

## Lecture 24: Theme 24 (Mon 20 Oct 2025 18:58)

### Limit when $x \rightarrow \infty$

[Example](page 161){this is form  $\infty - \infty$ } give :  $\lim_{x \rightarrow -\infty} (x^3 - 2x) = \lim_{x \rightarrow -\infty} x^3 \left(1 - \frac{2}{x^2}\right) = -\infty$  because  $\begin{cases} \lim_{x \rightarrow -\infty} x^3 = -\infty \\ \lim_{x \rightarrow -\infty} \left(1 - \frac{2}{x^2}\right) = 1 > 0 \end{cases}$   
 $\Rightarrow -\infty \cdot 1 = -\infty$  should  $\lim_{x \rightarrow -\infty} (x^3 - 2x) = -\infty$

**Example (page 162) form  $\frac{\infty}{\infty}$**

$$(1) \lim_{x \rightarrow +\infty} \frac{2x^2+x-1}{x^2-3x+1} = \lim_{x \rightarrow +\infty} \frac{\frac{2x^2}{x^2} + \frac{x}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{3x}{x^2} + \frac{1}{x^2}} = \lim_{x \rightarrow +\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{3}{x} + \frac{1}{x^2}} = \frac{2+0+0}{1-0+0} = 2$$

**Method 2:**

$\frac{2x^2}{x^2} = 2$  this is quick tip if degree of the numerator = degree of the denominator

$$(2) \lim_{x \rightarrow -\infty} \frac{-x+1}{2x^2+x-1} = \frac{\frac{-x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{x}{x^2} - \frac{1}{x^2}} = \frac{\frac{-1}{x} + \frac{1}{x^2}}{2 + \frac{1}{x} - \frac{1}{x^2}} = \frac{0+0}{2+0-0} = 0$$

$$(3) \lim_{x \rightarrow +\infty} \frac{x^3+3x^2+2}{2x+1} = \frac{\frac{x^3}{x^3} + \frac{3x^2}{x^3} + \frac{2}{x^3}}{\frac{2x}{x^3} + \frac{1}{x^3}} = \frac{1 + \frac{3}{x} + \frac{2}{x^3}}{\frac{2}{x^2} + \frac{1}{x^3}} = \frac{1+0+0}{0+0} = +\infty$$

### Summary (Quick Tip)

You only need to compare the highest powers (degrees) of the numerator and denominator:

- **If the degree of the numerator equals the degree of the denominator:**  
The limit equals the ratio of the leading coefficients of the numerator and the denominator.  
(As in Example 1)
- **If the degree of the numerator is less than the degree of the denominator:**  
The limit equals 0.  
(As in Example 2)
- **If the degree of the numerator is greater than the degree of the denominator:**  
The limit equals  $\infty$  or  $-\infty$ .  
(As in Example 3)

**Example 1(page 162):**

$$(1) \lim_{x \rightarrow -\infty} \frac{x^3 - 3x + 1}{5 - 2x} = -\infty;$$

$$(2) \lim_{x \rightarrow -\infty} \frac{1 - 3x^2 - x^3}{4x^2 + 1} = +\infty;$$

$$(3) \lim_{x \rightarrow +\infty} \frac{3x^2 - 2x + 1}{2x^2 - x + 5} = \frac{3}{2};$$

$$(4) \lim_{x \rightarrow -\infty} \frac{x + 1}{2x^2 - x + 1} = 0;$$

**Example 2(page 163){Note:  $\sqrt{x^2} = |x|$ }**

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 - x + 1}}{x + 1} &= \frac{\frac{\sqrt{4x^2 - x + 1}}{|x|}}{\frac{x + 1}{|x|}} \\ \implies \frac{\sqrt{4x^2 - x + 1}}{x} &= \sqrt{\frac{x^2 \left( \frac{4x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2} \right)}{x}} = \frac{|x| \sqrt{\left( 4 - \frac{1}{x} + \frac{1}{x^2} \right)}}{x} = \frac{-x \sqrt{\left( 4 - \frac{1}{x} + \frac{1}{x^2} \right)}}{x} = \\ -\sqrt{\left( 4 - \frac{1}{x} + \frac{1}{x^2} \right)} &= -\sqrt{4} = -2 \\ \implies \frac{x + 1}{x} &= 1 + \frac{1}{x} = 1 \\ \implies \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 - x + 1}}{x + 1} &= \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt{4x^2 - x + 1}}{|x|}}{\frac{x + 1}{|x|}} = -\frac{2}{1} = -2 \end{aligned}$$

**Example 3(page 163):{Note:  $\sqrt{x^2} = |x|$ }**

$$\begin{aligned} (1) \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2 + 1} + x}{x + 1} &= \lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{x^2 + 1} + x}{|x|}}{\frac{x + 1}{|x|}} \\ \implies \frac{\sqrt{x^2 + 1} + x}{x} &= \sqrt{\frac{x^2 \left( 1 + \frac{1}{x^2} \right) + 1}{x}} = \frac{|x| \sqrt{\left( 1 + \frac{1}{x^2} \right) + 1}}{x} = \frac{x \sqrt{\left( 1 + \frac{1}{x^2} \right) + 1}}{x} = \sqrt{\left( 1 + \frac{1}{x^2} \right) + 1} = \\ 1 &= \sqrt{\left( 1 + 0 \right)} + 1 = 2 \\ \implies \frac{x + 1}{x} &= 1 + \frac{1}{x} = 1 \\ \implies \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2 + 1} + x}{x + 1} &= \lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{x^2 + 1} + x}{|x|}}{\frac{x + 1}{|x|}} = \frac{2}{1} = 2 \\ (2) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 - x} - \sqrt{4x^2 + 1}}{2x + 3} &= \frac{\frac{\sqrt{x^2 - x} - \sqrt{4x^2 + 1}}{|x|}}{\frac{2x + 3}{|x|}} \\ \implies \frac{\sqrt{x^2 - x}}{x} &= \sqrt{\frac{x^2 \left( 1 - \frac{1}{x} \right)}{x}} = \frac{|x| \sqrt{\left( 1 - \frac{1}{x} \right)}}{x} = \frac{-x \sqrt{\left( 1 - \frac{1}{x} \right)}}{x} = -\sqrt{\left( 1 - \frac{1}{x} \right)} = \\ -\sqrt{1} &= -1 \\ \implies \frac{-\sqrt{4x^2 + 1}}{x} &= \frac{-\sqrt{x^2 \left( 4 + \frac{1}{x^2} \right)}}{x} = \frac{-|x| \sqrt{\left( 4 + \frac{1}{x^2} \right)}}{x} = \frac{-(-x) \sqrt{\left( 4 + \frac{1}{x^2} \right)}}{x} = \frac{x \sqrt{\left( 4 + \frac{1}{x^2} \right)}}{x} = \\ \sqrt{\left( 4 + \frac{1}{x^2} \right)} &= 2 \\ \frac{2x + 3}{x} &= 2 + \frac{3}{x} = 2 \\ \implies \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 - x} - \sqrt{4x^2 + 1}}{2x + 3} &= \frac{\frac{\sqrt{x^2 - x} - \sqrt{4x^2 + 1}}{|x|}}{\frac{2x + 3}{|x|}} = \frac{-1 + 2}{2} = \frac{1}{2} \end{aligned}$$

**Example 4(page 163)** Give  $\lim_{x \rightarrow +\infty} \frac{ax^2 + bx + 3}{cx^3 + 3x + 2} = 2$ ; know  $a, b, c \in \mathbb{R}$ . Calculate  $a + b + c$ ?

**Solve:**

Let degree of the numerator = A;

Let degree of the denominator = B;

We know :

Case 1:  $A = B \implies$  The limit equals the ratio the leading coefficients of the A and B;

Case 2:  $A < B \implies$  The limit = 0;

Case 3:  $A > B \implies$  The limit =  $\infty$  or  $-\infty$

Given :  $\lim_{x \rightarrow +\infty} \frac{ax^2+bx+3}{cx^3+3x+2} = 2$  only applies to case 1 (A = B)

$\implies c = 0 \implies \lim_{x \rightarrow +\infty} \frac{ax^2+bx+3}{3x+2} = 2$  Not yet consistent with the given condition, so let  $a = 0$

$$\implies \lim_{x \rightarrow +\infty} \frac{bx+3}{3x+2} = 2 \implies \text{satisfy given}$$

$$\implies \lim_{x \rightarrow +\infty} \frac{bx+3}{3x+2} = 2 \implies \frac{b}{3} = 2 \implies b = 6$$

$$\implies a + b + c = 6$$

### Additional exercises(page: 163)

Find limit:

$$(1) \lim_{x \rightarrow -\infty} \frac{x^2+x+1}{2x^3+2x+5} = 0 \text{ (Case 2)}$$

$$(2) \lim_{x \rightarrow +\infty} \frac{2x^2+1}{x^3-3x^2+2} = 0 \text{ (Case 2)}$$

$$(3) \lim_{x \rightarrow +\infty} \frac{\sqrt[3]{x^2-x+1}}{5x^2-1} = 0 \text{ (Note: } \sqrt[n]{x^m} = x^{\frac{m}{n}}\text{)}$$

$$(4) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x+1}}{2x+1} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x+1}}{\frac{2x+1}{x}}$$

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x+1}}{x} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(1-\frac{1}{x}+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{|x|\sqrt{(1-\frac{1}{x}+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{-x\sqrt{(1-\frac{1}{x}+\frac{1}{x^2})}}{x} =$$

$$\lim_{x \rightarrow -\infty} -\sqrt{(1-\frac{1}{x}+\frac{1}{x^2})} = -1$$

$$\implies \lim_{x \rightarrow -\infty} \frac{2x+1}{x} = \lim_{x \rightarrow -\infty} \frac{2x}{x} + \frac{1}{x} = \lim_{x \rightarrow -\infty} 2 + \frac{1}{x} = 2$$

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x+1}}{2x+1} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2-x+1}}{\frac{2x+1}{x}} = -\frac{1}{2}$$

$$(5) \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3-1}}{\sqrt{2x^2+1}} = \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3-1}}{\sqrt{2x^2+1}}$$

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3-1}}{x} = \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3(1-\frac{1}{x^3})}}{x} = \lim_{x \rightarrow -\infty} \frac{x\sqrt[3]{(1-\frac{1}{x^3})}}{x} = \lim_{x \rightarrow -\infty} \sqrt[3]{(1-\frac{1}{x^3})} =$$

1

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2+1}}{x} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(2+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{|x|\sqrt{(2+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{-x\sqrt{(2+\frac{1}{x^2})}}{x} =$$

$$\lim_{x \rightarrow -\infty} -\sqrt{(2+\frac{1}{x^2})} = -\sqrt{2}$$

$$\implies \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3-1}}{\sqrt{2x^2+1}} = \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^3-1}}{\sqrt{2x^2+1}} = \frac{1}{-\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

$$\begin{aligned}
(6) \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt{2x^2+1}} &= \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2}}{\frac{\sqrt{2x^2+1}}{x^2}} \\
&\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2} = \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt[3]{x^6}} = \lim_{x \rightarrow -\infty} \sqrt[3]{\frac{x^6+x^4+x^2+1}{x^6}} = 1 \\
&\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2+1}}{x^2} = \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2+1}}{\sqrt{x^4}} = \lim_{x \rightarrow -\infty} \sqrt{\frac{2x^2+1}{x^4}} = \lim_{x \rightarrow -\infty} \sqrt{\frac{2}{x^2} + \frac{1}{x^4}} = 0 \\
&\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x^6+x^4+x^2+1}}{\sqrt{2x^2+1}} = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt[3]{x^6+x^4+x^2+1}}{x^2}}{\frac{\sqrt{2x^2+1}}{x^2}} = \frac{1}{0} = +\infty
\end{aligned}$$

### Method 2:

We can see numerator the greatest is:  $\sqrt[3]{x^6} = x^2$  and all of them are positive.  $\Rightarrow$  numerator  $= +\infty$

We can see denominator is  $\sqrt{2x^2+1} \Rightarrow$  always positive

We know:  $x^2 > x^1$  (The degree of the numerator is greater than the degree of the denominator)

$$\Rightarrow \frac{\text{positive}}{\text{positive}} = +\infty$$

### Method 3:

The greatest degree of numerator is:  $\sqrt[3]{x^6} = x^2$

The greatest degree of denominator is:  $\sqrt{2x^2} = \sqrt{2} \cdot \sqrt{x^2}$  (because  $x \rightarrow -\infty$  should  $|x| = -x$ )

$$\Rightarrow \text{The limit after rewriting is: } \lim_{x \rightarrow -\infty} \frac{x^2}{\sqrt{2}(-x)} = \frac{x}{-\sqrt{2}} = \frac{\text{negative}}{\text{negative}} = +\infty$$

### Method 4: Using Casio f(x) 580 :>

$$\begin{aligned}
(7) \lim_{x \rightarrow -\infty} \frac{x-\sqrt{2x^2+1}}{2x+3\sqrt{x^2+1}} &= \lim_{x \rightarrow -\infty} \frac{\frac{x-\sqrt{2x^2+1}}{x}}{\frac{2x+3\sqrt{x^2+1}}{x}} \\
&\Rightarrow \lim_{x \rightarrow -\infty} \frac{x-\sqrt{2x^2+1}}{x} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{2x^2+1}+x}{x} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{x^2(2+\frac{1}{x^2})}+x}{x} = \lim_{x \rightarrow -\infty} \frac{-|x|\sqrt{(2+\frac{1}{x^2})}+x}{x} = \\
&\lim_{x \rightarrow -\infty} \frac{x\sqrt{(2+\frac{1}{x^2})}+x}{x} = \lim_{x \rightarrow -\infty} \frac{x\sqrt{(2+\frac{1}{x^2})}}{x} + \frac{x}{x} = \lim_{x \rightarrow -\infty} \sqrt{(2+\frac{1}{x^2})} + 1 = 1 + \sqrt{2} \\
&\Rightarrow \lim_{x \rightarrow -\infty} \frac{2x+3\sqrt{x^2+1}}{x} = \lim_{x \rightarrow -\infty} \frac{2x}{x} + \frac{3\sqrt{x^2+1}}{x} = \lim_{x \rightarrow -\infty} 2 + \frac{3\sqrt{x^2+1}}{x} = \\
&\lim_{x \rightarrow -\infty} 2 + \frac{3\sqrt{x^2(1+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{3|x|\sqrt{(1+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} \frac{-3x\sqrt{(1+\frac{1}{x^2})}}{x} = \lim_{x \rightarrow -\infty} 2 + \\
&-3 \cdot \sqrt{(1+\frac{1}{x^2})} = -1 \\
&\Rightarrow \lim_{x \rightarrow -\infty} \frac{x-\sqrt{2x^2+1}}{2x+3\sqrt{x^2+1}} = \lim_{x \rightarrow -\infty} \frac{\frac{x-\sqrt{2x^2+1}}{x}}{\frac{2x+3\sqrt{x^2+1}}{x}} = \frac{1+\sqrt{2}}{-1} = -1 - \sqrt{2}
\end{aligned}$$

$$(8) \lim_{x \rightarrow +\infty} \sqrt{\frac{x^3+1}{2x^3+5}} = \lim_{x \rightarrow +\infty} \sqrt{\frac{\frac{x^3+1}{x^3}}{\frac{2x^3+5}{x^3}}} = \lim_{x \rightarrow +\infty} \sqrt{\frac{1+\frac{1}{x^3}}{2+\frac{5}{x^3}}} = \sqrt{\frac{1+0}{2+0}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

### Method 2 (Case 1) {degree numerator = degree denominator}

$$\sqrt{\frac{x^3}{2x^3}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

[Example] (page 164) form  $\infty - \infty$

$$\begin{aligned}
(1) \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 2x} - x) &= \lim_{x \rightarrow +\infty} \frac{(\sqrt{x^2 + 2x} - x)(\sqrt{x^2 + 2x} + x)}{(\sqrt{x^2 + 2x} + x)} = \lim_{x \rightarrow +\infty} \frac{x^2 + 2x - x}{(\sqrt{x^2 + 2x} + x)} = \\
&\lim_{x \rightarrow +\infty} \frac{2x}{\sqrt{x^2 + 2x} + x} = \lim_{x \rightarrow +\infty} \frac{\frac{2x}{x}}{\sqrt{x^2 \left(1 + \frac{2}{x}\right)} + \frac{x}{x}} = \lim_{x \rightarrow +\infty} \frac{2}{|x| \sqrt{\left(1 + \frac{2}{x}\right)} + 1} = \lim_{x \rightarrow +\infty} \frac{2}{x \sqrt{\left(1 + \frac{2}{x}\right)} + 1} = \\
&\lim_{x \rightarrow +\infty} \frac{2}{\sqrt{\left(1 + \frac{2}{x}\right)} + 1} = \frac{2}{\sqrt{1+0+1}} = \frac{2}{2} = 1
\end{aligned}$$

$$\begin{aligned}
(2) \lim_{x \rightarrow +\infty} (\sqrt{x+1} - \sqrt{x}) &= \lim_{x \rightarrow +\infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{(\sqrt{x+1} + \sqrt{x})} = \lim_{x \rightarrow +\infty} \frac{1}{(\sqrt{x+1} + \sqrt{x})} = \\
&\frac{1}{+\infty} = 0
\end{aligned}$$

$$\begin{aligned}
(3) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + x + 1} + x) &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 + x + 1} + x)(\sqrt{x^2 + x + 1} - x)}{(\sqrt{x^2 + x + 1} - x)} = \lim_{x \rightarrow -\infty} \frac{x^2 + x + 1 - x^2}{(\sqrt{x^2 + x + 1} - x)} = \\
&\lim_{x \rightarrow -\infty} \frac{x+1}{(\sqrt{x^2 + x + 1} - x)} = \lim_{x \rightarrow -\infty} \frac{\frac{x+1}{x}}{\sqrt{x^2 \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)} - x} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{|x| \sqrt{\left(1 + \frac{1}{x}\right)} - x} = \\
&\lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{-x \left(\sqrt{1 + \frac{1}{x}}\right)^{-1}} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{-x \sqrt{1 + \frac{1}{x}} - x} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{-\sqrt{1 + \frac{1}{x}} - 1} = \frac{1+0}{-\sqrt{1+0-1}} = -\frac{1}{2}
\end{aligned}$$

**Example 1(page 165):**

$$\begin{aligned}
(1) \lim_{x \rightarrow +\infty} (\sqrt{x+3} - \sqrt{x}) &= \lim_{x \rightarrow +\infty} \frac{(\sqrt{x+3} - \sqrt{x})(\sqrt{x+3} + \sqrt{x})}{(\sqrt{x+3} + \sqrt{x})} = \lim_{x \rightarrow +\infty} \frac{3}{(\sqrt{x+3} + \sqrt{x})} = \\
&0
\end{aligned}$$

$$\begin{aligned}
(2) \lim_{x \rightarrow +\infty} \left( \frac{4}{\sqrt{x+2} - \sqrt{x-2}} \right) &= \lim_{x \rightarrow +\infty} \frac{4(\sqrt{x+2} + \sqrt{x-2})}{(\sqrt{x+2} - \sqrt{x-2})(\sqrt{x+2} + \sqrt{x-2})} = \lim_{x \rightarrow +\infty} \frac{4(\sqrt{x+2} + \sqrt{x-2})}{(x+2) - (x-2)} = \\
&\lim_{x \rightarrow +\infty} (\sqrt{x+2} + \sqrt{x-2}) = +\infty
\end{aligned}$$

$$\begin{aligned}
(3) \lim_{x \rightarrow -\infty} (\sqrt{x^2 - 3x} + x) &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 - 3x} + x)(\sqrt{x^2 - 3x} - x)}{(\sqrt{x^2 - 3x} - x)} = \lim_{x \rightarrow -\infty} \frac{-3x}{(\sqrt{x^2 - 3x} - x)} = \\
&\lim_{x \rightarrow -\infty} \frac{\frac{-3x}{x}}{\sqrt{x^2 - 3x} - x} = \lim_{x \rightarrow -\infty} \frac{-3}{\sqrt{x^2 - 3x} - x} = \lim_{x \rightarrow -\infty} \frac{-3}{\sqrt{x^2 \left(1 - \frac{3}{x}\right)} - 1} = \lim_{x \rightarrow -\infty} \frac{-3}{\sqrt{x^2 \left(1 - \frac{3}{x}\right)} - 1} = \\
&\lim_{x \rightarrow -\infty} \frac{-3}{|x| \sqrt{1 - \frac{3}{x}} - 1} = \lim_{x \rightarrow -\infty} \frac{-3}{-x \sqrt{1 - \frac{3}{x}} - 1} = \lim_{x \rightarrow -\infty} \frac{-3}{-\sqrt{1 - \frac{3}{x}} - 1} = \frac{-3}{-\sqrt{1-0-1}} = \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
(4) \lim_{x \rightarrow +\infty} (2x - \sqrt{4x^2 + 2x - 1}) &= \lim_{x \rightarrow +\infty} \frac{(2x - \sqrt{4x^2 + 2x - 1})(2x + \sqrt{4x^2 + 2x - 1})}{(2x + \sqrt{4x^2 + 2x - 1})} = \\
&\lim_{x \rightarrow +\infty} \frac{6x^2 - 2x + 1}{(2x + \sqrt{4x^2 + 2x - 1})} = +\infty
\end{aligned}$$

**Example 2(page 165): Value of  $A = \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4x + 5} + x)$  =?**

$$\begin{aligned}
A &= \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4x + 5} + x) = \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 + 4x + 5} + x)(\sqrt{x^2 + 4x + 5} - x)}{(\sqrt{x^2 + 4x + 5} - x)} = \lim_{x \rightarrow -\infty} \frac{4x + 5}{(\sqrt{x^2 + 4x + 5} - x)} = \\
&\lim_{x \rightarrow -\infty} \frac{\frac{4x + 5}{x}}{\left(\sqrt{x^2 + 4x + 5} - x\right)} = \lim_{x \rightarrow -\infty} \frac{\frac{4x + 5}{x}}{\sqrt{x^2 + 4x + 5} - x} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{\sqrt{x^2 \left(1 + \frac{4}{x} + \frac{5}{x^2}\right)} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{\sqrt{x^2 \left(1 + \frac{4}{x} + \frac{5}{x^2}\right)} - 1} = \\
&\lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{|x| \sqrt{\left(1 + \frac{4}{x} + \frac{5}{x^2}\right)} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{-x \sqrt{\left(1 + \frac{4}{x} + \frac{5}{x^2}\right)} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{5}{x}}{-\sqrt{1 + \frac{4}{x} + \frac{5}{x^2}} - 1} = \frac{4}{-2} = \\
&-2
\end{aligned}$$

$$\implies A = -2$$

### Example 3(page 165):

Give  $\lim_{x \rightarrow +\infty} (\sqrt{x^2 + ax} - x) = 8$ . Find  $a, a \in \mathbb{R}$

$$\begin{aligned} \lim_{x \rightarrow +\infty} (\sqrt{x^2 + ax} - x) &= \lim_{x \rightarrow +\infty} \frac{(\sqrt{x^2 + ax} - x)(\sqrt{x^2 + ax} + x)}{(\sqrt{x^2 + ax} + x)} = \lim_{x \rightarrow +\infty} \frac{ax}{(\sqrt{x^2 + ax} + x)} = \\ \lim_{x \rightarrow +\infty} \frac{\frac{ax}{x}}{\frac{\sqrt{x^2 + ax} + x}{x}} &= \lim_{x \rightarrow +\infty} \frac{a}{\frac{\sqrt{x^2 + ax} + x}{x}} = \lim_{x \rightarrow +\infty} \frac{a}{\frac{\sqrt{x^2(1 + \frac{a}{x})}}{x} + 1} = \lim_{x \rightarrow +\infty} \frac{a}{\frac{|x|\sqrt{(1 + \frac{a}{x})}}{x} + 1} = \\ \lim_{x \rightarrow +\infty} \frac{a}{\frac{|x|\sqrt{(1 + \frac{a}{x})}}{x} + 1} &= \lim_{x \rightarrow +\infty} \frac{a}{\sqrt{1 + \frac{a}{x}} + 1} = \frac{a}{\sqrt{1 + 0} + 1} = \frac{a}{2} = 8 \implies a = 16 \end{aligned}$$

### Additional Exercises(page 165):

$$\begin{aligned} (1) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4x + 1} + x) &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 + 4x + 1} + x)(\sqrt{x^2 + 4x + 1} - x)}{(\sqrt{x^2 + 4x + 1} - x)} = \lim_{x \rightarrow -\infty} \frac{4x + 1}{\sqrt{x^2 + 4x + 1} - x} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{4x + 1}{x}}{\frac{\sqrt{x^2 + 4x + 1} - x}{x}} &= \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{\sqrt{x^2 + 4x + 1} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{\frac{\sqrt{x^2 + 4x + 1}}{x} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{\frac{\sqrt{x^2(1 + \frac{4}{x} + \frac{1}{x^2})}}{x} - 1} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{4 + \frac{1}{x}}{|x|\sqrt{(1 + \frac{4}{x} + \frac{1}{x^2})}}}{\frac{x}{x} - 1} &= \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{\frac{-x\sqrt{(1 + \frac{4}{x} + \frac{1}{x^2})}}{x} - 1} = \lim_{x \rightarrow -\infty} \frac{4 + \frac{1}{x}}{-\sqrt{(1 + \frac{4}{x} + \frac{1}{x^2})} - 1} = \frac{4 + 0}{-\sqrt{1 + 0 + 0} - 1} = \\ \frac{4}{-2} &= -2 \end{aligned}$$

$$\begin{aligned} (2) \lim_{x \rightarrow \infty} (\sqrt{x^2 - 4x + 2} - x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 - 4x + 2} - x)(\sqrt{x^2 - 4x + 2} + x)}{\sqrt{x^2 - 4x + 2} + x} = \lim_{x \rightarrow \infty} \frac{-4x + 2}{\sqrt{x^2 - 4x + 2} + x} = \\ \lim_{x \rightarrow \infty} \frac{\frac{-4x + 2}{x}}{\frac{\sqrt{x^2 - 4x + 2} + x}{x}} &= \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\sqrt{x^2 - 4x + 2} + 1} = \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\frac{\sqrt{x^2(1 - \frac{4}{x} + \frac{2}{x^2})}}{x} + 1} = \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\frac{|x|\sqrt{1 - \frac{4}{x} + \frac{2}{x^2}}}{x} + 1} = \\ \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\frac{\sqrt{1 - \frac{4}{x} + \frac{2}{x^2}}}{x} + 1} &= \lim_{x \rightarrow \infty} \frac{-4 + \frac{2}{x}}{\sqrt{1 - \frac{4}{x} + \frac{2}{x^2}} + 1} = \frac{-4 + 0}{\sqrt{1 - 0 + 0} + 1} = -2 \end{aligned}$$

$$(3) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + x} - x) = +\infty - (-\infty) = +\infty + \infty = +\infty \{ \text{be careful it not form } \infty - \infty \}$$

$$\begin{aligned} (4) \lim_{x \rightarrow -\infty} (\sqrt{4x^2 + x} + 2x) &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{4x^2 + x} + 2x)(\sqrt{4x^2 + x} - 2x)}{\sqrt{4x^2 + x} - 2x} = \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{4x^2 + x} - 2x} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{x}{x}}{\frac{\sqrt{4x^2 + x} - 2x}{x}} &= \lim_{x \rightarrow -\infty} \frac{1}{\frac{\sqrt{4x^2 + x}}{x} - \frac{2x}{x}} = \lim_{x \rightarrow -\infty} \frac{1}{\frac{\sqrt{x^2(4 + \frac{1}{x})}}{x} - 2} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{1}{|x|\sqrt{4 + \frac{1}{x}}}}{\frac{x}{x} - 2} &= \lim_{x \rightarrow -\infty} \frac{1}{\frac{-x\sqrt{4 + \frac{1}{x}}}{x} - 2} = \lim_{x \rightarrow -\infty} \frac{1}{-\sqrt{4 + \frac{1}{x}} - 2} = \\ \frac{1}{-\sqrt{4 + 0} - 2} &= -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} (5) \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 - 1}) &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2 + 2x} - \sqrt{x^2 - 1})(\sqrt{x^2 + 2x} + \sqrt{x^2 - 1})}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 1}} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{(x^2 + 2x) - (x^2 - 1)}{x^2 + 2x + \sqrt{x^2 - 1}}}{\frac{2x + 1}{x^2 + 2x + \sqrt{x^2 - 1}}} &= \lim_{x \rightarrow -\infty} \frac{2x + 1}{\sqrt{x^2(1 + \frac{2}{x})} + \sqrt{x^2(1 - \frac{1}{x^2})}} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{x(2 + \frac{1}{x})}{|x|\sqrt{1 + \frac{2}{x}} + |x|\sqrt{1 - \frac{1}{x^2}}}}{\frac{2x + 1}{x^2 + 2x + \sqrt{x^2 - 1}}} &= \lim_{x \rightarrow -\infty} \frac{x(2 + \frac{1}{x})}{-x(\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{1}{x^2}})} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{1}{x}}{-(\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{1}{x^2}})} = \\ \frac{2 + 0}{-(\sqrt{1 + 0} + \sqrt{1 - 0})} &= \frac{2}{-(1 + 1)} = \frac{2}{-2} = -1 \end{aligned}$$

$$\begin{aligned} (6) \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + x - 2} + x}{3x - 1} &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{4x^2 + x - 2} + x)(\sqrt{4x^2 + x - 2} - x)}{(3x - 1)(\sqrt{4x^2 + x - 2} - x)} = \lim_{x \rightarrow -\infty} \frac{3x^2 + x - 2}{(3x - 1)(\sqrt{4x^2 + x - 2} - x)} = \\ \lim_{x \rightarrow -\infty} \frac{\frac{3x^2 + x - 2}{x^2}}{\frac{(3x - 1)(\sqrt{4x^2 + x - 2} - x)}{x^2}} &= \lim_{x \rightarrow -\infty} \frac{\frac{3 + \frac{1}{x} - \frac{2}{x^2}}{x^2}}{\frac{(3x - 1)(\sqrt{4x^2 + x - 2})}{x^2} - \frac{x}{x^2}} = \lim_{x \rightarrow -\infty} \frac{\frac{3 + \frac{1}{x} - \frac{2}{x^2}}{x^2}}{\frac{(3x - 1)(\sqrt{4x^2 + x - 2})}{x^2} - \frac{1}{x}} = \end{aligned}$$

$$\begin{aligned}
& \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{(3x-1)\left(\sqrt{x^2\left(4+\frac{1}{x}-\frac{2}{x^2}\right)}\right) - \frac{1}{x}} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{(3x-1)|x|\sqrt{\left(4+\frac{1}{x}-\frac{2}{x^2}\right)} - \frac{1}{x}} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{\frac{(3x-1)(-x\sqrt{\left(4+\frac{1}{x}-\frac{2}{x^2}\right)})}{x^2} - \frac{1}{x}} = \\
& \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{\frac{-3x^2\sqrt{\left(4+\frac{1}{x}-\frac{2}{x^2}\right)}}{x^2} - \frac{1}{x}} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{1}{x} - \frac{2}{x^2}}{-3\sqrt{\left(4+\frac{1}{x}-\frac{2}{x^2}\right)} - \frac{1}{x}} = \frac{3+0-0}{-3\sqrt{(4+0-0)-0}} = -\frac{1}{2} \\
(7) \quad & \lim_{x \rightarrow +\infty} \left\{ \frac{1}{x+1} \cdot (\sqrt{x^2+2x} - x) \right\} = \lim_{x \rightarrow +\infty} \frac{1}{x+1} \cdot \left( \frac{\sqrt{x^2+2x}-x(\sqrt{x^2+2x}+x)}{(\sqrt{x^2+2x}+x)} \right) = \\
& \lim_{x \rightarrow +\infty} \frac{1}{x+1} \cdot \left( \frac{2x}{\sqrt{x^2+2x}+x} \right) = \lim_{x \rightarrow +\infty} \frac{2x}{(x+1)\sqrt{x^2+2x}+x} = 0 \text{ (numerator degree < denominator degree)} \\
(8) \quad & \lim_{x \rightarrow +\infty} (\sqrt{x^2+3x} - \sqrt{x^2-3x}) = \lim_{x \rightarrow +\infty} \frac{(\sqrt{x^2+3x}-\sqrt{x^2-3x})(\sqrt{x^2+3x}+\sqrt{x^2-3x})}{(\sqrt{x^2+3x}+\sqrt{x^2-3x})} = \\
& \lim_{x \rightarrow +\infty} \frac{6x}{(\sqrt{x^2+3x}+\sqrt{x^2-3x})} = \lim_{x \rightarrow +\infty} \frac{\frac{6x}{x}}{\frac{\sqrt{x^2+3x}}{x} + \frac{\sqrt{x^2-3x}}{x}} = \lim_{x \rightarrow +\infty} \frac{6}{\frac{\sqrt{x^2\left(1+\frac{3}{x}\right)}}{x} + \frac{\sqrt{x^2\left(1-\frac{3}{x}\right)}}{x}} = \\
& \lim_{x \rightarrow +\infty} \frac{6}{\frac{|x|\sqrt{\left(1+\frac{3}{x}\right)}}{x} + \frac{|x|\sqrt{\left(1-\frac{3}{x}\right)}}{x}} = \lim_{x \rightarrow +\infty} \frac{6}{\frac{x\sqrt{\left(1+\frac{3}{x}\right)}}{x} + \frac{x\sqrt{\left(1-\frac{3}{x}\right)}}{x}} = \lim_{x \rightarrow +\infty} \frac{6}{\sqrt{\left(1+\frac{3}{x}\right)} + \sqrt{\left(1-\frac{3}{x}\right)}} = \\
& \frac{6}{\sqrt{1+0} + \sqrt{1-0}} = 3
\end{aligned}$$