Considering a simpler Langevin dynamics under the foloowing equations

$$\begin{cases} \dot{x} &= v \\ \dot{v} &= \frac{F(x)}{m} - \gamma v + \frac{\xi(t)}{m} \end{cases}$$

where (x, v) are position and velocity. $F(x) = -\partial_x U(x)$ which is the position-dependent force. The m, γ are mass and friction coefficient. The $\xi(t)$ represents randomness defines the Wiener noise in the Ito convention:

$$\langle \xi(t)\xi(t')\rangle = \frac{2m\gamma}{\beta}\langle dW_t dW_{t'}\rangle = \frac{2m\gamma}{\beta}\delta(t-t')$$

This equation describes a dynamics equivalent to an evolution of probability density f(x,v;t) through Fokker-Planck equation:

$$\frac{\partial f(x,v;t)}{\partial t} = -\hat{L} f(x,v;t)$$

where

$$\hat{L} \equiv \frac{F(x)}{m}\frac{\partial}{\partial v} + v\frac{\partial}{\partial x} - \gamma\left(\frac{\partial}{\partial v}v + \frac{1}{m\beta}\frac{\partial^2}{\partial v^2}\right)$$

The probabaility density f evolves through

$$f(x,v,t+\Delta t) = e^{-\Delta t \hat{L}} f(x,v;t)$$

The operator $\hat{L}$ is separated into three parts, and the interation is approximated through Lie-Trotter formula,

$$\begin{aligned} e^{-\Delta t \hat{L}} &\approx e^{-\Delta t \hat{L}_x} e^{-\Delta t \hat{L}_v} e^{-\Delta t \hat{L}_\gamma} \\ &\approx e^{-(\Delta t/2)\hat{L}_\gamma} e^{-(\Delta t/2)\hat{L}_v} e^{-(\Delta t/2)\hat{L}_x} e^{-(\Delta t/2)\hat{L}_x} e^{-(\Delta t/2)\hat{L}_v} e^{-(\Delta t/2)\hat{L}_\gamma} \end{aligned}$$

where

$$\begin{aligned} \hat{L}_x &= v\frac{\partial}{\partial x} \\ \hat{L}_v &= \frac{F(x)}{m}\frac{\partial}{\partial v} \\ \hat{L}_\gamma &= -\gamma\left(\frac{\partial}{\partial v}v + \frac{1}{m\beta}\frac{\partial^2}{\partial v^2}\right) \end{aligned}$$

The integration can then be taken into two parts.

I. $\hat{L}_\gamma$ - Diffusion in momentum space

This first part is a diffusion in momentum space

$$f(v^+ \equiv v(t^+), t^+) = e^{-(\Delta t/2)\hat{L}_\gamma} f(v,t)$$

which corresponds to the evolution of f

$$\frac{\partial f}{\partial t} = -\frac{\hat{L}_\gamma}{2} f = \frac{\gamma}{2}\left[\frac{\partial}{\partial v}(vf) + \frac{1}{m\beta}\frac{\partial^2 f}{\partial v^2}\right]$$

which has a solution [“The Fokker-Planck Equation Second Edition”, H. Risken] of

$$f(v^+, t^+|v,t) \propto \exp\left[-\frac{m\beta}{2}\frac{\left(v^+ - ve^{-\frac{\gamma}{2}\Delta t}\right)^2}{1-e^{-\gamma\Delta t}}\right] = \exp\left[-\frac{1}{2}\left(\frac{v^+ - \mu_v}{\sigma_v}\right)^2\right]$$

This step can then be performed by sampling from a Normal distribution $N(\mu_v, \sigma_v)$,

$$\begin{aligned} v(t^+) &\sim N(\mu_v, \sigma_v) \\ &= \mu_v + \sigma_v N(0,1) \\ &= v(t)e^{-\frac{\gamma}{2}\Delta t} + \sqrt{\frac{1}{m\beta}\left(1-e^{-\gamma\Delta t}\right)}N(0,1) \\ &= c_1 v(t) + c_2 N(0,1) \end{aligned}$$

I. $(\hat{L}_x, \hat{L}_v)$ **- Deterministic canonical transformation in phase space**

Both $\hat{L}_v$ and $\hat{L}_x$ are deterministic canonical transformation in phase space. Applying $f(v', t') = f(v(t+\Delta t), t+\Delta t) = e^{-(\Delta t/2)\hat{L}_v} f(v(t), t)$, and observing that by doing the transformation $p = t + \frac{F(x)}{2m} v$ and $q = \frac{F(x)}{2m} t - v$, we have

$$\frac{\partial f}{\partial t} + \frac{F(x)}{2m}\frac{\partial f}{\partial v} = \left[1 + \left(\frac{F(x)}{2m}\right)^2\right] \left.\frac{\partial f}{\partial p}\right|_q = 0$$

The solution is then $f(v,t) = f(q) = f(\frac{F(x)}{2m} t - v)$. With the boundary condition $v(0) = v$, the solution must be a delta function

$$f(v,t) = f(\frac{F(x)}{2m} t - v) = \delta(\frac{F(x)}{2m} t - v)$$
$$\Delta v = \frac{F(x)}{2m} \Delta t$$

Similarly, the operation of $\hat{L}_x$ during $\Delta t/2$ is equivalent to $\Delta x = v(\Delta t/2)$.