Q39 (a) Given, $J_1 = \frac{1}{n_1 n_2} \sum_{y_i \in Y_1} \sum_{y_j \in Y_2} (y_i - y_j)^2 = LHS$

To prove,
$$J_1 = (m_1 - m_2)^2 + \frac{1}{n_1} S_1^2 + \frac{1}{n_2} S_2^2 = RHS$$

here $S_i^2 = \sum_K (y_K - m_i)^2$

$$LHS = \frac{1}{n_1 n_2} \sum_{y_i \in Y_1} \sum_{y_j \in Y_2} (y_i - y_j)^2$$

$$= \frac{1}{n_1} \sum_{y_i \in Y_1} \left(\frac{1}{n_2} \sum_{y_j} y_i^2 + \frac{1}{n_2} \sum_{y_j} y_j^2 - \frac{1}{n_2} \sum_{y_j} 2 y_i y_j \right)$$

$$= \frac{1}{n_1} \sum_{y_i} \left(y_i^2 + \frac{1}{n_2} \sum_{y_j} y_j^2 - 2 y_i m_2 \right)$$

$$= \frac{1}{n_1} \sum_{y_i} y_i^2 + \frac{1}{n_1} \sum \left(\frac{1}{n_2} \sum_{y_j} y_j^2 \right) - \left(\frac{1}{n_1} \sum 2 y_i m_2 \right)$$

$$= \frac{1}{n_1} \sum_{y_i} y_i^2 + \frac{1}{n_2} \sum_{y_j} y_j^2 - 2 m_1 m_2 \qquad \text{— (1)}$$

Now, $RHS = (m_1 - m_2)^2 + \frac{1}{n_1} S_1^2 + \frac{1}{n_2} S_2^2$

$$= m_1^2 + m_2^2 - 2 m_1 m_2 + \frac{1}{n_1} \left(\sum_{y_i} (y_i - m_1)^2 \right) + \frac{1}{n_2} \left(\sum_{y_j} (y_j - m_2)^2 \right)$$

$$= m_1^2 + m_2^2 - 2 m_1 m_2 + \frac{1}{n_1} \sum_{y_i} (y_i^2 + m_1^2 - 2 y_i m_1)$$
$$+ \frac{1}{n_2} \sum_{y_j} (y_j^2 + m_2^2 - 2 y_j m_2)$$

$$= m_1^2 + m_2^2 - 2m_1m_2 + \frac{1}{n_1}\sum_{y_i} y_i^2 + \frac{1}{n_1}\sum_{y_i} m_1^2 - \frac{1}{n_1}\sum_{y_i} 2y_i m_1$$

$$+ \frac{1}{n_2}\sum_{y_j} y_j^2 + \frac{1}{n_2}\sum_{y_j} m_2^2 - \frac{1}{n_2}\sum 2y_j m_2$$

$$= \cancel{m_1^2} + \cancel{m_2^2} - 2m_1m_2 + \frac{1}{n}\sum_{y_i} y_i^2 + \cancel{m_1^2} - \cancel{2m_1^2}$$

$$+ \frac{1}{n_2}\sum_{y_j} y_j^2 + \cancel{m_2^2} - \cancel{2m_2^2}$$

$$= \frac{1}{n}\sum_{y_i} y_i^2 + \frac{1}{n_2}\sum_{y_j} y_j^2 - 2m_1m_2 \quad \text{--- (2)}$$

From (1) and (2), LHS = RHS

Hence proved.

(b) For within class scatter,

From above part (a),

Total Scatter, $J_1 = (m_1 - m_2)^2 + \frac{1}{n_1}S_1^2 + \frac{1}{n_2}S_2^2$ —(3)

We know,

Total scatter $= S_B + S_W$ —(4)

also, between class scatter for two class

$S_B = (m_1 - m_2)^2$ —(5)

Hence from (3), (4) and (5),

$$S_W = \frac{1}{n_1}S_1^2 + \frac{1}{n_2}S_2^2$$

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Q40 (a) Given, $|S_B - \lambda S_W| = 0$

$S_B e_i = \lambda_i S_W e_i$ here e_i are eigenvectors.

$e_i^t S_W e_j = \delta_{ij}$ —— (1)

$e_i^t S_B e_j = \lambda_i \delta_{ij}$ —— (2)

From question, if $\tilde{S}_W = W^t S_W W$

(here W has n eigenvectors in columns).

i.e., $$\tilde{S}_W = \begin{bmatrix} e_1^t \\ e_2^t \\ \vdots \end{bmatrix} \begin{bmatrix} & S_W & \end{bmatrix} \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$$

$$= \begin{bmatrix} e_1^t S_W e_1 & e_1^t S_W e_2 & \cdots & e_1^t S_W e_n \\ \vdots & & & \\ e_n^t S_W e_1 & \cdots & \cdots & e_n^t S_W e_n \end{bmatrix}$$

$$= \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \vdots & & & \\ \delta_{n1} & \cdots & \cdots & \delta_{nn} \end{bmatrix}$$ (using eqn (1))

if we assume, $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$ — (3)

Then $\boxed{\tilde{S}_W = I}$ — (4)

Now, $\tilde{S}_B = W^t S_B W$

$$= \begin{bmatrix} e_1^t \\ e_2^t \\ \vdots \end{bmatrix} \begin{bmatrix} & S_B & \end{bmatrix} \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$$

$$= \begin{bmatrix} e_1^t S_B e_1 & \cdots & e_1^t S_B e_n \\ \vdots & & \vdots \\ e_n^t S_B e_1 & & e_n^t S_B e_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \delta_{11} & \cdots & \lambda_1 \delta_{1n} \\ \vdots & & \vdots \\ \lambda_n \delta_{n1} & \cdots & \lambda_n \delta_{nn} \end{bmatrix} \quad \text{(using (2))}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ & \lambda_2 & 0 & \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{(using (3))}$$

$\Rightarrow \tilde{S}_B$ = diagonal matrix with eigen values as diagonal elements — (5)

Hence proved.

(b) From part (a),

$$|\hat{S}_B| = \begin{vmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ & & & \lambda_n \end{vmatrix}$$

$$= (\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n)$$

and $|\hat{S}_W| = |I| = 1$

Then, $$J = \frac{|\hat{S}_B|}{|\hat{S}_W|}$$

$$= \lambda_1 \lambda_2 \cdots \lambda_n$$

(c) If we perform transformation, then rotation, using D (a diagonal matrix) and Q (orthogonal matrix) respectively.

i.e., $y' = QDy$.

$y' = QD(W^T x)$ $\qquad (\because y = W^T x$ given)

$y' = W'^T x$

here $(W')^T = QDW^T$

Now, $S_B' = (W')^T S_B (W')$

$= QDW^T S_B W^T DQ^T$ $\qquad (\because (QDW^T)^T = W^T DQ^T)$

$= QD \underbrace{W^T S_B W} DQ^T$

$$|S_B'| = |QDW^T S_B W D Q^T|$$

$$= |Q||D|\,|W^T S_B W|\,|D||Q^T|$$

$$= |QQ^T|\,|D|^2\,|\lambda_1\lambda_2\lambda_3\cdots\lambda_n| \quad \text{(using eqn (5))}$$

$$|S_B'| = |D|^2|\lambda_1\lambda_2\cdots\lambda_n| \quad (\because Q \text{ is orthogonal})$$

for

$$|S_W'| = |W'^T S_W W'|$$

$$= |QDW^T S_W W D Q^T|$$

$$= |QQ^T||D|^2|W^T S_W W|$$

$$= |QQ^T||D|^2 \times 1 \quad \text{(using eqn (4).)}$$

$$= |D|^2$$

Now,

$$J' = \frac{|S_B'|}{|S_W'|}.$$

$$= \frac{|D|^2|\lambda_1\lambda_2\cdots\lambda_n|}{|D|^2}$$

$$= |\lambda_1\lambda_2\lambda_3\cdots\lambda_n|$$

$$= J$$

Hence we can state that due to the transformation $y' = QDy$, J is invariant of the transformation.

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In case of HMM, given a sequence of length T, each observation has possibility of 'c' states.

Hence for each cell, at a single observation, we need to search for previous layers values, i.e., observation's value. Hence $O(c)$ for one possibility of an observation.

For total 'c' observation, we require computation $O(c^2)$.

Finally for a total of T observations, complexity is $T \times O(c^2)$

$= O(c^2 T)$.

$$\alpha_j(t) = \begin{cases} 0 & t=0 \text{ \& } j \neq \text{initial} \\ 1 & t=0 \text{ \& } j = \text{initial} \\ \left[\sum_i \alpha_i(t-1)\, a_{ij}\right] b_{jk} v(t) & \text{else} \end{cases}$$