Area Energy

Dienstag, 15. Juli 2025 08:16

Computation for Area Force at $N_v > 2$ and $k=2$

- Area of a cell: $A_C = \frac{1}{2} \sum_{j=1}^{N_v} (v_j^x v_{j+1}^y - v_{j+1}^x v_j^y)$
- Area energy: $A_E(C) = \frac{1}{2} |A_c - A_d|^2$
- Area gradient: $\nabla_{\vec{v}_j} A_E(C) = \frac{1}{2}(A_c - A_d) \begin{pmatrix} v_{j+1}^y - v_{j-1}^y \\ v_{j-1}^x - v_{j+1}^x \end{pmatrix} =: \Psi_j(\vec{v}_1, \dots, \vec{v}_{N_v})$

$$\mu^{N_c}(A) = \frac{1}{N_c} \sum_{i=1}^{N_c} \delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(A), \quad \int_{\mathbb{R}^{2 \cdot N_v}} \phi(x) \, d\delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(x) = \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)$$

Let $\phi \in C_c^\infty(\mathbb{R}^{2 \cdot N_v}, \mathbb{R})$. $\phi(\vec{w}_1, \dots, \vec{w}_{N_v})$

$$\frac{d}{dt} \int \phi \, d\mu^N = \frac{d}{dt} \frac{1}{N_c} \sum_{i=1}^{N_c} \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)$$

$$= \sum_{j=1}^{N_v} \frac{1}{N_c} \sum_{i=1}^{N_c} \nabla_{\vec{v}_j^i} \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i) \cdot \frac{d\vec{v}_j^i}{dt}$$

$$= - \sum_{j=1}^{N_v} \frac{1}{N_c} \sum_{i=1}^{N_c} \nabla_{\vec{v}_j^i} \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i) \cdot \Psi_j(\vec{v}_{j-1}, \vec{v}_j, \vec{v}_{j+1})$$

Plug in all $(\vec{v}_j^i)_{j=1}^{N_v}$

$$= - \int \sum_{j=1}^{N_v} \frac{1}{N_c} \sum_{i=1}^{N_c} \nabla_{\vec{w}_j} \phi(x) \cdot \Psi_j(x) \, d\delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(x)$$

$$= - \int \sum_{j=1}^{N_v} \nabla_{\vec{w}_j} \phi(x) \cdot \Psi_j(x) \, d\mu^N(x)$$

$$= \int \phi(x) \cdot \left(\sum_{j=1}^{N_v} \nabla_{\vec{w}_j} \cdot (\Psi_j \cdot \mu^N)(x) \right) dx$$

$$\Rightarrow \partial_t \rho = \sum_{j=1}^{N_v} \nabla_{\vec{w}_j} \cdot (\Psi_j \rho)$$

rho maps from the same space as mu^N

$\rho : \mathbb{R}^2 \to \mathbb{R}_+$

$\mathcal{B}(\mathbb{R}^2)$

$$\nabla_{\vec{v}_2} \cdot (\rho \Psi_2) = (\nabla_{\vec{v}_2} \cdot \Psi_2) \rho + \Psi_2 \cdot \nabla_{\vec{v}_2} \rho$$

Why are you computing this?

$$2\cdot\nabla_{\vec{v}_2}\cdot\Psi_2 = \frac{d}{dv_2^x}\left[(A_c-A_d)\,(v_3^y-v_1^y)\right] + \frac{d}{dv_2^y}\left[(A_c-A_d)(v_1^x-v_3^x)\right]$$

$$\partial v_2^x(A_c-A_d) = \partial v_2^x A_c = \frac{1}{2}(v_3^y-v_1^y),\quad \partial v_2^y(A_c-A_d) = \frac{1}{2}(v_1^x-v_3^x)$$

$$= \frac{1}{2}(v_3^y-v_1^y)^2 \quad + \quad \frac{1}{2}(v_1^x-v_3^x)^2$$

$$= \frac{1}{2}\|\vec{v}_3-\vec{v}_1\|_2^2$$

This seems correct, but at the end of the day you want something of the form:
d_t rho = div(rho * F), so it is okay to keep it in the above form (conservation law)
this form is more instructive — F is the velocity / force field; this becomes less obvious when you expand the divergence

$$\Longrightarrow \nabla_{\vec{v}_2}\cdot\Psi_2 = \frac{1}{4}\|\vec{v}_3-\vec{v}_1\|_2^2$$

Edge Energy

Montag, 14. Juli 2025 08:51

- $\mu^{N_c} \in \mathcal{P}(\mathbb{R}^2)$, $\mu^{N_c}(A) = \underbrace{\frac{1}{N_c}\sum_{i=1}^{N_c}}_{=: \sum_i} \delta_{\vec{x}_i(t)}(A)$

Why N_C? Don't you want edges of one cell first?

I thought we compute for $N_C \in \mathbb{N}$ and later let $N_C \to \infty$, but I'm not sure whether that's correct

- For an energy $E: \mathbb{R}^2 \to \mathbb{R}_+$, we define the dynamic: $\frac{d\vec{x}_i}{dt} = -\nabla_{\vec{x}_i} E(C)$

This is the general procedure, yes! Nothing here depends on your model ✓

- We take a test function $\phi \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$ and compute

$$
\begin{aligned}
\frac{d}{dt}\int \phi \, d\mu^N &= \frac{d}{dt} \sum_i \phi(\vec{x}_i) \\
&= \sum_i \nabla\phi(\vec{x}_i) \cdot \frac{d\vec{x}_i}{dt} \\
&= -\sum_i \nabla\phi(\vec{x}_i) \cdot \nabla_{\vec{x}_i} E(C) \\
&= -\sum_i \int \nabla\phi(x) \cdot \nabla_{\vec{x}_i} E(x) \, d\delta_{\vec{x}_i} \\
&= -\int \nabla\phi(x) \cdot \nabla E(x) \, d\mu^N \, dx \\
&\overset{\text{IBP}}{=} \overset{0}{\cancel{\left[\phi(x) \cdot \nabla_{\vec{x}_i} E(x)\right]}} + \int \phi(x) \, \nabla \cdot (\mu^N(x) \nabla E(x)) \, dx
\end{aligned}
$$

$$\Rightarrow \frac{d}{dt}\int \phi(x) \, d\mu^N(x) = \int \phi(x) \, \nabla \cdot (\mu^N(x) (\nabla E(x))) \, dx$$ ✓

$\Rightarrow$ As a distribution: $\partial_t \mu^N = \nabla \cdot (\mu^N \nabla E)$ (F1)

see, you get a general conservation law with velocity -grad (which was the discrete velocity in the red line above)

Under suitable conditions:

1. $E(x)$ is sufficiently regular (Lipschitz, or locally smooth and confining)
2. Initial empirical measure $\mu^N(x,0)$ converges weakly to a prob. dens. $\rho_0(x) \in L^1(\mathbb{R}^2)$
3. $E(x)$ satisfies mean field scaling

$\Longrightarrow \mu^N(x,t) \xrightarrow{N\to\infty} \rho(x,t) \in L^1$

Computation for Edge Force at $N_v = 2$ and $k=2$

Edge energy: $E_2(C) = \frac{1}{2}\left|\overbrace{\|\vec{v}_1 - \vec{v}_2\|_2}^{=E_1} - E_d\right|^2$

$$
\begin{aligned}
\nabla_{\vec{v}_1} E_2(C) &= \operatorname{sgn}(E_1 - E_d) \frac{|E_1 - E_d|}{E_1} \begin{pmatrix} v_1^x - v_2^x \\ v_1^y - v_2^y \end{pmatrix} \\
&= \frac{E_1 - E_d}{E_1}(\vec{v}_1 - \vec{v}_2) =: \Psi(\vec{v}_1, \vec{v}_2) \qquad \Psi: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2
\end{aligned}
$$

$$\nabla_{\vec v_2} E_2(C) = -\nabla_{\vec v_1} E_2(C) = \frac{E_1 - E_d}{E_1}\left(\vec v_2 - \vec v_1\right) = \Psi(\vec v_1, \vec v_2)$$

Example: $\vec v_1 = \begin{pmatrix}0\\0\end{pmatrix}$, $\vec v_2 = \begin{pmatrix}1\\0\end{pmatrix}$, $E_1 = 1$, $E_d = 2$

$\Rightarrow \nabla_{\vec v_1} E_2(C) = -\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$\Rightarrow$ In the dynamic $-\nabla_{\vec v_1} E_2(C)$, $\vec v_1$ drifts to the left to increase edge length

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$$\mu^{N_c}(A) = \frac{1}{N_c}\sum_{i=1}^{N_c} \overset{\delta(\vec v_1,\dots,\vec v_{N_v})}{\delta_{(\vec v_1^i, \vec v_2^i)}}(A), \quad \int_{\mathbb{R}^{2\times 2}} \phi(x)\, d\delta_{(\vec v_1^i, \vec v_2^i)}(x) = \phi(\vec v_1^i, \vec v_2^i)$$

Let $\phi \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$. $\phi(\vec w_1, \vec w_2)$

We have to watch out with dimensions, since $\nabla_{\vec v_j}\phi \in \mathbb{R}^{2\times 2}$ and $\frac{d\vec v}{dt} \in \mathbb{R}^2$.
Convention for now: whenever I write $a \cdot b$ for a, b in $\mathbb{R}^x$ or $\mathbb{R}^{x,y}$
I hope there is a correct way of multiplicating them

$$\frac{d}{dt}\int \phi \, d\mu^N = \frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N} \phi(\vec v_1^i, \vec v_2^i)$$

$$= \sum_i \nabla_{\vec v_1^i}\phi(\vec v_1^i, \vec v_2^i)\cdot\frac{d\vec v_1^i}{dt} + \sum_i \nabla_{\vec v_2^i}\phi(\vec v_1^i, \vec v_2^i)\cdot\frac{d\vec v_2^i}{dt}$$

$$= -\sum_i \nabla_{\vec v_1^i}\phi(\vec v_1^i, \vec v_2^i)\cdot\underbrace{\frac{E_1^i - E_d}{E_1^i}(\vec v_1^i - \vec v_2^i)}_{\Psi(\vec v_1, \vec v_2)} - \sum_i \nabla_{\vec v_2^i}\phi(\vec v_1^i, \vec v_2^i)\cdot\underbrace{\frac{E_1^i - E_d}{E_1^i}(\vec v_2^i - \vec v_1^i)}_{-\Psi(\vec v_1, \vec v_2)}$$

$$= -\int \frac{1}{N}\sum_{i=1}^{N}\nabla_{\vec v_1^i}\phi(x)\cdot\Psi(x) + \frac{1}{N}\sum_{i=1}^{N}\nabla_{\vec v_2^i}\phi(x)(-\Psi(x))\, d\delta_{(\vec v_1^i, \vec v_2^i)}$$

$$= -\int \nabla_{\vec w_1}\phi(x)\cdot\Psi(x) + \nabla_{\vec w_2}\phi(x)\cdot(-\Psi(x))\, d\mu^N(x)$$

$$= -\int \Psi(x)\cdot\left(\nabla_{\vec w_1}\phi(x) - \nabla_{\vec w_2}\phi(x)\right)\, d\mu^N(x)$$

$$= \int \phi(x)\cdot\left(\nabla_{\vec w_1}\cdot(\Psi\cdot\mu^N)(x) - \nabla_{\vec w_2}\cdot(\Psi\cdot\mu^N)(x)\right)\, dx$$

$$\Rightarrow \partial_t \mu^N = \nabla_{\vec w_1}\cdot(\Psi\mu^N) - \nabla_{\vec w_2}\cdot(\Psi\mu^N)$$

seems similar to what we did on the blackboard?
in particular, looks good

Yes, I repeated that for practicing

Computation for Edge Force at $N_v = 2$ and $k = 2$

Edge energy: $$E_2(C) = \sum_{j=1}^{N_v}\frac{1}{2}\left|\overset{\|\vec v_j - \vec v_{j+1}\|_2}{E_C^j} - \underset{\text{desired length for all edges}}{E_d}\right|^2$$

For all $1 \le j \le N_v$

$$-\frac{d\vec v_j}{dt} = \nabla_{\vec v_j} E_2(C) = \frac{E^{j-1} - E_d}{E^{j-1}}(\vec v_j - \vec v_{j-1}) + \frac{E^j - E^d}{E^j}(\vec v_j - \vec v_{j+1}) =: \Psi_j(\vec v_{j-1}, \vec v_j, \vec v_{j+1}) : (\mathbb{R}^2)^3 \to \mathbb{R}^2$$

I did not check the second equality yet, seems okay

we use E^{j-1}, that requires $\vec v^{j-1}, \vec v^j, \vec v^{j+1}$

to compute E^{j}, E^{j+1} you also just need. v_{j-1},v_j, v_{j+1}?

For E^j, E^{j+1} we'd need $\vec v^j, \vec v^{j+1}, \vec v^{j+2}$

$$\mu^{N_c}(A) = \frac{1}{N_c}\sum_{i=1}^{N_c}\delta_{(\vec v_1^i,\dots,\vec v_{N_v}^i)}(A), \quad \int_{\mathbb{R}^{2\cdot N_v}}\phi(x)\, d\delta_{(\vec v_1^i,\dots,\vec v_{N_v}^i)}(x) = \phi(\vec v_1^i,\dots,\vec v_{N_v}^i)$$

Let $\phi \in C_c^\infty(\mathbb{R}^{2\cdot N_v}, \mathbb{R})$. $\phi(\vec w_1, \dots, \vec w_{N_v})$

$$\frac{d}{dt}\int \phi \, d\mu^{N} = \frac{d}{dt}\frac{1}{N_c}\sum_{i=1}^{N_c}\phi(\vec{v}_1^{\,i},\dots,\vec{v}_{N_v}^{\,i})$$

$$= \sum_{j=1}^{N_v}\frac{1}{N_c}\sum_{i=1}^{N_c}\nabla_{\vec{v}_j^{\,i}}\phi(\vec{v}_1^{\,i},\dots,\vec{v}_{N_v}^{\,i})\cdot\frac{d\vec{v}_j^{\,i}}{dt}$$

better to keep the i sum outside

$$= - \sum_{j=1}^{N_v}\frac{1}{N_c}\sum_{i=1}^{N_c}\nabla_{\vec{v}_j^{\,i}}\phi(\vec{v}_1^{\,i},\dots,\vec{v}_{N_v}^{\,i})\cdot\Psi_j(\vec{v}_{j-1},\vec{v}_j,\vec{v}_{j+1})$$

Plug in all $(\vec{v}_j^{\,i})_{j=1}^{N_v}$

this notation is abusive.
i know what you are trying to do but
the better way is to define \Psi: R^{2N_v} -> R^{2N_V}
(v_1, …., v_N) \mapsto \Psi(.),
where \Psi_j(.) = [your psi_j which should
have a different name then]

I think we used $\Psi_j(\vec{v}_1,\dots,\vec{v}_{N_v})$

that j tells to consider the gradient $\nabla_{\vec{v}_j}(\dots)$

$$= -\int \sum_{j=1}^{N_v}\frac{1}{N_c}\sum_{i=1}^{N_c}\nabla_{\vec{w}_j}\phi(x)\cdot\Psi_j(x)\, d\delta_{(\vec{v}_1^{\,i},\dots,\vec{v}_{N_v}^{\,i})}(x)$$

$$= -\int \sum_{j=1}^{N_v}\nabla_{\vec{w}_j}\phi(x)\cdot\Psi_j(x)\, d\mu^{N}(x)$$

$$= \int \phi(x)\cdot\left(\sum_{j=1}^{N_v}\nabla_{\vec{w}_j}\cdot(\Psi_j\,\mu^{N})(x)\right) dx$$

then you can write everything in a massive divergence:
div_(v_1, …., v_N) (\rho \Psi)

$$\Rightarrow \partial_t \rho = \sum_{j=1}^{N_v}\nabla_{\vec{w}_j}\cdot(\Psi_j\,\rho)$$

$\rho: \mathbb{R}^2 \to \mathbb{R}_+$

why does rho always map from R^2???

comments cf abvoe

I thought the density $\rho = \lim_{N_c\to\infty}\mu^{N_c}$ actually lives on $\mathbb{R}^2$. But I'm not really familiar with those probability computations and objects

$$\nabla_{\vec{v}_2}\cdot(\rho\,\Psi_2) = (\nabla_{\vec{v}_2}\cdot\Psi_2)\rho + \Psi_2\cdot\nabla_{\vec{v}_2}\rho$$

$$\frac{d}{dv_2^x}\|\vec{v}_1-\vec{v}_2\|_2 = \frac{d}{dv_2^x}\left[(v_1^x-v_2^x)^2+(v_1^y-v_2^y)^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{2\|\vec{v}_1-\vec{v}_2\|}\cdot\frac{d}{dv_2^x}\left[(v_1^x-v_2^x)^2+(v_1^y-v_2^y)^2\right]$$

$$= -\frac{v_1^x-v_2^x}{\|\vec{v}_1-\vec{v}_2\|}$$

$$= \frac{v_2^x-v_1^x}{\|\vec{v}_1-\vec{v}_2\|}$$

$$\frac{d}{dv_2^x}\|\vec{v}_2-\vec{v}_3\|_2 = \frac{v_2^x-v_3^x}{\|\vec{v}_2-\vec{v}_3\|_2}$$

$$\frac{d}{dv_2^y}\|\vec{v}_1-\vec{v}_2\|_2 = \frac{v_2^y-v_1^y}{\|\vec{v}_1-\vec{v}_2\|}$$

$$\frac{d}{dv_2^y}\|\vec{v}_2-\vec{v}_3\|_2 = \frac{v_2^y-v_3^y}{\|\vec{v}_1-\vec{v}_3\|}$$

did not check these…

$$\frac{d}{dv_2^x}\left[\frac{E^1-E^d}{E^1}\right] = \frac{d}{dv_2^x}\left[1-\frac{E^d}{E^1}\right] = -E^d\frac{d}{dv_2^x}\left[E^{1\,(-1)}\right]$$

$$= -E^d\left(-\frac{1}{(E^1)^2}\cdot\frac{d}{dv_2^x}E^1\right)$$

$$= \frac{E^d}{(E^1)^2}\,\frac{v_2^x-v_1^x}{E^1}$$

$$= \frac{E^d}{(E^1)^3}(v_2^x-v_1^x)$$

$$\frac{d}{dv_2^x}\left[\frac{E^2-E^d}{E^2}\right] = \frac{E^d}{(E^2)^3}(v_2^x-v_3^x)$$

$$\frac{d}{dv_2^y}\left[\frac{E^1-E^d}{E^1}\right] = \frac{E^d}{(E^1)^3}(v_2^y-v_1^y)$$

$$\frac{d}{dv_2^y}\left[\frac{E^2-E^d}{E^2}\right] = \frac{E^d}{(E^2)^3}(v_2^y-v_3^y)$$

$$\nabla_{\vec{v}_2}\cdot\Psi_2 = \frac{d}{dv_2^x}\left[\frac{E^1-E_d}{E^1}(v_2^x-v_1^x)+\frac{E^2-E^d}{E^2}(v_2^x-v_3^x)\right]+\frac{d}{dv_2^y}\left[\frac{E^1-E_d}{E^1}(v_2^y-v_1^y)+\frac{E^2-E^d}{E^2}(v_2^y-v_3^y)\right]$$

$$= \partial_{v_2^x}\left[\frac{E^1-E^d}{E^1}\right](v_2^x-v_1^x)+\frac{E^1-E_d}{E^1}\partial_{v_2^x}[v_2^x-v_1^x]+\partial_{v_2^x}\left[\frac{E^2-E^d}{E^2}\right](v_2^x-v_3^x)+\frac{E^2-E_d}{E^2}\partial_{v_2^x}[v_2^x-v_3^x]$$

$$+\partial_{v_2^y}\left[\frac{E^1-E^d}{E^1}\right](v_2^y-v_1^y)+\frac{E^1-E_d}{E^1}\partial_{v_2^y}[v_2^y-v_1^y]+\partial_{v_2^y}\left[\frac{E^2-E^d}{E^2}\right](v_2^y-v_3^y)+\frac{E^2-E_d}{E^2}\partial_{v_2^y}[v_2^y-v_3^y]$$

$$= \frac{E^d}{(E^1)^3}(v_2^x-v_1^x)^2+\frac{E^1-E^d}{E^1}+\frac{E^d}{(E^2)^3}(v_2^x-v_3^x)^2+\frac{E^2-E_d}{E^2}$$
$$+\frac{E^d}{(E^1)^3}(v_2^y-v_1^y)^2+\frac{E^1-E^d}{E^1}+\frac{E^d}{(E^2)^3}(v_2^y-v_3^y)^2+\frac{E^2-E^d}{E^2}$$

$$= \frac{E^d}{(E^1)^3}\left[\underbrace{(v_2^x - v_1^x)^2 + (v_2^y - v_1^y)^2}_{=(E^1)^2}\right] + \frac{2(E^1 - E^d)}{E^1} + \frac{E^d}{(E^2)^3}\left[\underbrace{(v_2^x - v_3^x)^2 + (v_2^y - v_3^y)^2}_{=(E^2)^2}\right] + \frac{2(E^2 - E^d)}{E^2}$$

$$= \frac{E^d + 2E^1 - 2E^d}{E^1} + \frac{E^d + 2E^2 - 2E^d}{E^2}$$

$$= \frac{2E^1 - E^d}{E^1} + \frac{2E^2 - E^d}{E^2}$$

$$= 2 - \frac{E^d}{E^1} + 2 - \frac{E^d}{E^2}$$

$$= 4 - \frac{E^d}{E^1} - \frac{E^d}{E^2}$$

what can we do with these?

Interior Angle Energy

Dienstag, 15. Juli 2025 08:16

Computation for Interior Angle Force at $N_v = 2$ and $k=2$

- interior angle at $\vec{v}_j$: $I^j = \left[\arctan2(\vec{v}_{j-1} - \vec{v}_j) - \arctan2(\vec{v}_{j+1} - \vec{v}_j)\right]_{[0,2\pi)}$ $\left(\arctan2(\vec{v}) = \arctan\left(\frac{v^y}{v^x}\right) + \text{constant}\right)$
- int angle energy: $I_2(C) = \sum_{j=1}^{N_v} \frac{1}{2} |I_c^j - I_d|^2$
- gradient:

$$\nabla_{\vec{v}_j} I_2(C) = (I_c^{j-1} - I_d)\left(-\frac{1}{\|\vec{v}_j - \vec{v}_{j-1}\|^2}\begin{pmatrix} v_j^y - v_{j-1}^y \\ v_{j-1}^x - v_j^x \end{pmatrix}\right) = \Psi_j^{(a)}$$

$$+ (I_c^j - I_d)\left(\frac{1}{\|\vec{v}_{j-1} - \vec{v}_j\|^2}\begin{pmatrix} v_{j-1}^y - v_j^y \\ v_j^x - v_{j-1}^x \end{pmatrix} - \frac{1}{\|\vec{v}_{j+1} - \vec{v}_j\|^2}\begin{pmatrix} v_{j+1}^y - v_j^y \\ v_j^x - v_{j+1}^x \end{pmatrix}\right) = \Psi_j^{(b)}$$

$$+ (I_c^{j+1} - I_d)\left(\frac{1}{\|\vec{v}_j - \vec{v}_{j+1}\|^2}\begin{pmatrix} v_j^y - v_{j+1}^y \\ v_{j+1}^x - v_j^x \end{pmatrix}\right) = \Psi_j^{(c)}$$

$$=: \Psi_j(\vec{v}_1, \dots, \vec{v}_2) = \Psi_j^{(a)} + \Psi_j^{(b)} + \Psi_j^{(c)}$$

Is $(I_c^j - I_d)$ the correct formula? (Problem: periodicity)

We need: a) correct sign: $+ \iff$ angle is too big

$- \iff$ angle is too small

b) correct absolute value: $I_c \longleftrightarrow I_d \Rightarrow$ big absolute value

Do we have to add $\pm 2\pi$ to an angle in some cases?

$\rightsquigarrow I_c, I_d \in [0, 2\pi)$

I_c	I_d	want value	$(I_c - I_d)$
0	2π	-2π	-2π
2π	0	2π	2π

$\Rightarrow$ Seems perfect

$\Rightarrow$ I am now sure that $(I_c - I_d)$ is the correct term.

If we would consider periodicity, the setup:

$I_c = 0 \quad I_d = 2\pi \quad , \quad [I_c - I_d]_{[0,2\pi)} = 0$

would yield no dynamic, but I_c must increase harshly, so

$(I_c - I_d) = -2\pi$

is perfect.

$$\mu^{N_c}(A) = \frac{1}{N_c}\sum_{i=1}^{N_c} \delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(A), \quad \int_{\mathbb{R}^{2 \cdot N_v}} \phi(x)\, d\delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(x) = \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)$$

Let $\phi \in C_c^\infty(\mathbb{R}^{2 \cdot N_v}, \mathbb{R})$. $\phi(\vec{w}_1, \dots, \vec{w}_{N_v})$

$$\frac{d}{dt}\int \phi \, d\mu^N = \frac{d}{dt}\frac{1}{N_c}\sum_{i=1}^{N_c} \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)$$

$$= \sum_{j=1}^{N_v}\frac{1}{N_c}\sum_{i=1}^{N_c} \nabla_{\vec{v}_j^i}\phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i) \cdot \frac{d\vec{v}_j^i}{dt}$$

$$= -\sum_{j=1}^{N_v}\frac{1}{N_c}\sum_{i=1}^{N_c} \nabla_{\vec{v}_j^i}\phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i) \cdot \Psi_j(\vec{v}_{j-1}, \vec{v}_j, \vec{v}_{j+1})$$

Plug in all $(\vec{v}_j^i)_{j=1}^{N_v}$

$$= -\int \sum_{j=1}^{N_v}\frac{1}{N_c}\sum_{i=1}^{N_c} \nabla_{\vec{w}_j}\phi(x) \cdot \Psi_j(x) \, d\delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(x)$$

$$= -\int \sum_{j=1}^{N_v} \nabla_{\vec{w}_j}\phi(x) \cdot \Psi_j(x)\, d\mu^N(x)$$

$$= \int \phi(x) \cdot \left(\sum_{j=1}^{N_v} \nabla_{\vec{w}_j} \cdot (\Psi_j \cdot \mu^N)(x)\right) dx$$

$$\Rightarrow \partial_t \rho = \sum_{j=1}^{N_v} \nabla_{\vec{w}_j} \cdot (\Psi_j \rho)$$

$\rho : \mathbb{R}^2 \to \mathbb{R}_+$

$$\nabla_{\vec{v}_2} \cdot (\rho \Psi_2) = (\nabla_{\vec{v}_2} \cdot \Psi_2)\rho + \Psi_2 \cdot \nabla_{\vec{v}_2}\rho$$

$$\nabla_{\vec{v}_2} \cdot \Psi_2^{(a)} = \frac{d}{dv_2^x}\left[(I_c^1 - I_d) \cdot \left(-\frac{1}{E_1^2}(v_2^y - v_1^y)\right)\right] + \frac{d}{dv_2^y}\left[(I_c^1 - I_d) \cdot \left(-\frac{1}{E_1^2}(v_1^x - v_2^x)\right)\right]$$

$\partial_{v_2^x}[I_c^1 - I_d] = \partial_{v_2^x}[I_c^1] = \underline{-\frac{1}{E_1^2}(v_2^y - v_1^y)}$, $\partial_{v_2^y}(I_c^1 - I_d) = \underline{-\frac{1}{E_1^2}(v_1^x - v_2^x)}$

$\partial_{v_2^x}\left[(E_1)^{-2}(v_2^y - v_1^y)\right] = (v_2^y - v_1^y)(-2E_1^{-3} \cdot \partial_{v_2^x} E_1) = (v_2^y - v_1^y)\left(-2E_1^{-3} \cdot \frac{1}{2E_1} \cdot \partial_{v_2^x}\left[(v_1^x - v_2^x)^2\right]\right)$

$\partial_{v_2^y}$ $(v_1^x - v_2^x)$ $\partial_{v_2^y}\left[(v_1^y - v_2^y)^2\right]$

$= (v_2^y - v_1^y)\left(-2E_1^{-3}\frac{1}{2E_1}\cdot\left(-2(v_1^x - v_2^x)\right)\right)$ $(v_1^y - v_2^y)$

$= (v_2^y - v_1^y)\left(\frac{2}{E_1^4}(v_1^x - v_2^x)\right)$ $(v_2^y - v_1^y)$

$= \underline{\frac{2}{E_1^4}(v_1^x - v_2^x)(v_2^y - v_1^y)}$ $(v_2^y - v_1^y)$

|| lol

$\partial_{v_2^y}\left[(E_1)^{-2}(v_1^x - v_2^x)\right] = \underline{\frac{2}{E_1^4}(v_2^y - v_1^y)(v_1^x - v_2^x)}$

$$\nabla_{\vec{v}_2} \Psi_2^{(a)}(C) = \left(-\frac{1}{E_1^2}(v_2^y - v_1^y)\right)^2 - (I_c^1 - I_d)\frac{2}{E_1^4}(v_1^x - v_2^x)(v_2^y - v_1^y)$$

$$+ \left(-\frac{1}{E_1^2}(v_1^x - v_2^x)\right)^2 - (I_c^1 - I^d)\frac{2}{E_1^4}(v_1^x - v_2^x)(v_2^y - v_1^y)$$

$$= \frac{1}{E_1^4}\Big[\underbrace{(v_2^y - v_1^y)^2 + (v_1^x - v_2^x)^2}_{= E_1^2} - 4(I_c^1 - I_d)(v_1^x - v_2^x)(v_2^y - v_1^y)\Big]$$

$$= \frac{1}{E_1^2} - \frac{4}{E_1^4}(I_c^1 - I_d)(v_1^x - v_2^x)(v_2^y - v_1^y)$$

$$\nabla_{\vec v_2}\cdot \Psi_2^{(c)} = \frac{d}{dv_2^x}\Big[(I_c^3 - I_d)\cdot\Big(\frac{1}{E_2^2}(v_2^y - v_3^y)\Big)\Big] + \frac{d}{dv_2^y}\Big[(I_c^3 - I_d)\cdot\Big(\frac{1}{E_2^2}(v_3^x - v_2^x)\Big)\Big]$$

$$\partial_{v_2^x}[I_c^3 - I_d] = \partial_{v_2^x}[I_c^3] = -\frac{1}{E_2^2}(v_2^y - v_3^y), \quad \partial_{v_2^y}(I_c^3 - I_d) = -\frac{1}{E_2^2}(v_3^x - v_2^x)$$

$$\partial_{v_2^x}\big[(E_2)^{-2}(v_2^y - v_3^y)\big] = (v_2^y - v_3^y)(-2E_2^{-3}\cdot\partial_{v_2^x}E_2) = (v_2^y - v_3^y)\Big(-2E_2^{-3}\cdot\frac{1}{2E_2}\cdot\partial_{v_2^x}\big[(v_3^x - v_2^x)^2\big]\Big)$$

$\partial_{v_2^y}$ $(v_1^x - v_2^x)$ $\partial_{v_2^y}((v_1^y - v_2^y)^2)$

$$= (v_2^y - v_3^y)\Big(-2E_2^{-3}\frac{1}{2E_2}\cdot(-2(v_3^x - v_2^x))\Big)$$

$(v_1^y - v_2^y)$

$$= (v_2^y - v_3^y)\Big(\frac{2}{E_2^4}(v_3^x - v_2^x)\Big)$$

$(v_2^y - v_1^y)$

$$= \frac{2}{E_2^4}(v_3^x - v_2^x)(v_2^y - v_3^y)$$

$(v_2^y - v_1^y)$

$$\partial_{v_2^y}\big[(E_1)^{-2}(v_1^x - v_2^x)\big] = \frac{2}{E_2^4}(v_2^y - v_3^y)(v_3^x - v_2^x)$$

$$\nabla_{\vec v_2}\cdot\Psi_2^{(c)}(C) = \frac{1}{E_2^2} + \frac{4}{E_2^4}(I_c^3 - I_d)(v_3^x - v_2^x)(v_2^y - v_3^y)$$

$$\nabla_{\vec v_j} I_2(C) = (I_c^{j-1} - I_d)\Big(-\frac{1}{\|\vec v_j - \vec v_{j-1}\|^2}\begin{pmatrix} v_j^y - v_{j-1}^y \\ v_{j-1}^x - v_j^x\end{pmatrix}\Big) = \Psi_j^{(a)}$$

$$+ (I_c^j - I_d)\Big(\frac{1}{\|\vec v_{j-1} - \vec v_j\|^2}\begin{pmatrix} v_{j-1}^y - v_j^y \\ v_j^x - v_{j-1}^x\end{pmatrix} - \frac{1}{\|\vec v_{j+1} - \vec v_j\|^2}\begin{pmatrix} v_{j+1}^y - v_j^y \\ v_j^x - v_{j+1}^x\end{pmatrix}\Big) = \Psi_j^{(b)}$$

$$+ (I_c^{j+1} - I_d)\Big(\frac{1}{\|\vec v_j - \vec v_{j+1}\|^2}\begin{pmatrix} v_j^y - v_{j+1}^y \\ v_{j+1}^x - v_j^x\end{pmatrix}\Big) = \Psi_j^{(c)}$$

$$=: \Psi_j(\vec v_1, \dots, \vec v_z) = \Psi_j^{(a)} + \Psi_j^{(b)} + \Psi_j^{(c)}$$

$$\nabla_{\vec v_2}\cdot\Psi_2^{(b)}(C) = \nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\Big(\frac{1}{E_1^2}\begin{pmatrix} v_1^y - v_2^y \\ v_2^x - v_1^x\end{pmatrix} - \frac{1}{E_2^2}\begin{pmatrix} v_3^y - v_2^y \\ v_2^x - v_3^x\end{pmatrix}\Big)\Big]$$

$$= \nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\underbrace{\frac{1}{E_1^2}\begin{pmatrix} v_1^y - v_2^y \\ v_2^x - v_1^x\end{pmatrix}}_{\nearrow \psi^{(a)}}\Big] - \nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\underbrace{\frac{1}{E_2^2}\begin{pmatrix} v_3^y - v_2^y \\ v_2^x - v_3^x\end{pmatrix}}_{\to -\psi^{(c)}}\Big]$$

$$\nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\frac{1}{E_1^2}\begin{pmatrix} v_1^y - v_2^y \\ v_2^x - v_1^x\end{pmatrix}\Big] = \partial_{v_2^x}\Big[(I_c^2 - I_d)\frac{1}{E_1^2}(v_1^y - v_2^y)\Big] + \partial_{v_2^y}\Big[(I_c^2 - I_d)\frac{1}{E_1^2}(v_2^x - v_1^x)\Big]$$

$$\partial_{v_2^x}\big[(E_1)^{-2}(v_2^y - v_1^y)\big] = (v_2^y - v_1^y)(-2E_1^{-3}\cdot\partial_{v_2^x}E_1) = (v_2^y - v_1^y)\Big(-2E_1^{-3}\cdot\frac{1}{2E_1}\cdot\partial_{v_2^x}\big[(v_1^x - v_2^x)^2\big]\Big)$$

$\partial_{v_2^y}$ $(v_1^x - v_2^x)$ $(v_1^y - v_2^y)$ $\partial_{v_2^y}((v_1^y - v_2^y)^2]$

$$= (v_2^y - v_1^y)\Big(-2E_1^{-3}\frac{1}{2E_1}\cdot(-2(v_1^x - v_2^x))\Big)$$

$(v_2^y - v_1^y)$

$$= (v_2^y - v_1^y)\Big(\frac{2}{E_1^4}(v_1^x - v_2^x)\Big)$$

$(v_2^y - v_1^y)$

$$= \frac{2}{E_1^4}(v_1^x - v_2^x)(v_2^y - v_1^y)$$

$\parallel$ lol

$$\partial_{v_2^y}\big[(E_1)^{-2}(v_1^x - v_2^x)\big] = \frac{2}{E_1^4}(v_2^y - v_1^y)(v_1^x - v_2^x)$$

$$\nabla_{\vec v_2} I_c^2 = \frac{1}{E_1^2}\begin{pmatrix} v_1^y - v_2^y \\ v_2^x - v_1^x\end{pmatrix} - \frac{1}{E_2^2}\begin{pmatrix} v_3^y - v_2^y \\ v_2^x - v_3^x\end{pmatrix}$$

$$\partial_{v_2^x}\Big[(I_c^2 - I_d)\frac{1}{E_1^2}(v_1^y - v_2^y)\Big] = \Big(\frac{1}{E_1^2}(v_1^y - v_2^y) - \frac{1}{E_2^2}(v_3^y - v_2^y)\Big)\frac{1}{E_1^2}(v_1^y - v_2^y)$$

$$+ (I_c^2 - I_d)\Big(-\frac{2}{E_1^4}(v_1^x - v_2^x)(v_2^y - v_1^y)\Big)$$

$$= \frac{1}{E_1^4}(v_1^y - v_2^y)^2 - \frac{1}{E_1^2E_2^2}(v_1^y - v_2^y)(v_3^y - v_2^y) - (I_c^2 - I_d)\Big(\frac{2}{E_1^4}(v_1^x - v_2^x)(v_2^y - v_1^y)\Big)$$

$$\partial_{v_2^y}\Big[(I_c^2 - I_d)\frac{1}{E_1^2}(v_2^x - v_1^x)\Big] = \Big(\frac{1}{E_1^2}(v_2^x - v_1^x) - \frac{1}{E_2^2}(v_2^x - v_3^x)\Big)\frac{1}{E_1^2}(v_2^x - v_1^x)$$

$$+ (I_c^2 - I_d)\Big(-\frac{2}{E_1^4}(v_2^y - v_1^y)(v_1^x - v_2^x)\Big)$$

$$= \frac{1}{E_1^4}(v_2^x - v_1^x)^2 - \frac{1}{E_1^2E_2^2}(v_2^x - v_1^x)(v_2^x - v_3^x) - (I_c^2 - I_d)\Big(\frac{2}{E_1^4}(v_2^y - v_1^y)(v_1^x - v_2^x)\Big)$$

$$\nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\frac{1}{E_1^2}\begin{pmatrix} v_1^y - v_2^y \\ v_2^x - v_1^x\end{pmatrix}\Big] = \frac{1}{E_1^4}(v_1^y - v_2^y)^2 - \frac{1}{E_1^2E_2^2}(v_1^y - v_2^y)(v_3^y - v_2^y) - (I_c^2 - I_d)\Big(\frac{2}{E_1^4}(v_1^x - v_2^x)(v_2^y - v_1^y)\Big)$$

$$+ \frac{1}{E_1^4}(v_2^x - v_1^x)^2 - \frac{1}{E_1^2E_2^2}(v_2^x - v_1^x)(v_2^x - v_3^x) - (I_c^2 - I_d)\Big(\frac{2}{E_1^4}(v_2^y - v_1^y)(v_1^x - v_2^x)\Big)$$

$$= \frac{1}{E_1^4}\Big[\underbrace{(v_1^x - v_2^x)^2 + (v_1^y - v_2^y)^2}_{= E_1^2}\Big] - \frac{1}{E_1^2E_2^2}\Big[(v_1^y - v_2^y)(v_3^y - v_2^y) + (v_2^x - v_1^x)(v_2^x - v_3^x)\Big]$$

$$- (I_c^2 - I_d)\Big(\frac{2}{E_1^4}\Big)\Big[\underline{(v_1^x - v_2^x)(v_2^y - v_1^y)} + \underline{(v_2^y - v_1^y)(v_1^x - v_2^x)}\Big]$$

$$= \frac{1}{E_1^2} - \frac{1}{E_1^2E_2^2}\Big[(v_1^y - v_2^y)(v_3^y - v_2^y) + (v_2^x - v_1^x)(v_2^x - v_3^x)\Big] - (I_c^2 - I_d)\Big(\frac{4}{E_1^4}\Big)\Big[(v_1^x - v_2^x)(v_2^y - v_1^y)\Big]$$

$$\nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\frac{1}{E_2^2}\begin{pmatrix} v_3^y - v_2^y \\ v_2^x - v_3^x\end{pmatrix}\Big] = \partial_{v_2^x}\Big[(I_c^2 - I_d)\frac{1}{E_2^2}(v_3^y - v_2^y)\Big] + \partial_{v_2^y}\Big[(I_c^2 - I_d)\frac{1}{E_2^2}(v_2^x - v_3^x)\Big]$$

$\to -\psi^{(c)}$

$$\partial_{v_2^x}[I_c^3 - I_d] = \partial_{v_2^x}[I_c^3] = -\frac{1}{E_2^2}(v_2^y - v_3^y), \quad \partial_{v_2^y}(I_c^3 - I_d) = -\frac{1}{E_2^2}(v_3^x - v_2^x)$$

$$\partial_{v_2^x}\big[(E_2)^{-2}(v_2^y - v_3^y)\big] = (v_2^y - v_3^y)(-2E_2^{-3}\cdot\partial_{v_2^x}E_2) = (v_2^y - v_3^y)\Big(-2E_2^{-3}\cdot\frac{1}{2E_2}\cdot\partial_{v_2^x}\big[(v_3^x - v_2^x)^2\big]\Big)$$

$\partial_{v_2^y}$ $(v_1^x - v_2^x)$ $\partial_{v_2^y}((v_1^y - v_2^y)^2]$

$$= (v_2^y - v_3^y)\Big(-2E_2^{-3}\frac{1}{2E_2}\cdot(-2(v_3^x - v_2^x))\Big)$$

$(v_1^y - v_2^y)$

$$= (v_2^y - v_3^y)\Big(\frac{2}{E_2^4}(v_3^x - v_2^x)\Big)$$

$(v_2^y - v_1^y)$

$$= \frac{2}{E_2^4}(v_3^x - v_2^x)(v_2^y - v_3^y)$$

$(v_2^y - v_1^y)$

$$\partial_{v_2^y}\big[(E_1)^{-2}(v_1^x - v_2^x)\big] = \frac{2}{E_2^4}(v_2^y - v_3^y)(v_3^x - v_2^x)$$

$$\partial_{v_2^x}\Big[(I_c^2 - I_d)\frac{1}{E_2^2}(v_3^y - v_2^y)\Big] = \Big(\frac{1}{E_1^2}(v_1^y - v_2^y) - \frac{1}{E_2^2}(v_3^y - v_2^y)\Big)\frac{1}{E_2^2}(v_3^y - v_2^y)$$

$$+ (I_c^2 - I_d)\Big(-\frac{2}{E_2^4}(v_3^x - v_2^x)(v_2^y - v_3^y)\Big)$$

$$= -\frac{1}{E_2^4}(v_3^y - v_2^y)^2 + \frac{1}{E_1^2E_2^2}(v_1^y - v_2^y)(v_3^y - v_2^y) - (I_c^2 - I_d)\Big(\frac{2}{E_2^4}(v_3^x - v_2^x)(v_2^y - v_3^y)\Big)$$

$$\partial_{v_2^y}\Big[(I_c^2 - I_d)\frac{1}{E_2^2}(v_2^x - v_3^x)\Big] = \Big(\frac{1}{E_1^2}(v_2^x - v_1^x) - \frac{1}{E_2^2}(v_2^x - v_3^x)\Big)\frac{1}{E_2^2}(v_2^x - v_3^x)$$

$$+ (I_c^2 - I_d)\Big(-\frac{2}{E_2^4}(v_2^y - v_3^y)(v_3^x - v_2^x)\Big)$$

$$= -\frac{1}{E_2^4}(v_2^x - v_3^x)^2 + \frac{1}{E_1^2E_2^2}(v_2^x - v_1^x)(v_2^x - v_3^x) - (I_c^2 - I_d)\Big(\frac{2}{E_2^4}(v_2^y - v_3^y)(v_3^x - v_2^x)\Big)$$

$$\nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\frac{1}{E_2^2}\begin{pmatrix} v_3^y - v_2^y \\ v_2^x - v_3^x\end{pmatrix}\Big] = -\frac{1}{E_2^4}(v_3^y - v_2^y)^2 + \frac{1}{E_1^2E_2^2}(v_1^y - v_2^y)(v_3^y - v_2^y) - (I_c^2 - I_d)\Big(\frac{2}{E_2^4}(v_3^x - v_2^x)(v_2^y - v_3^y)\Big)$$

$$- \frac{1}{E_2^4}(v_2^x - v_3^x)^2 + \frac{1}{E_1^2E_2^2}(v_2^x - v_1^x)(v_2^x - v_3^x) - (I_c^2 - I_d)\Big(\frac{2}{E_2^4}(v_2^y - v_3^y)(v_3^x - v_2^x)\Big)$$

$$= -\frac{1}{E_2^4}\Big[(v_2^x - v_3^x)^2 + (v_2^y - v_3^y)^2\Big] + \frac{1}{E_1^2E_2^2}\Big[(v_1^y - v_2^y)(v_3^y - v_2^y) + (v_2^x - v_1^x)(v_2^x - v_3^x)\Big]$$

$$- (I_c^2 - I_d)\Big(\frac{2}{E_2^4}\Big)\Big((v_3^x - v_2^x)(v_2^y - v_3^y) + (v_2^y - v_3^y)(v_3^x - v_2^x)\Big)$$

$$= -\frac{1}{E_2^2} + \frac{1}{E_1^2E_2^2}\Big[(v_1^y - v_2^y)(v_3^y - v_2^y) + (v_2^x - v_1^x)(v_2^x - v_3^x)\Big] - (I_c^2 - I_d)\Big(\frac{4}{E_2^4}\Big)\Big((v_3^x - v_2^x)(v_2^y - v_3^y)\Big)$$

$$\nabla_{\vec v_2}\cdot\Psi_2^{(b)}(C) = \nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\frac{1}{E_1^2}\begin{pmatrix} v_1^y - v_2^y \\ v_2^x - v_1^x\end{pmatrix}\Big] - \nabla_{\vec v_2}\cdot\Big[(I_c^2 - I_d)\frac{1}{E_2^2}\begin{pmatrix} v_3^y - v_2^y \\ v_2^x - v_3^x\end{pmatrix}\Big]$$

$$= \frac{1}{E_1^2} - \frac{1}{E_1^2E_2^2}\Big[(v_1^y - v_2^y)(v_3^y - v_2^y) + (v_2^x - v_1^x)(v_2^x - v_3^x)\Big] - (I_c^2 - I_d)\Big(\frac{4}{E_1^4}\Big)\Big[(v_1^x - v_2^x)(v_2^y - v_1^y)\Big]$$

$$- \Big[-\frac{1}{E_2^2} + \frac{1}{E_1^2E_2^2}\Big((v_1^y - v_2^y)(v_3^y - v_2^y) + (v_2^x - v_1^x)(v_2^x - v_3^x)\Big) - (I_c^2 - I_d)\Big(\frac{4}{E_2^4}\Big)\Big((v_3^x - v_2^x)(v_2^y - v_3^y)\Big)\Big]$$

$$= \frac{1}{E_1^2} + \frac{1}{E_2^2} - \frac{2}{E_1^2E_2^2}\Big[(v_1^y - v_2^y)(v_3^y - v_2^y) + (v_2^x - v_1^x)(v_2^x - v_3^x)\Big] + (I_c^2 - I_d)\Big[\Big(\frac{4}{E_2^4}\Big)(v_3^x - v_2^x)(v_2^y - v_3^y) - \Big(\frac{4}{E_1^4}\Big)(v_1^x - v_2^x)(v_2^y - v_1^y)\Big]$$

$$\nabla_{\vec{v}_2} \cdot \Psi_2(C) = \nabla_{\vec{v}_2} \cdot \Psi_2^{(a)}(C) + \nabla_{\vec{v}_2} \cdot \Psi^{(b)}(C) + \nabla_{\vec{v}_2} \cdot \Psi^{(c)}(C)$$

$$= \frac{1}{\varepsilon_1^2} - \frac{4}{\varepsilon_1^4}(I_c^1 - I_d)(v_1^x - v_2^x)(v_2^y - v_1^y)$$

$$+ \frac{1}{\varepsilon_1^2} + \frac{1}{\varepsilon_2^2} - \frac{2}{\varepsilon_1^2 \varepsilon_2^2}\Big[(v_1^y - v_2^y)(v_3^y - v_2^y) + (v_2^x - v_1^x)(v_2^x - v_3^x)\Big] + (I_c^2 - I_d)\left[\left(\frac{4}{\varepsilon_2^4}\right)(v_3^x - v_2^x)(v_2^y - v_3^y) - \left(\frac{4}{\varepsilon_1^4}\right)(v_1^x - v_2^x)(v_2^y - v_1^y)\right]$$

$$+ \frac{1}{\varepsilon_2^2} + \frac{4}{\varepsilon_2^4}(I_c^3 - I_d)(v_3^x - v_2^x)(v_2^y - v_3^y)$$

Overlap Energy

Dienstag, 15. Juli 2025 08:16

Computation for Overlap Force at $N_v > 2$ and $k=1$

- Area of a cell: $A_C = \frac{1}{2} \sum_{j=1}^{N_v} (v_j^x v_{j+1}^y - v_{j+1}^x v_j^y)$
- Area energy: $A_2(C) = \frac{1}{2} | A_c - A_d |^2$
- Area gradient: $\nabla_{\vec{v}_j} A_2(C) = \frac{1}{2} (A_c - A_d) \begin{pmatrix} v_{j+1}^y - v_{j-1}^y \\ v_{j-1}^x - v_{j+1}^x \end{pmatrix}$

- Overlap energy: $O_1(\vec{C}) = \sum_{i=1}^{N_c} \left(\sum_{m=i+1}^{N_c} \left(\sum_{D \in \Omega_{i,m}} \frac{1}{k} A_D \right) \right)$

  where $\Omega_{i,m} = \{ D \subset \mathbb{R}^2 \mid D$ is overlap between cells i and $m \}$
- Overlap gradient: $\nabla_{\vec{v}_j^i} O_1(\vec{C}) = \sum_{m=1, m \neq i}^{N_c} \left(\sum_{D \in \Omega_{im}} \delta_{\vec{v}_j^i}(D) \; \frac{1}{2} \begin{pmatrix} d_{j+1}^y - d_{j-1}^y \\ d_{j-1}^x - d_{j+1}^x \end{pmatrix} \right) =: \Psi_j(\vec{v}_1, \dots, \vec{v}_{N_v})$

where $\vec{d}_{j+1}, \vec{d}_{j-1}$ are the neighboring vertices of $\vec{v}_j^i$ in the overlap cell.

$$\mu^{N_c}(A) = \frac{1}{N_c} \sum_{i=1}^{N_c} \delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(A) \; , \qquad \int_{\mathbb{R}^{2 \cdot N_v}} \phi(x) \, d\delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(x) = \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)$$

Let $\phi \in C_c^\infty(\mathbb{R}^{2 \cdot N_v}, \mathbb{R})$. $\phi(\vec{w}_1, \dots, \vec{w}_{N_v})$

$$\begin{aligned}
\frac{d}{dt} \int \phi \; d\mu^N &= \frac{d}{dt} \frac{1}{N_c} \sum_{i=1}^{N_c} \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i) \\
&= \sum_{j=1}^{N_v} \frac{1}{N_c} \sum_{i=1}^{N_c} \nabla_{\vec{v}_j^i} \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i) \cdot \frac{d\vec{v}_j^i}{dt} \\
&= - \sum_{j=1}^{N_v} \frac{1}{N_c} \sum_{i=1}^{N_c} \nabla_{\vec{v}_j^i} \phi(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i) \cdot \Psi_j(\vec{v}_{j-1}, \vec{v}_j, \vec{v}_{j+1}) \\
&\qquad \text{Plug in all } (\vec{v}_j^i)_{j=1}^{N_v} \\
&= - \int \sum_{j=1}^{N_v} \frac{1}{N_c} \sum_{i=1}^{N_c} \nabla_{\vec{w}_j} \phi(x) \cdot \Psi_j(x) \; d\delta_{(\vec{v}_1^i, \dots, \vec{v}_{N_v}^i)}(x) \\
&= - \int \sum_{j=1}^{N_v} \nabla_{\vec{w}_j} \phi(x) \cdot \Psi_j(x) \; d\mu^N(x) \\
&= \int \phi(x) \cdot \left(\sum_{j=1}^{N_v} \nabla_{\vec{w}_j} \cdot (\Psi_j \cdot \mu^N)(x) \right) dx
\end{aligned}$$

$$\Rightarrow \partial_t \rho = \sum_{j=1}^{N_c} \nabla_{\vec{w}_j} \cdot (\psi_j \rho) \qquad \rho : \mathbb{R}^2 \to \mathbb{R}_+$$

$$\nabla_{\vec{v}_2} \cdot \left(\rho \, \psi_2 \right) = (\nabla_{\vec{v}_2} \cdot \psi_2) \rho + \psi_2 \cdot \nabla_{\vec{v}_2} \rho$$

$$\nabla_{\vec{v}_2^i} \cdot \psi_2 = \nabla_{\vec{v}_2^i} \cdot \sum_{m=1, m\neq i}^{N_c} \left(\sum_{D \in \Omega_{im}} \sigma_{\vec{v}_j^i}(D) \ \frac{1}{2} \begin{pmatrix} d_{j+1}^y - d_{j-1}^y \\ d_{j-1}^x - d_{j+1}^x \end{pmatrix} \right)$$

$$= \partial_{v_2^{x,i}} \left[\sum_{m=1, m\neq i}^{N_c} \left(\sum_{D \in \Omega_{im}} \sigma_{\vec{v}_j^i}(D) \ \frac{1}{2} \left(d_{j+1}^y - d_{j-1}^y \right) \right) \right]$$

$$+ \ \partial_{v_2^{y,i}} \left[\sum_{m=1, m\neq i}^{N_c} \left(\sum_{D \in \Omega_{im}} \sigma_{\vec{v}_j^i}(D) \ \frac{1}{2} \left(d_{j-1}^x - d_{j+1}^x \right) \right) \right]$$

Assume: Any small directional change of $\vec{v}_j$ does not create a new overlap and all old overlaps don't resolve.

$$\Rightarrow \nabla_{\vec{v}_j} \sigma_{\vec{v}_j}(D) = 0,$$

$$\nabla_{\vec{v}_j} \sigma_{\vec{v}_j^i}(D) \ \frac{1}{2} \left(d_{j-1}^x - d_{j+1}^x \right) = 0$$

If changing $\vec{v}_j$ causes a change of the set $\Omega_{i,m}$ for any m:
($\Leftrightarrow$ an old overlap disappears or a new overlap arises)

$\Rightarrow \nabla_{\vec{v}_j} \ldots \sum_{D \in \Omega_{im}} \ldots$ produces jumps
which means that the gradient is discontinuous
and the divergence is not defined in a classical sense.

How can we find a work around?