned}$$

Method 2:

We can see numerator the greatest is: $\sqrt[3]{x^6} = x^2$ and all of them are positive. \Rightarrow numerator = $+\infty$

We can see denominator is $\sqrt{2x^2+1} \Rightarrow$ always positive

We know : $x^2 > x^1$ (The degree of the numerator is greater than the degree of the denominator)

$$\Rightarrow \frac{\text{positive}}{\text{positive}} = +\infty$$

Method 3:

The greatest degree of numerator is : $\sqrt[3]{x^6} = x^2$

The greatest degree of denominator is : $\sqrt{2x^2} = \sqrt{2} \cdot \sqrt{x^2}$ (because $x \rightarrow -\infty$ should $|x| = -x$)

$$\Rightarrow \text{The limit after rewriting is : } \lim_{x \rightarrow -\infty} \frac{x^2}{\sqrt{2}(-x)} = \frac{x}{-\sqrt{2}} = \frac{\text{negative}}{\text{negative}} = +\infty$$

Method 4: Using Casio f(x) 580 :>

$$\begin{aligned}
(7) \quad \lim_{x \rightarrow -\infty} \frac{x-\sqrt{2x^2+1}}{2x+3\sqrt{x^2+1}} &= \lim_{x \rightarrow -\infty} \frac{\frac{x-\sqrt{2x^2+1}}{x}}{\frac{2x+3\sqrt{x^2+1}}{x}} \\
\Rightarrow \lim_{x \rightarrow -\infty} \frac{x-\sqrt{2x^2+1}}{x} &= \lim_{x \rightarrow -\infty} \frac{-\sqrt{2x^2+1}+x}{x} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{x^2(2+\frac{1}{x^2})}+x}{x} = \lim_{x \rightarrow -\infty} \frac{-|x|\sqrt{(2+\frac{1}{x^2})}+x}{x} = \\
\lim_{x \rightarrow -\infty} \frac{x\sqrt{(2+\frac{1}{x^2})}+x}{x} &= \lim_{x \rightarrow -\infty} \frac{x\sqrt{(2+\frac{1}{x^2})}}{x} + \frac{x}{x} = \lim_{x \rightarrow -\infty} \sqrt{(2+\frac{1}{x^2})} + 1 = 1 + \sqrt{2} \\
\Rightarrow \lim_{x \rightarrow -\infty} \frac{2x+3\sqrt{x^2+1}}{x} &= \lim_{x \rightarrow -\infty} \frac{2x}{x} + \frac{3\sqrt{x^2+1}}{x} = \lim_{x \rightarrow -\infty} 2 + \frac{3\sqrt{x^2+1}}{x} = \\
\lim_{x \rightarrow -\infty} 2 + \frac{3\sqrt{x^2(1+\frac{1}{x^2})}}{x} &= \lim_{x \rightarrow -\infty} \frac{3|x|\sqrt{(1+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{-3x\sqrt{(1+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} 2 + \\
-3 \cdot \sqrt{(1+\frac{1}{x^2})} &= -1
\end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow -\infty} \frac{x-\sqrt{2x^2+1}}{2x+3\sqrt{x^2+1}} = \lim_{x \rightarrow -\infty} \frac{\frac{x-\sqrt{2x^2+1}}{x}}{\frac{2x+3\sqrt{x^2+1}}{x}} = \frac{1+\sqrt{2}}{-1} = -1 - \sqrt{2}$$

$$(8) \quad \lim_{x \rightarrow +\infty} \sqrt{\frac{x^3+1}{2x^3+5}} = \lim_{x \rightarrow +\infty} \sqrt{\frac{\frac{x^3+1}{x^3}}{\frac{2x^3+5}{x^3}}} = \lim_{x \rightarrow +\infty} \sqrt{\frac{1+\frac{1}{x^3}}{2+\frac{5}{x^3}}} = \sqrt{\frac{1+0}{2+0}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

Method 2(Case 1){degree numerator = degree denominator}

$$\sqrt{\frac{x^3}{2x^3}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

[Example] (page 164) form $\infty - \infty$

$$(1) \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 2x} - x) = \lim_{x \rightarrow +\infty} \frac{(\sqrt{x^2 + 2x} - x)(\sqrt{x^2 + 2x} + x)}{(\sqrt{x^2 + 2x} + x)} = \lim_{x \rightarrow +\infty} \frac{x^2 + 2x - x^2}{(\sqrt{x^2 + 2x} + x)} =$$

$$\lim_{x \rightarrow +\infty} \frac{2x}{\sqrt{x^2 + 2x} + x} = \lim_{x \rightarrow +\infty} \frac{\frac{2x}{x}}{\sqrt{x^2 \left(1 + \frac{2}{x}\right)} + \frac{x}{x}} = \lim_{x \rightarrow +\infty} \frac{2}{|x| \sqrt{\left(1 + \frac{2}{x}\right)} + 1} = \lim_{x \rightarrow +\infty} \frac{2}{\frac{x}{x} \sqrt{\left(1 + \frac{2}{x}\right)} + 1} =$$

$$\lim_{x \rightarrow +\infty} \frac{2}{\sqrt{\left(1 + \frac{2}{x}\right)} + 1} = \frac{2}{\sqrt{1+0} + 1} = \frac{2}{2} = 1$$

$$(2) \lim_{x \rightarrow +\infty} (\sqrt{x+1} - \sqrt{x}) = \lim_{x \rightarrow +\infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{(\sqrt{x+1} + \sqrt{x})} = \lim_{x \rightarrow +\infty} \frac{1}{(\sqrt{x+1} + \sqrt{x})} =$$

$$\frac{1}{+\infty} = 0$$

$$(3) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x) = \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 + x + 1} + x)(\sqrt{x^2 + x + 1} - x)}{(\sqrt{x^2 + x + 1} - x)} = \lim_{x \rightarrow -\infty} \frac{x^2 + x + 1 - x^2}{(\sqrt{x^2 + x + 1} - x)} =$$

$$\lim_{x \rightarrow -\infty} \frac{x+1}{(\sqrt{x^2 + x + 1} - x)} = \lim_{x \rightarrow -\infty} \frac{\frac{x+1}{x}}{\frac{\sqrt{x^2 + x + 1}}{x} - \frac{x}{x}} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{\sqrt{x^2 \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)} - x} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{|x| \sqrt{\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)} - x} =$$

$$\lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{-x \left(\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - x\right)} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{-x \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - \frac{x}{x}} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{-\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - 1} = \frac{1+0}{-\sqrt{1+0}-1} = -\frac{1}{2}$$

Example 1(page 165):

$$(1) \lim_{x \rightarrow +\infty} (\sqrt{x+3} - \sqrt{x}) = \lim_{x \rightarrow +\infty} \frac{(\sqrt{x+3} - \sqrt{x})(\sqrt{x+3} + \sqrt{x})}{(\sqrt{x+3} + \sqrt{x})} = \lim_{x \rightarrow +\infty} \frac{3}{(\sqrt{x+3} + \sqrt{x})} =$$

$$0$$

$$(2) \lim_{x \rightarrow +\infty} \left(\frac{4}{\sqrt{x+2} - \sqrt{x-2}} \right) = \lim_{x \rightarrow +\infty} \frac{4(\sqrt{x+2} + \sqrt{x-2})}{(\sqrt{x+2} - \sqrt{x-2})(\sqrt{x+2} + \sqrt{x-2})} = \lim_{x \rightarrow +\infty} \frac{4(\sqrt{x+2} + \sqrt{x-2})}{(x+2) - (x-2)} =$$

$$\lim_{x \rightarrow +\infty} (\sqrt{x+2} + \sqrt{x-2}) = +\infty$$

$$(3) \lim_{x \rightarrow -\infty} (\sqrt{x^2 - 3x} + x) = \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 - 3x} + x)(\sqrt{x^2 - 3x} - x)}{(\sqrt{x^2 - 3x} - x)} = \lim_{x \rightarrow -\infty} \frac{-3x}{(\sqrt{x^2 - 3x} - x)} =$$

$$\lim_{x \rightarrow -\infty} \frac{\frac{-3x}{x}}{\frac{\sqrt{x^2 - 3x}}{x} - \frac{x}{x}} = \lim_{x \rightarrow -\infty} \frac{-3}{\sqrt{x^2 - 3x} - x} = \lim_{x \rightarrow -\infty} \frac{-3}{\sqrt{x^2 \left(1 - \frac{3}{x}\right)} - x} = \lim_{x \rightarrow -\infty} \frac{-3}{|x| \sqrt{\left(1 - \frac{3}{x}\right)} - x} =$$

$$\lim_{x \rightarrow -\infty} \frac{-3}{|x| \sqrt{1 - \frac{3}{x}} - 1} = \lim_{x \rightarrow -\infty} \frac{-3}{-x \sqrt{1 - \frac{3}{x}} - 1} = \lim_{x \rightarrow -\infty} \frac{-3}{-\sqrt{1 - \frac{3}{x}} - 1} = \frac{-3}{-\sqrt{1-0}-1} = \frac{3}{2}$$

$$(4) \lim_{x \rightarrow +\infty} (2x - \sqrt{4x^2 + 2x - 1}) = \lim_{x \rightarrow +\infty} \frac{(2x - \sqrt{4x^2 + 2x - 1})(2x + \sqrt{4x^2 + 2x - 1})}{(2x + \sqrt{4x^2 + 2x - 1})} =$$

$$\lim_{x \rightarrow +\infty} \frac{6x^2 - 2x + 1}{(2x + \sqrt{4x^2 + 2x - 1})} = +\infty$$

Example 2(page 165):Value of $A = \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4x + 5} + x) = ?$

$$A = \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4x + 5} + x) = \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 + 4x + 5} + x)(\sqrt{x^2 + 4x + 5} - x)}{(\sqrt{x^2 + 4x + 5} - x)} = \lim_{x \rightarrow -\infty} \frac{4x+5}{(\sqrt{x^2 + 4x + 5} - x)} =$$

$$\lim_{x \rightarrow -\infty} \frac{\frac{4x+5}{x}}{\frac{\sqrt{x^2 + 4x + 5}}{x} - \frac{x}{x}} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{\sqrt{x^2 + 4x + 5} - x} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{\sqrt{x^2 \left(1 + \frac{4}{x} + \frac{5}{x^2}\right)} - x} =$$

$$\lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{|x| \sqrt{\left(1 + \frac{4}{x} + \frac{5}{x^2}\right)} - x} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{-x \sqrt{\left(1 + \frac{4}{x} + \frac{5}{x^2}\right)} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{-\sqrt{1 + \frac{4}{x} + \frac{5}{x^2}} - 1} = \frac{4}{-2} =$$

-2

$$\Rightarrow A = -2$$

Example 3(page 165):

Give $\lim_{x \rightarrow +\infty} (\sqrt{x^2 + ax} - x) = 8$. Find $a, a \in \mathbb{R}$

$$\begin{aligned} \lim_{x \rightarrow +\infty} (\sqrt{x^2 + ax} - x) &= \lim_{x \rightarrow +\infty} \frac{(\sqrt{x^2 + ax} - x)(\sqrt{x^2 + ax} + x)}{(\sqrt{x^2 + ax} + x)} = \lim_{x \rightarrow +\infty} \frac{ax}{(\sqrt{x^2 + ax} + x)} = \\ \lim_{x \rightarrow +\infty} \frac{\frac{ax}{x}}{\frac{\sqrt{x^2 + ax} + x}{x}} &= \lim_{x \rightarrow +\infty} \frac{a}{\frac{\sqrt{x^2 + ax}}{x} + \frac{x}{x}} = \lim_{x \rightarrow +\infty} \frac{a}{\sqrt{x^2(1 + \frac{a}{x})} + 1} = \lim_{x \rightarrow +\infty} \frac{a}{|x|\sqrt{(1 + \frac{a}{x})} + 1} = \\ \lim_{x \rightarrow +\infty} \frac{a}{x\sqrt{(1 + \frac{a}{x})} + 1} &= \lim_{x \rightarrow +\infty} \frac{a}{\sqrt{1 + \frac{a}{x}} + 1} = \frac{a}{\sqrt{1 + 0} + 1} = \frac{a}{2} = 8 \implies a = 16 \end{aligned}$$

Additional Exercises(page 165):

$$\begin{aligned} (1) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4x + 1} + x) &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 + 4x + 1} + x)(\sqrt{x^2 + 4x + 1} - x)}{(\sqrt{x^2 + 4x + 1} - x)} = \lim_{x \rightarrow -\infty} \frac{4x + 1}{\sqrt{x^2 + 4x + 1} - x} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{4x + 1}{x}}{\frac{\sqrt{x^2 + 4x + 1} - x}{x}} &= \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{\frac{\sqrt{x^2 + 4x + 1}}{x} - \frac{x}{x}} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{\frac{\sqrt{x^2 + 4x + 1}}{x} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{\sqrt{x^2(1 + \frac{4}{x} + \frac{1}{x^2})} - 1} = \\ \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{|x|\sqrt{(1 + \frac{4}{x} + \frac{1}{x^2})} - 1} &= \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{-x\sqrt{(1 + \frac{4}{x} + \frac{1}{x^2})} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{-\sqrt{(1 + \frac{4}{x} + \frac{1}{x^2})} - 1} = \frac{4 + 0}{-\sqrt{1 + 0 + 0} - 1} = \\ \frac{4}{-2} &= -2 \end{aligned}$$

$$\begin{aligned} (2) \lim_{x \rightarrow \infty} (\sqrt{x^2 - 4x + 2} - x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 - 4x + 2} - x)(\sqrt{x^2 - 4x + 2} + x)}{(\sqrt{x^2 - 4x + 2} + x)} = \lim_{x \rightarrow \infty} \frac{-4x + 2}{\sqrt{x^2 - 4x + 2} + x} = \\ = \lim_{x \rightarrow \infty} \frac{\frac{-4x + 2}{x}}{\frac{\sqrt{x^2 - 4x + 2} + x}{x}} &= \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\frac{\sqrt{x^2 - 4x + 2}}{x} + \frac{x}{x}} = \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\frac{\sqrt{x^2 - 4x + 2}}{x} + 1} = \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\sqrt{x^2(1 - \frac{4}{x} + \frac{2}{x^2})} + 1} = \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{|x|\sqrt{(1 - \frac{4}{x} + \frac{2}{x^2})} + 1} = \\ = \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{x\sqrt{(1 - \frac{4}{x} + \frac{2}{x^2})} + 1} &= \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\sqrt{1 - \frac{4}{x} + \frac{2}{x^2}} + 1} = \frac{-4 + 0}{\sqrt{1 - 0 + 0} + 1} = -2 \end{aligned}$$

$$(3) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + x} - x) = +\infty - (-\infty) = +\infty + \infty = +\infty \text{ \{be careful it not form } \infty - \infty\}}$$

$$\begin{aligned} (4) \lim_{x \rightarrow -\infty} (\sqrt{4x^2 + x} + 2x) &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{4x^2 + x} + 2x)(\sqrt{4x^2 + x} - 2x)}{(\sqrt{4x^2 + x} - 2x)} = \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{4x^2 + x} - 2x} = \\ = \lim_{x \rightarrow -\infty} \frac{\frac{x}{x}}{\frac{\sqrt{4x^2 + x} - 2x}{x}} &= \lim_{x \rightarrow -\infty} \frac{1}{\frac{\sqrt{4x^2 + x}}{x} - \frac{2x}{x}} = \lim_{x \rightarrow -\infty} \frac{1}{\frac{\sqrt{x^2(4 + \frac{1}{x})}}{x} - 2} = \\ = \lim_{x \rightarrow -\infty} \frac{1}{\frac{|x|\sqrt{4 + \frac{1}{x}}}{x} - 2} &= \lim_{x \rightarrow -\infty} \frac{1}{-x\sqrt{4 + \frac{1}{x}} - 2} = \lim_{x \rightarrow -\infty} \frac{1}{-\sqrt{4 + \frac{1}{x}} - 2} = \\ = \frac{1}{-\sqrt{4 + 0} - 2} &= -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} (5) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 - 1}) &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 + 2x} - \sqrt{x^2 - 1})(\sqrt{x^2 + 2x} + \sqrt{x^2 - 1})}{(\sqrt{x^2 + 2x} + \sqrt{x^2 - 1})} = \\ \lim_{x \rightarrow -\infty} \frac{(x^2 + 2x) - (x^2 - 1)}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 1}} &= \lim_{x \rightarrow -\infty} \frac{2x + 1}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 1}} = \lim_{x \rightarrow -\infty} \frac{x(2 + \frac{1}{x})}{\sqrt{x^2(1 + \frac{2}{x})} + \sqrt{x^2(1 - \frac{1}{x^2})}} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{x(2 + \frac{1}{x})}{x}}{\frac{|\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{1}{x^2}}|}{x}} &= \lim_{x \rightarrow -\infty} \frac{x(2 + \frac{1}{x})}{-x(\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{1}{x^2}})} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{1}{x}}{-(\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{1}{x^2}})} = \\ \frac{2 + 0}{-(\sqrt{1 + 0} + \sqrt{1 - 0})} &= \frac{2}{-(1 + 1)} = \frac{2}{-2} = -1 \end{aligned}$$

$$\begin{aligned} (6) \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + x} - 2x}{3x - 1} &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{4x^2 + x} - 2x)(\sqrt{4x^2 + x} + 2x)}{(3x - 1)(\sqrt{4x^2 + x} + 2x)} = \lim_{x \rightarrow -\infty} \frac{3x^2 + x - 2}{(3x - 1)(\sqrt{4x^2 + x} + 2x)} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{3x^2 + x - 2}{x^2}}{(3x - 1)\frac{(\sqrt{4x^2 + x} + 2x)}{x^2}} &= \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{(3x - 1)(\frac{\sqrt{4x^2 + x}}{x^2} + \frac{2x}{x^2})} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{(3x - 1)(\sqrt{4 + \frac{1}{x}} + \frac{2}{x})} = \end{aligned}$$

$$\begin{aligned}
& \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{(3x-1)\left(\sqrt{x^2\left(4 + \frac{1}{x} - \frac{2}{x^2}\right)}\right) - \frac{1}{x}} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{(3x-1)|x|\sqrt{\left(4 + \frac{1}{x} - \frac{2}{x^2}\right)} - \frac{1}{x}} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{(3x-1)\left(-x\sqrt{\left(4 + \frac{1}{x} - \frac{2}{x^2}\right)}\right) - \frac{1}{x}} = \\
& \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{\frac{-3x^2\sqrt{\left(4 + \frac{1}{x} - \frac{2}{x^2}\right)} - \frac{1}{x}}{x^2}} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{-3\sqrt{\left(4 + \frac{1}{x} - \frac{2}{x^2}\right)} - \frac{1}{x}} = \frac{3+0-0}{-3\sqrt{(4+0-0)}-0} = -\frac{1}{2} \\
(7) \quad & \lim_{x \rightarrow +\infty} \left\{ \frac{1}{x+1} \cdot (\sqrt{x^2 + 2x} - x) \right\} = \lim_{x \rightarrow +\infty} \frac{1}{x+1} \cdot \left(\frac{\sqrt{x^2+2x}-x(\sqrt{x^2+2x+x})}{(\sqrt{x^2+2x+x})} \right) = \\
& \lim_{x \rightarrow +\infty} \frac{1}{x+1} \cdot \left(\frac{2x}{\sqrt{x^2+2x+x}} \right) = \lim_{x \rightarrow +\infty} \frac{2x}{(x+1)\sqrt{x^2+2x+x}} = 0 (\text{numerator degree} < \text{denominator degree}) \\
(8) \quad & \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 3x} - \sqrt{x^2 - 3x}) = \lim_{x \rightarrow +\infty} \frac{(\sqrt{x^2+3x}-\sqrt{x^2-3x})(\sqrt{x^2+3x}+\sqrt{x^2-3x})}{(\sqrt{x^2+3x}+\sqrt{x^2-3x})} = \\
& \lim_{x \rightarrow +\infty} \frac{6x}{(\sqrt{x^2+3x}+\sqrt{x^2-3x})} = \lim_{x \rightarrow +\infty} \frac{\frac{6x}{x}}{\frac{\sqrt{x^2+3x}}{x} + \frac{\sqrt{x^2-3x}}{x}} = \lim_{x \rightarrow +\infty} \frac{6}{\sqrt{x^2\left(1+\frac{3}{x}\right)} + \sqrt{x^2\left(1-\frac{3}{x}\right)}} = \\
& \lim_{x \rightarrow +\infty} \frac{6}{\frac{|x|\sqrt{\left(1+\frac{3}{x}\right)}}{x} + \frac{|x|\sqrt{\left(1-\frac{3}{x}\right)}}{x}} = \lim_{x \rightarrow +\infty} \frac{6}{\frac{x\sqrt{\left(1+\frac{3}{x}\right)}}{x} + \frac{x\sqrt{\left(1-\frac{3}{x}\right)}}{x}} = \lim_{x \rightarrow +\infty} \frac{6}{\sqrt{\left(1+\frac{3}{x}\right)} + \sqrt{\left(1-\frac{3}{x}\right)}} = \\
& \frac{6}{\sqrt{1+0} + \sqrt{1-0}} = 3
\end{aligned}$$