HEIGHTS OF CM CYCLES AND DERIVATIVES OF L-SERIES

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ABSTRACT. We extend the work of S. Zhang and Yuan-Zhang-Zhang to obtain a Gross-Zagier formula for modular forms of even weight in terms of an arithmetic intersection pairing of CM-cycles on Kuga-Sato varieties over Shimura curves. Combined with a result of the first author and de Vera-Piquero adapting Kolyvagin's method of Euler systems to this setting, we bound the associated Selmer and Tate-Shafarevich groups, assuming the non-vanishing of the derivative of the L-function at the central point.

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1. INTRODUCTION

Given an imaginary quadratic field, the classical Gross-Zagier formula [9] relates the first derivative of the L-function of a modular form of weight two to the Néron-Tate height pairing of a suitable Heegner point on the modular curve, whose existence is guaranteed by a certain Heegner hypothesis. This result was generalized by S. Zhang [23] to the setting of modular forms of positive even weight, and by Yuan-Zhang-Zhang [21] to the setting of imaginary quadratic extensions of totally real number fields, with a relaxed Heegner hypothesis. These results contribute to the conjecture of Birch and Swinnerton-Dyer and its generalization due to Beilinson, Bloch and Kato. Combined with Kolyvagin's method of Euler systems [11] subsequently adapted by Nekovář in both settings [13, 14], they imply a bound on the rank of the associated Selmer and Tate-Shafarevich groups, assuming the non-vanishing of the derivative at the central point of the L-function.

In this paper, we simultaneously extend the results of [21] and [23] to obtain a Gross-Zagier formula for Hilbert modular forms of even parallel weight, under a relaxed Heegner hypothesis. More precisely, we relate the central value of the

Date: August 7, 2017.
2010 *Mathematics Subject Classification.* 11F12, and 11F85.
Key words and phrases. Gross-Zagier formula, Modular forms, Shimura varieties, CM cycles.

first derivative of the associated Rankin-Selberg L-function to the intersection of suitable CM cycles on Kuga-Sato varieties over Shimura curves. Combining this with a recent result of the first author and de Vera-Piquero adapting Kolyvagin and Nekovar's Euler system argument to this setting [7], we provide a bound on the rank of the associated Selmer and Tate-Shafarevich groups.

Let F be a totally real number field. Let π be a cuspidal automorphic representation of an indefinite quaternion algebra $\mathbf{B}^\times$ over a totally real number field F associated, by the Jacquet-Langlands correspondence, to a cuspidal automorphic representation σ of parallel weight $2k$, and $\tilde{\pi}$ its contragredient representation. Let K be an imaginary quadratic extension of F with character η, such that finite primes dividing the discriminant of $\mathbf{B}$ are inert in K over F. Given a Hecke character χ of K, we denote by Z_χ, the χ-eigencomponent of a suitable CM-cycle Z on the Kuga-Sato variety W over the Shimura curve over F, and by T a suitable map from $\pi \otimes \tilde{\pi}$ to the group of homomorphisms of the $k-1$-th Chow group of W as in (2.24).

Taking first $F = \mathbf{Q}$, our generalization for the Gross-Zagier formula can be stated as follows:

Theorem 1.0.1. *For $f_1 \otimes f_2$ in $\pi \otimes \tilde{\pi}$, the following equality holds*

$$\langle T(f_1 \otimes f_2) Z_\chi, Z_{\chi^{-1}} \rangle_{\mathrm{GS}} = \frac{\zeta_F(2) L'(\frac{1}{2}, \pi, \chi)}{4L(1,\eta)^2 L(1,\pi,\mathrm{ad})} \alpha(f_1 \otimes f_2), \tag{1.1}$$

where

$$T : \bigoplus_{\pi \in \mathcal{A}(\mathbf{B}^\times)} \pi \otimes \tilde{\pi} \hookrightarrow \mathrm{Hom}(\mathrm{CH}^{k-1}(W), \mathrm{CH}^{k-1}(W))_{\mathbf{C}}, \tag{1.2}$$

is a suitable correspondence defined in Lemma (2.7.2), $\langle \cdot, \cdot \rangle_{\mathrm{GS}}$ is the Gillet-Soulé pairing on arithmetic Chow groups, $\alpha(\cdot \otimes \cdot)$ is the Petersson inner product, and $L(s, \pi, \mathrm{ad})$ is the usual adjoint L-function.

The following two corollaries correspond to the prediction of the Beilinson, Bloch and Kato conjectures. Still assume $F = \mathbf{Q}$, and consider f in π. Let p be an odd prime not dividing $N(2k!)$, and let $\wp$ be a prime ideal dividing p in the number field E generated over $\mathbf{Q}$ by the Fourier coefficients of f. We denote by $O_\wp$ the ring of integers of E, localized at $\wp$. Consider the p-adic Abel–Jacobi map

$$\Phi : \mathrm{CH}^k(W_{k-1}/K)_0 \otimes \mathcal{O}_\wp \longrightarrow \mathrm{Sel}_\wp^{(\infty)}(f, K),$$

where $\mathrm{CH}^k(W_{k-1}/K)_0$ is the k-th Chow group of a suitable Kuga-Sato variety W_{k-1} over K, and $\mathrm{Sel}_\wp^{(\infty)}(f, K)$ is the Selmer group associated to f and K defined by (7.1.1). Its cokernel

$$Ш_{\wp^\infty} := \mathrm{coker}(\Phi) = \mathrm{Sel}_\wp^{(\infty)}(f, K)/\mathrm{Im}(\Phi).$$

is the $\wp$-primary part of the Shafarevich–Tate group.

On the one hand, assuming the injectivity of the p-adic Abel Jacobi map, if the order of vanishing of the L-functions associated to π and χ at the central point is 1, then $\mathrm{Im}(\Phi)$ has rank 1 over $\mathbf{Q}$ as in the next corollary.

Corollary 1.0.2. *Assume that the p-adic Abel Jacobi map Φ (7.3) is injective. Then*

$$L'(\frac{1}{2}, \pi, \chi) \neq 0 \implies Ш_{\wp^\infty} \text{ is finite, and } \mathrm{Im}(\Phi) \otimes \mathbf{Q} \text{ has rank } 1. \tag{1.3}$$

On the other hand, assuming the Gillet-Soulé conjecture, if $\mathrm{Im}(\Phi)$ has rank 1 over $\mathbf{Q}$, then the L-function associated to π and χ has order of vanishing 1 at the central point.

Corollary 1.0.3. *Suppose the Gillet-Soulé conjecture (7.2.2) holds. Then*

$$\mathrm{Im}(\Phi) \otimes \mathbf{Q} \text{ has rank } 1 \implies L'(\frac{1}{2}, \pi, \chi) \neq 0. \tag{1.4}$$

Our work also sets the stage for the general case of F a totally real field, depending on the existence of suitable CM cycles . Granting the existence of such cycles, our methods give the analogous Gross-Zagier formula in the form of Theorem 1.0.1 with little modifications. Also, assuming, as one would expect, that the CM cycles, under the p-adic Abel-Jacobi map form an Euler system as in [7], the analogues of Corollaries 1.0.2 and 1.0.3 will also then follow for the associated Selmer groups over F.

1.1. **Outline.** In Section (2), we introduce the basic objects of our study and give a complete statement of Theorem (1.0.1). Our proof of the theorem follows closely the method of [21], which in turn follows the proof of the Waldspurger formula [19], inspired by the philosophy of Kudla.

In Section (3), we construct the analytic and geometric kernels respectively corresponding to the right and left sides of the main identity (1.1). We reformulate it as a kernel identity,

$$2(\tilde{Z}(\cdot, \chi, \Phi), \varphi)_{\mathrm{Pet}} = (I'(0, \cdot, \chi, \Phi), \varphi)_{\mathrm{Pet}} \tag{1.5}$$

which can be viewed as an arithmetic theta lifting. In particular, the geometric kernel relates to a generating series of CM cycles. In Section (4), we show that this generating series defines an automorphic form on $GL_2(\mathbf{A})$ as this is an essential ingredient in the comparison of kernels. We also prove an arithmetic theta lifting,

$$\tilde{Z}(\Phi \otimes \varphi) = \frac{L(1, \pi, \mathrm{ad})}{2\zeta_F(2)} T(\theta(\Phi \otimes \varphi)), \tag{1.6}$$

relating the generating series to the correspondence. The key observation is that the automorphy of the generating series can be essentially deduced from the weight 2 case, while the arithmetic theta lifting argument relies on the local Siegel-Weil formulae, which are independent of weight at nonarchidmean places.

In Section (5), we decompose the global kernels into local components. In Section (6), we prove the kernel identity. We use the Gillet-Soulé intersection theory and Green's currents as developed in [23]. We restrict ourselves to the class of degenerate Schwartz functions introduced in [21], which simplifies the local computations. This completes the proof of the main identity, which is combined in Section (7) with [7, Theorem 1.1] to prove Corollary (1.0.2), and with the Gillet-Soulé conjecture (7.2.2) to prove Corollary (1.0.3).

1.2. **Notation and measures.** We fix the following notation:

- F is a totally real number field with adele ring $\mathbf{A}_F = \mathbf{A}$, and a fixed embedding $\tau : F \hookrightarrow \mathbf{C}$. Also, $F_\infty = \prod_{v|\infty} F_v$, and $\mathbf{A}^S = \prod'_{v \notin S} F_v$.
- If π is an irreducible admissible complex representation of $\mathbf{B}^\times$, then by the Jacquet-Langlands correspondence we associate a cuspidal automorphic representation σ of $GL_2(\mathbf{A})$, discrete of parallel weight $2k$ at all infinite places.

- K is an imaginary quadratic extension of F with adele ring $\mathbf{A}_K$ and quadratic character $\eta : K^\times\backslash\mathbf{A}_K^\times \longrightarrow \mathbf{C}^\times$, such that finite primes dividing the discriminant of $\mathbf{B}$ are inert in K over F. Also consider a character $\chi : \mathrm{Gal}(K^{ab}/K) \longrightarrow \mathbf{C}^\times$.
- $\mathrm{CH}^k(X)$ is the Chow group of codimension k cycles of the variety X.
- $\mathbf{V}$ is the orthogonal space $\mathbf{B}$ with reduced norm q.
- For any algebraic group T, we denote the adelic quotient by $[T] = T_F\backslash T_{\mathbf{A}}/Z_{\mathbf{A}}$, and fix $G = GL(2)$.

1.2.1. *Local measures.* In general, our choice of measures will be fixed so as to be consistent with that of [21]. Given a local field F_v, we will choose the additive Haar measure dx to be self-dual, that is, $\hat{\hat{f}}(x) = f(-x)$ with respect to the Fourier transform

$$\hat{f}(y) = \int_{F_v} f(x)\psi(xy)dx, \tag{1.7}$$

where the additive character $\psi(x)$ is defined by

$$\psi(x) = \begin{cases} \exp(2\pi i x) & F_v \simeq \mathbf{R} \\ \exp(4\pi i \mathrm{Re}(x)) & F_v \simeq \mathbf{C} \\ \exp(-2\pi i \mathrm{Tr}_{F_v/\mathbf{Q}_p}(x)) & F_v \text{ finite extension of } \mathbf{Q}_p. \end{cases} \tag{1.8}$$

Thus, dx is the Lebesgue measure if $F_v \simeq \mathbf{R}$, twice the Lebesgue measure if $F_v \simeq \mathbf{C}$, and such that $\mathrm{vol}(\mathcal{O}_v) = |\mathfrak{d}|^{1/2}$ where $\mathfrak{d}$ is the different of F_v over $\mathbf{Q}_p$ if F_v is a finite extension of $\mathbf{Q}_p$. Fix the multiplicative measure $d^\times x$ on $F_v^\times$ to be $d^\times x = \zeta_{F_v}(1)|x|^{-1}dx$, where $\zeta_{F_v}(s) = \pi^{-s/2}\Gamma(s/2)$ for $F_v \simeq \mathbf{R}$, $2(2\pi)^{-s}\Gamma(s)$ for $F_v \simeq \mathbf{C}$, and $(1-q_v^{-s})^{-1}$ for F_v nonarchimedean, with residue field of cardinality q_v.

Let (V, q) be a quadratic space of a local field F_v. The self-dual measure on V with respect to ψ is the unique Haar measure dx such that the Fourier transform

$$\hat{f}(y) = \int_{F_v} f(x)\psi(\langle x, y\rangle)dx \tag{1.9}$$

satisfies $\hat{\hat{f}}(x) = f(-x)$, where $\langle x, y\rangle = q(x+y) - q(x) - q(y)$ is the associated bilinear pairing.

Let E be a quadratic étale algebra extension, or a quaternion algebra over F_v with reduced norm q. Endow K with the self-dual Haar measure with respect to (E, q), and $E^\times$ with the multiplicative Haar measure

$$d^\times x = \zeta_E(1)|q(x)|^{-j}dx \tag{1.10}$$

where $j = 1$ if E is a quadratic étale algebra extension, and $j = 2$ if it is a quaternion algebra. Let E^1 be set of elements in $E^\times$ with reduced norm 1. If $E = K$ a quadratic extension, set

$$\mathrm{vol}(K^1) = \begin{cases} 2 & K = \mathbf{C},\ F_v = \mathbf{R} \\ |\mathfrak{d}|^{\frac{1}{2}} & K/F_v \text{ nonsplit, unramified,} \\ 2|D|^{\frac{1}{2}}|\mathfrak{d}|^{\frac{1}{2}} & K/F_v \text{ ramified} \end{cases} \tag{1.11}$$

where D is the discriminant of K in F_v, and $\mathrm{vol}(\mathcal{O}_K) = |D|^{\frac{1}{2}}|\mathfrak{d}|$.

If E is a quaternion algebra, we have for F_v nonarchimedean,

$$\mathrm{vol}(G(\mathcal{O}_v)) = \zeta_{F_v}(2)^{-1}\mathrm{vol}(\mathcal{O}_v)^4, \qquad \mathrm{vol}(SL(2, \mathcal{O}_v)) = \zeta_{F_v}(2)^{-1}\mathrm{vol}(\mathcal{O}_v)^3, \tag{1.12}$$

whereas for $F_v = \mathbf{R}$ and E equal to the Hamiltonian algebra, we have $\mathrm{vol}(E^1) = 4\pi^2$.

1.2.2. *Global measures.* Let G be a reductive group over a global field F, with center Z. Assume that the volume of the adelic quotient $[G] = G(F)\backslash G(\mathbf{A}_F)/Z(\mathbf{A}_F)$ is finite under some Haar measure dg of $G(\mathbf{A}_F)/Z(\mathbf{A}_F)$. Denote by G^1 the norm one subgroup of G. Hence, taking the Tamagawa measure we have $\mathrm{vol}(K^1\backslash\mathbf{A}_K^1) = 2L(1,\eta)$. On the other hand, the Tamagawa measure on the quaternion algebra B over F induces

$$\mathrm{vol}(B^1\backslash\mathbf{B}^1) = 1, \text{ and } \qquad \mathrm{vol}(B^\times\backslash\mathbf{B}^\times/Z(\mathbf{A})) = 2, \tag{1.13}$$

where B^1 denotes the elements of $B^\times$ with reduced norm one.

Define the modulus function

$$\delta_v: \ P(F_v) \longrightarrow \mathbf{R}^\times : \begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto \left|\frac{a}{d}\right|_v^{1/2}, \tag{1.14}$$

where $P(F_v)$ is the parabolic group of F_v. This extends to a map δ_v from $G_2(F_v)$ to $\mathbf{R}^\times$. Let $\delta = \prod_v \delta_v$. Define the regularized integral

$$\int_{[G]}^* f(g)dg = \int_{[G]} \frac{1}{\mathrm{vol}(Z(\mathbf{A}))} \int_{Z(\mathbf{A})} f(zg)dz\, dg, \tag{1.15}$$

for any automorphic function f on $G(\mathbf{A})$ invariant under $Z(F_\infty)$.

Also, denote by $(f_1, f_2)_{\mathrm{Pet}}$ the Petersson pairing for automorphic forms f_1, f_2 on $G(\mathbf{A})$ with respect to the Tamagawa measure on G

$$(f_1, f_2)_{\mathrm{Pet}} = \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} f_1(g)\overline{f_2(g)}dg, \tag{1.16}$$

which is defined when at least one of the two automorphic forms is cuspidal.

2. Kuga-Sato varieties over Shimura curves

In this section we introduce the main objects of our study and some of their properties, and we give a more precise statement of Theorem 1.0.1.

2.1. **Quaternion algebras.** Recall that F is a totally real number field with adele ring $\mathbf{A} = \mathbf{A}_F$. Let S be a finite set of places of F of odd cardinality containing the archimedean places S_∞. Let $\mathbf{B}$ be a (incoherent) quaternion algebra over $\mathbf{A}$, such that $\mathbf{B}_v = \mathbf{B} \otimes_{\mathbf{A}} F_v$ is isomorphic to $M_2(F_v)$ if $v \notin S$ and to the unique division quaternion algebra over F_v for $v \in S$. Note that $\mathbf{B}$ is totally definite at infinity, that is, $\mathbf{B}_v$ is the Hamiltonian algebra for every $v \in S_\infty$.

For a place v of F, we denote by $B(v)$ the nearby quaternion algebra over F obtained from $\mathbf{B}$ by changing the Hasse invariant at v, and we fix an identification $B(v) \otimes_F \mathbf{A}_F \simeq \mathbf{B}_f$. The ramification set of $B(v)$ has distance one from S, that is, it is either $S \cup \{v\}$ or $S - \{v\}$ depending on whether $\mathbf{B}$ is ramified at v.

2.2. **The Kuga-Sato variety.** For every open compact subgroup U of $\mathbf{B}_f^\times = (\mathbf{B} \otimes_{\mathbf{A}_F} \mathbf{A}_f)^\times$, we have a compactified quaternionic Shimura curve X_U over F. Over $\mathbf{C}$, given an embedding $\tau : F \hookrightarrow \mathbf{C}$, and the quaternion algebra $\mathbf{B}(\tau)$ over F ramified exactly at $S - \{\tau\}$, we have a uniformization

$$X_U = (\mathbf{B}(\tau)^\times\backslash\mathbf{H}^\pm \times \mathbf{B}_f^\times)/U \tag{2.1}$$

where $\mathbf{H}^{\pm} = \mathbf{C} - \mathbf{R}$. The set of cusps is empty (and hence X_U is compact) since we exclude the setting where $F = \mathbf{Q}$ and $S = \{\infty\}$ already considered in [23], in which X_U is the classical modular curve.

Fix $F = \mathbf{Q}$. We consider the projective limit

$$X = \varprojlim_U X_U, \tag{2.2}$$

over compatible systems of open compact subgroups U. The Shimura curve X_U admits a model over $\mathbf{Q}$ which is the coarse moduli scheme classifying abelian varieties A with quaternionic multiplication (QM) by U, that is, an optimal embedding $\iota_U : U \hookrightarrow \mathrm{End}(A)$. Given an integer $M \geq 3$, consider the fine moduli scheme classifying pairs (A, ι) endowed with level M structure, that is, an embedding $v : U/MU \hookrightarrow A[M]$, and let $\mathscr{A}_U$ be a universal abelian variety with QM by U. Let $\mathscr{A} = \varinjlim_U \mathscr{A}_U$.

The Kuga-Sato variety W_U over the Shimura curve X_U is defined as the $(k-1)$-th fibre product of $\mathscr{A}_U$ over X_U,

$$W_U = \mathscr{A}_U \times_{X_U} \cdots \times_{X_U} \mathscr{A}_U \to X_U. \tag{2.3}$$

It is a nonsingular projective variety of dimension $2k-1$. In the same way, we define the Kuga-Sato variety W over the Shimura curve X to be the $(k-1)$-fold fibre product of $\mathscr{A}$ over X, or, equivalently,

$$W = \varprojlim_U W_U. \tag{2.4}$$

For F totally real, the Shimura curve X_U is not a moduli space of abelian varieties, but is related to one that is, i.e., the Shimura curve associated to the group GU of unitary similitudes, as in [5], and over these the analogue of the Kuga-Sato variety was introduced by T. Saito in [17, Section 6.3].

2.3. **CM cycles over Shimura curves.** Complex multiplication cycles on Kuga-Sato varieties arose in the work of S. Zhang [23] over the classical modular curve and in the work of Iovita and Spieß, Besser, Chida, and de Vera Piquero and the first author [10, 2, 6, 7] over the Shimura curve. In this subsection, we describe the construction of CM cycles in the case $F = \mathbf{Q}$. The construction in the setting $F \neq \mathbf{Q}$ is the subject of forthcoming work of Chida and Hsieh.

Recall that K_c denotes the ring class field of K of conductor c. An abelian surface A with QM has by U has complex multiplication (CM) by K_c if the ring of endomorphisms of A commuting with the optimal embedding $\tau : U \hookrightarrow \mathrm{End}(A)$ denoted by $\mathrm{End}(A, \tau)$ is isomorphic to K_c. The complex multiplication points of conductor c of the Shimura curve X are in bijection with isomorphism classes of abelian surfaces A with QM by U and CM by K_c. Such an abelian surface is isomorphic to a product of elliptic curves with CM by K_c

$$A \simeq E \times E_{\mathfrak{a}}, \tag{2.5}$$

where $E(\mathbf{C}) = \mathbf{C}/K_c$, $E_{\mathfrak{a}}(\mathbf{C}) = \mathbf{C}/\mathfrak{a}$, and $\mathfrak{a}$ is an ideal of K_c defined by $U = K_c + e\mathfrak{a}$ for some e in $\mathbf{B}$.

Given a map $\gamma : E \longrightarrow E_{\mathfrak{a}}$, we denote by

$$Z_\gamma = \mathrm{graph}(\gamma) - (E \times 0) - \deg\gamma\ (0 \times E_{\mathfrak{a}}) \ \subseteq E \times E_{\mathfrak{a}}. \tag{2.6}$$

Writing $\mathfrak{a} = \mathbf{Z}a + \mathbf{Z}\sqrt{-D}$ for a in $\mathbf{Z}$, the Néron-Severi group $NS(A)$ of A is given by

$$NS(A) = < E \times 0,\ 0 \times E_{\mathfrak{a}},\ Z_a,\ Z_{\sqrt{-D}} > . \tag{2.7}$$

We define $Z_A = Z_{\sqrt{-D}} \subseteq A$. Then $Z = Z_A^{k-1}$ is a fiber of W with dimension $k-1$. Note that its construction depends on a given CM point on X, which arises from an embedding of the imaginary quadratic field $K \hookrightarrow \mathbf{B}^\times$. This embedding exists because primes dividing the discriminant of $\mathbf{B}$ are inert in K.

2.4. **Chow groups and height pairings.** Given a regular arithmetic scheme Y of dimension d over $\mathcal{O}_F$, and an integer $p \geq 0$, we denote by $A^{p,p}(Y)$ the real vector space of real differential forms of type (p,p) on $\mathbf{C}$, such that $\tau^*\alpha = (-1)^p\alpha$. Here, τ stands for complex conjugation. Similarly, $D^{p,p}(Y)$ refers to currents satisfying these properties. A cycle $Z = \sum_\alpha r_\alpha Z_\alpha$, where Z_α are closed irreducible subvarieties of codimention p on Y defines a current of integration δ_Z. For a form η of complementary degree, $\delta_Z(\eta) = \sum_\alpha r_\alpha \int_{Z_\alpha^{ns}(\mathbf{C})} \eta$, where $Z_\alpha^{ns} = Z_\alpha - Z_\alpha^{sing}$ is the smooth part of Z_α. A green current for Z is any current g in $D^{p-1,p-1}(Y_\mathbf{R})$ such that the curvature $h_Z = \delta_Z - \dfrac{\delta\bar{\delta}}{\pi i} g$ is a smooth form in $A^{p,p}(Y)$.

Consider the Chow group $\mathrm{CH}^k(X)_0$ of codimension k null-homologous algebraic cycles on X modulo rational equivalence. Let $\widehat{\mathrm{CH}}^k(X)_0$ be the group generated by pairs (Z,g) where Z is in $\mathrm{CH}^k(X)_0$, and g is a green current for Z. There is an associative and commutative intersection product due to Gillet and Soulé

$$\widehat{\mathrm{CH}}^k(X)_0 \times \widehat{\mathrm{CH}}^{k'}(X)_0 \longrightarrow \widehat{\mathrm{CH}}^{k+k'}(X)_0 \tag{2.8}$$

mapping cycles $\hat{Z}_1 = (Z_1, g_1)$ in $\widehat{\mathrm{CH}}^k(X)_0$ and $\hat{Z}_2 = (Z_2, g_2)$ in $\widehat{\mathrm{CH}}^{k'}(X)_0$ with $\mathrm{codim}(Z_1 \cap Z_2) = k + k'$ to $(Z_1 \cdot Z_2, g_2\delta_{Z_1} + h_2 g_1)$. This defines a height pairing by

$$\langle Z_1, Z_2 \rangle_{\mathrm{GS}} = (-1)^k \hat{Z}_1 \cdot \hat{Z}_2, \tag{2.9}$$

where the dot indicates taking intersection in $\mathrm{CH}^k(X) \times \mathrm{CH}^{k'}(X)$. In Section (7) we discuss the nondegeneracy of the Gillet-Soulé pairing.

When the arithmetic cycles $\hat{Z}_1 = (Z_1, g_1)$ and $\hat{Z}_2 = (Z_2, g_2)$ have codimension p and $d - p$ in Y, with Z_1 and Z_2 irreducible intersecting properly, then

$$\hat{Z}_1 \cdot \hat{Z}_2 = \log |\Gamma(Z_1 \cdot Z_2, 0)| + \int_{Z_2(\mathbf{C})} g_1 + \int_{Y(\mathbf{C})} g_2 h_{Z_1}. \tag{2.10}$$

2.5. **Hodge structures.** Denote by $\mathcal{A}(\mathbf{B}^\times)$ the set of equivalence classes of irreducible admissible representations π of $\mathbf{B}^\times$ that correspond by Jacquet-Langlands to cuspidal automorphic representations of $GL_2(\mathbf{A}_F)$, discrete of parallel weight $2k$ at infinity. By [2, Theorem 5.9], we have a Hodge decomposition of the étale cohomology of W_U depending on the fixed embedding τ of F,

$$H^{k-1}(W_U(\mathbf{C}), \mathbf{Q}) \otimes_\mathbf{Q} \mathbf{C} \simeq \bigoplus_{1 \leq i \leq k-1} H^i(W_U(\mathbf{C}), R^{k-1-i}\pi_{U*}\mathbf{Q}) \otimes_\mathbf{Q} \mathbf{C}, \tag{2.11}$$

where $\pi_U : W_U \longrightarrow X_U$, and a natural isomorphism

$$H^{k-1,0}(W_U, \mathbf{C}) \simeq \bigoplus_{\pi \in \mathcal{A}(\mathbf{B}^\times)} \pi^U, \tag{2.12}$$

where π^U are the fixed vectors of π by U. Note that π^U is nonzero for only finitely many π, and appears with multiplicity at most one. By complex conjugation we also have a decomposition of $H^{0,k-1}(W_U, \mathbf{C})$ into $\bigoplus_{\pi\in\mathcal{A}(\mathbf{B}^\times)} \bar{\pi}^U$, whereas the contragredient representation gives a decomposition into the dual spaces $\bigoplus_{\pi\in\mathcal{A}(\mathbf{B}^\times)} \tilde{\pi}^U$, with the usual pairing

$$\pi \times \tilde{\pi} \to \mathbf{C}. \tag{2.13}$$

Taking direct limits, we obtain similar decompositions of the cohomology of W in terms of $\pi, \bar{\pi}$ and $\tilde{\pi}$.

2.6. **Hodge class.** The action of U on $\mathscr{A}_U$ induces an action of $\mathbf{B}^\times$ on $R^q\pi_*\mathbf{Q}_p$, for $q \geq 1$. Consider the p-adic sheaf

$$\mathscr{L}_{U,2} := \bigcap_{b\in\mathbf{B}} \ker(b - \mathrm{n}(b) : R^2\pi_*\mathbf{Q}_p \to R^2\pi_*\mathbf{Q}_p),$$

which is a 3-dimensional local system on the Shimura curve. The non-degenerate pairing

$$(\ ,\ ) : \mathscr{L}_{U,2} \otimes \mathscr{L}_{U,2} \hookrightarrow R^2\pi_*\mathbf{Q}_p \otimes R^2\pi_*\mathbf{Q}_p \xrightarrow{\cup} R^4\pi_*\mathbf{Q}_p \xrightarrow{tr} \mathbf{Q}_p(-2)$$

induces a Laplacian operator

$$\Delta_{k-1} : \mathrm{Sym}^{k-1}\mathscr{L}_{U,2} \longrightarrow (\mathrm{Sym}^{k-3}\mathscr{L}_{U,2})(-2),$$

and $\mathscr{L}_{U,2k-2} \subseteq \mathrm{Sym}^{k-1}\mathscr{L}_{U,2}$ is defined as the kernel of Δ_{k-1}.

We consider the projector constructed in [2, Theorem 5.8(iii)],

$$\mathscr{P}_U \in \mathrm{Corr}^0(W_U, W_U) = \mathrm{CH}^{2k-1}(W_U \times W_U) \tag{2.14}$$

encoding the construction of the line bundle $\mathscr{L}_{U,2k-2}$. Its global sections are holomorphic modular forms of weight $2k$ and level U [10, Section 10.1]. We denote by $\deg(\mathscr{P}_U)$ the degree of the correspondence.

In what follows, we index the connected components of W_U by α in $\pi_0(W_{U,\bar{F}})$, and denote by $\mathscr{P}_{U,\alpha}$ the restriction of $\mathscr{P}_U$ to a connected component α of W_U.

2.7. **Hecke correspondences.** Let R_x be the action of x in $\mathbf{B}^\times$ on X by right translation, so that X_U is the quotient of X by the action of $\{R_x : x \in U\}$. This gives an isomorphism $X_{x^{-1}Ux} \to X_U$ for each U, inducing a map $W \longrightarrow W$ which we still denote by R_x. The Hecke algebra $\mathscr{H} = C_c^\infty(\mathbf{B}_f^\times)$ is defined as the smooth and compactly supported functions from $\mathbf{B}_f^\times$ to $\mathbf{C}$. Given $\phi \in \mathscr{H}$, define a correspondence on W given by

$$R(\phi) = \int_{\mathbf{B}_f^\times} \phi(x) R_x \, dx. \tag{2.15}$$

At level U, define the Hecke algebra $\mathscr{H}_U = \{\phi \in \mathscr{H} \mid \phi(UxU) = \phi(x),\ x \in \mathbf{B}_f^\times\}$.

Let $Z(x) \subset W \times W$ be the graph of R_x, and $Z(x)_U$ its image in $W_U \times W_U$. Define the linear combination

$$R(\phi)_U = \sum_{x\in U\backslash\mathbf{B}^\times/U} \phi(x) Z(x)_U \tag{2.16}$$

in $\mathrm{CH}^{2k-1}(W_U \times W_U)_{\mathbf{C}}$. We have a map

$$\mathrm{CH}^{2k-1}(W_U \times W_U) \to \mathrm{Hom}(\mathrm{CH}^{k-1}(W_U), \mathrm{CH}^{k-1}(W_U)), \tag{2.17}$$

given by pushforward of correspondences,

$$c \mapsto \big(c_1 \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^{-1}(c_1)\cdot c)\big). \tag{2.18}$$

The direct systems are compatible as U varies by the same argument as in [21, Lemma 3.1.2], observing that [12, Prop 1.13] holds in our setting. Consider the diagonal subspace

$$\iota_U : \bigoplus_{\pi \in \mathcal{A}(\mathbf{B}^\times)} \pi^U \otimes \tilde{\pi}^U \hookrightarrow \mathrm{Hom}(H^{k-1,0}(W_U, \mathbf{C}), H^{k-1,0}(W_U, \mathbf{C})). \tag{2.19}$$

The following lemma allows us to identify its image with an induced morphism on Chow groups.

Lemma 2.7.1. *For π in $\mathcal{A}(\mathbf{B}^\times)$, there is a function ϕ in $\mathscr{H}_U$ such that*

$$\iota_U(\pi^U \otimes \tilde{\pi}^U) = R(\phi)^*_U. \tag{2.20}$$

The proof of the lemma is as in [21, Prop 3.14(1)]. By the lemma, it follows that the image of ι_U is contained in the image of the inclusion

$$\mathrm{Hom}_F(\mathrm{CH}^{k-1}(W_U), \mathrm{CH}^{k-1}(W_U))_{\mathbf{C}} \hookrightarrow \mathrm{Hom}(H^{k-1,0}(W_U, \mathbf{C}), H^{k-1,0}(W_U, \mathbf{C})), \tag{2.21}$$

since the image of (2.19) is represented by the induced morphism $R(\phi)^*_U$ on (2.17). This defines a map

$$T_U : \bigoplus_{\pi \in \mathcal{A}(\mathbf{B}^\times)} \pi^U \otimes \tilde{\pi}^U \hookrightarrow \mathrm{Hom}_F(\mathrm{CH}^{k-1}(W_U), \mathrm{CH}^{k-1}(W_U))_{\mathbf{C}}. \tag{2.22}$$

Lemma 2.7.2. *We have a direct system*

$$T = \{\deg(\mathscr{P}_U) T_U\}_U, \tag{2.23}$$

defining a map

$$T : \bigoplus_{\pi \in \mathcal{A}(\mathbf{B}^\times)} \pi \otimes \tilde{\pi} \hookrightarrow \mathrm{Hom}^0(\mathrm{CH}^{k-1}(W), \mathrm{CH}^{k-1}(W))_{\mathbf{C}}. \tag{2.24}$$

Proof. The key property of the normalizing volume factor $\mathrm{vol}(X_U)$ in [21, Prop 3.14(2)] is that it is computed by the degree of the Hodge bundle $\deg(L_U)$. We normalize instead by $\deg(\mathscr{P}_U)$ and use the pullback and pushforward maps $p^*_{U',U}$ and p_{U',U_*} on the spaces $H^{k-1,0}(W_U, \mathbf{C})$ and $H^{k-1,0}(W_{U'}, \mathbf{C})$ corresponding to the inclusion of open compact sets $U' \subset U$, the lemma follows. □

2.8. **The main identity.** We formulate the main identity in terms of projectors. Recall that K is an imaginary quadratic extension of F with a fixed embedding $\mathbf{A}_K \hookrightarrow \mathbf{B}^\times$ over $\mathbf{A}_F$ defining a torus T over F and an associated quadratic character $\eta : K^\times \backslash \mathbf{A}_K^\times \longrightarrow \mathbf{C}^\times$. Also, χ is a character $\chi : \mathrm{Gal}(K^{ab}/K) \longrightarrow \mathbf{C}^\times$ with $\omega_\pi \cdot \chi\big|_{\mathbf{A}_\mathbf{F}^\times} = 1$, where ω_π is the central character of π. By the reciprocity law, it can be viewed as a character of $K^\times \backslash \mathbf{A}_K^\times \longrightarrow \mathbf{C}^\times$.

Denote by $(\cdot,\cdot) :\ \pi_v \otimes \tilde{\pi}_v \longrightarrow \mathbf{C}$ the pairing of contragredient representations. Consider the period integral $\alpha = \otimes_v \alpha_v :\ \pi \otimes \tilde{\pi} \longrightarrow \mathbf{C}$ where α_v is defined by the linear forms

$$\alpha_v(f_1, f_2) = \frac{L(1, \eta_v) L(1, \pi_v, \mathrm{ad})}{\zeta_{F_v}(2) L(k - \frac{1}{2}, \pi_v, \chi_v)} \int_{K^\times/F^\times} (\pi_v(t) f_1, f_2) \chi_v(t) dt \tag{2.25}$$

for f_1 in π_v and f_2 in $\tilde{\pi}_v$.

Consider the CM-cycle Z arising from the CM point of conductor 1 on the Shimura curve corresponding to the embedding $K \hookrightarrow \mathbf{B}^\times$. Viewing R_t as a correspondence in $\mathrm{CH}^{2k-1}(W \times W)$, the χ-eigencomponent Z_χ of Z, given by

$$Z_\chi = \mathrm{vol}^{-1}([T]) \int_{[T]} R_t(Z)\chi(t)dt, \tag{2.26}$$

belongs to $\mathrm{CH}^{2k-1}(W)_0(\bar{F})_{\mathbf{C}}$.

We state the main theorem.

Theorem 2.8.1. *For $f_1 \otimes f_2$ in $\pi \otimes \tilde{\pi}$, we have*

$$L'(\frac{1}{2}, \pi, \chi) = \frac{4L(1,\eta)^2 L(1,\pi,\mathrm{ad})}{\zeta_F(2)\alpha(f_1 \otimes f_2)} \langle T(f_1 \otimes f_2) Z_\chi, Z_{\chi^{-1}} \rangle_{\mathrm{GS}} \tag{2.27}$$

where $\langle \cdot, \cdot \rangle_{\mathrm{GS}}$ is the Gillet-Soulé pairing on arithmetic Chow groups.

The Gross-Zagier formula obtained in [23, Corollary 0.3.1] under the Heegner hypothesis is given by

$$L'(k, f, \chi) = \frac{2^{4k-1}\pi^{2k}(f,f)_{\mathrm{Pet}}}{(2k-2)!u^2 h\sqrt{|D|}} \langle s'_{\chi,f}, s'_{\chi,f} \rangle_{\mathrm{GS}}, \tag{2.28}$$

where f is a weight $2k$ modular form and $s'_{\chi,f}$ is a suitable Heegner cycle on the Kuga-Sato variety fibered over the classical modular curve. It can be compared to our formula by a careful use of the explicit computations in [4] for the fudge factors in the Gross-Zagier and Waldspurger formulae. More precisely, the classical Petersson inner product on weight $2k$ modular forms $(f,f)_{\mathrm{Pet}}$ is proportional to $L(1,\pi,\mathrm{ad})/\zeta_F(2)$ (see [4, Proposition 1.11]), the constant $(2k-2)!$ arises from the Gamma factors of the archimedean L-values, and the arithmetic constants are given by the usual class number formula for $L(1,\eta)$.

3. Analytic and geometric kernels

3.1. **Preliminaries.** In this section, we describe the Schwartz and Fock spaces, Waldspurger's Weil representation, and Shimizu's theta lift which we use to define the analytic and geometric kernels.

Let $(\mathbf{V}, q)$ be an even-dimensional quadratic space over $\mathbf{A}$, where q is the reduced norm. Similarly, denote by (V_v, q_v) the local quadratic spaces. Given an embedding $\mathbf{A}_K \hookrightarrow \mathbf{B}$, we have an orthogonal decomposition

$$\mathbf{B} = \mathbf{A}_K + j\mathbf{A}_K, \tag{3.1}$$

where $j^2 \in \mathbf{A}_F^\times$. The space $\mathbf{V}$ is coherent if it has a model over F, and incoherent otherwise. Note that $\mathbf{V}_1 = \mathbf{A}_K$ is coherent, while $\mathbf{V}_2 = j\mathbf{A}_K$ is incoherent if and only if S is odd.

3.1.1. *The Weil representation.* Let F_v be a nonarchimedean local field, and consider (V_v, q_v) over F_v. Given $u \in F_v^\times$, we can associate to F_v a quadratic space with new norm (V_v, uq_v). Let $GO(\mathbf{V})$ be the general orthogonal group of $\mathbf{V}$ with similitude character $\nu : GO(\mathbf{V}) \to \mathbf{G}_m$, and denote by $S(V_v \times F_v^\times)$ the Schwartz space, that is, compactly supported, locally constant, complex-valued functions on $V_v \times F_v^\times$.

We denote

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \; w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$m(a) = \mathrm{diag}(a, a^{-1})$, and $d(a) = \mathrm{diag}(1, a)$. The Weil representation of $G(F_v) \times GO(V_v)$ on $S(V_v \times F_v^\times)$ is defined as follows:

$$r(n(b), \cdot)\Phi(x, u) = \psi(buq(x))\Phi(x, u),\ b \in F_v^\times \tag{3.2}$$

$$r(m(a), \cdot)\Phi(x, u) = \chi_{(V_v, uq_v)}(a)|a|^{\dim V/2}\Phi(ax, u),\ a \in F_v^\times \tag{3.3}$$

$$r(d(a), \cdot)\Phi(x, u) = |a|^{-\dim V/4}\Phi(x, a^{-1}u),\ a \in F_v^\times \tag{3.4}$$

$$r(w, \cdot)\Phi(x, u) = \gamma_{(V_v, uq_v)}\hat{\Phi}(x, u) \tag{3.5}$$

$$r(\cdot, h)\Phi(x, u) = \Phi(h^{-1}x, \nu(h)u),\ h \in GO(V_v) \tag{3.6}$$

where $\chi_{(V_v, uq_v)}$ is the quadratic character attached to V_v, $\gamma_{(V_v, uq_v)}$ is a fourth root of unity depending on V_v,, and $\hat{\Phi}$ is the Fourier transform of $\Phi(x, u)$ in the x variable, with respect to the self-dual measure determined by ψ (1.8). Set $\bar{\mathscr{S}}(V_v \times F_v^\times) = S(V_v \times F_v^\times)$ in this case.

Consider F_v real archimedean, and assume V_v is positive definite. Define $S(V_v \times \mathbf{R}^\times)$ to be the Fock space, which is the space of functions spanned by

$$P(x)e^{-2\pi|u|q_v(x)}H(u), \tag{3.7}$$

where H is a compactly supported smooth function on $\mathbf{R}^\times$ and P is a complex polynomial function on V_v. This space is not stable under the action of $G(\mathbf{R})$, but under the restriction of the induced $(\mathfrak{gl}_2(\mathbf{R}), O_2(\mathbf{R}))$ action on the usual Schwartz space [21, §2.1.2]. Also define the reduced Fock space $\bar{\mathscr{S}}(V_v \times \mathbf{R}^\times)$, spanned by functions

$$P_1(uq_v(x)) + \mathrm{sgn}(u)P_2(uq_v(x))e^{-2\pi|u|q_v(x)}, \tag{3.8}$$

where P_1, P_2 are polynomial functions, which admits a surjective quotient map $S(V_v \times \mathbf{R}^\times) \to \bar{\mathscr{S}}(V_v \times \mathbf{R}^\times)$.

Suppose that $\mathbf{V}_\infty = \prod_{v|\infty} V_v$ is positive-definite. We define the restricted tensor product $S(\mathbf{V} \times \mathbf{A}^\times) = \otimes' S(V_v \times F_v^\times)$ with respect to the spherical functions

$$\Phi_v(x, u) = 1_{\mathcal{V}_v}(x)1_{\varpi_v^{n_v}}(u) \tag{3.9}$$

for nonarchimedean v, where $\mathcal{V}_v \subset V_v$ is an $\mathcal{O}_{F_v}$-lattice, n_v is the level of ψ_v, and ϖ_v is a uniformizer of $\mathcal{O}_{F_v}$. We call $\Phi_v(x, u)$ the standard Schwartz function at v. The reduced space $\bar{\mathscr{S}}(V_v \times \mathbf{R}^\times)$ is similarly defined. The Weil representation gives an action of $(\mathfrak{gl}_2(F_\infty), O_2(V_\infty)) \times G(\mathbf{A}_f) \times GO(\mathbf{V}_f)$ on both spaces.

We introduce in particular at real archimedean places the Schwartz function $\Phi_k(x) = P_k(x)e^{-2\pi|u|q_v(x)}$ of weight $-2k$,

$$P_k(x) = \sum_{j=0}^{k} \frac{(-4\pi|u|)^j j!}{(2j)!}\binom{k}{j}(x_1 - x_2)^{2j} \tag{3.10}$$

so that $r(k_\theta)\Phi_k(x) = e^{-2\pi ik\theta}\Phi_k(x)$ for all $k_\theta \in SO_2(\mathbf{R})$ [15, Proposition 2.5.5].

Remark 3.1.1. If $\mathbf{V}$ is the orthogonal space $(\mathbf{B}, q)$, we can identify $S(\mathbf{V} \times \mathbf{A}^\times)$ with the space of Schwartz functions bi-invariant under an open compact subgroup U of $\mathbf{B}_f^\times$, and we identify $\bar{\mathscr{S}}(\mathbf{V} \times \mathbf{A}^\times)$ with the maximal $\mathbf{B}_\infty^\times \times \mathbf{B}_\infty^\times$ quotient. Given an element Φ in $S(\mathbf{V} \times \mathbf{A}^\times)$, its image in $\bar{\mathscr{S}}(\mathbf{V} \times \mathbf{A}^\times)$ can be obtained by the integration

$$\int_{F_\infty^\times \backslash (\mathbf{B}_\infty^\times \times \mathbf{B}_\infty^\times)} r(1, h_\infty)\Phi(x, u)dh_\infty, \tag{3.11}$$

where $F_\infty^\times$ is diagonally embedded into $\mathbf{B}_\infty^\times \times \mathbf{B}_\infty^\times$.

3.1.2. *Shimizu's theta lift.* Let σ be a cuspidal automorphic representation of $G(\mathbf{A})$ which is a discrete series at all places in S, and let π be the representation of $\mathbf{B}^\times$ associated to σ by the Jacquet-Langlands correspondence. Consider the embedding of $(\mathbf{B}^\times \times \mathbf{B}^\times)/\mathbf{A}^\times$ in $GO(\mathbf{V})$ where $h = (h_1, h_2)$ in $(\mathbf{B}^\times \times \mathbf{B}^\times)/\mathbf{A}^\times$ acts on x in V by $(h_1, h_2)x = h_1 x h_2^{-1}$. For g in $G(\mathbf{A})$, h in $(\mathbf{B}^\times \times \mathbf{B}^\times)/\mathbf{A}^\times$, and Φ in $S(\mathbf{V} \times \mathbf{A}^\times)$, define the theta series

$$\theta(g, h, \Phi) = \sum_{u \in F^\times} \sum_{x \in V} r(g, h)\Phi(x, u). \tag{3.12}$$

The incoherent (global) Shimizu lift of φ in σ, given by

$$\theta(\varphi \otimes \Phi)(h) = \frac{\zeta_F(2)}{2L(1, \pi, \mathrm{ad})} \int_{G(F)\backslash G(\mathbf{A})} \varphi(g)\theta(g, h, \Phi) dg, \tag{3.13}$$

is an element in $\mathrm{Hom}_{G(\mathbf{A}) \times \mathbf{B}^\times \mathbf{B}^\times}(\sigma \otimes S(\mathbf{V} \times \mathbf{A}^\times), \pi \otimes \tilde{\pi})$.

3.2. **Analytic kernel.** Our expression for the analytic kernel follows from the derivation of [21, §2], based on the Waldspurger formula [19], by observing that the proofs of the necessary statements hold for general cuspidal automorphic forms in GL_2.

For g in $G(\mathbf{A})$, and Φ in $S(\mathbf{V} \times \mathbf{A}^\times)$, define the mixed Eisenstein-theta series

$$I(s, g, \Phi) = \sum_{\gamma \in P(F)\backslash G(F)} \delta(\gamma g)^s \sum_{(x_1, u) \in V_1 \times F^\times} r(\gamma g, 1)\Phi(x_1, u). \tag{3.14}$$

The *analytic kernel* is its χ-component

$$I(s, g, \chi, \Phi) = \int_{T(F)\backslash T(\mathbf{A})} I(s, g, r(t, 1)\Phi)\chi(t) dt. \tag{3.15}$$

Given $\varphi \in \sigma$, consider the Petersson inner product

$$P(s, \chi, \Phi, \varphi) := (I(s, g, \chi, \Phi), \varphi(g))_{\mathrm{Pet}} = \int_{[G]} \varphi(g) I(s, g, \chi, \Phi) dg. \tag{3.16}$$

By a result of Waldspurger [21, Prop 2.5], we have $P(s, \chi, \Phi, \varphi) = \prod_v P_v(s, \chi_v, \Phi_v, \varphi_v)$, where $P_v(s, \chi_v, \Phi_v, \varphi_v)$ is given by the local integral representation

$$P_v(s, \chi_v, \Phi_v, \varphi_v) = \int_{Z_v\backslash T_v} \chi(t) dt \int_{N_v\backslash G_v} \delta(g)^s W_{-1,v}(g) r(g, 1)\Phi_v(t^{-1}, q_v(t)) dg, \tag{3.17}$$

and $W_{-1,v}$ is the Whittaker function of φ_v with respect to $\bar{\psi}_v$. Moreover, for v unramified, we have

$$P_v(s, \chi_v, \Phi_v, \varphi_v) L(1 + s, \eta_v) = L\big(\frac{1+s}{2}, \pi_v, \chi_v\big). \tag{3.18}$$

Hence, we normalize the local integrals by the functional α_v in $\mathrm{Hom}(\pi_v \otimes \tilde{\pi}_v, \mathbf{C})$, given by the integration of matrix coefficients:

$$\alpha_v(f_1 \otimes f_2) = \frac{L(1, \eta_v) L(1, \pi_v, \mathrm{ad})}{\zeta_{F_v}(2) L(\frac{1}{2}, \pi_v, \chi_v)} \int_{T(F_v)/Z(F_v)} (\pi_v(t) f_1, f_2)\chi_v(t) dt, \tag{3.19}$$

where as usual the torus T is defined by the quadratic extension K.

This leads to the integral representation of the L-function:

Proposition 3.2.1 ([21, Prop 2.6]). *If $|S|$ is odd, then $L(\frac{1}{2}, \pi, \chi) = 0$ and the value of the derivative of $P(s, \chi, \Phi, \varphi)$ at $s = 0$ is*

$$(I'(0, \chi, \Phi, \varphi), \varphi(g))_{\mathrm{Pet}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{2L(1, \eta)} \prod_v \alpha_v(\theta(\Phi_v \otimes \varphi_v)). \tag{3.20}$$

This analytic kernel will be compared with the geometric kernel, which we introduce next.

3.3. **The geometric kernel.** In order to define the geometric kernel, we first define the generating series of algebraic cycles, which will be the focus of our study.

3.3.1. *The generating series.* Our goal is to define a generating series in

$$\mathrm{Hom}_{(\mathbf{B}^\times \times \mathbf{B}^\times \times G(\mathbf{A}))}(S(V, \mathbf{A}^\times),\ C^\infty(G(F)\backslash G(\mathbf{A}), \mathrm{CH}^{2k-1}(W \times W))). \tag{3.21}$$

First, observe that $\mathrm{CH}^{2k-1}(W \times W)$ is invariant under $\mathbf{B}_\infty^\times \times \mathbf{B}_\infty^\times$. Thus we may work with functions in the quotient $\bar{\mathscr{S}}(\mathbf{V} \times \mathbf{A}^\times)$.

For g in $GL_2(\mathbf{A})$, u in $F^\times$, and Φ in $\bar{\mathscr{S}}(\mathbf{V} \times \mathbf{A}^\times)$, define the Siegel Eisenstein series by

$$E(s, g, u, \Phi) = \sum_{\gamma \in P^1(F)\backslash SL_2(F)} \delta(\gamma g)^s r(\gamma g, 1)\Phi(0, u). \tag{3.22}$$

The constant term of its Fourier expansion is given in terms of Whittaker functions

$$E_0(s, g, u, \Phi) = \delta(g)^s r(g, 1)\Phi(0, u) + W_0(s, g, u, \Phi), \tag{3.23}$$

where

$$W_a(s, g, u, \Phi) = \int_{N(\mathbf{A})} \delta(wng)^s r(wng, 1)\Phi(0, u)\psi(-an)dn \tag{3.24}$$

for $a \in F^\times$. Here we have written $n = n(b)$. The so-called intertwining part $W_0(s, g, u, \Phi)$ has analytic continuation to $s = 0$, is nonzero only if $F = \mathbf{Q}$ and $S = \{\infty\}$, and satisfies

$$W_0(0, g, u, \Phi) = W_0(0, 1, u, r(g, 1)\Phi). \tag{3.25}$$

Let $\mu_U = F^\times \cap U$ which can be viewed as a finite index subgroup of $\mathcal{O}_F^\times$, and let $w_U = |\{-1, 1\} \cap U|$ which is equal to 1 if U is small enough, otherwise it is 2.

We define the generating series

$$Z(g, \Phi)_U = Z_0(g, \Phi)_U + Z_*(g, \Phi)_U, \tag{3.26}$$

where

$$Z_*(g, \Phi)_U = w_U \sum_{a \in F^\times} \sum_{x \in U\backslash \mathbf{B}_f^\times/U} r(g)\Phi(x, aq(x^{-1}))Z(x)_U, \tag{3.27}$$

and

$$Z_0(g, \Phi)_U = -\sum_\alpha \sum_{u \in \mu_U^2\backslash F^\times} E_0(0, 1, \alpha^{-1}u, r(g)\Phi)\mathscr{P}_{U,\alpha}. \tag{3.28}$$

We refer to $Z_0(g, \Phi)_U$ as the constant part of $Z(g, \Phi)_U$.

We normalize the generating series as follows

$$\tilde{Z}(g, \Phi)_U = \frac{2^{[F:\mathbf{Q}]-1} h_F |d_F|^{-1/2}}{[\mathcal{O}_F^\times : \mu_U^2]} Z(g, \Phi)_U, \tag{3.29}$$

where h_F is the class number of F, and d_F the discriminant of F. Note that the quotient equals 2 if $F = \mathbf{Q}$. Also, the denominator is finite since μ_U^2 is a subgroup of finite index, and ensures that the right-hand side is compatible under pull-back from different levels.

Lemma 3.3.1. *The system $\tilde{Z}(g, \Phi) = \{\tilde{Z}(g, \Phi)_U\}_U$ is compatible with pull-back and defines an element of $\mathrm{CH}^{2k-1}(W \times W)_{\mathbf{C}}$.*

Proof. The proof follows [21, Lemma 3.18] with the pull-back maps obtained instead from the morphism of Kuga-Sato varieties

$$W_{U'} \times W_{U'} \to W_U \times W_U \tag{3.30}$$

induced from right multiplication of (h_1, h_2) in $\mathbf{B}_f^\times \times \mathbf{B}_f^\times$ on the underlying Shimura curves, under the condition that $h_i U' h_i^{-1} \subset U$ and U acts trivially on Φ. □

3.3.2. *The geometric kernel.* Define the height series

$$\tilde{Z}(g, (h_1, h_2), \Phi) = \langle \tilde{Z}(g, \Phi) R_{h_1}(Z), R_{h_2}(Z) \rangle_{\mathrm{GS}} \tag{3.31}$$

for h_1, h_2 in $\mathbf{B}^\times$. As in [21, Lemma 3.19], the height series $\tilde{Z}(g, (h_1, h_2), \Phi)$ is independent of Z and invariant under the left action of $T_F \times T_F$ on (h_1, h_2), and it is a cusp form on $GL_2(\mathbf{A})$.

The *geometric kernel* is the χ-component of $\tilde{Z}(g, (h_1, h_2), \Phi)$ given by

$$\tilde{Z}(g, \chi, \Phi) = \int_{[T]}^{*} \tilde{Z}(g, (t, 1), \Phi) \chi(t) dt. \tag{3.32}$$

Lemma 3.3.2. *The geometric kernel $\tilde{Z}(g, \chi, \Phi)$ is a cuspidal automorphic representation of $GL_2(\mathbf{A})$, and satisfies*

$$\tilde{Z}(g, \chi, r(t_1, t_2)\Phi) = \chi(t_1^{-1} t_2) \tilde{Z}(g, \chi, \Phi) \tag{3.33}$$

for $t_1, t_2 \in \mathbf{A}_E^\times$.

Proof. See [21, Lemma 3.20]. □

3.4. **The kernel identity.** We describe the arithmetic theta lifting and use it to show that the main theorem is equivalent to a kernel identity that can be viewed as an arithmetic Siegel-Weil formula according to the philosophy of Kudla. We give two versions, the first in terms of a function Φ in $S(\mathbf{V} \times \mathbf{A}^\times)$, and the second in terms of its image ϕ in $\bar{\mathscr{S}}(\mathbf{V} \times \mathbf{A}^\times)$.

3.4.1. *Kernel identity I.* Shimizu's theta lift gives a map $\theta : S(\mathbf{V} \times \mathbf{A}^\times) \times \sigma \to \pi \otimes \tilde{\pi}$, which we compose with a Hecke correspondence $T : \pi \otimes \tilde{\pi} \to \mathrm{Hom}_F(\mathrm{CH}^{k-1}, \mathrm{CH}^{k-1})$. For φ in σ and Φ in $S(\mathbf{V} \times \mathbf{A}^\times)$, the arithmetic theta lifting asserts that

$$\tilde{Z}(\varphi \otimes \Phi) := \int_{[G]}^{*} \varphi(g) \tilde{Z}(g, \Phi) dg \tag{3.34}$$

is proportional to the Shimizu lift

$$\tilde{Z}(\varphi \otimes \Phi) = \frac{L(1, \pi, \mathrm{ad})}{2\zeta_F(2)} T(\theta(\varphi \otimes \Phi)), \tag{3.35}$$

see Theorem (4.2.1) for a precise formulation.

Then the first from of the kernel identity to be established is the following:

Theorem 3.4.1. *Consider Φ in $S(\mathbf{V}\times\mathbf{A}^\times)$. Then the identity*

$$(I'(0,\cdot,\chi,\Phi),\varphi)_{\mathrm{Pet}} = 2(\tilde{Z}(\cdot,\chi,\Phi),\varphi)_{\mathrm{Pet}} \tag{3.36}$$

for all φ in σ, together with the arithmetic theta lifting (3.35), implies Theorem (2.8.1).

Proof. The Shimizu lift θ is surjective since $\pi\otimes\tilde{\pi}$ is an irreducible representation of $\mathbf{B}^\times\times\mathbf{B}^\times$. Therefore, by linearity, we may assume that

$$f_1\otimes f_2 = \theta(\varphi\otimes\Phi) \tag{3.37}$$

for some φ in σ and Φ in $S(\mathbf{B}\times\mathbf{A}^\times)$. Applying Proposition (3.2.1) to the kernel identity, we have

$$\frac{L'(\frac{1}{2},\pi,\chi)}{2L(1,\eta)}\alpha(\theta(\varphi\otimes\Phi)) = 2(\tilde{Z}(\cdot,\chi,\Phi),\varphi)_{\mathrm{Pet}}. \tag{3.38}$$

It then suffices to prove that

$$(\tilde{Z}(\cdot,\chi,\Phi),\varphi)_{\mathrm{Pet}} = \frac{L(1,\pi,\mathrm{ad})L(1,\eta)}{\zeta_F(2)}\langle T(f_1\otimes f_2)Z_\chi, Z_{\chi^{-1}}\rangle_{\mathrm{GS}}. \tag{3.39}$$

Using the relation

$$\langle\tilde{Z}(g,\Phi)R_t(Z),R_1(Z)\rangle_{\mathrm{GS}} = \langle\tilde{Z}(g,\Phi)R_{t't}(Z),R_{t'}(Z)\rangle_{\mathrm{GS}} \tag{3.40}$$

for t' in $T(\mathbf{A})$, which follows by viewing the action of $T(F)\backslash T(\mathbf{A})\simeq K^\times\backslash\mathbf{A}_K^\times$ on the cycle Z as a Galois action by class field theory, and after a change of variables, we obtain

$$\tilde{Z}(g,\chi,\Phi) = 2L(1,\eta)\langle\tilde{Z}(g,\Phi)Z_\chi, Z_{\chi^{-1}}\rangle_{\mathrm{GS}}, \tag{3.41}$$

where $2L(1,\eta)$ arises from the class number formula for $T(F)$.

Hence, we have

$$(\tilde{Z}(\cdot,\chi,\Phi),\varphi)_{\mathrm{Pet}} = 2L(1,\eta)\langle\tilde{Z}(\varphi\otimes\Phi)Z_\chi, Z_{\chi^{-1}}\rangle_{\mathrm{GS}}. \tag{3.42}$$

The main identity follows from the arithmetic theta lift (3.35). ∎

3.4.2. *Kernel identity II.* We restate the kernel identity for ϕ in $\bar{\mathscr{S}}(\mathbf{V}\times\mathbf{A}^\times)$ after rewriting the analytic kernel in terms of ϕ (the geometric kernel does not change).

We define the mixed theta-Eisenstein series for ϕ with respect to U,

$$I(s,g,\phi)_U = w_U\sum_{\gamma\in P^1(F)\backslash SL_2(F)}\delta(\gamma g)^s\sum_{x_1\in E}\sum_{u\in\mu_U^2\backslash F^\times} r(\gamma g,1)\phi(x_1,u), \tag{3.43}$$

where $\mu_U = F^\times\cap U$, as before. It defines an automorphic form on $GL_2(\mathbf{A})$. If $w_U = 1$, which is the case if U is small enough, we may rewrite it as

$$\sum_{\gamma\in P(F)\backslash GL_2(F)}\delta(\gamma g)^s\sum_{(x_1,u)\in\mu_U\backslash(E\times F^\times)} r(\gamma g,1)\phi(x_1,u). \tag{3.44}$$

Define the twisted average

$$I(s,g,\chi,\phi)_U = \int_{[T]}^{*} I(s,g,r(t,1)\phi)_U\chi(t)dt. \tag{3.45}$$

Note that both $\chi(t)$ and $I(s,g,r(t,1),\phi)_U$ are invariant under $T(F)$ and $T(F_\infty)$ as functions of t in $T(\mathbf{A})$, so that the regularized integral is well-defined. Also note that for different levels $U'\subset U$, we have the relation

$$I(s,g,\phi)_{U'} = [\mu_U^2:\mu_{U'}^2]I(s,g,\phi)_U, \tag{3.46}$$

and similarly for $I(s,g,\chi,\phi)_{U'}$.

We state the second version of the kernel identity, which is the one we prove:

Theorem 3.4.2. *Let ϕ be a Schwartz function in $\bar{\mathscr{S}}(\mathbf{V}\times\mathbf{A}^\times)$ bi-invariant under a fixed $U\subset\mathbf{B}_f^\times$, and let Φ be its preimage in $S(\mathbf{V}\times\mathbf{A}^\times)$. Then*

$$(I'(0,\cdot,\chi,\phi)_U,\varphi)_{\mathrm{Pet}} = 2(\tilde{Z}(\cdot,\chi,\phi),\varphi)_{\mathrm{Pet}} \tag{3.47}$$

is equivalent to (3.36).

Proof. The equivalence follows from the comparison of normalizations, namely from the identity

$$I(s,g,\chi,\Phi) = \frac{2^{[F;\mathbf{Q}]-1}h_F}{[\mathcal{O}_F^\times:\mu_U^2]\sqrt{d_F}} I(s,g,\chi,\Phi)_U, \tag{3.48}$$

clear from the definitions. □

4. Modularity and arithmetic theta lift

In this section, we establish two key properties of the generating series of CM cycles. First, we show that the series defines an automorphic form on $G(\mathbf{A})$. This plays a key role in the comparison of local analytic and geometric terms at bad finite primes. For this argument, we essentially use the modularity proved on the Shimura curve by [21]. Second, we prove the arithmetic theta lift identity, which will relate the geometric kernel to the correspondence $T(f_1\otimes f_2)$ that appears in the main identity. In this case, our argument still follows that of [21], but is complicated by the appearance of higher cohomological degrees.

4.1. **Modularity of the generating series.** We say that the generating series $Z(r(g)\Phi)_U$ is *modular* if for any linear functional $\ell:\mathrm{CH}^{k-1}(W_U\times W_U)_{\mathbf{C}}\to\mathbf{C}$, $\ell(Z(r(g)\Phi)_U)$ converges absolutely and defines an automorphic form on $G(\mathbf{A})$ with coefficients in $\mathrm{CH}^{2k-1}(W_U\times W_U)_{\mathbf{C}}$.

A version of modularity for a generating series of CM-cycles over Kuga-Sato varieties over Shimura curves was proven in the setting $F=\mathbf{Q}$ by [22] using Borcherd's method of the singular theta lift, which is at present not available for totally real fields F. Our method is closer to Kudla's method of arithmetic theta lifting.

Theorem 4.1.1 (Modularity of generating series)**.** *Given g in $G(\mathbf{A})$ and ϕ in $\bar{\mathscr{S}}(\mathbf{V}\times\mathbf{A}^\times)$, the generating series $Z(g,\phi)_U$ converges absolutely, and is an automorphic form on $GL_2(\mathbf{A}_F)$ with values in $\mathrm{CH}^{2k-1}(W_U\times W_U)_{\mathbf{C}}$.*

Proof. We use the modularity of the generating series constructed in [21]. Given g in $G(\mathbf{A})$ and ϕ in $\bar{\mathscr{S}}(\mathbf{V}\times\mathbf{A}^\times)$, we recall the generating series of special divisors on $X_U\times X_U$ taking values in $\mathrm{Pic}(X_U\times X_U)_{\mathbf{C}}$

$$\begin{aligned} Z'(g,\phi)_U = &-\sum_{\alpha}\sum_{u\in\mu_U^2\backslash F^\times} E_0(\alpha^{-1}u, r(g)\phi)\mathscr{L}_{K,\alpha} \\ &+ w_U\sum_{a\in F^\times}\sum_{x\in K\backslash\mathbf{B}_f^\times} r(g,1)\phi(x,aq(x)^{-1})Z'(x)_U, \end{aligned} \tag{4.1}$$

where $K=U\times U$, and $\mathscr{L}_K$ is the Hodge bundle on $X_U\times X_U$.

We use the fact that the non-constant terms in $Z'(g,\phi)_U$ are constructed from $Z'(x)_U$, where $Z'(x)_U$ is defined by the correspondence in $\mathrm{Pic}(X_U\times X_U)$ induced by the action of R_x on X_U. In our setting, the correspondence $Z(x)_U$ is induced by the same action R_x on W_U. Consider the map

$$Z'(g,\phi)_U \mapsto Z(g,\phi)_U, \tag{4.2}$$

for fixed g and ϕ, obtained by

$$Z'(x)_U \mapsto Z(x)_U,\ \mathscr{L}_{K,\alpha} \mapsto \mathscr{P}_{K,\alpha}. \tag{4.3}$$

Extending it linearly on multiples of $Z'(g,\phi)_U$ in $\mathrm{Pic}(X_U \times X_U)$, and by 0 otherwise, we see that the map is well-defined. Then given a linear functional ℓ on $\mathrm{CH}^{2k-1}(W_U \times W_U)$, denote by ℓ' the composition of ℓ with the map above. It defines a linear functional on $\mathrm{Pic}(X_U \times X_U)$. Thus

$$\ell(Z(g,\phi)) = \ell'(Z'(g,\phi)) \tag{4.4}$$

is an automorphic form on $G(\mathbf{A})$ by the modularity of the left-hand side [21, Theorem 3.4.1].

□

4.2. **Arithmetic theta lifting.** Let G be an open compact subgroup of $GO(\mathbf{A}_f)$ such that the restriction Φ_f of Φ to $S(V \times \mathbf{A}_f^\times)$ is invariant under the action of G by Weil representation. Let $\mu_G = F^\times \cap G$, and define a theta series as follows

$$\theta(g,\Phi)_G = \sum_{u \in \mu_G^2 \backslash F^\times} \sum_{x \in V} r(g)\Phi(x,u). \tag{4.5}$$

For convenience, we recall the geometric kernel,

$$\tilde{Z}(\Phi \otimes \varphi) = (\tilde{Z}(g,\Phi),\varphi)_{\mathrm{Pet}} = \int_{[G]}^{*} \tilde{Z}(r(g)\Phi)\varphi(g)dg. \tag{4.6}$$

We prove the arithmetic theta lifting for the geometric kernel.

Theorem 4.2.1. *For Φ in $S(\mathbf{V} \times \mathbf{A}^\times)$ and φ in σ, we have*

$$\tilde{Z}(\Phi \otimes \varphi) = \frac{L(1,\pi,\mathrm{ad})}{2\zeta_F(2)} T(\theta(\Phi \otimes \varphi)) \tag{4.7}$$

in $\mathrm{Hom}_F(\mathrm{CH}^{k-1}, \mathrm{CH}^{k-1})_{\mathbf{C}}$.

Proof. First, observe that the identity is true up to a constant. Indeed, both $\tilde{Z}$ and $T \circ \theta$ are invariant under the diagonal action of $GL_2(\mathbf{A})$ on $\Phi \otimes \varphi$, and the action of $\mathbf{B}^\times \times \mathbf{B}^\times$ on Φ. Therefore, they must be multiples of the Shimizu lift, and hence belong to the one-dimensional space

$$\mathrm{Hom}_{GL_2(\mathbf{A}) \times \mathbf{B}^\times \times \mathbf{B}^\times}(\mathscr{S}(\mathbf{V} \times \mathbf{A}^\times) \otimes \sigma, \pi \otimes \tilde{\pi}). \tag{4.8}$$

Viewing this as an identity of operators on $H^{2k-1,0}(W_U) = H^{2k-1,0}(W_{U,\tau}(\mathbf{C}), \mathbf{C})$, it suffices to compare the traces of each operator.

The computation of the trace of the projector follows that of [21, Section 4.4]. By definition, we have

$$T(f_1 \otimes f_2) = \deg(\mathscr{P}_U) T(f_1 \otimes f_2)_U \tag{4.9}$$

where the action of $T(f_1 \otimes f_2)_U$ on $H^{k,0}(W_U)$ is given by $f \mapsto (f, f_2) f_1$. It follows that

$$\mathrm{tr}(T(\theta(\Phi \otimes \varphi)) | H^{k,0}(W_U)) = \deg(\mathscr{P}_U) \mathscr{F}\theta(\Phi \otimes \varphi), \tag{4.10}$$

where $\mathscr{F} = (\cdot,\cdot) : \pi \otimes \tilde{\pi} \to \mathbf{C}$. The Shimizu lift is normalized [19, Proposition 5] so that

$$\mathscr{F}\theta(\Phi \otimes \varphi) = \prod_v \frac{\zeta_v(2)}{L_v(1,\pi,\mathrm{ad})} \int_{N_v\backslash G_v} W_{\varphi_v,-1}(g) r(g,1)\Phi_v(1,1)dg. \tag{4.11}$$

We compute the trace of the correspondence $\tilde{Z}(\Phi \otimes \varphi)_U$ by an application of the Lefschetz trace formula

$$\deg \Delta^* \tilde{Z}(\Phi \otimes \varphi)_U = \sum_{i=0}^{2k-1} (-1)^i \mathrm{tr}(\tilde{Z}(g,\Phi)_U | H^i(W_U)), \tag{4.12}$$

where $\Delta : W_U \to W_U \times W_U$ is the diagonal embedding. The left-hand side is equal to the fixed points of the correspondence $\tilde{Z}(\Phi \otimes \varphi)_U$.

We have that $\mathrm{tr}(\tilde{Z}(g,\Phi)_U | H^i(W_U)) = 0$ unless $i = k-1$ since the correspondence $\tilde{Z}(\Phi \otimes \varphi)_U$ only acts in middle degree. By the inclusions (2.17) and (2.21), it acts on the top filtration

$$H^{k-1,0}(W_U) = \Gamma(W_U, \Omega^{k-1}_{W_U}) \otimes_F \mathbf{C}, \tag{4.13}$$

where $\Gamma(W_U, \Omega^{k-1}_{W_U})$ are the global sections of the $(k-1)$-th tensor power of the canonical bundle Ω_{W_U} over W_U. Since F is totally real, the action is fixed by complex conjugation, and is equal to the trace on $H^{0,k-1}(W_U)$. The Lefschetz formula reduces to

$$\mathrm{tr}(\tilde{Z}(\Phi \otimes \varphi)_U | H^{k-1,0}(W_U)) = -\frac{1}{2} \deg \Delta^* \tilde{Z}(\Phi \otimes \varphi)_U. \tag{4.14}$$

Note that the vanishing of the trace on the top and bottom degrees can also be computed explicitly as in [21, Proposition 4.3].

We show next that

$$\deg \Delta^* \tilde{Z}(\Phi \otimes \varphi)_U = -\deg(\mathscr{P}_U) \frac{L(1,\pi,\mathrm{ad})}{\zeta_F(2)} \mathscr{F}\theta(\Phi \otimes \varphi). \tag{4.15}$$

Combined with the identity (4.10), this completes the proof of the theorem. $\square$

4.2.1. *Degree of the pullback.* The rest of this section is devoted to proving the formula (4.15). The key strategy is to express the degree of the pullback in terms of an incoherent Eisenstein series. It suffices to consider the case $|S| > 1$ in which our computations are simplified by compactness, in contrast with [21].

The next proposition implies Formula (4.15) by carefully unfolding the integrals as in the argument following [21, Proposition 4.5].

Proposition 4.2.2. *Let ϕ in $\bar{S}(\mathbf{A} \times \mathbf{A}^\times)$ be invariant under $U \times U$. For g in $G(\mathbf{A})$,*

$$\deg \Delta^* Z(\Phi \otimes \varphi)_U = -\deg(\mathscr{P}_U) J(0,g,\phi)_U, \tag{4.16}$$

where

$$J(s,g,\phi)_U = \sum_{\gamma \in P_F\backslash G_F} \delta(\gamma g)^s \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{x \in F} r(\gamma g, 1)\phi(x,u) \tag{4.17}$$

is the mixed Eisenstein-theta series.

In order to prove the proposition, we require an explicit parametrization of the CM cycles on W_U, which by construction depend on CM cycles on the base Shimura curve. Using this, we obtain a formula for the pullback, then compute its degree.

Let $\mathbf{B}_0$ be the set of elements of $\mathbf{B}$ of trace 0, and denote by Ad the conjugation action of $\mathbf{B}^\times$ on $\mathbf{B}_0$ so that $\mathrm{Ad}(h)\circ x = hxh^{-1}$ for h in $\mathbf{B}^\times$ and x in $\mathbf{B}_0$. There is an orthogonal decomposition of quadratic spaces $\mathbf{B} = \mathbf{A}\oplus\mathbf{B}_0$. Fixing an archimedean place τ of F, we denote by $B = B(\tau)$ the nearby quaternion algebra, and define similarly the local decomposition $B = F\oplus B_0$.

For y in B_0 with $q(y)\neq 0$, we denote by B_y the centralizer of y in B. Then $B_0 = F[y]$ is a quadratic extension of F which is CM if and only if y is in the subset $B_{0,+}$ of elements of B_0 with totally positive norm. Let τ_y be the unique point of $\mathbf{H}^+$ fixed by the action of $B_y^\times \subset B_\tau(\mathbf{R})$, and $\bar{\tau}_y$ the fixed point by the action of $B_y^\times$ in $\mathbf{H}^-$. For h in $B_f^\times$, consider the pushforward $c(y,h)_U$

$$\tau_y^\pm \times B_y^\times\backslash \mathbf{B}_{f,y}^\times/U_h \longrightarrow X_U = B^\times\backslash\mathbf{H}^\pm\times\mathbf{B}_f^\times/U :\ (\tau,b)\mapsto(\tau,bh), \tag{4.18}$$

where $U_h = \mathbf{B}_{f,y}^\times \cap hUh^{-1}$.

Consider the image $C(y,h)_U$ of $c(y,h)_U$ by the map associating to each *CM* point $c(y,h)_U$ a CM cycle in $\mathrm{CH}^{k-1}(W_U)$ by the construction in Section (2.3). By [21, Lemma 4.6], given y in $\mathbf{B}_{f,0}^{\mathrm{ad}} = \{x\in\mathbf{B}_{f,0}\mid q(x)\in F_+^\times\}$, we may define $C(y)_U = C(y_0,h)_U$, where $y = h^{-1}y_0h$, y_0 in $B_{0,+}$, and h in $\mathbf{B}_f^\times$.

We have the following pull-back formula.

Proposition 4.2.3. *Let $\phi = \phi^0\otimes\phi_0$ under the orthogonal decomposition $\mathbf{B} = \mathbf{A}\oplus\mathbf{B_0}$. Then*

$$\Delta^* Z(g,\phi)_U = \sum_{u\in\mu_U^2\backslash F^\times} \theta(g,u,\phi^0)C(g,u,\phi_0)_U, \tag{4.19}$$

where $C(g,u,\phi_0)_U$ is given by

$$-r(g)\phi_0(0,u)\mathscr{P}_U + \sum_{y\in Ad(U)\backslash\mathbf{B}_{f,0}^{ad}} [F[y]^\times\cap U:\mu_U]^{-1} r(g)\phi_0(y,u)C(y)_U. \tag{4.20}$$

Here $F[y] = F+Fy$ is the CM extension of F generated by y, and $r(g)\phi_0(y,u) = r(g)\phi(1,uq(yx))$.

Proof. Let x be in $\mathbf{B}_f^\times$. We first claim that if x is in $F^\times U$, then $\Delta^*Z(x)_U = -\mathscr{P}_U$, and if x is not in $F^\times U$, then

$$\Delta^* Z(x)_U = \sum_{y\in Ad(U)\backslash\mathbf{B}_{f,0}^{ad}/F^\times} [F[y]^\times U\cap UxU:U]C(y)_U. \tag{4.21}$$

Indeed, if $x\in F^\times U$, then $Z(x)_U = Z(1)_U = \Delta$, and the identity follows from the definition of $\mathscr{P}_U$, which encodes the construction of the bundle $\mathscr{L}_{U,2k-2}$. If $x\notin F^\times U$, then $\Delta\cdot Z(x)_U$ is a proper intersection, and the proof follows that of [21, Lemma 4.7(2)].

We divide the summation into the constant term, the sum over x in $F^\times U$ and the sum over x not in $F^\times U$, and we write the pullback as $\Delta^*Z(g,\phi)_U = P+Q+R$, where

$$P = -\sum_{\alpha} \sum_{u \in \mu_U^2 \backslash F^\times} \phi(0, \alpha^{-1}u) \Delta^* \mathscr{P}_{K,\alpha} \tag{4.22}$$

$$Q = w_U \sum_{a \in F^\times} \sum_{x \in U \backslash F^\times U/U} \phi(x)_a \Delta^* Z(x)_U \tag{4.23}$$

$$R = w_U \sum_{a \in F^\times} \sum_{x \in U \backslash (\mathbf{B}_f^\times - F^\times U)/U} \phi(x)_a \Delta^* Z(x)_U. \tag{4.24}$$

On the one hand, the pullback $\Delta^* \mathscr{P}_{K,\alpha}$ is nontrivial if and only if $\mathscr{P}_{K,\alpha}$ lies in the same component as the diagonal Δ, that is, if α is the connected component of the identity. Then, since $\Delta^* \mathscr{P}_{K,1} = \mathscr{P}_U$, it follows that

$$P = -\sum_{u \in \mu_U^2 \backslash F^\times} \phi(0, u) \mathscr{P}_U. \tag{4.25}$$

On the other hand, by the preceding claim, we have

$$Q = -\sum_{u \in \mu_U^2 \backslash F^\times} \sum_{x \in F^\times} \phi(x, u) \mathscr{P}_U, \tag{4.26}$$

so that the decomposition $\phi = \phi^0 \otimes \phi_0$ implies

$$P + Q = -\sum_{u \in \mu_U^2 \backslash F^\times} \theta(g, u, \phi^0) \phi_0(0, u) \mathscr{P}_U. \tag{4.27}$$

As for the R term, we choose for simplicity U small enough so that the all the ramification indices $[F[y]^\times \cap U : \mu_U]$ equal 1. Then following [21, Proposition 4.8], by Equality (4.21),

$$R = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{y \in \mathrm{Ad}(U) \backslash \mathbf{B}_{f,0}^{\mathrm{ad}}/F^\times} \sum_{z \in F + F^\times y} \phi(z, u) C(y)_U \tag{4.28}$$

leading to formula (4.19). □

We use the formula for the pullback to compute the degree. We compute the degree for g in $SL_2(\mathbf{A})$, which extends to $G(\mathbf{A})$ by the diagonal action of the center $Z(\mathbf{A})$.

Proposition 4.2.4. *Assume $|S| > 1$. We have*

$$\deg C(g, u, \phi_0)_U = -\deg(\mathscr{P}_U) E(0, g, u, \phi_0), \tag{4.29}$$

where $E(0, g, u, \phi_0)$ is the Eisenstein series given by

$$E(s, g, u, \phi_0) = \sum_{\gamma \in P^1(F)/SL_2(F)} \delta(\gamma g)^s r(\gamma g) \phi_0(0, u), \tag{4.30}$$

and g is in $\tilde{SL}_2(\mathbf{A})$.

Proof. By the pullback formula (4.21) on $W_U \times W_U$ and the modularity of $Z(g, \phi)_U$ of Theorem (4.1.1), it follows that $C(g, u, \phi_0)_U$ also defines an automorphic form on $\tilde{SL}_2(\mathbf{A})$ with coefficients in $\mathrm{CH}^{k-1}(W_U)_{\mathbf{C}}$. Both $E(0, g, u, \phi_0)$ and $\deg C(g, u, \phi_0)_U$ define elements in the one-dimensional space

$$\mathrm{Hom}_{SO(\mathbf{B}_0) \times \tilde{SL}_{2k}(\mathbf{A})}(S(\mathbf{B}_0), \mathbf{C}^\infty(SL_2(F) \backslash \tilde{SL}_{2k}(\mathbf{A}))). \tag{4.31}$$

They are hence proportional and related by the quotient of their constant terms. The formula

(4.32)
$$E(0,g,u,\phi_0) = r(g)\phi_0(0,u) - \sum_{a\in F^\times} L(1,\eta_{-ua}) \int_{\mathbf{B}_{y_a}^\times\backslash \mathbf{B}^\times} r(g,(h,h))\phi_0(y_a,u)dh$$

for the incoherent Eisenstein series of weight $\frac{3}{2}$ given in [21, Proposition 2.10(3)(a)] implies

$$\deg C_0(g,u,\phi_0)_U = -\deg(\mathscr{P}_U)E_0(0,g,u,\phi_0). \tag{4.33}$$

□

Remark 4.2.5. Though not used in this paper, we briefly note that the above argument can also be extended to the classical case of the Kuga-Sato variety over the modular curve, that is, where $k > 1$, $F = \mathbf{Q}$, and $|S| = 1$. In this setting, the Kuga-Sato variety is again compact after desingularization and compactification, but the incoherent Eisenstein series has now an additional non-holomorphic part, described in [21, Proposition 2.10(3)(b)].

The contribution of the non-holomorphic part to the degree makes up one part of the archimedean contribution of the extra series $B(g,\phi)$ described in [21, Theorem 4.15], but we observe that this contribution is in fact a Schwartz function and in particular nonsingular, so that we may indeed use the general fact that the theta lift of any cuspidal automorphic representation of $G(\mathbf{A})$ to $GSO(V_{\mathrm{hyp}}(\mathbf{A}))$ is zero. Thus this contribution is zero.

5. Local decomposition of kernels

In this section, we describe the class of *degenerate* Schwartz functions. We show that if the kernel identity holds for this class of functions, then it holds in general. Then, we decompose the analytic and geometric kernels into local components, and make case by case computations. In the next section, these kernels will be compared term to term at each place.

5.1. **Degenerate Schwartz functions.** We consider degenerate Schwartz functions introduced in [21, §5.2].

Write the set of all places S_F of F as

$$S_F = S_\infty \cup S_{\mathrm{nonsplit}} \cup S_{\mathrm{split}}, \tag{5.1}$$

where S_∞ is the set of archimedean places of F, S_{nonsplit} is the set of non-archimedean places of F nonsplit in E, and S_{split} is the set of non-archimedean places of F split in E. Denote by S_1 a finite subset of S_{nonsplit} containing all places ramified over $\mathbf{Q}$, in $\mathbf{B}$, K, σ or χ, and by S_2 the two places in S_{split} at which χ and σ are ramified.

The *standard* Schwartz function ϕ in $\bar{\mathscr{S}}(V\times\mathbf{R}^\times)$ is the Gaussian

$$\phi(x,u) = P_k(x)e^{-2\pi uq(x)}1_{\mathbf{R}^+}(u). \tag{5.2}$$

where $1_{\mathbf{R}^+}$ is the characteristic function of $(0,\infty)$. The space $\bar{\mathscr{S}}(V\times\mathbf{R}^\times)$ gives the weight $2k$ discrete series of $G(\mathbf{R})$. The standard Schwartz function corresponds to the element of lowest weight. The action of $\mathfrak{gl}_2(\mathbf{R})$ on it generates the entire space, so that it is sufficient to consider it.

We make the following assumptions on the Schwartz functions, which simplify the computations at many places.

- The Schwartz function $\phi = \otimes\phi_v \in \bar{\mathscr{S}}(\mathbf{B}\times\mathbf{A}^\times)$ is a pure tensor and ϕ_v is standard for any v in S_∞.
- For v in S_1, ϕ_v lies in $\bar{\mathscr{S}}^1(\mathbf{B}_v \times F_v^\times)$ defined by

(5.3)
$$\{\phi_v \in \bar{\mathscr{S}}(\mathbf{B}_v \times F_v^\times)|\phi_v(x,u) = 0 \text{ if } v(uq(x)) \geq -v(d_v) \text{ or } v(uq(x_2)) \geq -v(d_v)\},$$

  where dv is the local different of F at v, and x_2 denotes the orthogonal projection of x in $V_{2,v} = E_v j_v$.
- For any v in S_2, ϕ_v lies in $\bar{\mathscr{S}}^2(\mathbf{B}_v \times F_v^\times)$ defined by

$$\{\phi_v \in \bar{\mathscr{S}}(\mathbf{B}_v \times F_v^\times)|r(g)\phi_v(0,u) = 0,\ \forall g \in G(F_v),\ u \in F_v^\times\}. \tag{5.4}$$

- For any v in $S_{\text{nonsplit}} - S_1$, assume that ϕ_v is the standard characteristic function of $\mathcal{O}_{\mathbf{B}_v} \times \mathcal{O}_{F_v}^\times$.
- Let $U = \prod_v U_v$ be an open compact subgroup of $\mathbf{B}_f^\times$ such that:
  - ϕ is invariant under the action of $U \times U$,
  - χ is invariant under the action of $U_T = U \cap T(\mathbf{A}_f)$,
  - U_v is of the form $(1 + \bar{w}_v^r \mathcal{O}_{\mathbf{B}_v})^\times$ for some $r \geq 0$ for every finite place v,
  - U_v is maximal for all v in $S_{\text{nonsplit}} - S_1$, and v in S_2,
  - U does not contain -1,
  - U is small enough that that each connected component of the complex points of X_U is an unramified quotient of $\mathcal{H}$ by the complex uniformization.

Theorem 5.1.1. *The kernel identity is true for all (ϕ, U) if and only if its true for all (ϕ, U) satisfying the assumptions.*

Proof. It suffices to show that there exists a Schwartz function Φ in $\mathscr{S}(\mathbf{B} \times \mathbf{A}^\times)$ such that its image ϕ in $\bar{\mathscr{S}}(\mathbf{B} \times \mathbf{A}^\times)$ satisfies the assumptions and the pairing $\alpha(\theta(\phi_f \otimes \varphi_f))$ is nonzero for some φ in σ. These properties follow from Proposition 5.11 and Proposition 5.15 of [21] for v in S_1 and v in S_2 respectively. □

We record the following properties of the degenerate Schwartz function which we make use of.

Corollary 5.1.2. *Consider ϕ_v in $\bar{\mathscr{S}}(B_v \times F_v^\times)$.*

(1) Let v be in $S_{nonsplit}$, assume moreover there is a constant $c > 0$ such that $\phi_v(x,u) = 0$ for all (x,u) with $v(uq(x_2)) \geq 2$. Then

$$\phi_v = \sum_{i=1}^{r} \phi_{1,v}^i \otimes \phi_{2,v}^i \tag{5.5}$$

where for each i, $\phi_{1,v}^i$ in $\bar{\mathscr{S}}(E_v \times F_v^\times)$ and $\phi_{2,v}^i$ in $\bar{\mathscr{S}}(E_v j_v \times F_v^\times)$ satisfy $\phi_{2,v}^i(x_2,u) = 0$ for all (x_2,u) with $v(uq(x_2)) \geq c$.

(2) Let v be a finite place of F. Then $r(g)\phi_v(0,u) = 0$ for all g in $G(F_v)$, and u in $F_v^\times$ if and only if the average

$$\int_{B_v(a)} r(g,h)\phi_v(x,u)dx = 0 \tag{5.6}$$

for all g in $G(F_v)$, h in $B_v^\times \times B_v^\times$, and a, u in $F_v^\times$.

Proof. See Lemma 5.9 and Lemma 5.10 of [21]. □

5.2. **Derivative of the analytic kernel.** In this section, we decompose the derivative of the analytic kernel $I(s, g, \phi)$ into a sum of local components. As discussed, the analytic kernel is essentially the one considered in [21, §6], which follows Waldspurger's integral representation for the L-function. We summarize the main results we use.

5.2.1. *Decomposition of the test function.* Using the decomposition $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$, we may write ϕ in $\mathscr{S}(\mathbf{V} \times \mathbf{A}^\times)$ as a finite linear combination of

$$\phi_1 \otimes \phi_2 \text{ in } \bar{\mathscr{S}}(\mathbf{V}_1 \times \mathbf{A}^\times) \times \bar{\mathscr{S}}(\mathbf{V}_2 \times \mathbf{A}^\times), \tag{5.7}$$

where $(\phi_1 \otimes \phi_2)(x_1 + x_2, u) := \phi_1(x_1, u)\phi_2(x_2, u)$. This decomposition preserves the Weil representation, that is,

$$r(g, (t_1, t_2))(\phi_1 \otimes \phi_2)(x, u) = r_1(g, (t_1, t_2))\phi_1(x_1, u) r_2(g, (t_1, t_2))\phi_2(x_2, u) \tag{5.8}$$

for g in $GL_2(\mathbf{A})$, and (t_1, t_2) in $\mathbf{A}_E^\times \times \mathbf{A}_E^\times$. Here, r_1, r_2 denote the Weil representations associated to $\mathbf{V}_1, \mathbf{V}_2$ respectively, and (t_1, t_2) acts on $x_i \in \mathbf{V}_i$ by $(t_1, t_2)x_i = t_1 x_i t_2^{-1}$.

As a consequence, we can express the analytic kernel (3.43) as a product of a theta series and an Eisenstein series

$$I(s, g, \phi)_U = \sum_{u \in \mu_U^2 \backslash F^\times} \theta(g, u, \phi_1) E(s, g, u, \phi_2), \tag{5.9}$$

for g in $GL_2(\mathbf{A})$, where

$$\theta(g, u, \phi_1) = \sum_{x_1 \in E} r(g, 1)\phi_1(x_1, u) \tag{5.10}$$

and $E(s, g, u, \phi_2)$ is defined as in (3.22). Note that this definition of $\theta(g, u, \phi_1)$ differs from that given in (3.12), as the summation over u is dropped here.

5.2.2. *Decomposition of the derivative.* Given u in $F^\times$ and a place v of F, denote by $F_u(v)$ the set of a in $F^\times$ represented by $(\mathbf{V}_2^v, uq)$. Note that $F_u(v)$ is non-empty only if E is nonsplit at v, and $W_{a,v}(0, g, u, \phi) = 0$ for a in $F_u(v)$.

Fix a place v of F nonsplit in E. For a in $F_u(v)$, we can evaluate the derivative $E'(s, g, u, \phi_2)$ at $s = 0$ using the Fourier expansion of $E(s, g, u, \phi_2)$ in terms of Whittaker functions to obtain

$$E'(0, g, u, \phi_2) = E'_0(0, g, u, \phi_2) - \sum_{v \text{ nonsplit}} E'_v(0, g, u, \phi_2), \tag{5.11}$$

where $E'_0(0, g, u, \phi_2)$ is the derivative at $s = 0$ of the usual constant term of the Siegel Eisenstein series $E_0(s, g, u, \phi_2)$, and

$$E'_v(0, g, u, \phi_2) := \sum_{a \in F_u(v)} W'_{a,v}(0, g, u, \phi_2) W_a^v(0, g, u, \phi_2) \tag{5.12}$$

for suitable local Whittaker functions $W_{a,v}(0, g, u, \phi_2)$ and $W_a^v(0, g, u, \phi_2)$ translated by the Weil index $\gamma_{u,v}$.

The decomposition of the analytic kernel follows from the preceding definitions. We separate the places v in F that are nonsplit and split in E.

$$I'(0, g, \phi)_U = -\sum_{v \text{ nonsplit}} I'_v(0, g, \phi)_U + \sum_{u \in \mu_U^2 \backslash F^\times} \theta(g, u, \phi_1) E'_0(0, g, u, \phi_2) \tag{5.13}$$

where

$$I'_v(0,g,\phi)_U := \sum_{u\in\mu_U^2\backslash F^\times} \theta(g,u,\phi_1)E'_v(0,g,u,\phi_2). \tag{5.14}$$

We will see that the contribution at the split places will vanish by our assumption on the Schwartz function.

5.2.3. *Local components.* Let v be a place of F nonsplit in E. If v is non-archimedean, we introduce the pseudo-theta series

$$\mathcal{K}_\phi^{(v)}(g,(t_1,t_2)) := \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in V_2} k_{r(t_1,t_2)\phi_v}(g,y,u)r(g,(t_1,t_2))\phi^v(y,u), \tag{5.15}$$

where

$$k_{\phi_v}(g,y,u) := \frac{L(1,\eta_v)}{\mathrm{vol}(E_v^1)} r(g)\phi_{1,v}(y_1,u)W'_{uq(y_2),v}(0,g,u,\phi_{2,v}) \tag{5.16}$$

for $\phi_v = \phi_{1,v}\otimes\phi_{2,v}$, and $y = y_1+y_2$ in V_v with $y_2\neq 0$. The identity in the next lemma uses the fact that $\mathcal{K}_\phi^{(v)}(g,(t_1,t_2))$ approximates the function $\theta(g,(t_1,t_2),k_{\phi_v}\otimes\phi^v)$. By the modularity of the pseudo-theta series, this implies that the two series are equal, which is not true in general.

Lemma 5.2.1. *Consider v in S_1 such that ϕ_v in $\bar{\mathscr{S}}^1(B_v\times F_v^\times)$ satisfies Assumption* (5.3)*. Then*

$$\mathcal{K}_\phi^{(v)}(g,(t_1,t_2)) = \theta(g,(t_1,t_2),k_{\phi_v}\otimes\phi^v) \tag{5.17}$$

for g in $P(F_{S_1})GL_2(\mathbf{A}^{S_1})$, and t_1,t_2 in $T(\mathbf{A})$.

Proof. See [21, Corollary 6.9]. □

If v is archimedean, define

$$\mathcal{K}_\phi^{(v)}(g,(t_1,t_2)) := \widetilde{\lim}_{s\to 0}\sum_{a\in F^\times}\sum_{y\in\mu_U\backslash(B(v)_+^\times - E^\times)} k_{v,s}(y)r(g,(t_1,t_2))\phi(y,aq(y)^{-1}), \tag{5.18}$$

where

$$k_{v,s}(y) := \frac{\Gamma(s+2k-1)}{2(4\pi)^s\Gamma(2k-1)}\int_1^\infty (t-1)^k t^{-k-1}(1-tuq(y_2))^{-s-k}dt, \tag{5.19}$$

and by the limit $\widetilde{\lim}$ refers to the constant term of the Laurent expansion at $s=0$.[1]

5.2.4. *Holomorphic projection.* Consider the orthogonal projection

$$\mathrm{Pr} : \mathcal{A}(G(\mathbf{A}),\omega) \to \mathcal{A}_0^{(2k)}(G(\mathbf{A}),\omega) \tag{5.20}$$

from the space of automorphic forms of central character ω to the space of holomorphic cusp forms of parallel weight $2k$ mapping f in $\mathcal{A}(G(\mathbf{A}),\omega)$ to the unique form $\mathrm{Pr}(f)$ in $\mathcal{A}_0^{(2k)}(G(\mathbf{A}),\omega)$ such that

$$(\mathrm{Pr}(f),\varphi)_{\mathrm{Pet}} = (f,\varphi)_{\mathrm{Pet}} \tag{5.21}$$

for all φ in $\mathcal{A}_0^{(2k)}(G(\mathbf{A}),\omega)$. Refer to [9, Proposition 5.1] for a classical formulation of the holomorphic projection onto weight $2k$ forms.

[1] Note that in [21], the analogous function is denoted by $\bar{\mathcal{K}}_\phi^{(v)}$. Our choice of notation is intended to simplify the following formulae.

For any automorphic form f for $GL(2,\mathbf{A})$, we define a Whittaker function $f_{\psi,s}(g)$ as follows:

$$\left(\frac{(4\pi)^{2k-1}}{\Gamma(2k-1)}\right)^{[F:\mathbf{Q}]} W^{(2k)}(g_\infty)\int_{Z(F_\infty)N(F_\infty)\backslash G(F_\infty)} \delta(h)^s f_\psi(gh)\overline{W^{(2k)}(h)}dh, \tag{5.22}$$

where $W^{(2k)}$ is the standard holomorphic Whittaker function of weight $2k$ at infinity, and f_ψ is the Whittaker function of f with respect to the additive character ψ. That is,

$$W^{(2k)}(g) = \begin{cases} \omega(z)|y|^k e^{2ik\theta} e^{2\pi i(x+iy)} & y>0 \\ 0 & y\le 0 \end{cases} \tag{5.23}$$

where as usual

$$g = z\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} \tag{5.24}$$

by the Iwasawa decomposition, with $y>0$ and $z\in\mathbf{R}^\times$.

The following growth condition is a key ingredient in our projection formula.

Lemma 5.2.2. *Consider f in $\mathcal{A}(G(\mathbf{A}),\omega)$ such that*

$$f\Big(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\Big) = O_g(|a|^{1-\epsilon}) \tag{5.25}$$

as $|a|\to\infty$ for a in $\mathbf{A}^\times$ and some $\epsilon>0$. The holomorphic projection $\mathrm{Pr}(f)$ has Whittaker function given by

$$\mathrm{Pr}(f)_\psi(g) = \lim_{s\to 0} f_{\psi,s}(g). \tag{5.26}$$

Proof. The proof follows that of [21, Proposition 6.12], except that we use the Whittaker function of weight $2k$ inducing instead

$$\int_{Z(\mathbf{R})N(\mathbf{R})\backslash G(\mathbf{R})} |W^{(2k)}(g)|^2 dg = \int_0^\infty y^{2k-1}e^{-4\pi y}\frac{dy}{y} \tag{5.27}$$

$$= 4\pi^{-2k+1}(2k-1)! \tag{5.28}$$

at each archimedean place. □

We introduce the notation

$$\mathrm{Pr}'(f)(g) = \sum_{a\in F^\times} \widetilde{\lim_{s\to 0}} f_{\psi,s}\Big(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\Big). \tag{5.29}$$

Under the growth condition (5.25), we have an equality $\mathrm{Pr}(f)=\mathrm{Pr}'(f)$.

We decompose the derivative of the analytic kernel into local components.

Proposition 5.2.3. *Under Assumptions (5.3) and (5.4), the holomorphic projection of the analytic kernel is given by*

$$\mathrm{Pr}I'(0,g,\chi,\phi) = \int_{[T]}^* \mathrm{Pr}'I'(0,g,r(t,1)\phi)\chi(t)dt, \tag{5.30}$$

where $\mathrm{Pr}'I'(0,g,\phi)$ can be expressed as

$$\mathrm{Pr}'I'(0,g,\phi) = -\sum_{v\ \text{nonsplit}} 2\mathrm{vol}([T])^{-1}\int_{[T]} \mathcal{K}_\phi^{(v)}(g,(t,t))dt, \tag{5.31}$$

and $\mathcal{K}_\phi^{(v)}(g,(t,t))$ *is respectively described by* (5.18) *and* (5.17) *for archimedean and non-archimedean* v*.*

Proof. For non-archimedean v, under Assumption (5.3), we have

$$I'(0,g,\phi)_U = -\sum_{v \text{ nonsplit}} I'_v(0,g,\phi)_U \tag{5.32}$$

for g in $P(F_{S_1})GL_2(\mathbf{A}^{S_1})$ by [21, Proposition 6.7], and Identity (5.31) follows.

For archimedean v, we first observe that under Assumption (5.4), our kernel function satisfies the condition

$$I'(0, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g, \chi, \phi) = O(e^{-\epsilon_g |a|}) \tag{5.33}$$

for some $\epsilon_g > 0$ as in [21, Proposition 6.14]. [21, Proposition 6.12] then leads to Identity (5.30). More precisely, we express the integrand as in (5.31) by [21, Prop 6.15] for $\mathcal{K}_\phi^{(v)}(g,(t,t))$ given by Lemma (5.2.1). We replace $k_{\phi_v}(g,y,u)$ by

$$\tilde{k}_{\phi_v,s}(g,y,u) = \frac{4\pi^{2k-1}}{\Gamma(2k-1)} W^{(2k)}(g_v) \int_{F_{v,+}} a^{s+k-1} e^{-2\pi a} k_{\phi_v}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, y, u) \frac{da}{a}, \tag{5.34}$$

and take the quasi-limit $\widetilde{\lim}$. Assume $uq(y)=1$ and $u>0$ for simplicity. Consider the function

$$q_k(a) = \int_1^\infty e^{-at}(t-1)^k t^{-k-1} dt \tag{5.35}$$

for $a > 0$ defined in [9, Proposition 3.3(e)]. Then [20, Lemma 2.3(3)] implies that for a in $F_u(v)$, or equivalently, for $ua < 0$, we have

$$W_{a,v}{}'(0,1,u) = -\pi e^{2\pi a} q_k(-4\pi a) \tag{5.36}$$

if $u > 0$, and $W_{a,v}{}'(0,1,u) = 0$ otherwise. Note that $q_0(a) = -\mathrm{Ei}(-a)$ is the exponential integral. By properties of the Weil representation, we have

$$k_{\phi_v}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, y, u) = -\frac{1}{2} q_k(4\pi u q(y_2) a)|a|^k e^{2\pi u q(y) a}. \tag{5.37}$$

It suffices to consider $uq(y) = 1$. Taking $u > 0$, we evaluate the integral as follows:

$$\int_{F_{v,+}} a^{s+k-1} e^{-2\pi a} k_{\phi_v}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, y, u) \frac{da}{a} \tag{5.38}$$

$$= -\frac{1}{2} \int_0^\infty a^{s+k-1} e^{-2\pi a} q_k(4\pi u q(y_2) a) a^k e^{2\pi a} \frac{da}{a} \tag{5.39}$$

$$= \frac{1}{2} \int_0^\infty a^{s+2k-1} e^{-4\pi a} \int_1^\infty e^{-4\pi u q(y_2) a t}(t-1)^k t^{-k-1} dt \, \frac{da}{a} \tag{5.40}$$

$$= \frac{1}{2} \int_1^\infty (t-1)^k t^{-k-1} \int_0^\infty a^{s+2k-1} e^{-4\pi(1-uq(y_2)t)a} \frac{da}{a} \, dt. \tag{5.41}$$

$$= \frac{\Gamma(s+2k-1)}{2(4\pi)^{s+2k-1}} \int_1^\infty (t-1)^k t^{-k-1} (1 - tuq(y_2))^{-s-k} dt \tag{5.42}$$

Hence we conclude that $\tilde{k}_{\phi_v,s}(g,y,u)$ is given by

$$\frac{\Gamma(s+2k-1)}{2(4\pi)^s \Gamma(2k-1)} W^{(2k)}(g_v) \int_1^\infty (t-1)^k t^{-k-1} (1 - tuq(y_2))^{-s-k} dt, \tag{5.43}$$

which equals $W^{(2k)}(g_v)k_{v,s}(y)$ by definition. □

5.3. **Decomposition of the geometric kernel.** Consider arithmetic cycles $\hat{Z}_1 = (Z_1, g_1)$ and $\hat{Z}_2 = (Z_2, g_2)$ with codimension p and $d-p$ in a regular arithmetic scheme Y of dimension d, with Z_1 and Z_2 irreducible intersecting properly. If Z_1, Z_2 are disjoint at the generic fiber then the intersection (Z_1, Z_2) with support defines an element in $\mathrm{CH}^d_{|Z_1|\cap|Z_2|}(Y)$, as in [23, §1.2]. So we can write $(Z_1, Z_2) = \sum_v x_v \mathscr{P}_v$, for x_v in $\mathrm{CH}^d_{|Y\otimes k(v)|}(Y)$. Define

$$\begin{cases} (\hat{Z}_1, \hat{Z}_2)_v = \deg(x_v) \text{ if } v \text{ is finite,} \\ (\hat{Z}_1, \hat{Z}_2)_v = \int_{Z_{2v}(\mathbf{C})} g_1 + \int_{Y_v(\mathbf{C})} g_2 h_1 \text{ if } v \text{ is infinite,} \end{cases} \tag{5.44}$$

where $Y_v(\mathbf{C}) = Y \otimes_{O_F,\sigma} \mathbf{C}$, $\sigma : F \to \mathbf{C}$ inducing v and $Z_{2v}(\mathbf{C})$ is the pullback on Z_2 of Y_v.

If Z_1, Z_2 are not disjoint at the generic fiber, assume there is a morphism

$$\pi : Y \to X, \tag{5.45}$$

where X is a regular arithmetic surface such that both Z_1 and Z_2 are contained in a fiber Y_D of Y over an integral divisor D of X, and the morphism $\pi_D : Y_D \to D$ is smooth. Let t be a local coordinate for D_F in X_F. Let η be a differential form such that $d\eta = \bar{d}\eta = 0$ on each fiber $Y(\mathbf{C})q$ when q is near $|D|(\mathbf{C})$ and with restriction $\delta_{Z_2}(\mathbf{C})$ on $Y(\mathbf{C})_{D(\mathbf{C})}$ modulo $\mathrm{Im}(\delta)$ and $\mathrm{Im}(\bar{\delta})$. Let $\mathrm{div}(d_D t)$ denote $(\frac{1}{e}\mathrm{div}(t^e) - D)|_D$ and let $\mathrm{ord}_v d_D t$ be the rational number such that the pushforward of the 0-cycle $\mathrm{div}(d_D t)$ to $\mathrm{Spec}(O_F)$ is $\sum_v \mathrm{ord}_v d_D t[v]$. Then

$$\begin{cases} (\hat{Z}_1, \hat{Z}_2)_v = -(Z_{1,F}, Z_{2,F})_{Y_{D,F}} \mathrm{ord}_v d_D t \text{ if } v \text{ is finite,} \\ (\hat{Z}_1, \hat{Z}_2)_v = G(D_\sigma(\mathbf{C})) + \int_{Y_v(\mathbf{C})} g_2 h_{Z_1} \text{ if } v \text{ is infinite,} \end{cases} \tag{5.46}$$

where

$$G(p) = \lim_{q\to p} \Big(\int_{Y(\mathbf{C})q} g_1 \eta + \log|t(q)| (Z_{1,F}, Z_{2,F})_{Y_{D,F}} \Big). \tag{5.47}$$

5.3.1. *Decomposition.* We use the local decomposition of the arithmetic intersection pairing to decompose the geometric kernel as follows. Recall that by Lemma (3.3.2), the generating series $Z(g,(t_1,t_2),\phi)_U$ is a cuspidal automorphic form on $G(\mathbf{A})$, so only its nonconstant part Z_* is nonzero. We have

$$Z(g,(t_1,t_2),\phi)_U = \langle Z_*(g,\phi)_U R_{t_1} Z_U, R_{t_2} Z_U \rangle_{\mathrm{GS}}, \tag{5.48}$$

for t_1, t_2 in $\mathbf{B}^\times$. By Assumption (5.4), the pairing $\langle Z_*(g,\phi)_U R_{t_1} Z_U, R_{t_2} Z_U \rangle_{\mathrm{GS}}$ has no self-intersections for any g in $1_{S_1} G(\mathbf{A}^{S_1})$. It decomposes into a local sum

$$\sum_v i_v(Z_*(g,\phi)t_1, t_2), \text{ where } i_v(Z_*(g,\phi)t_1,t_2) := (Z_*(g,\phi)_U R_{t_1} Z_U, R_{t_2} Z_U)_v. \tag{5.49}$$

By class field theory, we have a surjection $T(F)\backslash T(\mathbf{A}) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$. Hence, we obtain the following expression for the local decomposition of the geometric kernel.

Proposition 5.3.1.

$$Z(g,(t_1,t_2),\phi)_U = -\sum_v \mathrm{vol}([T]) \int_{[T]} i_{\bar{v}}(Z_*(g,\phi)tt_1, tt_2)dt \tag{5.50}$$

where

$$i_{\bar v}(Z_*(g,\phi)t_1,t_2)=\sum_{a\in F^\times}\sum_{x\in \mathbf{B}_f^\times/U} r(g)\phi(x,aq(x)^{-1})\sum_v i_{\bar v}(t_1x,t_2), \tag{5.51}$$

and $i_{\bar v}(t_1x,t_2)=(t_1x,t_2)_v$.

6. Proof of kernel identity

In this section, we use the local decomposition of the analytic and geometric kernels to prove the kernel identity under the assumptions on the Schwartz functions in (5.1). The structure of our arguments follows that of [21], except that we must use the arithmetic intersection pairings on the geometric side, rather than the Néron-Tate height pairings.

6.1. The main local theorem.

The kernel identity will follow from the local computations listed below. We first show that they imply the main theorem, then the rest of the section is devoted to these local assertions.

Theorem 6.1.1. *Suppose the assumptions of Section (5.1) are satisfied. For all t_1, t_2 in $T(\mathbf{A}_f)$, and g in $1_{S_1}G(\mathbf{A}^S)$, suppose that the following conditions are satisfied.*

1. *If $v\in S_{split}$, then $i_{\bar v}(Z_*(g,\phi)t_1,t_2)=0$.*
2. *If $v\in S_\infty$, then $\mathcal{K}_\phi^{(v)}(g,(t_1,t_2))=i_{\bar v}(Z_*(g,\phi)t_2,t_2)$.*
3. *If $v\in S_{nonsplit}-S_1$, then $\mathcal{K}_\phi^{(v)}(g,(t_1,t_2))=i_{\bar v}(Z_*(g,\phi)t_2,t_2)\log N_v$.*
4. *If $v\in S_1$, then there exist Schwartz functions $k_{\phi_v},m_{\phi_v}\in\bar{\mathscr{S}}(B(v)_v\times F_v^\times)$ depending on ϕ_v and U_v, such that*

$$\mathcal{K}_\phi^{(v)}(g,(t_1,t_2))=\theta(g,(t_1,t_2),k_{\phi_v}\otimes\phi^v), \tag{6.1}$$

$$i_{\bar v}(Z_*(g,\phi)t_1,t_2)=\theta(g,(t_1,t_2),m_{\phi_v}\otimes\phi^v). \tag{6.2}$$

Then, the kernel identity (3.47) holds.

Proof. By holomorphic projection, it suffices to prove

$$(\mathrm{Pr}I'(0,g,\chi,\phi)-2Z(g,\chi,\phi),\ \varphi(g))_{\mathrm{Pet}}=0 \tag{6.3}$$

Decomposing the analytic kernel using Proposition (5.2.3), and the geometric kernel using Proposition (5.3.1), we have

$$\mathrm{Pr}I'(0,g,\chi,\phi)-2Z(g,\chi,\phi)=2\mathrm{vol}([T])^{-1}\int^*_{[T]}\int_{[T]}D(g,(t_2t_1,t_2),\phi)\chi(t_1)dt_1dt_2 \tag{6.4}$$

where $D(g,(t_1,t_2),\phi)$ is given by

$$\sum_{v|\infty}\mathcal{K}_\phi^{(v)}(g,(t_1,t_2))+\sum_{v<\infty\text{ nonsplit}}\mathcal{K}_\phi^{(v)}(g,(t_1,t_2))-\sum_v i_{\bar v}(Z_*(g,\phi)t_1,t_2)\log N_v. \tag{6.5}$$

By the assumptions on the Schwartz function ϕ, and using the identities (1) to (4), only the terms corresponding to $v\in S_1$ remain, hence

$$D(g,(t_1,t_2),\phi)=\sum_{v\in S_1}\theta(g,(t_1,t_2),d_{\phi_v}\otimes\phi^v) \tag{6.6}$$

where $d_{\phi_v}=k_{\phi_v}-m_{\phi_v}\log N_v$ is a Schwartz function in $\bar{\mathscr{S}}(B(v)_v\times F_v^\times)$. The equality is true for g in $1_{S_1}G(\mathbf{A}^S)$. By modularity, it extends to g in $G(F)G(\mathbf{A}^S)$, and by density, it extends to $G(\mathbf{A})$.

We define $\Phi(v) = \Phi_\infty \otimes d_{\phi_v} \otimes \phi_f^v$ in $S(\mathbf{B}(v) \times \mathbf{A}^\times)$, where Φ_∞ is such that its image in $\bar{\mathscr{S}}(B_\infty \times F_\infty^\times)$ is the standard Schwartz function. Working with this Schwartz function instead, we define

$$I(0, g, \chi, \Phi(v)) = \int_{T(F)\backslash T(\mathbf{A})} \int_{[T]} \theta(g, (t_2 t_1, t_2), \Phi(v))\chi(t_1) dt_1 dt_2, \tag{6.7}$$

where

$$\theta(g, (t_1, t_2), \Phi(v)) = \sum_{u \in F^\times} \sum_{y \in B(v)} r(g, (t_1, t_2))\Phi(v)(y, u). \tag{6.8}$$

Since $I(0, g, \chi, \Phi(v))$ and $I(0, g, \chi, d_{\phi_v} \otimes \phi^v)$ are equal up to a constant, it suffices to show that

$$(I(0, g, \chi, \Phi(v)),\ \varphi(g))_{\mathrm{Pet}} = 0 \tag{6.9}$$

for all v in S_1. The series $\theta(g, (t_1, t_2), \Phi(v))$ is defined on the nearby quaternion algebra. Therefore, by the same argument as in [21, §7.4.3], we interpret the left-hand side as an integral of a Shimizu lifting. Since the ramification set of $B(v)$ is different from S, the right-hand sum is perpendicular to σ by an application of the theorem of Tunnell and Saito [18, 16] stating that

$$\mathrm{Hom}_{K^\times}(\pi(v) \otimes \chi, \mathbf{C}) = 0. \tag{6.10}$$

Hence, Equality (6.9) follows. This proves the kernel identity. □

Next, we prove the local assertions of Theorem (6.1.1). We separate our analysis according to the reduction type of the Shimura variety at various places, and treat archimedean places disjointly. The computations at archimedean and ordinary primes are essentially those of [21] and [23]. At supersingular and superspecial primes, the computations involve computations of the intersection pairing rather than the Néron-Tate height. In all cases, we use the notion of geometric pairing with multiplicity function introduced in [24], which depends only on the relative position of the CM cycles.

6.2. **Archimedean Case.** As in [23, §3.4], consider the Legendre function of the second kind defined for $t > 1$ by

$$Q_{s+k}(t) = \int_0^\infty (t + \sqrt{t^2 - 1}\cosh u)^{-k-s} du. \tag{6.11}$$

For z_1, z_2 in $\mathbf{H}$ and $\mathrm{Re}(s) > 0$, let

$$m_s(z_1, z_2) = -2Q_{s+k}\big(1 + \frac{|z_1 - z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}\big), \tag{6.12}$$

where the argument of Q_{s+k} is the hyperbolic cosine of the distance between z_1 and z_2. Let $\mathbf{B}(v)$ be the adelic nearby quaternion algebra. For $(z_1, \beta_1), (z_2, \beta_2)$ in $\mathbf{H} \times \mathbf{B}(v)_f^\times$, the local height of the induced CM cycles is defined by

$$i_{\bar{v}}((z_1, \beta_1), (z_2, \beta_2)) = \tilde{\lim_{s\to 0}} \sum_{\gamma \in \mu_U \backslash B_+^\times} m_s(z_1, \gamma z_2) 1_U(\beta_1^{-1}\gamma\beta_2). \tag{6.13}$$

In particular, the archimedean pairing used in [24, Proposition 3.4.1] is the same as in the weight 2 case, except that s is replaced with $s + k$. In the case of CM points, we have for any $\gamma \in B_{v,+}^{\times} - E_v^{\times}$,

$$1 + \frac{|z_0 - \gamma z_0|^2}{2\mathrm{Im}(z_0)\mathrm{Im}(\gamma z_0)} = 1 - 2\lambda(\gamma) \tag{6.14}$$

where $\lambda(\gamma) = q(\gamma_2)/q(\gamma)$. Thus we denote $m_s(\gamma) = Q_{s+k}(1 - 2\lambda(\gamma))$.

Lemma 6.2.1. *Let v be an infinite place of F. Then*

$$\mathcal{K}_\phi^{(v)}(g, (t_1, t_2)) = i_{\bar{v}}(Z_*(g)t_1, t_2). \tag{6.15}$$

Proof. For two distinct CM points given by $(z_1, \beta_1), (\gamma z_1, \beta_2)$, we can write

$$\sum_{\gamma \in \mu_U \backslash B_+^{\times}} m_s(z_1, \gamma z_2) 1_U(\beta_1^{-1}\gamma\beta_2) = \sum_{\gamma \in \mu_U \backslash B_+^{\times}} m_s(\gamma) 1_U(\beta_1^{-1}\gamma\beta_2). \tag{6.16}$$

We may replace the summation index by $B_+^{\times} - E^{\times}$ since $\beta_1 \neq \beta_2$ implies that $\beta_1^{-1}\gamma\beta_2 \notin U$ for all $\gamma \in E^{\times}$. Thus we can write the local height (6.13) as

$$i_{\bar{v}}((z_1, \beta_1), (z_2, \beta_2)) = \widetilde{\lim_{s \to 0}} \sum_{\gamma \in \mu_U \backslash B_+^{\times} - E^{\times}} m_s(\gamma) 1_U(\beta_1^{-1}\gamma\beta_2). \tag{6.17}$$

By [21, Proposition 8.1], we obtain

$$i_{\bar{v}}(Z_*(g)t_1, t_2) = \sum_{a \in F^{\times}} \widetilde{\lim_{s \to 0}} \sum_{\gamma \in \mu_U \backslash B_+^{\times} - E^{\times}} r(g, (t_1, t_2))\phi(\gamma + aq(\gamma)^{-1}) m_s(\gamma). \tag{6.18}$$

To compare this to

$$\mathcal{K}_\phi^{(v)}(g, (t_1, t_2)) = \sum_{a \in F^{\times}} \widetilde{\lim_{s \to 0}} \sum_{\gamma \in \mu_U \backslash B_+^{\times} - E^{\times}} r(g, (t_1, t_2))\phi(\gamma + aq(\gamma)^{-1}) k_{v,s}(\gamma) \tag{6.19}$$

given in Proposition (5.2.3), it is enough to compare $m_s(\gamma)$ with $k_{v,s}(\gamma)$. Recall that

$$k_{v,s}(\gamma) = \frac{\Gamma(s + 2k - 1)}{2(4\pi)^s \Gamma(2k - 1)} \int_1^{\infty} (t - 1)^k t^{-k-1} (1 + \lambda(y)t)^{-s-k} dt. \tag{6.20}$$

Referring to the computation in [9, p.293], we may rewrite the integrand by partial fraction decomposition as a linear combination of t^{-j} and $(1 + \lambda(y)t)^{-s-j}$ with $1 \leq j \leq k$. Then setting $z = 1 + \lambda(y)$ and making the substitution $t = 1 + \sqrt{\frac{z-1}{z+1}} e^u$ we obtain

$$\int_1^{\infty} (t - 1)^{k-1} t^{-k} (1 + \frac{z - 1}{2} x)^{-k} dt = \int_{-\infty}^{\infty} (z + \sqrt{z^2 - 1} \cosh u)^{-k} du = 2Q_k(z). \tag{6.21}$$

The equality follows. This proves Assumption (2) of Theorem (6.1.1). $\square$

In the non-archimedean setting, given a fixed place v of F, the multiplicity function m on CM points x, y in $K^{\times} \backslash \mathbf{B}_f^{\times} / U$ will be defined by

$$m(x, y) = m_0(x, y) m_1(x, y), \tag{6.22}$$

where $m_0(x, y)$ is the multiplicity function at v defined on CM points in [23, Section 5], and $m_1(x, y)$ denotes the local intersection at v of the CM cycles $S_k(x)$ and $S_k(y)$ attached to x and y.

6.3. **Supersingular Case.** Let v be a finite prime of F nonsplit in E but split in B. The local pairing depends on the local integral model of the Shimura curve. We first study the multiplicity function m_1.

6.3.1. *Local intersections.* The Shimura curve X can be viewed as the fine moduli space classifying triples (A, ι, v) where A is an abelian surface endowed with maps

$$\iota : U \hookrightarrow A, \text{ and } v : (U/MU)_S \hookrightarrow A[M], \tag{6.23}$$

and $A[M]$ denotes the M-division points of A. Let W be a complete discrete valuation ring with characteristic zero field of fraction. Let x, y be in $X \otimes W$ and let W_x, W_y be the normalizations of their structure rings. Denote by w a uniformizer of W_x. Define $\mathrm{Isom}_n(x, y)$ as the set of couples (f, g) of an embedding $f : \mathrm{Spec}\ W_x/w^n \longrightarrow \mathrm{Spec}\ W_y$ over W, and a homomorphism g of group schemes over W_x/w^n making the following diagrams commutative.

$$\begin{tikzcd} U \arrow[r, "\iota_x"] \arrow[dr, "\iota_y"'] & A_x \arrow[d, "g"] \\ & A_y \end{tikzcd} \qquad \begin{tikzcd} U/MU \arrow[r, "v_x"] \arrow[dr, "v_y"'] & A_x[M] \arrow[d, "g"] \\ & A_y[M] \end{tikzcd}$$

As in [23, Proposition 4.1.1(a)] and [23, Equation 3.3.1], one computes

$$\langle S_k(x), S_k(y)\rangle_v = \frac{(-1)^k}{2} \sum_{\substack{n\geq 0,\\ \psi\in \mathrm{Isom}_n(x,y)}} S_k(y)_0 \cdot \psi_0^* S_k(x)_0 \tag{6.24}$$

$$= \langle x, y\rangle_v \ S_k(y)_0 \cdot \psi_0^* S_k(x)_0, \tag{6.25}$$

where $\psi_0^* S_k(x)_0$ is the pullback of $S_k(x)_0$, the operation $\cdot$ indicates taking intersection product in $A_y^k \otimes_g (W_x/w)$, and $\langle x, y\rangle_v$ is the local length at v of the intersection of x with y. Next, we write $S_k(y) \cdot S_k(x)$ for $S_k(y)_0 \cdot \psi_0^* S_k(x)_0$ to simplify notation. This is possible because $|\mathrm{Isom}_0(x, y)| = 1$. We remind the reader that by our assumptions we need not consider self-intersections.

6.3.2. *Multiplicity function.* Given an isogeny between x and y, we have an induced endomorphism ϕ^* on the Shimura curve X given by a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Lemma 6.3.1. *Let $P_{k-1}(t)$ denote a constant multiple of $\frac{d^{k-1}}{dt^{k-1}}(t^2-1)^{k-1}$ such that $P_{k-1}(1) = 1$, then*

$$S_k(x)_0 \cdot \phi^* S_k(y)_0 = (-\det M)^{k-1} P_{k-1}\Big(\frac{bc + ad}{\det M}\Big). \tag{6.26}$$

Proof. It suffices to follow the proof in [23, Proposition 3.3.3] using the moduli interpretation of the Shimura curve X in terms of abelian surfaces as in Section (6.3.1), and the fact that $\dim H^2_{et}(A, \mathbf{Q}_\ell) = 2$ for a smooth abelian surface A. □

Given CM points x and y in $K^\times\backslash \mathbf{B}_f^\times/U$ with same reduction in $F_+^\times\backslash \mathbf{B}_f^\times/\det U$, the map

$$K^\times\backslash \mathbf{B}_f^\times/U \longrightarrow B\backslash F_v^\times \times \mathbf{B}_f^v/U : \ g \longrightarrow (\det g_v, g^\wp) \tag{6.27}$$

associates to them the supersingular points $(\det g_v, g^v)$ and $(\det h_v, h^v)$ in $B\backslash F_v^\times \times \mathbf{B}_f^v/U$. Since x and y lie in the same connected component, there is γ in $\mathbf{B}_f^\times$ such that

$$\gamma = \gamma_v\gamma^v, \quad h^v = \gamma^v g^v, \quad \det(h_v) = \det(\gamma_v)\det(g_v), \tag{6.28}$$

where γ_v lies in $\mathbf{B}_v^\times$, and γ^v lies in $\mathbf{B}_f^{\times,v}$. Let B_0 be the trace zero elements of $B^\times$. The isomorphism

$$B_0\backslash\mathbf{B}_f^v/U^v \simeq B\backslash B_f/U^{'}, \tag{6.29}$$

where $U^{'} = \mathcal{O}_{B,v}^\times \cdot U^v$, allows to view γ^v in $B\backslash B_f^\times/U^{'}$ and further localize it at v to obtain the element g_1 in $B_v^\times$. The intersection $S_k(x)\cdot S_k(y)$ of the CM cycles associated to x and y at v only depends on g_1 and $g_2 = g_v h_v^{-1}$. We will hence denote it by $m_1(g_1, g_2)$.

Given a pair (a, b) in

$$\mathcal{H}_{U_v} := B_v^\times \times_{K_v^\times} \mathbf{B}_v^\times/U_v \tag{6.30}$$

satisfying $\det(a)\det(b) = 1$, we denote by $m_0(a, b)$ the multiplicity function defined in [23, Section 5.5]. The multiplicity function m is an intersection pairing on $\mathcal{H}_{U_v}$ defined by

$$m(a, b) = m_0(a, b)\ m_1(a, b). \tag{6.31}$$

Using Definition (6.31), we have by [21, Lemma 8.2] the expression

$$i_{\bar{v}}(\beta_1, \beta_2) = \sum_{\gamma\in\mu_U\backslash B^\times} m(\gamma\beta_{2v}, \beta_{1v}^{-1})1_{U^v}((\beta_1^v)^{-1}\gamma\beta_2^v). \tag{6.32}$$

As a consequence, by [21, Proposition 8.4], we obtain

$$i_{\bar{v}}(Z_*(g)t_1, t_2) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B-E} r(g,(t_1,t_2))\phi^v(y,u)m_{r(g,(t_1,t_2))\phi_v}(y,u), \tag{6.33}$$

for g in $P(F_{S_1})G(\mathbf{A}^{S_1})$, where

$$m_{\phi_v}(y,u) = \int_{\mathbf{B}_v^\times/U_v} m(y, x^{-1})\phi_v(x, uq(y)/q(x))dx, \tag{6.34}$$

for y in $B_v - E_v$, and u in $F_v^\times$.

As in [21, Section 8.2], in the unramified case, one can rewrite $m_{\phi_v}(y,u)$ as

$$m_{\phi_v}(y,u) = \sum_{x\in M_2(\mathcal{O}_{F_v})_n/U_v} m(y^{-1}, x), \tag{6.35}$$

where $M_2(\mathcal{O}_{F_v})_n$ denotes the set of integral matrices whose determinants have valuation $n = \mathrm{ord}_v(q(y))$. Consider the decomposition

$$GL_2(F_v) = \coprod_{c=0}^{\infty} K_v^\times h_c GL_2(\mathcal{O}_{F_v}), \tag{6.36}$$

where $h_c = \begin{pmatrix}1 & 0\\ 0 & w^c\end{pmatrix}$. The set $K_v^\times h_c GL_2(\mathcal{O}_{F_v}) \cap M_2(\mathcal{O}_{F_v})_n$ is non-empty only if $n-c$ is non-negative and even. It is given then by $w^{\frac{n-c}{2}}\mathcal{O}_{K_v}^\times h_c U_v$. Therefore, letting

$$x_c = \begin{pmatrix} w^{\frac{n-c}{2}} & 0 \\ 0 & w^{\frac{n+c}{2}} \end{pmatrix}, \tag{6.37}$$

we have

$$m_{\phi_v}(y,u) = \sum_{c=0}^{\infty} m(y^{-1}, x_c)\mathrm{vol}(K_v^\times h_c GL_2(\mathcal{O}_{F_v}) \cap M_2(\mathcal{O}_{F_v})_n). \tag{6.38}$$

Consider $\psi = \begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix}$. Then $\psi^{\frac{n-c}{2}-1}$ is an isogeny between x_c and x_n in $G(F_v)$. Hence,

$$m_1(y, x_c) = m_1(y, x_n) \tag{6.39}$$

by Lemma (6.3.1). Consider the isogeny $\psi_n = \begin{pmatrix} w^{-\frac{n}{2}} & 0 \\ 0 & w^{\frac{n}{2}} \end{pmatrix}$ between $w^{-\frac{n}{2}}x_n$ and 1 in $\mathbf{B}_v^\times$, then

$$m_1(y, x_n) = m_1(w^{\frac{n}{2}}y, w^{-\frac{n}{2}}x_n) = m_1(w^{\frac{n}{2}}y, 1) = 1. \tag{6.40}$$

The first equality follows from the property $m(y^{-1}, a^{-1}x) = m((ay)^{-1}, x)$ for a in $F_v^\times$, and the second and third equalities follow from Lemma (6.3.1) respectively applied to the matrices ψ_n and 1.

As explained in [21, Section 8.2.3], for $c \neq 0$,

$$m_0(y^{-1}, x_c)\mathrm{vol}(K_v^\times h_c GL_2(\mathcal{O}_{F_v}) \cap M_2(\mathcal{O}_{F_v})_n) = 1, \tag{6.41}$$

implying that

$$m(y^{-1}, x_c)\mathrm{vol}(K_v^\times h_c GL_2(\mathcal{O}_{F_v}) \cap M_2(\mathcal{O}_{F_v})_n) = 1 \tag{6.42}$$

by the identities (6.39) and (6.40). Hence, we have

$$m_{\phi_v}(y,u) = \frac{1}{2}(\mathrm{ord}_v(q(y_2)) + 1), \tag{6.43}$$

as in [21, Proposition 8.7].

6.3.3. *Local comparisons.* We remind the reader that the assumptions on the degenerate Schwartz functions are in force. We divide the cases into the unramified case where ϕ_v is the characteristic function of $M_2(\mathcal{O}_v) \times \mathcal{O}_v^\times$ and U_v is the maximal compact subgroup of $G(\mathcal{O}_v)$, and the ramified case where U_v is general. These cases correspond to nonsplit v not in S_1 and v in S_1 respectively.

Lemma 6.3.2. *Let v be nonsplit in E but split in $\mathbf{B}$. In the unramified case, we have*

$$k_{r(t_1,t_2)\phi_v}(g,y,u) = m_{r(g,(t_1,t_2))\phi_v}(y,u)\log N_v. \tag{6.44}$$

and thus

$$\mathcal{K}_\phi^{(v)}(g,(t_1,t_2)) = i_{\bar{v}}(Z_*(g)t_1, t_2)\log N_v. \tag{6.45}$$

In the ramified case, given ϕ_v in $\bar{\mathscr{S}}^1(B_v \times F_v^\times)$ invariant under the right action of U_v, the function $m_{\phi_v}(y,u)$ extends to a Schwartz function on $B_v \times F^\times$, and thus

$$i_{\bar{v}}(Z_*(g,\phi)t_1, t_2) = \theta(g,(t_1,t_2), m_{\phi_v} \otimes \phi^v). \tag{6.46}$$

for all g in $1_{S_1}G(\mathbf{A}^{S_1})$.

Proof. We first consider the unramified setting. The case $(g, t_1, t_2) = (1, 1, 1)$ follows from (6.43) and the formula

$$k_{\phi_v}(1, y, u) = 1_{\mathcal{O}_{B_v} \times \mathcal{O}_{F_v}^\times}(y, u) \frac{1}{2}(1 + v(q(y_2))) \log N_v \tag{6.47}$$

by [21, Proposition 6.8], since the Whittaker functions at non-archimedean places are the same as in the weight 2 case. The result extends to g in $G(\mathcal{O}_{F_v})$ since for standard Schwartz functions ϕ_v, the kernel functions are the same. More generally, apply the action of $P(F_v)$ and $E_v^\times \times E_v^\times$, observing that k_{ϕ_v} and m_{ϕ_v} transform like the Weil representation respectively by [21, Lemma 6.6] and by directly verifying that

$$m_{r(g,(t_1,t_2))\phi_v}(y, u) = r(g, (t_1, t_2)) m_{\phi_v}(y, u) \tag{6.48}$$

for g in $P(F_v)$ and t_1, t_2 in $E_v^\times$. Note that this uses the property

$$m(y^{-1}, ax) = m(ay^{-1}, x) \tag{6.49}$$

for a in $F_v^\times$.

As for the ramified case, we defer the proof to Lemma (6.4.1). $\square$

This completes the proof of (3) in Theorem (6.1.1).

6.4. **Superspecial Case.** The approach in the superspecial case is very similar to the one in the supersingular case. Let v be nonsplit in E and $\mathbf{B}$. As in the supersingular case, we will view the multiplicity function m as a function on $\mathcal{H}_{U_v} = B_v^\times \times_{E_v^\times} \mathbf{B}_v^\times / U_v$.

Lemma 6.4.1. *Given ϕ_v in $\bar{\mathscr{S}}^1(B_v \times F_v^\times)$ invariant under the right action of U_v, the function $m_{\phi_v}(y, u)$ extends to a Schwartz function on $B_v \times F_v^\times$, thus*

$$i_{\bar{v}}(Z_*(g, \phi) t_1, t_2) = \theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi^v), \tag{6.50}$$

for g in $1_{S_1} G(\mathbf{A}^{S_1})$.

Proof. The lemma follows from the key observation that the multiplicity function $m(\beta_1, \beta_2)$ vanishes if $m_0(\beta_1, \beta_2)$ vanishes. That is, we have an inclusion of supports

$$\operatorname{supp}(m(\beta_1, \beta_2)) \subset \operatorname{supp}(m_0(\beta_1, \beta_2)). \tag{6.51}$$

Then, by the arguments of [21, Proposition 8.9] and [21, Proposition 8.13], $m_{\phi_v}(y, u)$ extends to a Schwartz function on $B_v \times F_v^\times$. This allows us to approximate

$$i_{\bar{v}}(Z_*(g, \phi) t_1, t_2)$$

by the theta series

$$\theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi^v) = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{y \in V} r(g, (t_1, t_2)) m_{\phi_v}(y, u) r(g, (t_1, t_2)) \phi^v(y, u), \tag{6.52}$$

giving rise to Identity (6.50). $\square$

This completes the proof of (4) in Theorem (6.1.1).

6.5. **Ordinary Case.** Let v be a finite prime of F split in K. The reduction map from CM points to ordinary CM points is given by

$$K^\times\backslash \mathbf{B}_f^\times/U \longrightarrow K^\times\backslash [N(F_v)\backslash \mathbf{B}_{f,v}^\times]\times \mathbf{B}_f^{v,\times}/U, \tag{6.53}$$

where $N(F_v)$ is the unipotent radical of $B^\times(F_v)$. Hence, ordinary CM points x and y in the same connected component can be represented by g, h in $\mathbf{B}_f^\times$ such that $h^v = g^v$ and $h_v = ng_v$ for n in $N(F_v)$, see [23, Section 5] for more details. In this case, we do not need an explicit form of the multiplicity function m. We fix an identification $\mathbf{B}_v \simeq M_2(F_v)$ under which $K_v = \mathrm{diag}(F_v, F_v)$. Consider the prime v_1 lying in K above v such that v_1 corresponds to the ideal $\mathrm{diag}(F_v, 0)$. The local height of two CM points $\beta_1, \beta_2 \in K^\times\backslash \mathbf{B}_f^\times/U$ is given by

$$i_{\bar{v}_1}(\beta_1,\beta_2) = \sum_{\gamma\in\mu_U\backslash K^\times} m_{v_1}(\beta_1,\gamma\beta_2)1_{U^v}(\beta_1^{-1}\gamma\beta_2). \tag{6.54}$$

The summation is independent of the nearby quaternion algebra, so using Assumption (5.3), by the same argument as in [21, Proposition 8.15], we have

$$i_{\bar{v}}(Z_*(g)t_1, t_2) = 0. \tag{6.55}$$

This proves (1) of Theorem (6.1.1), and concludes the proof.

7. Bounding the Selmer group

In this section, we relate the order of vanishing of the L-function of π at the central point to the size of the associated Selmer and Shafarevich-Tate groups. Under the conjectures on the injectivity of the p-adic étale Abel Jacobi map (7.3) and the Gillet-Soulé conjecture (7.2.2), we obtain an equivalence between order of vanishing 1 for the L-function and rank 1 for the Selmer group.

7.1. **Selmer and Tate-Shafarevich groups.** We define the Selmer and Tate-Shafarevich groups, quote the Beilinson-Bloch-Kato conjectures, and state a result of the first author and de Vera Piquero on the size of the Selmer group, assuming the non-vanishing of the CM cycle Z.

Fix $F = \mathbf{Q}$, and consider f in π. Let p be an odd prime not dividing $N\cdot(2k)!$, and let $\wp$ be a prime ideal dividing p in the number field E generated over $\mathbf{Q}$ by the Fourier coefficients of f. We view the Galois representation $V_\wp(f)$ attached to f, that is a 2-dimensional $E_\wp$-vector space, as a factor of the middle étale cohomology of the Kuga-Sato variety over the Shimura curve [2, 10]. Let $Y := V_\wp\otimes \mathbf{Q}_p/\mathbf{Z}_p$ and $Y_s := Y_{p^s} = V_\wp/p^sV_\wp$ for $s\geq 1$.

Definition 7.1.1. The (p^s-th) *Selmer group* $\mathrm{Sel}_\wp^{(s)}(f,K)\subseteq H^1(K,Y_s)$ is defined as

$$\{x\in H^1(K,Y_s) : x_v\in H^1_{\mathrm{ur}}(K_v,Y_s) \text{ for all } v\nmid Np \text{ and } x_v\in H^1_{\mathfrak{f}}(K_v,Y_s) \text{ for } v\mid p\},$$

where

$$H^1_{\mathrm{ur}}(K_v,Y_s) = H^1(\bar{K}_v/K_v^{ur},Y_s), \text{ and } H^1_{\mathfrak{f}}(K_v,Y_s) = \text{ finite part of } H^1(K_v,Y_s). \tag{7.1}$$

The p-adic étale Abel-Jacobi map induces a Hecke and Galois-equivariant map

$$\mathrm{CH}^k(W_{k-1}/K)_0\otimes E_\wp \longrightarrow H^1(K,V_\wp(f)) \tag{7.2}$$

and factors through the Selmer group

$$\mathrm{CH}^k(W_{k-1}/K)_0 \otimes \mathcal{O}_\wp/p^s\mathcal{O}_\wp \longrightarrow \mathrm{Sel}_\wp^{(s)}(f,K). \tag{7.3}$$

The Beilinson conjectures predict that the rank of the Chow group of a smooth projective variety X of dimension $2i+1$ over a number field K corresponds to the order of vanishing of the L-function attached to its étale cohomology

$$\mathrm{ord}_{s=i+1}\ L(H_{\mathrm{et}}^{2i+1}(X),\ s) = \dim_{\overline{\mathbf{Q}}}\ \mathrm{CH}^{i+1}(X)_0, \tag{7.4}$$

(see [1, Conjecture 5.9]). This conjecture can be refined by applying a suitable projector e to both sides of the equality. Letting M denote $H_{\mathrm{et}}^{2i+1}(X)$, Bloch and Kato predict that the Abel-Jacobi map

$$e\ \mathrm{CH}^{i+1}(X)_0 \otimes \mathbf{Q}_p \longrightarrow \mathrm{Sel}(K, e\ M) \tag{7.5}$$

is an isomorphism [3, Conjecture 5.3]. Hence, one expects that

$$\mathrm{ord}_{s=i+1}L(e\ M, s) = \mathrm{rank}\ \mathrm{Sel}(K, e\ M). \tag{7.6}$$

We denote

$$\mathrm{Sel}_\wp^{(\infty)}(f,K) := \varprojlim \mathrm{Sel}_\wp^{(s)}(f,K). \tag{7.7}$$

Taking the inductive limit of the Abel–Jacobi maps (7.3) one obtains a map

$$\Phi : \mathrm{CH}^k(W_{k-1}/K)_0 \otimes \mathcal{O}_\wp \longrightarrow \mathrm{Sel}_\wp^{(\infty)}(f,K) \subseteq H^1(K, V_\wp). \tag{7.8}$$

Its cokernel is by definiton the $\wp$-primary part of the *Shafarevich-Tate group*,

$$Ш_{\wp^\infty} := \mathrm{coker}(\Phi) = \mathrm{Sel}_\wp^{(\infty)}(f,K)/\mathrm{Im}(\Phi). \tag{7.9}$$

Theorem 7.1.2. [7, Theorem 1.1] *Suppose Z is non-torsion. Then* $\mathrm{Im}(\Phi)$ *has rank* 1 *and* $Ш_\wp$ *is finite. More precisely, we have*

$$(\mathrm{Im}(\Phi))^\varepsilon = 0 \quad \textit{and} \quad (\mathrm{Im}(\Phi))^{-\varepsilon} = E_\wp \cdot y_0.$$

In [7], the authors construct an Euler system of Kolyvagin cohomology classes c_n in $\mathrm{Sel}_\wp^{(s)}(f, K_n)$ where K_n is a ring class field of conductor n, that is a collection of cohomology classes satisfying suitable global and local compatibility properties. Kolyvagin's method is subsequently applied to bound the size of $\mathrm{Sel}_\wp^{(\infty)}(f,K)$ and $Ш_{\wp^\infty}$.

7.2. **Bridging analytic and algebraic invariants.** We combine Theorem (7.1.2) with our main theorem (2.8.1) to link analytic and algebraic invariants associated to π, under suitable assumptions.

Theorem 7.2.1. *Assume that the p-adic étale Abel-Jacobi map Φ is injective. Then*

$$L'(k,\pi) \neq 0 \implies \mathrm{Im}(\Phi) \otimes \mathbf{Q}\ \textit{has rank}\ 1,\ \textit{and}\ |Ш_{\wp^\infty}| < \infty. \tag{7.10}$$

Proof. If $L'(k,\pi) \neq 0$, then Z is non-torsion in $\mathrm{CH}^k(W_{k-1}/K)_0$ by Theorem(2.8.1) hence

$$Z_0 = \mathrm{cor}_{K_1/K} Z_\wp \tag{7.11}$$

is non-torsion in $H^1(K, V_\wp)$. The injectivity of the p-adic Abel Jacobi map predicted by Bloch and Kato [3] implies that $\Phi(Z_0)$ is non-torsion. By Theorem (7.1.2), we conclude that $\mathrm{Im}(\Phi) \otimes \mathbf{Q}$ has rank 1, and $Ш_{\wp^\infty}$ is finite. □

Next, we deduce analytic rank 1 from algebraic rank 1, under the Gillet-Soulé conjecture [8, Conjecture 2] which implies that the height pairing $\langle\cdot,\cdot\rangle_{\mathrm{GS}}$ is non-degenerate in our setting.

We state the Gillet-Soulé conjecture. Consider a line bundle $\mathscr{L}$ on a regular arithmetic scheme Y of dimension d over $\mathrm{Spec}(\mathcal{O}_F)$, equipped with a smooth hermitian metric invariant under the complex conjugation F_∞. One can associate to $\mathscr{L}$ a first Chern class $\hat{c}_1(\bar{\mathscr{L}})$ in $\widehat{\mathrm{CH}}^1(Y)_{\mathbf{R}}$, namely the class $(\mathrm{div}(s) - \log||s||)$, for a nonzero rational section s of $\mathscr{L}$ on Y. Denote by L the map given by

$$L\ :\ \widehat{\mathrm{CH}}^p(Y)_{\mathbf{R}} \longrightarrow \widehat{\mathrm{CH}}^{p+1}(Y)_{\mathbf{R}}\ :\ Y \mapsto (Y, \hat{c}_1(\bar{\mathscr{L}})). \tag{7.12}$$

The line bundle $\bar{\mathscr{L}}$ is *positive* if the following conditions are satisfied:

- it is ample on Y,
- the curvature $c_1(\bar{\mathscr{L}})$ is a positive $1-1$ form on $Y(\mathbf{C})$, and
- for any subvariety Y' of Y of dimension n, flat over $\mathrm{Spec}(\mathbf{Z})$, we have

$$(c_1(\bar{\mathscr{L}})^n, Y') > 0.$$

Conjecture 7.2.2 (Gillet-Soulé)**.** *Let $\bar{\mathscr{L}}$ be a positive hermitian line bundle on Y and x an arithmetic Chow cycle of codimension p. If $2p \leq d$, $x \neq 0$ and $L^{d-2p+2}(x) = 0$, then*

$$(-1)^p \deg(xL^{d-2p+1}x) > 0. \tag{7.13}$$

In our setting, $d = 2k-1$ and $p = k$, so that $d - 2p + 2 = 1$. The conjecture hence implies that if $x \neq 0$ and $(x, \hat{c}_1(\bar{\mathscr{L}})) = 0$, then $(-1)^k \deg(x \cdot x) > 0$, namely, the height pairing is non-degenerate.

Proposition 7.2.3. *For CM cycles Z_A, the intersection $\hat{c}_1(\mathscr{L}) \cdot Z_A$ is zero in $\widehat{\mathrm{CH}}(W)$.*

Proof. Consider a line bundle $\bar{\mathscr{L}}$ that is the sum of pull-backs $\mathscr{L}_i$, $1 \leq i \leq 2k-2$, of line bundles on $\mathbf{A}$ with a divisor supported on the unit section. The proof follows the proof of [23, Proposition 3.1.1] where the next lemma substitutes Lemma 3.1.3. □

Lemma 7.2.4. *We have*

$$(Z_A, \{0\} \times E_{\mathfrak{a}}) = 0 \tag{7.14}$$

$$(Z_A, E \times \{0\}) = 0. \tag{7.15}$$

Theorem 7.2.5. *Suppose the Gillet-Soulé conjecture* (7.2.2) *holds. Then*

$$\mathrm{Im}(\Phi) \otimes \mathbf{Q} \text{ has rank } 1 \quad \Longrightarrow \quad L'(k, \pi) \neq 0. \tag{7.16}$$

Proof. Let g be a generator of $\mathrm{Im}(\Phi) \otimes \mathbf{Q}$. Then $T(f_1 \otimes f_2)Z$ and Z can be written in terms of g. Since the height pairing $\langle\cdot,\cdot\rangle_{\mathrm{GS}}$ is non-degenerate by Conjecture (7.2.2) and Proposition (7.2.3), we have that $\langle T(f_1 \otimes f_2)Z, Z\rangle_{\mathrm{GS}}$ is non-zero. Hence, $L'(k, \pi) \neq 0$ by Theorem(2.8.1). □

As a consequence, we obtain the following corollary.

Corollary 7.2.6. *Assume that the p-adic Abel-Jacobi map Φ is injective and the Gillet-Soulé conjecture* (7.2.2) *holds. Then*

$$\mathrm{ord}_{s=k} L(s, \pi) = 1 \iff \mathrm{rank}\ (\mathrm{Im}(\Phi) \otimes \mathbf{Q}) = 1. \tag{7.17}$$

Remark 7.2.7. Chida [6] used the method of Euler systems to prove that, under certain assumptions, the non-vanishing of the central value of the Rankin-Selberg L-function associated with a modular form f and a ring class character implies the finiteness of the associated Selmer group.

Acknowledgments. This research was supported through the programme "Oberwolfach Leibniz Fellows" by the Mathematisches Forschungsinstitut Oberwolfach in 2017, and also by the Max Planck Institute for Mathematics, Bonn.

References

[1] Alexandre A. Beĭlinson. Height pairing between algebraic cycles. In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, volume 1289 of *Lecture Notes in Math.*, pages 1–25. Springer, Berlin, 1987.
[2] Amnon Besser. CM cycles over Shimura curves. *J. Algebraic Geom.*, 4(4):659–691, 1995.
[3] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.
[4] Li Cai, Jie Shu, and Ye Tian. Explicit Gross-Zagier and Waldspurger formulae. *Algebra Number Theory*, 8(10):2523–2572, 2014.
[5] Henri Carayol. Sur la mauvaise réduction des courbes de Shimura. *Compositio Math.*, 59(2):151–230, 1986.
[6] Masataka Chida. Selmer groups and central values of L-functions for modular forms. *To appear in Annales de lInstitut Fourier.*
[7] Yara Elias and Carlos de Vera-Piquero. CM cycles on Kuga-Sato varieties over Shimura curves and Selmer groups. *To appear in Forum Math.*
[8] Henri Gillet and Christophe Soulé. Arithmetic analogs of the standard conjectures. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 129–140. Amer. Math. Soc., Providence, RI, 1994.
[9] Benedict H. Gross and Don B. Zagier. Heegner points and derivatives of L-series. *Invent. Math.*, 84(2):225–320, 1986.
[10] Adrian Iovita and Michael Spieß. Derivatives of p-adic L-functions, Heegner cycles and monodromy modules attached to modular forms. *Invent. Math.*, 154(2):333–384, 2003.
[11] Victor A. Kolyvagin. *Euler Systems*, volume 2 of *The Grothendieck Festschrift.* Birkhäuser Boston, Boston, MA, 1990.
[12] David I. Lieberman. Higher Picard varieties. *Amer. J. Math.*, 90:1165–1199, 1968.
[13] Jan Nekovář. Kolyvagin's method for Chow groups of Kuga-Sato varieties. *Invent. Math.*, 107(1):99–125, 1992.
[14] Jan Nekovář. The Euler system method for CM points on Shimura curves. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 471–547. Cambridge Univ. Press, Cambridge, 2007.
[15] Alexandru A. Popa. Central values of Rankin L-series over real quadratic fields. *Compos. Math.*, 142(4):811–866, 2006.
[16] Hiroshi Saito. On Tunnell's formula for characters of GL(2). *Compositio Math.*, 85(1):99–108, 1993.
[17] Takeshi Saito. Hilbert modular forms and p-adic hodge theory. *Compositio Mathematica*, 145(5):1081–1113, 2009.
[18] Jerrold B. Tunnell. Local ϵ-factors and characters of GL(2). *Amer. J. Math.*, 105(6):1277–1307, 1983.
[19] Jean-Loup Waldspurger. Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie. *Compositio Math.*, 54(2):173–242, 1985.
[20] Tonghai Yang. Taylor expansion of an Eisenstein series. *Trans. Amer. Math. Soc.*, 355(7):2663–2674, 2003.
[21] Xinyi Yuan, Shouwu Zhang, and Wei Zhang. *The Gross–Zagier formula on Shimura curves.* Number 184. Princeton University Press, 2013.
[22] Shaul Zemel. A Gross-Kohnen-Zagier type theorem for higher-codimensional Heegner cycles. *Res. Number Theory*, 1:Art. 23, 44, 2015.

[23] Shouwu Zhang. Heights of Heegner cycles and derivatives of L-series. *Invent. Math.*, 130(1):99–152, 1997.
[24] Shouwu Zhang. Gross-Zagier formula for GL_2. *Asian J. Math.*, 5(2):183–290, 2001.

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