Lecture 23: Theme 23 (Mon 20 Oct 2025 18:58)

Limit when x $\to$ a

Example 1(page 155): Calculate $\lim\limits_{x\to 1} \dfrac{1}{|x-1|}$

We know $|x-1| = \begin{cases} x-1 (x \geq 1) \\ 1-x (1 > x) \end{cases}$

$\implies \frac{1}{|x-1|} = \begin{cases} \frac{1}{x-1} (x > 1) \\ \frac{1}{1-x} (x < 1) \end{cases}$

$\implies \lim\limits_{x\to 1^+} \frac{1}{x-1} = +\infty$ Explain: $\begin{cases} 1 > 0 \\ \lim\limits_{x\to 1^+} = 0^+ \end{cases}$

$\implies \lim\limits_{x\to 1^-} \frac{1}{1-x} = +\infty$ Explain: $\begin{cases} 1 > 0 \\ \lim\limits_{x\to 1^-} = 0^+ \end{cases}$

$\implies \lim\limits_{x\to 1} = +\infty$

Example 2(page 155): Calculate $\lim\limits_{x\to 0} \left(1 - \frac{1}{x^2}\right)$.

$\lim\limits_{x\to 0^+} (1 - \frac{1}{x^2}) = -\infty$

$\lim\limits_{x\to 0^-} (1 - \frac{1}{x^2}) = -\infty$

$\implies \lim\limits_{x\to 0} = -\infty$

Example 1(page 156): Calculate $\lim\limits_{x\to 3} (x^2 - 2x + 3)$

Method 1: Nature

$\lim\limits_{x\to 3} (x^2 - 2x + 3)$

$= \lim\limits_{x\to 3} x^2 - \lim\limits_{x\to 3} 2x + \lim\limits_{x\to 3} 3$

$= \lim\limits_{x\to 3} x \cdot \lim\limits_{x\to 3} x - 2 \lim\limits_{x\to 3} x + \lim\limits_{x\to 3} 3$

$= 3 \cdot 3 - 2 \cdot 3 + 3$

$= 6$

Method 2: Quick application:

$\lim\limits_{x\to 3} (x^2 - 2x + 3) = 3^2 - 2 \cdot 3 + 3 = 6$

Example 2(page 156): Calculate $\lim\limits_{x\to 1} \dfrac{2x}{x^2+4}$

$= \frac{\lim\limits_{x\to 1} 2x}{\lim\limits_{x\to 1} x^2 + 4}$

$= \frac{2}{5}$

Method 2 : Nature

$= \frac{2 \lim\limits_{x\to 1} x}{\lim\limits_{x\to 1} x^2 + \lim\limits_{x\to 1} 4}$

$= \frac{2\cdot 1}{1^2+4} = \frac{2}{5}$

Example 3(page 156): Calculate $\lim\limits_{x\to 3} \dfrac{x^2+1}{2\sqrt{x}}$

Method 1: Nature

$\lim\limits_{x\to 3} \frac{x^2+1}{2\sqrt{x}}$

$= \frac{\lim\limits_{x\to 3} x^2 + \lim\limits_{x\to 3} 1}{2\cdot \lim\limits_{x\to 3} \sqrt{x}}$

$= \frac{3^2+1}{2\cdot\sqrt{3}} = \frac{5\sqrt{3}}{3}$

Method 2:Quick application

$= \frac{3^2+1}{2\cdot\sqrt{3}} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$

Example 4(page 156):

a)

$\lim\limits_{x\to 2}[5f(x) + (g(x))^2]$

$= 5\lim\limits_{x\to 2} 3 + \lim\limits_{x\to 2} 4^2$

$= 5\cdot 3 + 4^2 = 31$

b)

$\lim\limits_{x\to 2} \frac{f(x)-4}{g(x)}$

$= \frac{3-4}{4} = -\frac{1}{4}$

Method 2: Nature

$\lim\limits_{x\to 2} \frac{f(x)-4}{g(x)} = \frac{\lim\limits_{x\to 2} f(x) - \lim\limits_{x\to 2} 4}{\lim\limits_{x\to 2} g(x)} = \frac{3-4}{4} = -\frac{1}{4}$

Example 5(page 156):

We have a linear equation in two variable, $f(x)$ and $g(x)$:

$\lim\limits_{x\to 2} 3f(x) - 2g(x) = 2$

$\lim\limits_{x\to 2} 2f(x) - g(x) = 3$

$$\iff \begin{cases} \lim\limits_{x\to 2} 3f(x) - 2\lim\limits_{x\to 2} g(x) = 2 \\ \lim\limits_{x\to 2} 2f(x) - \lim\limits_{x\to 2} g(x) = 3 \end{cases} \iff \begin{cases} f(x) = 4 \\ g(x) = 5 \end{cases}$$

$\implies \lim\limits_{x\to 2}\{f(x)+g(x)\} = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} g(x) = 4+5 = 9$

Example 1 (Page 158):

$\lim\limits_{x\to 2} \frac{x^2-4}{x^2-3x+2}$ with $x = 2 \implies$ form $\frac{0}{0}$

$\lim\limits_{x\to 2} \frac{x^2-4}{x^2-3x+3} = \frac{(x-2)(x+2)}{(x-2)(x-1)} = \frac{x+2}{x-1} = 4$

$\implies$ choose B.

Example 2(page 158):

The numerator : $(-1)^2 + (2\cdot -1) + 1 = 0$

The denominator : $2\cdot -1 + 2 = 0$

$\implies$ form $\frac{0}{0}$

$\lim\limits_{x\to -1}\frac{x^2+2x+1}{2x+2}=\lim\limits_{x\to -1}\frac{(x+1)^2}{2(x+1)}=\lim\limits_{x\to -1}\frac{x+1}{2}=\frac{-1+1}{2}=\frac{0}{2}=0$
$\implies$ choose B.
Example 3(page 158):
The numerator: $2\cdot(\sqrt{3})^2-6=0$
The denominator: $\sqrt{3}-\sqrt{3}=0$
$\implies$ form $\frac{0}{0}$
$I=\lim\limits_{x\to\sqrt{3}}\frac{2x^2-6}{x-\sqrt{3}}=\lim\limits_{x\to\sqrt{3}}\frac{2(x^2-3)}{x-\sqrt{3}}=\lim\limits_{x\to\sqrt{3}}\frac{2(x-\sqrt{3})(x+\sqrt{3})}{x-\sqrt{3}}=\lim\limits_{x\to\sqrt{3}}2\cdot(x+\sqrt{3})=2\cdot(\sqrt{3}+\sqrt{3})=4\sqrt{3}$
$\implies a^2+b^2=4^2+3^2=16+9=25\implies$ choose D.
Example 4(page 158):
The numerator : $1^2+(3a+2)\cdot 1-3a-3=0$
The denominator: $1-1=0$
$\implies$ from $\dfrac{0}{0}$
We transform the numerator :
$x^2+(3a+2)x-3a-3)$
$=x^2+3ax+2x-3a-3$
$=(x^2+2x-3)+(3ax-3a)$
$=(x-1)(x+3)+3a(x-1)$
$=(x-1)[(x+3)+3a]$
$=(x-1)(x+3a+3)$
$\implies\lim\limits_{x\to 1}\frac{x^2+(3a+2)x-3a-3}{x-1}=\lim\limits_{x\to 1}\frac{(x-1)(x+3a+3)}{x-1}=\lim\limits_{x\to 1}(x+3a+3)=1+3a+3=3a+4\implies$ choose C.
Example 5(page 158):
$\implies$ form $\dfrac{0}{0}$
$\implies$ form $\sqrt{A}-B\implies$ using technique "multiplying by the conjugate."
$\implies\lim\limits_{x\to 5}\frac{\sqrt{3x+1}-4}{3-\sqrt{x+4}}=\lim\limits_{x\to 5}\frac{(\sqrt{3x+1}-4)(\sqrt{3x+1}+4)(3+\sqrt{x+4})}{(\sqrt{3x+1}+4)(3-\sqrt{x+4})(3+\sqrt{x+4})}=\lim\limits_{x\to 5}\frac{(3x+1-4^2)(3+\sqrt{x+4})}{(\sqrt{3x+1}+4)[3^2-(x+4)]}=\lim\limits_{x\to 5}\frac{(3x-15)(3+\sqrt{x+4})}{(\sqrt{3x+1}+4)(5-x)}=\lim\limits_{x\to 5}\frac{3(x-5)(3+\sqrt{x+4})}{(\sqrt{3x+1}+4)(5-x)}=\lim\limits_{x\to 5}\frac{-3(3+\sqrt{x+4})}{\sqrt{3x+1}+4}=\frac{-3(3+\sqrt{5}+4)}{\sqrt{3\cdot 5+1}+4}=-\frac{9}{4}$
Example 6(page 158):
$\implies$ form $\frac{0}{0}$
we know : $\sqrt[3]{1}=1$
we know : $a^3+b^3=(a+b)(a^2-ab+b^2)$
we know : $a^3-a^3=(a-b)(a^2+ab+b^2)$
$\implies$ we have conjugate expression : $(\sqrt[3]{1+4x})^2+(\sqrt[3]{1+4x}\cdot 1)+1^2\implies$
$\lim\limits_{x\to 0}\frac{\sqrt[3]{1+4x}-1}{x}=\lim\limits_{x\to 0}\frac{(\sqrt[3]{1+4x}-1)\cdot(\sqrt[3]{1+4x})^2+(\sqrt[3]{1+4x}\cdot 1)+1^2}{x\cdot[(\sqrt[3]{1+4x})^2+(\sqrt[3]{1+4x}\cdot 1)+1^2]}=\lim\limits_{x\to 0}\frac{1+4x-1}{x\cdot[(\sqrt[3]{1+4x})^2+(\sqrt[3]{1+4x}\cdot 1)+1^2]}=\lim\limits_{x\to 0}\frac{4}{(\sqrt[3]{1+4x})^2+(\sqrt[3]{1+4x}\cdot 1)+1^2}=\frac{4}{3}$
Example 7(page 158):

$\frac{1}{2-2} \cdot (\frac{1}{2+4} - \frac{1}{2^2+2}) \implies$ form $0 \cdot \infty$

$\implies \left(\frac{1}{x+4} - \frac{1}{x^2+x}\right) = \left(\frac{1}{x+4} - \frac{1}{x(x+1)}\right) = \frac{x(x+1)}{(x+4)[x(x+1)]} - \frac{(x+4)}{(x+4)[x(x+1)]} = \frac{x^2-4}{(x+4)[x(x+1)]} = \frac{(x-2)(x+2)}{(x+4)[x(x+1)]}$

$\implies \lim\limits_{x\to 2} \frac{1}{x-2} \cdot \left(\frac{1}{x+4} - \frac{1}{x^2+x}\right) = \lim\limits_{x\to 2} \frac{1}{x-2} \cdot \left(\frac{(x-2)(x+2)}{(x+4)[x(x+1)]}\right) = \lim\limits_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x+4)[x(x+1)]}$
$= \lim\limits_{x\to 2} \frac{(x+2)}{(x+4)[x(x+1)]} = \frac{2+2}{(2+4)\cdot[2(2+1)]} = \frac{1}{9}$

Example 8(page 158):

We know : $\lim\limits_{x\to a} \{f(x) \cdot g(x)\} = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = \alpha \cdot \beta$

$\implies \lim\limits_{x\to 2} \frac{(x^2+2x)\cdot f(x)}{x-2} = 9 \iff \lim\limits_{x\to 2} (x^2+2x) \cdot \lim\limits_{x\to 2} \frac{f(x)}{x-2} = 9$

$\implies \lim\limits_{x\to 2} \frac{f(x)}{x-2} = \frac{9}{8}$

$\implies \lim\limits_{x\to 2} \frac{\left(x^3+4x+8\right)\cdot f(x)}{x-2} = \lim\limits_{x\to 2} (x^3+4x+8) \cdot \lim\limits_{x\to 2} \frac{f(x)}{x-2} = 24 \cdot \frac{9}{8} = 27$

Example 9(page 158):

$I = \lim\limits_{x\to 3} \frac{x-3}{\sqrt{x+a}+b} = 6$

We can see the numerator : $3-3=0 \implies I$ is definitely in the form $\frac{0}{0}$, because the $\frac{0}{0}$ form, after simplification, will result in a finite value.

$\implies \sqrt{x+a} + b = 0$

$\implies \sqrt{3+a} + b = 0$

$\implies b = -\sqrt{3+a}$

$\implies \lim\limits_{x\to 3} \frac{x-3}{\sqrt{x+a}+b} = \lim\limits_{x\to 3} \frac{x-3}{\sqrt{x+a}-\sqrt{3+a}} = \frac{(x-3)\cdot\left(\sqrt{x+a}+\sqrt{3+a}\right)}{\left(\sqrt{x+a}-\sqrt{3+a}\right)\cdot\left(\sqrt{x+a}+\sqrt{3+a}\right)} = \lim\limits_{x\to 3} \frac{(x-3)\cdot\left(\sqrt{x+a}+\sqrt{3+a}\right)}{(x+a)-(3+a)} =$
$\lim\limits_{x\to 3} \frac{(x-3)\cdot(\sqrt{x+a}+\sqrt{3+a})}{(x-3)} = \lim\limits_{x\to 3} \left(\sqrt{x+a} + \sqrt{3+a}\right) = 2\sqrt{3+a} = 6$

$\implies \sqrt{3+a} = 3$

$\implies 3+a = 9 \implies a = 6 \implies b = -3 \implies a-b = 6-(-3) = 9$

Additional practice exercises(page 159):

BON 1:

form $\frac{0}{0}$

$$\implies \lim_{x\to 1} \frac{x-1}{x^2+2x-3} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+3)} = \lim_{x\to 1} \frac{1}{x+3} = \frac{1}{4}$$

BON 2:

form $\frac{0}{0}$

$\implies \lim\limits_{x\to 2} \frac{x^3-8}{x-2} = \lim\limits_{x\to 2} \frac{x^3-2^3}{x-2} = \lim\limits_{x\to 2} \frac{(x-2)(x^2+2x+2^2)}{(x-2)} = \lim\limits_{x\to 2} x^2 + 2x + 2^2 = 2^2 + 2\cdot 2 + 4 = 12$

BON 3:

form $\frac{0}{0}$

$\implies \lim\limits_{x\to -1} \frac{x^3+2x^2+x}{x^2-1} = \lim\limits_{x\to -1} \frac{x(x^2+2x+1)}{(x-1)(x+1)} = \lim\limits_{x\to -1} \frac{x(x+1)^2}{(x-1)(x+1)} = \lim\limits_{x\to -1} \frac{x(x+1)}{(x-1)} = 0$

BON 4:

form $\frac{0}{0}$

$\implies \lim\limits_{x\to1} \frac{x-1}{\sqrt{x}-1} = \lim\limits_{x\to1} \frac{(x-1)(\sqrt{x}+1)}{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)} = \lim\limits_{x\to1} \frac{(x-1)\left(\sqrt{x}+1\right)}{(x-1)} = \lim\limits_{x\to1} (\sqrt{x}+1) = 2$

BON 5:

form $\frac{0}{0}$

$\implies \lim\limits_{x\to1} \frac{\sqrt{x+8}-3}{x^2-1} = \lim\limits_{x\to1} \frac{\left(\sqrt{x+8}-3\right)\left(\sqrt{x+8}+3\right)}{(x-1)(x+1)\left(\sqrt{x+8}+3\right)} = \lim\limits_{x\to1} \frac{(x-1)}{(x-1)(x+1)\left(\sqrt{x+8}+3\right)} = \lim\limits_{x\to1} \frac{1}{(x+1)\left(\sqrt{x+8}+3\right)} = \frac{1}{12}$

BON 6:

form $\infty \cdot 0 \implies$ convert to form $\frac{0}{0}$

$\implies \left(\frac{1}{\sqrt{x+1}} - 1\right) = \frac{1}{\sqrt{x+1}} - \frac{\sqrt{x+1}}{\sqrt{x+1}} = \frac{1-\sqrt{x+1}}{\sqrt{x+1}}$

$\implies \lim\limits_{x\to0} \frac{1}{x}\left(\frac{1}{\sqrt{x+1}} - 1\right) = \lim\limits_{x\to0} \frac{1}{x}\left(\frac{1-\sqrt{x+1}}{\sqrt{x+1}}\right) = \lim\limits_{x\to0} \frac{1-\sqrt{x+1}}{x\cdot(\sqrt{x+1})} \implies$ we have form $\frac{0}{0}$

$\implies \lim\limits_{x\to0} \frac{1-\sqrt{x+1}}{x(\sqrt{x+1})} = \lim\limits_{x\to0} \frac{\left(1-\sqrt{x+1}\right)\left(1+\sqrt{x+1}\right)}{x\left(\sqrt{x+1}\right)\left(1+\sqrt{x+1}\right)} = \lim\limits_{x\to0} \frac{1-(x+1)}{x\left(\sqrt{x+1}\right)\left(1+\sqrt{x+1}\right)} = \lim\limits_{x\to0} \frac{-x}{x\left(\sqrt{x+1}\right)\left(1+\sqrt{x+1}\right)} = \lim\limits_{x\to0} \frac{-1}{\left(\sqrt{x+1}\right)\left(1+\sqrt{x+1}\right)} = -\frac{1}{2}$

BON 7:

$I = \lim\limits_{x\to2} \frac{f(x)-3x}{x-2} = 3 = c \implies I$ have form $\frac{0}{0}$ after that , remove the indeterminate form $\frac{0}{0}$ to get a result of 3, hence, the constant $c = 3$

$\implies I$ Sure have form $\frac{0}{0}$

$\implies f(x) - 3x = 0 \implies f(2) = 6$

$\implies \lim\limits_{x\to2} \frac{[f(x)-3x]\cdot[f(x)+2]}{x^2-4} = \lim\limits_{x\to2} \frac{f(x)-3x}{(x-2)} \cdot \lim\limits_{x\to2} \frac{f(x)+2}{(x+2)} = 3 \cdot \frac{8}{4} = 6$

Note:

All the addition, subtraction, multiplication, and division properties of limits as $x \to a$ are valid only when the two individual limits both exist (and are finite numbers).

BON 8:

$I = \lim\limits_{x\to0} \frac{\sqrt{x^2+ax+b}-a}{\sqrt{x+a}-\sqrt{a-x}} = 1$

We can see denominator: $\sqrt{0+a} - \sqrt{a-0} = \sqrt{a} - \sqrt{a} = 0 \implies I$ is definitely in the form $\frac{0}{0}$, because the $\frac{0}{0}$ form, after simplification, will result in a finite value.

$\implies \sqrt{x^2+ax+b} - a = 0$

$\iff \sqrt{b} - a = 0$

$\implies a = \sqrt{b} \iff b = a^2$

$\implies I = \lim\limits_{x\to0} \frac{\sqrt{x^2+ax+a^2}-a}{\sqrt{x+a}-\sqrt{a-x}} = \lim\limits_{x\to0} \frac{\left(\sqrt{x^2+ax+a^2}-a\right)\left(\sqrt{x^2+ax+a^2}+a\right)\left(\sqrt{x+a}+\sqrt{a-x}\right)}{\left(\sqrt{x+a}-\sqrt{a-x}\right)\left(\sqrt{x+a}+\sqrt{a-x}\right)\left(\sqrt{x^2+ax+a^2}+a\right)} = \lim\limits_{x\to0} \frac{x(x+a)\left(\sqrt{x+a}+\sqrt{a-x}\right)}{2x\left(\sqrt{x^2+ax+a^2}+a\right)} = \lim\limits_{x\to0} \frac{(x+a)\left(\sqrt{x+a}+\sqrt{a-x}\right)}{2\left(\sqrt{x^2+ax+a^2}+a\right)} = \lim\limits_{x\to0} \frac{(0+a)\left(\sqrt{0+a}+\sqrt{a-0}\right)}{2\left(\sqrt{0^2+a\cdot0+a^2}+a\right)} = \frac{2a\sqrt{a}}{4a} = \frac{\sqrt{a}}{2} = 1 \implies a = 4 \implies b = 16 \implies a + b = 20$