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A Lookahead Matheuristic for the Unweighed Variable-Sized Two-dimensional Bin Packing Problem --Manuscript Draft--

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Abstract:	<p>The unweighed oriented variable-sized two-dimensional guillotine bin packing problem consists in packing without overlap small rectangular items into large non-identical rectangular bins, with the items obtained via guillotine cuts. It minimizes the waste of the used bins. It is herein approximately solved using a hybrid matheuristic that applies a sequence of low-level mixed-integer programs that reserve space for unpacked items and that are guided by feasibility constraints and upper bounds on the objective function. The embedded constraints constitute a lookahead mechanism that prohibits the investigation of infeasible directions and constrains the search to improving ones. The matheuristic further employs high-level diversification and intensification mechanisms. The diversification incorporates a sequential value correction algorithm that tags a pseudo-price to each item to govern the fitness functions of mixed integer programs and subsequently their solution construction process. The intensification is a local search that investigates the neighbourhood of promising solutions. The extensive computational experiments provide evidence of the good performance of the proposed matheuristic. For the variable-sized bin packing benchmark instances, the matheuristic matches and improves 90.8% of the upper bounds. For the single bin-size bin packing benchmark instances, the matheuristic further proves the optimality of 82.6% upper bounds while it matches 14.4% and improves 2.6% existing bounds.</p>
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March 21, 2021

Prof. Jose Oliveira
Editor-in-Chief, European Journal of Operational Research

Dear Prof. Oliveira,

Please find enclosed a manuscript entitled

“A Lookahead Matheuristic for the Unweighed Variable-Sized Two-dimensional Bin Packing Problem”.

I am submitting this manuscript on my behalf and that of Dr. S. Polyakovskiy for possible publication in the European Journal of Operational Research.

Looking forward to your reply,

Sincerely,

A handwritten signature in black ink, appearing to read "Rym M'Hallah".

Rym M'Hallah, Ph.D.

Highlights

for

A Lookahead Matheuristic for the Unweighed Variable-Sized Two-dimensional Bin Packing Problem

- Small oriented rectangles cut from large non-identical rectangles via guillotine cuts
- A matheuristic that uses lookahead mechanisms minimizes the area of used bins
- Mixed-integer programs reserve space for unpacked items
- The search is guided by feasibility constraints and its objective function value is upper bounded
- Matheuristic matches/improves 90.8% of existing upper bounds

A Lookahead Matheuristic for the Unweighed Variable-Sized Two-dimensional Bin Packing Problem

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Abstract

The unweighed oriented variable-sized two-dimensional guillotine bin packing problem consists in packing without overlap small rectangular items into large non-identical rectangular bins, with the items obtained via guillotine cuts. It minimizes the waste of the used bins. It is herein approximately solved using a hybrid matheuristic that applies a sequence of low-level mixed-integer programs that reserve space for unpacked items and that are guided by feasibility constraints and upper bounds on the objective function. The embedded constraints constitute a lookahead mechanism that prohibits the investigation of infeasible directions and constrains the search to improving ones. The matheuristic further employs high-level diversification and intensification mechanisms. The diversification incorporates a sequential value correction algorithm that tags a pseudo-price to each item to govern the fitness functions of mixed integer programs and subsequently their solution construction process. The intensification is a local search that investigates the neighbourhood of promising solutions. The extensive computational experiments provide evidence of the good performance of the proposed matheuristic. For the variable-sized bin packing benchmark instances, the matheuristic matches and improves 90.8% of the upper bounds. For the single bin-size bin packing benchmark instances, the matheuristic further proves the optimality of 82.6% upper bounds while it matches 14.4% and improves 2.6% existing bounds.

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1. Introduction

This paper addresses the unweighed oriented two-dimensional guillotine bin packing problem with variable bin sizes (VS2BP). This problem consists in

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packing without overlap a set $I = \{1, \dots, n\}$ of n distinct rectangular items into large non-identical rectangular bins with the items obtained via guillotine edge to edge cuts. An item $i \in I$ of size $a_i = l_i w_i$ is characterised by its length $l_i \in \mathbb{N}_{>0}$ and its width $w_i \in \mathbb{N}_{>0}$, where $\mathbb{N}_{>0}$ is the set of strictly positive integers. There are m types of bins. Each bin type $t \in T$, $T = \{1, \dots, m\}$, is specified by its length $L_t \in \mathbb{N}_{>0}$, weight $W_t \in \mathbb{N}_{>0}$, and either its size $A_t = L_t W_t$ or its cost that is proportional to its size. There are at least n bins of type $t \in T$. Any item $i \in I$ can fit in at least one bin type, and any bin type $t \in T$ can fit at least one item. The problem searches for a feasible packing that minimizes the total packing cost of the items. This total cost is simply the total area of the bins selected to pack the items. Equivalently, the problem maximizes the packing density, known also as the utilization of the bin, defined as the ratio of the sum of the area of the items to the sum of the area of the used bins.

The unweighed guillotine VS2BP is academically challenging. It is an extension of the one-dimensional variant, which is in turn strongly NP-hard [8]. Thus, it is unlikely that any efficient algorithm solves medium or large-sized instances of VS2BP to optimality in a reasonable time. Yet, many problems -including scheduling, cutting, and maintenance- are equivalent to VS2BP [9, 28] or contain VS2BP as a subproblem. For example, glass cutting becomes a VS2BP in the presence of defects [14, 22]. In addition to its academic pertinence, VS2BP arises in numerous industrial cases such as biscuit, wood, plastic, metal and paper manufacturing, newspaper paging, telecommunication, shipping and transportation.

The literature on VS2BP is rather limited. For the non-guillotine variant of the problem, Pisinger and Sigurd [23] adapted a column generation approach, and opted for an exact branch-and-price algorithm. Liu et al. [15] developed two dynamic programming-based heuristics that aggregate the state space to limit the storage requirements. Alvarez-Valdes et al. [1] presented a GRASP algorithm for the (un)weighed two- and three-dimensional multiple bin-size bin packing problems. They then applied path relinking to combine the best solutions obtained in the iterative process. They also proposed two simple lower bounds that improved those of Pisinger and Sigurd [23]. Wei et al. [27] proposed a goal-driven approach.

Ortmann et al. [21] considered the unweighed oriented guillotine version of VS2BP. They developed a two-stage heuristic that packs items into strips and strips into large bins. It then tries to reposition items from larger bins into smaller ones with the objective of increasing the filling ratio. Their results were improved by Hong et al. [12] who also considered the oriented case of VS2BP. They dotted a mixed bin packing heuristic that fills single bins with a backtracking capability. They then hybridized it with an iterative simulated annealing and binary search. Their method is hereafter denoted as SA. To the best of the authors' knowledge, Ortmann et al. [21] and Hong et al. [12] are the only references dealing with the same problem investigated in this paper.

VS2BP has a huge solution space with many alternative solutions whose packing configurations are symmetric. Consequently, exact approaches generally fail to solve its medium and large sized instances unless they employ symmetry

breaking constraints. In the presence of such constraints, exact approaches such as mixed-integer programming (MIP), constraint programming (CP) and decomposition techniques (e.g. column generation and Benders decomposition) may find optima for instances with up to 100 items. However, they may neither find the optima for instances with as few as 20 items nor guarantee a sufficiently good convergence to a global optimum.

VS2BP is an extension of the classical two-dimensional bin packing problem (2BP) [16]. Yet, the difficulty of 2BP restricted solution approaches to approximate ones that varied from simple best/first fit algorithms [6] to meta-heuristics [16] to hyper-heuristics [18] to multi-agent based algorithms [25] to a combination of CP and iterative packing heuristics [24]. Obviously, any heuristic that solves VS2BP solves 2BP, but not vice versa. In fact, VS2BP is harder than 2BP. Its larger number of variables and of constraints augment the problem's complexity. Subsequently, fewer VS2BP instances can be solved to optimality, and larger instances are much harder to solve than smaller ones. Thus, it is important to design an effective approximate approach for VS2BP. For this purpose, this paper proposes a matheuristic.

Matheuristics hybridize exact solution techniques with meta-heuristics. Their wider spectrum of application is due not only to the advance of mathematical solvers and computing technologies [20] but also to their success in identifying (near-) global optima in reduced run times [3]. In general, they limit the role of their mathematical programming component to iteratively solving a relaxed model or to improving an incumbent or to tackling one of the many aspects of the problem. This limitation is mainly imposed by the techniques employed by off-the-shelf solvers –such as branch and bound–. Such techniques explore the neighborhood of solutions obtained by the linear programming relaxation [14]. Thus, they tend to obtain integer solutions at leaf nodes; an aspect that makes them very time and storage consuming. Even in the presence of strong bounds / pruning rules, off-the-shelf solvers may not prove optimality or converge to a global optimum on large/difficult instances [14].

The proposed matheuristic (MH) avoids some of the limitations of MIP solvers in general and of the model of VS2BP in particular. It obtains partial solutions by iteratively pre-assigning some of the items, locks them, bounds the value of the objective function, augments the model with feasibility constraints, and complements solutions using a different mathematical program. It then explores the neighborhood of feasible integer solutions while restricting the search to improving directions. It iteratively imposes stricter upper bounds on the objective function value. That is, MH embeds its mathematical models within a heuristic that uses a proximity search [26] like concept. Yet, MH differs from the matheuristic of Kollsker and Malaguti [13] for the LEGO two-dimensional construction problem. Their matheuristic uses a constructive heuristic to pack large elements into regular patterns and the mathematical models to fit the small parts and ensure the stability of the solution. That is, as in many of the available matheuristics, they feed an initial solution to a modified simpler version of their mathematical model to address a component of their problem.

Because solving an instance of VS2BP is not trivial (i.e., every item has a

large number of alternative positions within the bins, and many solutions have equal costs), MH enhances the chances of converging to a good-quality feasible solution to VS2BP within a reasonable time by implementing three strategies to its packing procedures. The **first** strategy reserves a free space for all unpacked items while it explores all available free regions and alternative cuts to extract the packed items. This feature distinguishes the proposed packing procedures from most existing heuristics, which pack items sequentially into bins [16]. Even though it enlarges the search space and the size of the MIP, reserving free space dots the packing procedure with a futuristic vision. It makes the packing procedure consider only potentially feasible packing alternatives; thus, makes the packing procedure less greedy. In addition, it decreases the number of iterations necessary to pack all items at the cost of a slightly higher computational effort per cycle. The **second** strategy augments the basic MIP model with a set of feasibility constraints that prohibit partial (feasible) solutions that would lead to infeasible solutions in future cycles. These constraints constitute a lookahead mechanism that directs the search towards a feasible packing and enables current decisions to account for their impact on future ones. It prunes infeasible parts of the search space. The **third** strategy imposes an upper bound on the total packing cost. It forbids the use of new bins when the sum of their costs and the cost of already used bins exceeds the bound. Along with the second strategy, it limits the number of candidate solutions. In summary, MH uses a fast constructive heuristic and limits the MIP search space by (i) using bounds on the objective function value; (ii) using regions to pack items thus limiting the choices of positions for every item and reducing the symmetry embedded in the problem, and (iii) by reinforcing improving search directions.

This paper is organized as follows. Section 2 presents the three guillotine MIP-based packing procedures of MH. Section 3 details the lower and upper levels of MH detailing how the three packing procedures are used. Section 4 describes the computational results. Finally, Section 5 concludes the paper and presents future directions of research.

2. Guillotine MIP-based packing procedures

Section 2.1 describes the feasibility constraints used as a lookahead mechanism within MIP whereas Sections 2.2-2.4 detail the principles of three MIP-based guillotine packing procedures that approximately solve VS2BP.

2.1. Feasibility Constraints

In multi-dimensional bin packing, dual feasible functions (DFFs) are known to yield tight lower bounds on the minimal number of bins required to orthogonally pack a given set of oriented items [2]. When applied to a single region (which may be a bin), DFFs can detect the infeasibility of packing the items into that region. When DFFs detect an infeasible assignment, MH rules out the packing of this subset into this region from further examination.

A DFF is a function $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\sum_{q \in Q} q \leq 1 \Rightarrow \sum_{q \in Q} \varphi(q) \leq 1$ holds for any set $Q \subset \mathbb{R}_{\geq 0}$ of numbers. Let (φ_a, φ_b) be a pair of two DFFs.



Then $(\varphi_a(l_i/L), \varphi_b(w_i/W)) \in (0, 1]^2$ represents the transformed dimensions of item $i \in I$ calculated from its original dimensions (l_i, w_i) . A feasible packing into a single bin (L, W) exists if and only if the sum of the areas of the modified items does not exceed 1; that is, when the following constraint holds

$$\sum_{i \in I} \varphi_a\left(\frac{l_i}{L}\right) \varphi_b\left(\frac{w_i}{W}\right) \leq 1. \quad (1)$$

Various combinations of DFFs may yield different modified items. Let F_k be a set of distinct combinations of two DFFs, φ_a and φ_b , $k \in K$ a target region (or a bin) for packing of item $i \in I$, and K a set of regions. Then $\lambda_{ik}^f \in [0, 1]$ is a transformed area of item i obtained via the combination $f \in F_k$ for bin k . Applying the set F_k , one can derive a set of valid feasibility inequality constraints from Eq. (1). A feasibility constraint has the form

$$\sum_{i \in I} \lambda_{ik}^f x_{ik} \leq 1, \quad f \in F_k, k \in K, \quad (2)$$

where the binary decision variable $x_{ik} = 1$ if item i is assigned to region k and 0 otherwise.

Some of the $|F_k|$ constraints of inequality (2) may be redundant. A constraint $f, f \in F_k$, is redundant if either $\sum_{i \in I} \lambda_{ik}^f \leq 1$ or there exists $f', f' \in F_k, f \neq f'$, such that $\lambda_{ik}^f \leq \lambda_{ik}^{f'}$ for all $i \in I$. MH ignores redundant constraints. In fact, to maintain a reasonable computation time, it only augments its mathematical models with the strongest f_{max} non-redundant constraints for each region $k \in K$; i.e., with those that maximize the values of $\sum_{i \in I} \lambda_{ik}^f, f \in F_k, |F_k| \leq f_{max}$. To build the set F_k , MH applies the step functions proposed by Fekete and Schepers [11] for 2BP, in particular $u^{(1)}, U^{(\epsilon)}$, and $\phi^{(\epsilon)}$.

MH does not enumerate all possible combinations of DFFs for each region that the algorithm may encounter during the search. Despite its constant run time, frequently computing λ_{ik}^f for a given region $k \in K$ and item $i \in I$ increases the overall computational time considerably. To reduce its runtime, MH partitions the two-dimensional space (\bar{L}, \bar{W}) into $\mu \times \mu$ zones, where each zone is associated with a pair of indices (μ_l, μ_w) , where μ is an integer number of partitions of $\bar{L} = \max_{t \in T} \{L_t\}$ and $\bar{W} = \max_{t \in T} \{W_t\}$ such that $1 \leq \mu \leq \min\{\bar{L}, \bar{W}\}$. Based on this partition, zone (μ_l, μ_w) , $\mu_l = 0, \dots, \mu - 1$ and $\mu_w = 0, \dots, \mu - 1$, corresponds to a subset $K_{\mu_l, \mu_w} \in K$ of regions whose dimensions fall in the range of (μ_l, μ_w) ; i.e.,

$$K_{\mu_l, \mu_w} = \left\{ k \in K : \left\lfloor \frac{\mu_l \bar{L}}{\mu} \right\rfloor < L_k \leq \left\lfloor \frac{(\mu_l + 1) \bar{L}}{\mu} \right\rfloor, \left\lfloor \frac{\mu_w \bar{W}}{\mu} \right\rfloor < W_k \leq \left\lfloor \frac{(\mu_w + 1) \bar{W}}{\mu} \right\rfloor \right\}.$$

Regions $k \in K$ with $L_k > \left\lfloor \frac{(\mu-1) \bar{L}}{\mu} \right\rfloor$ belong to a zone with $\mu_l = \mu - 1$ whereas those with $W_k > \left\lfloor \frac{(\mu-1) \bar{W}}{\mu} \right\rfloor$ are in a zone with $\mu_w = \mu - 1$. Subsequently, MH affiliates the regions of set K with the zones according to the size. It considers



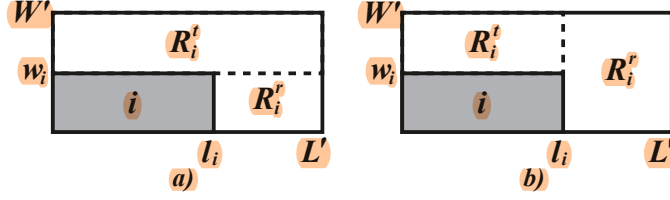


Figure 1: Illustrating the two cutting patterns resulting from a different sequence of cuts. The first cut is horizontal in pattern α (on the left) and vertical in pattern β (on the right).

each zone (μ_l, μ_w) as a potential region, and uses the upper bounds on this zone's dimensions as the input for DFFs. In this way, MH computes up to f_{max} best combinations of functions for items that fit in this zone (μ_l, μ_w) . During the search, MH builds F_k for region k using the combinations of DFFs of the zone that matches the dimensions of k .

2.2. The MAXSUM packing procedure

The first packing procedure, labeled MAXSUM, is iterative. Its iterations stop when it identifies either a feasible packing of the n items or an infeasible direction of the search. At each iteration, it feeds an MIP with a set \bar{I} of already packed items and their current cost of packing \bar{U} (i.e. the total size of the bins used in the partial solution), a set I' of not-yet-packed items such that $\bar{I} \cup I' = I$ and $\bar{I} \cap I' = \emptyset$, an upper bound U^* on the objective function cost, and three sets R , S , and T of available regions. Whereas T is the set of new bins, the sets R and S consist respectively of paired and simple active regions issued from the packing of the items of \bar{I} . They are dynamic sets obtained as explained below and updated in each call of MIP.

In guillotine packing, an item $i \in I$ can be obtained from a new bin of type $t \in T$ or from a two-dimensional region (L', W') , $L' \geq l_i$, $W' \geq w_i$, within a bin via a sequence of horizontal and vertical cuts.

- When i is the result of the first cut made **horizontally**, the region (L', W') is split -as shown in Figure 1.a- into two new regions: $R_i^{\alpha t}$ of size $(L', W' - w_i)$ above i and $R_i^{\alpha r}$ of size $(L' - l_i, w_i)$ to its right. In this case, i is the product of an α cutting pattern.
- When i is the result of the first cut made **vertically**, the region (L', W') is cut -as depicted in Figure 1.b- into two new regions: $R_i^{\beta t}$ of size $(l_i, W' - w_i)$ at the top of i and $R_i^{\beta r}$ of size $(L' - l_i, W')$ at the right of i . In this case, i is the product of a β cutting pattern.

That is, the first cut applied to item i exclusively determines the pair of generated active regions: Either $(R_i^{\alpha t}, R_i^{\alpha r})$ or $(R_i^{\beta t}, R_i^{\beta r})$. Regardless, the selected pair has one region above i and one to the right of i , with either or both being possibly of zero size. Let

- $R^{\bullet t} = \cup_{i \in \bar{I}} \{R_i^{\bullet t}\}$ (resp. $R^{\bullet r} = \cup_{i \in \bar{I}} \{R_i^{\bullet r}\}$) be a set of regions at the top (resp. at the right) of items cut via a $\bullet, \bullet = \alpha, \beta$, pattern; and
- $R = R^{\bullet t} \cup R^{\bullet r}$, $\bullet = \alpha, \beta$, be a set of regions derived from the packing of items of set \bar{I} .

When item $i \in I$ fits tightly in region (L', W') (i.e., $l_i = L'$ and $w_i = W'$), all four resulting regions $R_i^{\alpha t} = R_i^{\alpha r} = R_i^{\beta t} = R_i^{\beta r}$ are of zero size; thus, are eliminated from R . In addition, when $l_i = L'$ (resp. $w_i = W'$), $R_i^{\alpha t} = R_i^{\beta t}$ (resp. $R_i^{\alpha r} = R_i^{\beta r}$) because regions $R_i^{\alpha r} = R_i^{\beta r}$ (resp. $R_i^{\alpha t} = R_i^{\beta t}$) are of zero size. Thus, a single region results from the cutting of item i . This region, labeled ‘simple’, is excluded from R and is attached to S . Furthermore, when no item $j \in I'$ fits $R_i^{\alpha t}$ (resp. $R_i^{\beta r}$), region $R_i^{\beta t}$ (resp. $R_i^{\alpha r}$) is redundant, and the only useful region $R_i^{\beta r}$ (resp. $R_i^{\alpha t}$) leaves R and enters S . Finally, when $R_i^{\alpha t}$, $R_i^{\alpha r}$, $R_i^{\beta t}$, and $R_i^{\beta r}$ are of non-zero size but no item may fit in these regions, the four regions are removed from R .

Let $\hat{R} \subseteq R$ denote the subset of regions available for packing in the current call of MIP and let $\hat{I} \subseteq \bar{I}$ be the subset of packed items that generated \hat{R} . Similarly, let $\hat{S} \subseteq S$ be a subset of currently active simple regions. Let $K = \hat{R} \cup \hat{S} \cup T$ be the set of available regions and let $K_i \subseteq K$ denote the subset of regions that fit item $i \in I'$. Subsequently, let $I'_k \subseteq I'$ designate the subset of free items that fit into region $k \in K$.

Along with the above sets, MAXSUM feeds two parameters to MIP:

- γ_i , the modifier adjusting the assignment cost of item $i \in I$ for any region $k \in K_i$, and
- $\Omega = \sum_{i \in I'} (\gamma_i \sum_{k \in K_i} (a_i/A_k))$ the penalty paid for each item that MIP does not pack.

MAXSUM feeds the above input to an MIP that employs three categories of decision variables.

- The **first** category consists of three sets of binary variables:

- $x_{ik} = 1$ if item $i \in I'_k$ is assigned to region $k \in K$, and 0 otherwise;
- $y_{ik} = 1$ if item i reserves free space in region k , and 0 otherwise; and
- $e_i = 1$ if item i remains unassigned, and 0 otherwise.

While x_{ik} determines whether i is packed during the current call of MIP, y_{ik} allocates space that could be used to pack i during a subsequent call of MIP. In addition, x_{ik} contributes the cost of packing i into k to the objective function of MIP, while y_{ik} does not. Indeed, y_{ik} only reinforces the constraints. It is a lookahead mechanism that reserves space for the not yet packed items of I' . Indicator variable e_i signals when i is neither assigned to a bin nor reserved a space for future assignment.

- The **second** category consists of the non-negative integer decision variables z_t , $t \in T$, which determine the number of new bins of type t required in addition to the regions available in \hat{R} and \hat{S} .
- The **third** category consists of binary decision variables v_i , $i \in \hat{I}$, that select the cutting pattern (α or β) for every packed item i and activate the respective pair of regions: $v_i = 1$ if i is the product of an α pattern and its packing creates $(R_i^{\alpha t}, R_i^{\alpha r})$, and 0 if i is the product of a β pattern resulting in $(R_i^{\beta t}, R_i^{\beta r})$.

Using these three sets of variables, we model MIP as follows.

$$\max \sum_{i \in I'} \left(\gamma_i \sum_{k \in K_i} \frac{a_i}{A_k} x_{ik} - \Omega e_i \right) \quad (3)$$

$$\text{s.t. } e_i + \sum_{k \in K_i} (x_{ik} + y_{ik}) = 1 \quad i \in I' \quad (4)$$

$$\sum_{i \in I'_k} x_{ik} \leq 1 \quad k \in K \quad (5)$$

$$\sum_{i \in I'_k} a_i (x_{ik} + y_{ik}) \leq A_k \quad k \in \hat{R} \cup \hat{S} \quad (6)$$

$$\sum_{i \in I'_t} a_i (x_{it} + y_{it}) \leq A_t z_t \quad t \in T \quad (7)$$

$$\sum_{t \in T} A_t z_t \leq U^* - \bar{U} - 1 \quad (8)$$

$$\sum_{k \in \{R_i^{\alpha t}, R_i^{\alpha r}\}} \sum_{j \in I'_k} (x_{jk} + y_{jk}) \leq V_i^{\alpha} v_i \quad i \in \hat{I} \quad (9)$$

$$\sum_{k \in \{R_i^{\beta t}, R_i^{\beta r}\}} \sum_{j \in I'_k} (x_{jk} + y_{jk}) \leq V_i^{\beta} (1 - v_i) \quad i \in \hat{I} \quad (10)$$

$$\sum_{i \in I'_k} \lambda_{ik}^f (x_{ik} + y_{ik}) \leq 1 \quad k \in \hat{R} \cup \hat{S}, f \in F_k \quad (11)$$

$$\sum_{i \in I'_t} \lambda_{it}^f (x_{it} + y_{it}) \leq z_t \quad t \in T, f \in F_t \quad (12)$$

$$x_{ik}, y_{ik} \in \{0, 1\} \quad k \in K \quad (13)$$

$$e_i \in \{0, 1\} \quad i \in I' \quad (14)$$

$$z_t \in \mathbb{Z}_{\geq 0} \quad t \in T \quad (15)$$

$$v_i \in \{0, 1\} \quad i \in \hat{I} \quad (16)$$

Eq. (3) defines the objective function of MIP. It maximizes the net profit of the solution. This net profit is the difference of the profit earned by assigning items to regions and the penalty cost of non-assigned items. Eq. (4) either packs item $i \in I'$ to one of the regions (i.e., when $x_{ik} = 1$, $k \in K_i$) or assigns it

to a region that can potentially fit i in one of the later calls (i.e., when $y_{ik} = 1$, $k \in K_i$) or marks it as unassigned (i.e., when $e_i = 1$). That is, $x_{ik} \neq y_{ik}$ and item i is assigned to at most one region in K_i . Eq. (5) prohibits the packing of more than one item into region k , $k \in K$, during the current call. However, there is no constraint on the number of items that can be allocated to k during future calls. Eq. (6) ensures that the area A_k of region $k \in \hat{R} \cup \hat{S}$ is large enough to accommodate all items assigned to it. Similarly, Eq. (7) ensures that the total area $A_t z_t$ of any allocated bin of type t , $t \in T$, is sufficiently large for its assigned items. Obviously, if the bin is not used (i.e., $z_t = 0$), no item is assigned to the bin. Eq. (8) requires the total size of new bins to be less than the known upper bound U^* minus the size \bar{U} of previously used bins. It is a no-good constraint. Eqs. (9) and (10) select the cutting pattern for item $i \in \hat{I}$; thus, choose between the respective pairs of regions $(R_i^{\alpha t}, R_i^{\alpha r})$ and $(R_i^{\beta t}, R_i^{\beta r})$. When $v_i = 0$, the first applied cut for i is vertical; therefore, Eq. (9) forbids the assignment of any items into $R_i^{\alpha t}$ and $R_i^{\alpha r}$. Here, $V_i^\alpha = \sum_{k \in \{R_i^{\alpha t}, R_i^{\alpha r}\}} |I'_k|$, $i \in \hat{I}$, is a “Big-M” value used to model the disjunction. On the other hand, when $v_i = 1$, the first applied cut for i is horizontal; hence, Eq. (10) implies that no item can be assigned into $R_i^{\beta t}$ and $R_i^{\beta r}$. Similarly, $V_i^\beta = \sum_{k \in \{R_i^{\beta t}, R_i^{\beta r}\}} |I'_k|$, $i \in \hat{I}$, is a “Big-M” value required to implement the disjunction.

Eqs. (11) and (12) are feasibility constraints. They strengthen the search for a feasible packing and cut off futile partial solutions earlier during the search. They reinforce the geometric feasibility of the MIP’s solution. In fact, nowhere does MIP account for the geometric relationship. For instance, Eqs. (6) and (7) overlook the geometric relationship between pairs of items assigned into the same region. Jointly with Eq. (4), Eqs. (11) and (12) constitute an integral part of the lookahead strategy. They rely on the values of transformed areas λ_{ik}^f pre-computed for every region $k \in K_i$, an unpacked item $i \in I'_k$, and a combination of DFFs $f \in F_k$. Eq. (11) requires that the sum of the transformed areas of items being packed in the current MIP’s call (i.e., $x_{ik} = 1$) and those to be packed in future calls (i.e., $y_{ik} = 1$) in selected region k be bounded by 1. Similarly Eq. (12) binds the number of bin types imposing that the sum of transformed areas of selected bins of type $t \in T$ be less than the number z_t of allocated bins. Even though Eqs. (11) and (12) fathom many partial solutions that eventually lead to an infeasible packing, they do not guarantee that all not-yet-packed items get a feasible position during later calls. Finally, Eqs. (13)-(16) declare the types of the decision variables.

When MIP finds a positive objective function value, it returns to MAXSUM a feasible (partial) solution. Before calling MIP again, MAXSUM updates the set I' by moving the packed items from I' to \bar{I} . Let I'' denote the set of items packed during the current MIP’s call. MAXSUM then removes the regions that have been used in the current call (i.e., those whose $x_{ik} = 1$ for some item $i \in I'$) from \hat{S} and \hat{R} .

- It removes from \hat{S} any simple region k that has been assigned an item, and checks whether this packing has produced a pair of coupled regions or a simple

region. As a result, it adds new valid region(s) to the respective set \hat{R} or \hat{S} .

- It discards from \hat{S} any simple free region k that can't fit any of the items of the updated I' .
- It examines the four regions $\{R_i^{\alpha t}, R_i^{\alpha r}, R_i^{\beta t}, R_i^{\beta r}\}$ for each item $i \in \hat{I}$.
 - When any of the four regions is assigned an item $i \in \hat{I}$, it removes the four of them from \hat{R} , and determines the active pair: either $(R_i^{\alpha t}, R_i^{\alpha r})$ or $(R_i^{\beta t}, R_i^{\beta r})$. It further checks whether one or both regions of the active pair are used. In the former case, it inserts the free region into \hat{S} if the free region may pack an item $i \in I'$. For example, if $R_i^{\alpha t}$ contains an assigned item and $R_i^{\alpha r}$ is empty, then the latter is a candidate for inclusion into \hat{S} .
 - When none of the four regions are used, it proceeds depending on whether there is an item $i \in I'$ that fits into any of them.
 - * It excludes the four of them from \hat{R} when none fits any $i \in I'$.
 - * It keeps them in \hat{R} if each of them may fit an item $i \in I'$.
 - * If only one of the four regions can hold an item $i \in I'$, it moves that region to \hat{S} and deletes the three others from \hat{R} .

Next, **MAXSUM** checks the validity of the new pair of coupled regions $(R_i^{\alpha t}, R_i^{\alpha r})$ and $(R_i^{\beta t}, R_i^{\beta r})$ that emanated from packing item $i \in I'$. It discards all four regions when they are either empty or too small to fit any item of I' , but attaches them to \hat{R} when they may fit at least one item of I' . In addition, when either $R_i^{\alpha t}$ or exclusively $R_i^{\beta r}$ may pack an item in a future call, it adds this simple region to \hat{S} .

For any value $z_t > 0$, **MAXSUM** treats bin $t \in T$ as a single simple region. It tests whether packing item $i \in I'$ gives a pair of coupled regions or a simple region, and adds the valid region(s) to the sets \hat{R} and \hat{S} , respectively. It assimilates any $z_t > 1$ value to $z_t = 1$, $t \in T$, because Eq. (5) restricts the assignment to region t to at most one item. Indeed, variables z estimate the remaining space and halt the packing procedure as soon as they detect an infeasible solution (subject to the bound U^*). They further help MIP to locally select and adjust the right combination of bins that are required to pack the items in I' . Even though the current solution may have several z_t values greater than 1, not all the selected bin types are utilized immediately. Indeed, the packing procedure attempts to balance its greediness (using all allocated bins at once) and its providence (gradually adding new allocated bins allowing their combinations to change as its search progresses). Therefore, it sorts the selected bin types in the ascending order of their sizes and adds the largest one to the current (partial) solution. Each next largest bin is selected with a uniform probability χ . If the bin at hand is declined, then the remaining smaller bins are discarded straightaway. Thus, the probability of accepting the next chosen bin decreases.

MAXSUM iterates as long as the MIP solver returns a feasible solution (i.e., a positive objective value). It stops when $I' = \emptyset$ and returns a new incumbent solution. However, at one of the iterations, the solver may deem MIP infeasible; implying that some of the not-yet-packed items may not be packed without surpassing the current cost U^* . In such a case, any further search is futile. Therefore, the procedure terminates immediately signaling infeasibility.

MAXSUM implements a common greedy packing policy: The chance to find a suitable region for smaller items is higher than for larger ones, which therefore should be positioned first. Accordingly, the packing procedure treats larger items of VS2BP as complicated and strives to pack them earlier during the search. At the same time, using the y_{ik} assignment variables, it plans for space for other items too; thus, it aims to increase the overall packing density of items in the resulting solution. In practice, MAXSUM finds high-quality solutions in a very short time and facilitates the quick improvement of the upper bound U^* , which is fed back to the algorithm.

2.3. The MINSUM packing procedure

The second packing procedure, labeled MINSUM, generally outperforms MAXSUM in terms of solution quality but at the cost of a larger run time. Its MIP excludes the y_{ik} assignment variables of MAXSUM, but employs a real-valued variable $C \in \mathbb{R}_{\geq 0}$ that corresponds to the maximum total cost across all regions, where the cost associated with a region $k \in K_i$ is the cost of the items assigned to k . It uses parameter

$$M = \sum_{i \in I'} \gamma_i \left(\sum_{k \in K_i} \max\{a_i/A_k, \lambda_{ik}^1, \dots, \lambda_{ik}^{|F_k|}\} \right)$$

that represents the penalty cost of an unassigned item.

The model of MINSUM is as follows.

$$\min C + \sum_{i \in I'} M e_i \quad (17)$$

$$\text{s.t. } \sum_{i \in I'_k} \frac{\gamma_i a_i}{A_k} x_{ik} \leq C \quad k \in K \quad (18)$$

$$\sum_{i \in I'_k} (\gamma_i \lambda_{ik}^f) x_{ik} \leq C \quad k \in K, f \in F_k \quad (19)$$

$$\text{Eqs. (4) -- (16)}$$

$$C \in \mathbb{R}_{\geq 0} \quad (20)$$

Eq. (17) defines the objective function of MIP. It minimizes the sum of two terms. The first term is the maximum, across the available regions of K , of the sum of costs of items assigned to an available region. The second term is the penalty cost of unassigned items. Both Eqs. (18) and (19) ensure that the value of C is larger than or equal to the sum of item-to-region assignment costs of items packed in region k , for each $k \in K$. In Eq. (18), each cost is proportional to the packing density a_i/A_k of item i assigned to region k . On the other hand,

in Eq. (19), each cost is proportional to the transformed area of item i when i is packed in region k , with both a_i/A_k and λ_{ik}^f being reals in $(0, 1]$. Eqs. (4)-(16) are those of **MAXSUM**. Finally, Eq. (20) declares the type of variable C .

As for **MAXSUM**, the current call of MIP can either obtain a feasible solution or indicate infeasibility. When there is a feasible (partial) solution, **MINSUM** filters the unused regions and computes new ones for the next MIP's call. Among all the items assigned to a region $k \in K$ (i.e., $x_{ik} = 1$), it selects one that maximizes the item-to-region assignment cost ($\gamma_i a_i / A_k$) while it treats the other items as though they were not yet packed, keeping them in set I' for the next call. On the other hand, in the presence of an item i , $i \in I'$, whose $e_i = 1$, **MINSUM** fails to pack all items without violating the current cost U^* ; therefore, the search stops.

The packing procedure of **MINSUM** is diametrically opposed to that of **MAXSUM**. Rather being greedy and trying to fill as much free space of the available regions, **MINSUM** maximizes, over all regions of set K , the gap between the total size of items (related to their packing cost) and the size of the region that holds them. This rule strives to distribute the items evenly among the available regions and avoid their overfilling in later MIP calls. Despite the tightness of the bounds imposed by constraints (11) and (12), the would be needed space for packing items in later calls is larger than the reserved space. Thus, **MINSUM** creates a gap between the bound and the estimation to increase its chance of eventually obtaining a feasible packing solution.

2.4. The *MINMAX* packing procedure

The third packing procedure, labeled **MINMAX**, succeeds in many instances in finding hidden solutions that are reached by neither **MAXSUM** nor **MINSUM**. Of course, this occurs at the cost of increased computational effort. Its MIP formulation drops the y_{ik} assignment variables of **MAXSUM** while it adds the real-valued variables $c_k \in \mathbb{R}_{\geq 0}$, which represent the largest item-to-region assignment cost for region $k \in K$ among all items $i \in I'_k$. In addition to these variables, it uses parameter $\Theta = \sum_{i \in I'} \sum_{k \in K_i} \frac{A_k}{\gamma_i a_i}$ as the penalty of a non-assigned item. Using the above, **MINMAX** is given as follows.

$$\min \sum_{k \in K} c_k + \sum_{i \in I'} \Theta e_i \quad (21)$$

$$\text{s.t. } \frac{A_k}{\gamma_i a_i} x_{ik} \leq c_k \quad k \in K, i \in I'_k \quad (22)$$

$$\text{Eqs. (4) - (16)}$$

$$c_k \in \mathbb{R}_{\geq 0} \quad k \in K \quad (23)$$

Eq. (21) defines the objective function of **MINMAX**. It minimizes the sum of the maximal item-to-region assignment costs over all regions of set K and the penalty cost of unassigned items. Eq. (22) calculates the maximal assignment cost for each region $k \in K$: It sets c_k larger than or equal to the cost $\frac{A_k}{\gamma_i a_i}$ of item

$i \in I'_k$ when i is assigned to k . Eqs. (4)-(16) are those from MAXSUM. Finally, Eq. (23) declares the type of c_k variables.

As when solving MAXSUM, the MIP solver may identify either a feasible solution or deem the problem infeasible. In the former case, MINMAX uses the (partial) solution to determine a set of prospective regions for the next MIP's call. Furthermore, it chooses among all the items assigned to a region $k \in K$ (i.e., $x_{ik} = 1$) the one that gives the smallest item-to-region assignment cost $\frac{A_k}{\gamma_i a_i}$ while it discards the assignment of the other items to k and inserts them back into the set I' for the next call. In the latter case (i.e., if there is an item $i \in I'$ such that $e_i = 1$), MINMAX terminates the search as its MIP fails to pack at least one item without exceeding the current cost U^* .

While MAXSUM focuses on maximising the sum of item-to-region assignment costs of larger items, MINMAX strives to find better positions for items with low packing coefficients. Because it bases its reservation of free space for such items on a “worst-case assignment”, it enhances its chances of packing them during subsequent MIP calls, and eventually obtaining a feasible solution. In this sense, MINMAX is not as greedy as MAXSUM.

3. The Lookahead Matheuristic

MH is an iterative algorithm that searches for a minimal cost feasible guillotine packing of the items of I . Its search is a series of MIP-based packing procedures that apply diversification and intensification strategies. Section 3.1 details the solution construction process of MH. Section 3.2 describes a diversification mechanism, achieved through a sequential value correction heuristic (SVC).

3.1. The Solution Construction Process

MH, whose pseudo code is given in Algorithm 1, constructs a solution as follows. Its input is an instance of VS2BP, defined by I and T , and a maximal runtime limit τ . Its output is (P^*, U^*) where P^* is the incumbent solution expressed as a set of packed bins and U^* its corresponding cost. Its initialization phase sets (P^*, U^*) to (\emptyset, ∞) . Its iterative phase strives to obtain a (near-)global optimal guillotine packing. It is repeated as long as the runtime of MH does not exceed τ . At each iteration, it calls $\text{SEARCH}(I, U^*)$, a subroutine that attempts to build a feasible packing of the set I of items such that the packing's cost is less than the upper bound U^* . SEARCH returns solution (P, U) , where P is either a new feasible (incumbent) solution of cost $U < U^*$ or an infeasible solution recognized by its infinite cost U . In the former case, MH updates (P^*, U^*) while in the latter case, it simply discards P .

Subroutine SEARCH uses the input parameters (\tilde{I}, \tilde{U}) fed from MH and three iteration limits: η , η' , and η'' , which, respectively, constrain the total number of non-improving calls of packing procedures, and the number of times MINSUM and MINMAX are called. $\text{SEARCH}(I, U)$ is looking for a feasible guillotine packing of the set \tilde{I} such that the packing's cost \hat{U} is less than \tilde{U} . This packing can be

the result of one of the three packing procedures: **MAXSUM**, **MINSUM**, or **MINMAX**. **SEARCH** is recursive. At each recursive step, it goes through an initialization phase, an iterative phase, and an update phase.

The initialization phase of a recursive step of **SEARCH** sets its candidate solution (\hat{P}, \hat{U}) to (\emptyset, \hat{U}) . Initially, **SEARCH** opts for **MAXSUM** as its packing procedure: **PACK:=MAXSUM**. It then initialises the set B of densest packed bins to the empty set; i.e., no item has yet been assigned to any bin. Next, it initializes the set \tilde{S} of simple regions to the empty set and their maximal cost $\tilde{C}:=\hat{U}$. With $\tilde{S} = \emptyset$, **PACK** has no constraint on its selection mechanism of bins from T . Finally, **SEARCH** initializes the counter $q = 0$, which keeps track of the maximal number of bins in the packing solutions of **PACK**.

The iterative step of **SEARCH** repeats as long as the counter η of non-improving calls to the packing procedure **PACK** does not exceed η . It calls procedure **PACK** with three parameters: \tilde{I} , \tilde{U} , and \tilde{S} to obtain the solution (P, U) , where P is feasible if $U < \tilde{U}$, and infeasible otherwise.

- When P is feasible, **SEARCH** updates (\hat{P}, \hat{U}) and decrements the counter of iterations: $\eta = \eta - 1$. In addition, because $\hat{U} \leq U^*$, P is a new global incumbent solution.
- When P is infeasible, both the set \tilde{S} and the bound \tilde{C} are updated. Because **PACK** failed to identify a feasible packing whose cost is less than \hat{U} , **SEARCH** restarts **PACK** from a set of pre-allocated bins to guide the search towards feasibility. In this way, **SEARCH** acts in a “conservative” manner; making slower but steadier progress towards tighter bounds. To determine this specific set of bins, it uses subroutine **PREDEFINE**, which determines a set \tilde{S} whose cost C of bins is as close as possible to \tilde{C} .

Let \underline{C}_P be the cost of bins used in the infeasible solution P , \tilde{C} the maximal cost of the current \tilde{S} of simple regions. In addition, let the integer decision variable ζ_t , $t \in T$, denote the number of bins of type t that should be included in set \tilde{S} to ensure a cost lower than \tilde{C} . Using these parameters and decision variable, **PREDEFINE** follows.

$$\max C = \sum_{t \in T} A_t \zeta_t \quad (24)$$

$$\text{s.t. } \underline{C}_P \leq C \leq \tilde{C} - 1 \quad (25)$$

$$\zeta_t \in \mathbb{N}_{\geq 0} \quad t \in T \quad (26)$$

Eq. (24) maximizes C , the total cost of selected bins. Eq. (25) bounds the value of C to the discrete interval $[\underline{C}_P, \tilde{C}]$. Finally, Eq. (26) defines the type of ζ variables. When **PREDEFINE** has a feasible solution, **SEARCH** sets \tilde{S} to ζ_t bins of every type t , $t \in T$, and sets $\tilde{C} = C$. Otherwise, **SEARCH** resets \tilde{S} to the empty set and \tilde{C} to \hat{U} .

Independently of the feasibility of P , **SEARCH** records the maximum number q of bins filled across all **PACK**’s solutions: $q = \max\{q, |P|\}$. Next, it revises

its set B augmenting B with any densely packed bins of P . (The cardinality of B is bounded according to a policy that only retains high-quality packing patterns. That is, $|B|$ is rather small.) Having updating B , **SEARCH** recalculates the pseudo-costs of the items of \tilde{I} . Altering the pseudo-costs changes the structure of solutions obtained by future calls of **PACK**. That is, they are used for diversification purposes. They are set using a sequential value correction (SVC) heuristic as discussed in Section 3.2.

When η' successive calls to **PACK** fail to improve the local incumbent solution and **PACK**=MAXSUM, **SEARCH** changes **PACK** to MINSUM and resets $(\tilde{S}, \tilde{C}) = (\emptyset, \hat{U})$. Similarly, when the incumbent solution is still not improved after η'' calls and **PACK**=MINSUM, **SEARCH** switches **PACK** to MINMAX and resets $(\tilde{S}, \tilde{C}) = (\emptyset, \hat{U})$. (This change of packing procedure is equivalent to a diversification strategy.)

After η non-improving calls to **PACK**, if $q > 1$, **SEARCH** applies an intensification strategy that explores the neighbourhoods of the obtained solutions. It exploits the promising parts of the search space in attempt to build a feasible improving incumbent solution. Specifically, **SEARCH** iteratively fixes a highly dense bin $b \in B$, and repacks all other items. Let $I(b)$ denote the set of items packed in b and let $\text{COST}(b)$ be its cost. Then **SEARCH** calls itself recursively with parameters $(\tilde{I} \setminus I(b), \hat{U} - \text{COST}(b))$ to obtain a solution P of cost U . When P is feasible and $U < \hat{U} - \text{COST}(b)$, P improves the local incumbent \hat{P} . Therefore, (\hat{P}, \hat{U}) is updated. In this way, the recursive call not only decreases the solution's cost but also enhances the algorithm's performance. It investigates the most promising parts of the search space by keeping the most successful packing patterns fixed and rebuilding the solution's parts whose packing can be significantly improved. This strategy enhances the primal bound rapidly. Because it does not spend any effort/time on improving already dense bins, it solves fewer and smaller packing sub-problems; thus, requires less runtime while obtaining faster improvements. When $q \leq 1$, **SEARCH** applies its base case, where recursive calls are prohibited.

Finally, **SEARCH** returns its output to MH. If $\hat{P} \neq \emptyset$, it returns the local incumbent (\hat{P}, \hat{U}) ; otherwise, it returns the output (\emptyset, ∞) signaling the impossibility of improving the current incumbent.

3.2. Sequential Value Correction

As a diversification strategy, subroutine **SEARCH** applies SVC to the pseudo costs of the items. Initially, the pseudo-cost of item $i \in I$ is $\gamma_i = 1$. SVC alters the pseudo-costs of items to propagate information regarding the success of previous assignments to future iterations and to facilitate the packing of new dense bins. It then randomizes and updates each pseudo-cost at every iteration to reflect the unused area in the bin where the item is currently packed. These pseudo-costs are part of the objective function that **PACK** optimizes. Altering them makes **PACK** obtain different items to bins assignments. Their unequal values allow **PACK** to randomly pack items; thus, diversify the search and overcome the highly symmetric nature of bin packing solutions; an intrinsic feature to VS2BP [17]. This diversification has been found effective in strip packing [5], in one-

Algorithm 1 Solution process as a whole in MH for VS2BP

Input: A problem instance (I, T) , t , η , η' , η''

routine MAIN(I, T)

- 1: initialise the global incumbent solution $(P^*, U^*) := (\emptyset, \infty)$
- 2: **while** (τ is not exceeded) **do**
- 3: obtain solution $(P, U) := \text{SEARCH}(I, U^*)$
- 4: **if** ($U < U^*$) **then** update the incumbent solution $(P^*, U^*) := (P, U)$
- 5: return the incumbent solution (P^*, U^*)

subroutine SEARCH(\tilde{I}, \tilde{U})

- 1: initialise the local incumbent solution $(\hat{P}, \hat{U}) := (\emptyset, \tilde{U})$
 - 2: select the packing procedure $\text{PACK} := \text{MAXSUM}$
 - 3: initialise the set of most dense packed bins $B := \emptyset$
 - 4: initialise the set of predefined bins and their maximal cost $(\tilde{S}, \tilde{C}) := (\emptyset, \tilde{U})$
 - 5: initialise the counter $q := 0$
 - 6: **for** ($\eta = 1$; $\eta \leq \eta$; $\eta = \eta + 1$) **do**
 - 7: obtain solution $(P, U) := \text{PACK}(\tilde{I}, \hat{U}, \tilde{S})$
 - 8: **if** ($U < \hat{U}$) **then**
 - 9: update the local incumbent $(\hat{P}, \hat{U}) := (P, U)$
 - 10: set $\eta = \eta - 1$
 - 11: **else**
 - 12: update $(\tilde{S}, \tilde{C}) := \text{PREDEFINE}(T, \text{COST}(P), \tilde{C})$
 - 13: update $q := \max\{q, |P|\}$
 - 14: update the set B with regard to the packed bins of P
 - 15: update the pseudo-costs γ_i for every item $i \in \tilde{I}$
 - 16: **if** ($\text{PACK} = \text{MAXSUM}$ **and** $\hat{P} = \emptyset$ **and** $\eta' \leq \eta$) **then**
 - 17: select $\text{PACK} := \text{MINSUM}$ and set $(\tilde{S}, \tilde{C}) := (\emptyset, \hat{U})$
 - 18: **if** ($\text{PACK} = \text{MINSUM}$ **and** $\hat{P} = \emptyset$ **and** $\eta'' \leq \eta$) **then**
 - 19: select $\text{PACK} := \text{MINMAX}$ and set $(\tilde{S}, \tilde{C}) := (\emptyset, \hat{U})$
 - 20: **if** ($q > 1$) **then**
 - 21: **for** (each bin $b \in B$) **do**
 - 22: obtain solution $(P, U) := \text{SEARCH}(\tilde{I} \setminus \text{I}(b), \hat{U} - \text{COST}(b))$
 - 23: **if** ($U + \text{COST}(b) < \hat{U}$) **then**
 - 24: update the local incumbent $(\hat{P}, \hat{U}) := (P \cup \{b\}, U + \text{COST}(b))$
 - 25: **if** ($\hat{P} \neq \emptyset$) **then**
 - 26: return the local incumbent solution (\hat{P}, \hat{U})
 - 27: **else**
 - 28: return the infeasible solution $(\hat{P}, \hat{U}) := (\emptyset, \infty)$
-

and two-dimensional bin packing [4, 10], and in bi-objective bin packing with due-dates where the objective is to minimize both the number of bins and the

maximum lateness of items [19].

When defining the pseudo-cost γ_i of item $i \in I$, two factors are of relevance:

- the portion $\rho_i = \frac{a_i}{A_t}$ of the area of bin t that item i is using, and
- the relative size of item i with respect to the largest item in I , expressed as $\frac{a_i}{a_{max}}$ where $a_{max} = \max_{j \in I} \{a_j\}$.

Based on these two factors, the pseudo cost γ_i of an item $i \in I$ that has been recently positioned in a bin is reset to

$$\gamma_i = \delta_i \gamma_i + \left(1 + \frac{a_i}{a_{max}}\right)^{\varepsilon'} \frac{(1 - \delta_i)}{\rho_i}. \quad (27)$$

$\varepsilon' > 1$ is a constant that makes bigger items have higher costs. δ_i is generated randomly from the uniform $[0, \bar{\rho}_i)$, where $\bar{\rho}_i$ is the average packing density of bins where item i has been placed during the last κ iterations. That is, δ_i allows items with a historically larger packing density $\bar{\rho}_i$ to inherit a larger fraction of their previous pseudo-costs. The first term of Expr. (27) reinforces the information inherited from the previous patterns. On the other hand, the second term extracts information from the current pattern. The value of γ_i increases as ρ_i decreases; thus prioritizing the joint packing of an item i with other items when i has a low packing density. In addition, packing a large item is more challenging than packing a small one. Therefore, the class of large items is given a higher priority than the class of small items. This is further reinforced via parameter ε' .

On the other hand, the pseudo cost γ_i of an item $i \in I$ that has not yet been packed is reset at a higher value. This is to increase its chances of being packed in the next call of PACK. Specifically, its

$$\gamma_i = \frac{1}{\bar{\rho}_i} \left(1 + \frac{a_i}{a_{max}}\right)^{\varepsilon''}, \quad (28)$$

where $\varepsilon'' > \varepsilon'$.

4. Experiments and Results

The proposed MH is implemented in C# and run on a personal computer with 16GB of RAM and an i7 2.8 GHz Intel Core processor. It is tested on the benchmark set of [23]. The set has ten-classes. For classes 1-6, the item sizes l_i and w_i emanate from the same interval and the bins are squares as described by Table 1. Classes 7-10, which use bins of size $L = W = 100$, reflect a more realistic situation because their items emanate from a mixture of uniform distributions. Indeed, the items are classified into four types, and are mixed as indicated in Table 2. Uniformly distributed dimensions of items do not make problems easy to solve as it is the case for the one-dimensional case. The cost of a bin (L, W) is its area LW . The instances of the variable bin sizes data

set use the same items as those for the single bin type. However, each instance has five bin types where the bin dimensions are randomly generated from the $\text{Uniform}[L/2, L]$ and $[W/2, W]$, respectively, where L and W are the instance's original bin size obtained according to the class type. Regardless of the number of bin types of the data set, each class has five problem sizes: $n = 20, 40, 60, 80$, and 100, and ten instances per problem type.

Table 1: Description of the benchmark instances of classes 1 to 6

Class	l_i, w_i	$L = W$
1	$[1, 10]$	10
2	$[1, 10]$	30
3	$[1, 35]$	40
4	$[1, 35]$	100
5	$[1, 100]$	100
6	$[1, 100]$	$[1, 300]$

Table 2: Description of the benchmark instances of classes 7 to 10

Item type	l_i	w_i	Class			
			7	8	9	10
1	$[\frac{2}{3}L, L]$	$[1, \frac{1}{2}W]$	70%	10%	10%	10%
2	$[1, \frac{1}{2}L]$	$[\frac{2}{3}W, W]$	10%	70%	10%	10%
3	$[\frac{1}{2}L, L]$	$[\frac{1}{2}W, W]$	10%	10%	70%	10%
4	$[1, \frac{1}{2}L]$	$[1, \frac{1}{2}W]$	10%	10%	10%	70%

For all instances, the performance measure is the overall utilization of the used bins. The utilization, denoted hereafter u , is the ratio of the sum of the areas of the n items to the sum of the areas of the used bins. To obtain this measure, we run MH for $\tau = 1$ minute and for $\tau = 10$ minutes. Let u_{MH_1} and $u_{\text{MH}_{10}}$ denote the corresponding obtained utilization. Both u_{MH_1} and $u_{\text{MH}_{10}}$ are compared to the existing upper bound u_{SA} of [12] obtained using their SA based heuristic within a 1-minute runtime. Running MH requires the specification of its control parameters, which are detailed in Section 4.1. Section 4.2 presents and analyzes the results of MH for the variable bin types benchmark set. Finally, Section 4.3 indicates the applicability of MH to the identical bin type benchmark set too while proving the optimality of a large number of open problems.

4.1. Experimental Setup

The following settings, inferred from preliminary computational investigations, provide the best trade-off between MH's solution quality and runtime. MH evokes IBM CPLEX 12.10 to solve the MIP components of its three packing procedures. The MIP solver uses a single core. It is configured such that it generates more feasible solutions while optimizing the problem, at the cost of a slower proof of optimality. Its relative optimality gap is set to 0.01 in lieu of CPLEX's default value of 10^{-6} . This is consistent with the philosophy of MH. Its PACK procedures favour high-quality feasible MIP solutions to

optimal ones when the former are obtained faster than the latter. In addition, their MIPs rely on fitness functions that do not necessarily lead to VS2BP's optimum. Thus, the larger than the default relative optimality gap does not affect the overall quality of MH's solution. Moreover, MIP's node selection is set to prefer the node with the best progress toward integer feasibility. This setting is recommended when it is difficult to find feasible solutions or when a proof of optimality is not crucial. Furthermore, we limit the execution time of the MIPs in **PACK** subroutines to 3 seconds. This runtime limit does not affect the MIPs: Even the largest instances of the tested instances obtained a solution in less than 0.1 second. Finally, when MIP of **PACK** selects several bin types as candidates to enter the partial solution, it adds the bin of the largest size and decides on each subsequent bin with a uniform probability $\chi = 0.5$ (see Section 2.2 for details). All smaller bins are immediately discarded if the bin at hand is declined.

We generate the feasibility constraints of Eq. (2) according to the algorithm of Fekete and Schepers [11] whose control parameters (p, q) are selected from $\{0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5\}^2$. From the set of constraints generated for each entry K_{μ_l, μ_w} of the data structure of Section 2.1 (with partition $\mu = 10$), at most $f_{max} = 10$ strongest constraints are injected into MIP while weaker constraints are discarded. A larger set of constraints does not necessarily tighten the lower bound on the free space available for packing, but does increase the time needed to solve MIP.

The three iteration limits, η , η' , and η'' , of the **SEARCH** subroutine of MH are set, respectively, to 11, 5, and 9. That is, **MAXSUM** is executed at least 5 times. When **MAXSUM** is unsuccessful, **MINSUM** makes no less than 4 attempts to find a feasible packing solution. When both packing procedures fail, **MINMAX** tries at least twice to solve the problem. The total number of non-improving calls of packing procedures cannot exceed 11. When tested on the benchmark set, **MINMAX** took up to 1 second of computational time. Therefore, it was rarely called. Because its goal is to instantly improve the incumbent solution and the primal bound U^* when both **MAXSUM** and **MINSUM** fail, there was no need to increase its number of calls. The new bound U^* that **MINMAX** obtains helps **MAXSUM** and **MINSUM** to focus on the parts of the search space as guided by the findings of **MINMAX**.

SEARCH selects the size of the set B of densest packed bins adaptively. It uses a threshold parameter $\min\{0.97, 1.1 \sum_{i \in I} a_i / U^*\}$, where $\sum_{i \in I} a_i / U^*$ is the packing density of the current incumbent solution. When the packing ratio of the candidate bin is larger than this threshold parameter, **SEARCH** allows B to hold two best patterns; otherwise B holds only the best pattern. In this way, the threshold parameter forces **SEARCH** to launch more recursive calls when the bins in B have better than average density (i.e., when the bins in B exceed the packing density associated of P^*). Any patterns with packing density above 0.97 are considered high-quality. Therefore, **SEARCH** explores the promising parts of the solution space by conducting more recursive calls for items that are still unassigned and causing low-density bins. This rule allows

SEARCH to quickly improve the primal bound and enhances MH's performance: It focuses on smaller sub-problems that lead to faster improvements.

The upper bound $\bar{\rho}_i$ of the uniform distribution used to generate the weights δ_i for updating the pseudo costs is the average packing density of bins where item i has been placed during the last κ iterations. Herein, κ is set to 15. Finally, ε' is set to 2 in Expr. (27) whereas ε'' is set to 3 in Expr. (28). Both values assign higher costs to larger items; thus, prioritise the packing of larger items at the onset of the packing procedures. $\varepsilon'' > \varepsilon'$ so that items that remained unpacked during the previous run of PACK get an increased chance of being packed during the current call.

4.2. Multiple-Sized Bins

Table 3 summarizes the MH results. Columns 1 and 2 specify the class and number of items. Columns 3-5 report the average utilizations u_{SA} , u_{MH_1} and $u_{MH_{10}}$ over each set of ten instances. Columns 6-8 give the number of times u_{SA} is better, the same, and worse than u_{MH_1} whereas Columns 9-11 provide the same information but for u_{SA} versus $u_{MH_{10}}$. Finally, Column 12 indicates the number of time the longer runtime of MH improved its solution.

u_{SA} and u_{MH_1} are obtained within equal run times; yet, there is statistical evidence that the mean of u_{MH_1} is larger than the mean of u_{SA} at any level of confidence (p-value=0.0000), with a 1.42 average difference. This is clearly depicted in Figure 2, which displays –in addition to the histogram of the paired differences $u_{MH_1}-u_{SA}$ – the null hypothesis H_0 , which stipulates that the mean difference of $u_{MH_1}-u_{SA}$ is 0 versus being strictly positive, and the 95% one sided confidence interval for the mean $u_{MH_1}-u_{SA}$ with a 1.26 estimate of its lower bound.

Increasing the runtime of MH from 1 to 10 minutes further amplifies the aforementioned advantages. This is clearly depicted by the five-point summaries of $u_{MH_1}-u_{SA}$ and $u_{MH_{10}}-u_{SA}$. (A five point summary consists of the minimum, first, second and third quartiles, and the maximum observed difference over the 500 instances.) For $u_{MH_1}-u_{SA}$, the five point summary is $(-3.92, 0.00, 0.46, 2.57, 10.33)$. It is shifted to the right for $u_{MH_{10}}-u_{SA}$: $(-2.58, 0.00, 0.76, 3.14, 10.81)$. This shift is also depicted in Figure 3, which represents the same information as Figure 2 but for $u_{MH_{10}}-u_{SA}$. Thus, increasing the runtime of MH reduces the gap for those instances where $u_{SA} > u_{MH_1}$ while it enlarges it for cases where $u_{SA} < u_{MH_1}$. Solutions of instances with $u_{SA} = u_{MH_1} = u_{MH_{10}}$ are most likely optimal. Thus, further increasing the runtime of MH does not affect them. A paired statistical test confirms again that the mean of $u_{MH_{10}}$ is larger than the mean of u_{SA} at any level of confidence (p-value=0.0000), with a larger average difference of 1.67 versus 1.42, and a larger lower bound estimate of 1.50 of the lower bound of a 95% one sided confidence interval estimate of the mean $u_{MH_{10}}-u_{SA}$ (versus 1.26 for $u_{MH_1}-u_{SA}$).

Figure 4 further explains the impact of runtime on MH. It presents the box plots of the ratios $\frac{u_{MH_1}}{u_{SA}}$, $\frac{u_{MH_{10}}}{u_{SA}}$ and $\frac{u_{MH_{10}}}{u_{MH_1}}$, as a function of the class of the instances. The comparison of the box plots of $\frac{u_{MH_1}}{u_{SA}}$ and $\frac{u_{MH_{10}}}{u_{SA}}$ depicts

Table 3: Results of MH for VS2BP

Class	n	Utilization			u_{SA} vs. u_{MH_1}			Number of times			$u_{MH_1} < u_{MH_{10}}$
		u_{SA}	u_{MH_1}	$u_{MH_{10}}$	>	=	<	u_{SA} vs. $u_{MH_{10}}$	>	=	
1	20	90.18	90.74	90.74		8	2		8	2	
	40	93.70	95.17	95.32		2	8		2	8	
	60	91.95	95.14	95.70	1		9			10	
	80	90.25	94.87	95.19			10			10	
	100	91.47	96.01	96.36			10			10	
Overall		91.51	94.39	94.66	1	10	39	0	10	40	24
2	20	86.62	86.62	86.62			10			10	
	40	97.42	96.81	97.20	3	7		2	8		1
	60	98.90	98.50	98.63	3	7		2	8		1
	80	98.70	98.17	98.36	7	3		6	4		3
	100	99.62	99.17	99.41	8	2		5	5		3
Overall		96.25	95.85	96.04	21	29	0	15	35	0	8
3	20	84.38	86.09	86.09		4	6		4	6	
	40	88.08	89.93	90.04	2	1	7	1	1	8	4
	60	87.08	89.74	90.27	2		8	1		9	7
	80	85.55	90.32	90.81			10			10	6
	100	88.13	91.90	92.38			10			10	6
Overall		86.65	89.59	89.92	4	5	41	2	5	43	23
4	20	80.82	80.82	80.82			10			10	
	40	91.93	91.93	91.93			10			10	
	60	94.99	94.22	94.60	5	5		2	8		3
	80	95.12	94.50	94.86	6	2	2	4	3	3	2
	100	95.82	95.20	95.26	6	3	1	5	4	1	2
Overall		91.74	91.34	91.50	17	30	3	11	35	4	7
5	20	82.05	83.06	83.18	1	4	5		5	5	1
	40	85.64	87.40	87.85	2	2	6	1	2	7	6
	60	84.70	89.13	89.57	1		9			10	6
	80	84.49	89.33	89.80			10			10	8
	100	86.25	90.59	91.45			10			10	8
Overall		84.63	87.90	88.37	4	6	40	1	7	42	29
6	20	81.25	81.25	81.25			10			10	
	40	90.18	90.01	90.18	1	9				10	1
	60	92.18	92.10	92.33	1	9		1	8	1	2
	80	93.49	92.99	93.25	5	5		4	4	2	3
	100	94.09	94.06	94.14	4	3	3	3	4	3	2
Overall		90.24	90.08	90.23	11	36	3	8	36	6	8
7	20	83.99	84.13	84.17	1	5	4	1	5	4	1
	40	87.74	89.19	89.52		1	9		1	9	3
	60	89.52	90.83	91.07		1	9		1	9	3
	80	86.23	88.76	89.27			10			10	5
	100	87.19	89.94	91.17			10			10	8
Overall		86.93	88.57	89.04	1	7	42	1	7	42	20
8	20	82.27	83.64	83.64		6	4		6	4	
	40	86.04	87.67	87.79	1	1	8		1	9	2
	60	87.92	89.82	90.03	1		9	1		9	4
	80	86.66	90.28	90.41			10			10	3
	100	86.41	90.32	90.75			10			10	5
Overall		85.86	88.35	88.52	2	7	41	1	7	42	14
9	20	74.42	75.49	75.49		6	4		6	4	
	40	73.42	75.16	75.17		1	9		1	9	1
	60	73.07	74.15	74.15		1	9		1	9	
	80	74.67	75.84	75.84			10			10	
	100	75.84	77.23	77.27			10			10	1
Overall		74.29	75.58	75.58		8	42		8	42	2
10	20	83.46	84.81	85.33		6	4		5	5	1
	40	90.86	91.56	91.77	1	5	4		3	7	3
	60	92.20	92.26	92.39	3	3	4	4	2	4	2
	80	92.39	93.04	93.24	3		7	2		8	4
	100	92.64	93.18	93.60	1	1	8	1		9	6
Overall		90.31	90.97	91.27	8	15	27	7	10	33	16
Overall		87.84	89.26	89.51	69	153	278	46	160	294	151

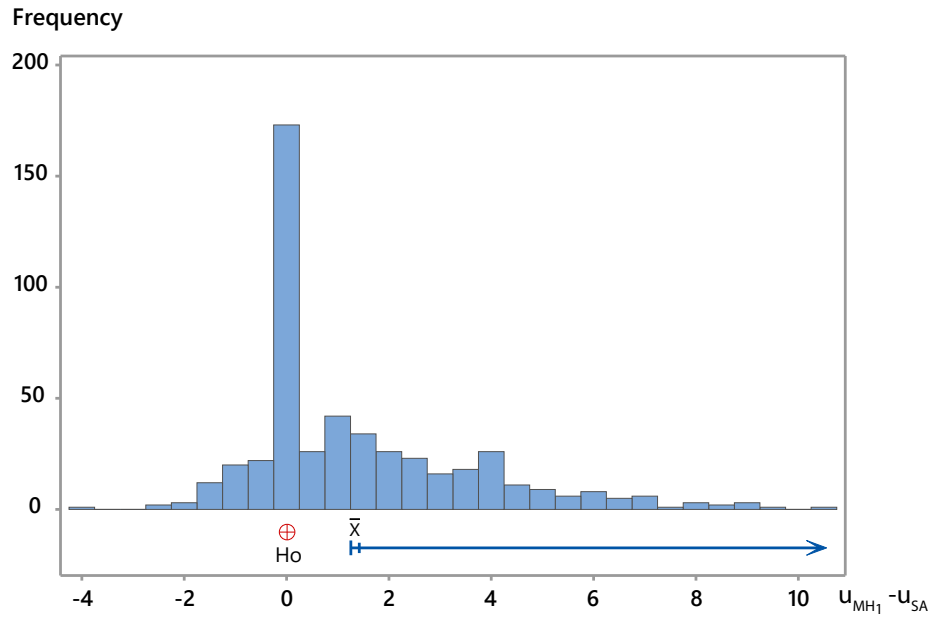


Figure 2: Histogram of $u_{MH1} - u_{SA}$ and one-sided 95% confidence interval of their mean differences

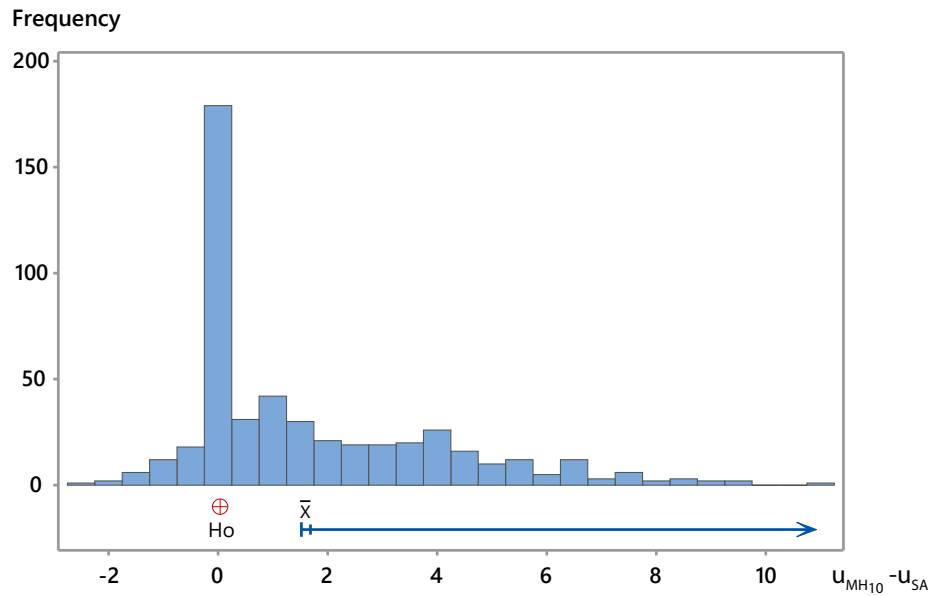


Figure 3: Histogram of $u_{MH10} - u_{SA}$ and one-sided 95% confidence interval of their mean differences

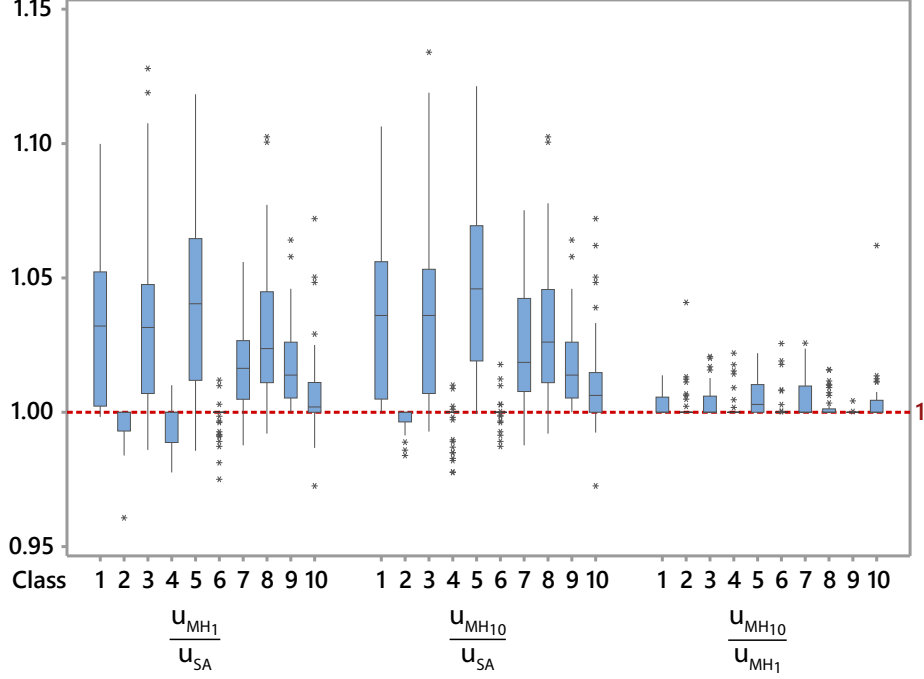


Figure 4: Box plots of $\frac{u_{MH_1}}{u_{SA}}$, $\frac{u_{MH_{10}}}{u_{SA}}$ and $\frac{u_{MH_{10}}}{u_{MH_1}}$

the impact of runtime on MH, and highlights the classes that are difficult for MH. These are, in general, classes 2 and 4, which use fewer bins than other classes. Therefore, the use of an extra bin or of a larger-sized bin makes a big difference on the result. For these classes, MH is limited by the behavior of the mathematical programming solver, which was observed to fail to identify feasible solutions that reduce the number of bins of its current solutions. In addition, DFF constraints appear loose for such instances. Their looseness weakens the lookahead mechanism and its power to guide the search process to feasible areas of the search space.

While $\frac{u_{MH_{10}}}{u_{MH_1}}$ assesses the solution enhancement brought to MH by the additional runtime for each class of instances, $\frac{u_{MH_{10}}}{u_{SA}}$ reflects the absolute solution quality enhancement with respect to the existing upper bound u_{SA} . The analysis of the cases where MH neither matches nor improves the existing upper bound u_{SA} indicates that those instances of classes 2 and 4 are independent of the problem size, as the box plots of Figure 5 further illustrate.

4.3. Single-Sized Bins

Table 4 gives the results of MH on the single bin size benchmark set, where b_{MT} denotes the best known bound on the number of identical bins needed (avail-

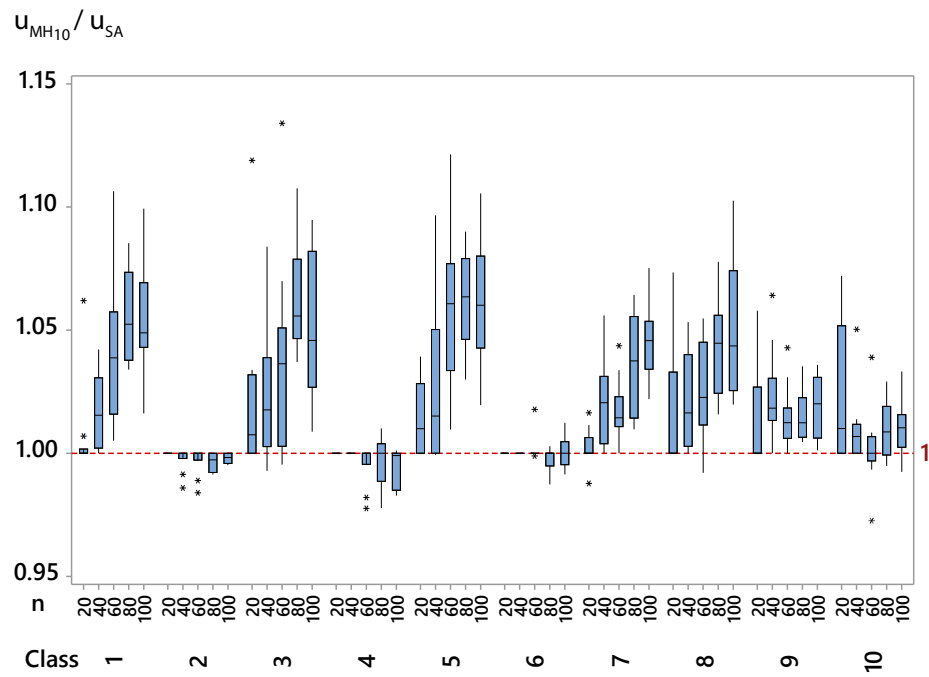


Figure 5: Box plots of $\frac{u_{MH10}}{u_{SA}}$ as a function of class and problem size

able from <http://or.dei.unibo.it/general-files/best-known-solution-and-lower-bound-each-instance>) while b_{CHBP} , b_{SA} and b_{MH} are the counterparts obtained by CHBP, the constructive heuristic followed by a biased sufficiency criterion and by post optimization of Charalambous and Fleszar [7], MH and SA, respectively. Columns 1 and 2 specify the class and number of items. Columns 3-6 report the sum of b_{MT} , b_{CHBP} , b_{SA} and b_{MH} over each set of ten instances. Columns 7 and 8 report the average utilization of u_{SA} and u_{MH} over each set of ten instances. Finally, Columns 9 and 10 indicate the number of instances where b_{MH} exceeds the lower bound b_{MT} of the number of bins and the number of times MH improves the solution of SA by using fewer bins; i.e., the number of times b_{SA} exceeds b_{MH} over each set of ten instances.

Table 4 shows that MH can be successfully applied to the case with identical bins. MH matches the lower bound on the number of bins for all but 85 instances, using one additional bin for all but the first instance of class 5 and $n = 100$ where it uses two additional bins. For all other 415 instances, MH solution is optimal: It uses the lowest number of bins b_{MT} . In addition, MH succeeds in matching all solutions of SA and enhances the best known solution of 13 instances.

5. Conclusion

This paper approximately solves the unweighed fixed-orientation variable-sized two-dimensional guillotine bin packing problem. It applies a hybrid matheuristic that diversifies its search via three mixed-integer programs whose objective function coefficients are determined by a sequential value correction algorithm. The mathematical programs are dotted with lookahead mechanisms that reserve space for unpacked items while being guided by dual feasibility function constraints and by no-good cuts imposed by upper bounds on the objective function. These mechanisms make the mathematical programs focus on the most ‘productive’ part of the search space. This matheuristic is innovative in terms of modeling and solution approach: It opts for item to region assignment type (without explicitly considering the geometrical aspect of the problem), and for many items to many bins (not necessarily identical) assignments in contrast to existing sequential heuristics which pack items one by one filling one bin at a time. It outperforms state-of-the-art methods. It proves the optimality of 415 solutions of the 500 identical-bin benchmark instances, matches the best known results of another 72 instances, and improves the other 13. For the variable-size bin packing, it matches 160 and improves 294 best known results (out of 500). This matheuristic represents a general framework that can be applied to other cutting and packing problems such as glass cutting with defects.

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Table 4: Results of MH for the identical bins case

Class	n	Number of bins				Utilization		Number of times	
		b_{MT}	b_{CHBP}	b_{SA}	b_{MH}	u_{SA}	u_{MH}	$b_{MT} < b_{MH}$	$b_{SA} > b_{MH}$
1	20	71	71	71	71	82.16	82.16		
	40	134	134	134	134	87.61	87.61		
	60	197	201	200	200	90.05	90.05	3	
	80	274	275	275	275	90.26	90.26	1	
	100	317	321	320	317	93.86	94.78		3
Overall		993	1002	1000	997	88.79	88.97	4	3
2	20	10	10	10	10	64.30	64.30		
	40	19	20	19	19	69.42	69.42		
	60	25	26	25	25	81.76	81.76		
	80	31	33	31	31	89.30	89.30		
	100	39	39	39	39	85.74	85.74		
Overall		124	128	124	124	78.10	78.10		
3	20	51	52	52	52	74.08	74.08	1	
	40	92	97	94	94	83.08	83.08	2	
	60	136	140	140	140	86.30	86.30	4	
	80	187	196	191	190	87.43	87.91	3	1
	100	221	230	227	225	88.92	89.79	4	2
Overall		687	715	704	701	83.96	84.23	14	3
4	20	10	10	10	10	60.92	60.92		
	40	19	19	19	19	66.15	66.15		
	60	23	25	25	25	78.77	78.77	2	
	80	30	33	32	32	83.99	83.99	2	
	100	37	39	38	38	85.09	85.09	1	
Overall		119	126	124	124	74.98	74.98	5	
5	20	65	65	65	65	74.31	74.31		
	40	119	121	119	119	82.40	82.40		
	60	179	183	181	181	84.21	84.21	2	
	80	241	247	247	247	84.96	84.96	6	
	100	279	288	287	285	88.40	89.05	5	2
Overall		883	904	899	897	82.85	82.99	13	2
6	20	10	10	10	10	53.15	53.15		
	40	15	19	18	18	62.37	62.37	3	
	60	21	22	22	22	77.56	77.56	1	
	80	30	30	30	30	77.67	77.67		
	100	32	35	35	35	81.07	81.07	3	
Overall		108	116	115	115	70.37	70.37	7	
7	20	55	55	55	55	77.95	77.95		
	40	109	112	111	111	82.61	82.61	2	
	60	156	160	159	159	84.43	84.43	3	
	80	224	233	232	232	83.44	83.44	8	
	100	269	275	273	273	85.55	85.55	4	
Overall		813	835	830	830	82.79	82.79	17	
8	20	58	58	58	58	75.42	75.42		
	40	112	114	113	113	81.59	81.59	1	
	60	159	163	162	161	84.11	84.59	2	1
	80	223	226	225	224	84.86	85.27	1	1
	100	274	279	279	278	84.76	85.02	4	1
Overall		826	840	837	834	82.15	82.38	8	3
10	20	42	44	43	43	75.54	75.54	1	
	40	74	74	74	74	85.56	85.56		
	60	98	103	102	102	88.34	88.34	4	
	80	123	130	130	129	90.25	90.93	6	1
	100	153	163	160	159	91.95	92.51	6	1
Overall		490	514	509	507	86.33	86.57	17	2
Overall		7173	7312	7272	7259	79.36	79.46	85	13

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