BIASES IN PRIME FACTORIZATIONS AND LIOUVILLE FUNCTIONS FOR ARITHMETIC PROGRESSIONS

PETER HUMPHRIES, SNEHAL M. SHEKATKAR, AND TIAN AN WONG

Abstract. We introduce a refinement of the classical Liouville function to primes in arithmetic progressions. Using this, we discover new biases in the appearances of primes in a given arithmetic progression in the prime factorizations of integers. For example, we observe that the primes of the form $4k+1$ tend to appear an even number of times in the prime factorization of a given integer, more so than for primes of the form $4k+3$. We are led to consider variants of Pólya's conjecture, supported by extensive numerical evidence, and its relation to other conjectures.

Contents

1. Introduction

1.1. **The Liouville function.** The classical Liouville function is the completely multiplicative function defined by $\lambda(p) = -1$ for any prime p. It can be expressed as

$$\lambda(n) = (-1)^{\Omega(n)} \tag{1.1}$$

where $\Omega(n)$ is the total number of prime factors of n. One sees that it is -1 if n has an odd number of prime factors, and 1 otherwise. By its relation to the Riemann zeta function

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}, \tag{1.2}$$

the Riemann hypothesis is known to be equivalent to the statement that

$$L(x) := \sum_{n \leq x} \lambda(n) = O_\epsilon(x^{1/2+\epsilon}) \tag{1.3}$$

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for any $\epsilon > 0$; whereas the prime number theorem is equivalent to the estimate $o(x)$. Indeed, the behaviour of the Liouville function, being a close relative of the more well-known Möbius function, is strongly connected to prime number theory. Also, we note that by the generalized Riemann hypothesis, one also expects (1.3) to hold for partial sums of $\lambda(n)$ restricted to arithmetic progressions.

In this paper, we introduce natural refinements of the Liouville function, which detect how primes in given arithmetic progressions appear in prime factorizations. Interestingly, we find that that these functions behave in somewhat unexpected ways, which is in turn related to certain subtleties of the original Liouville function. We now describe this briefly.

Define $\Omega(n; q, a)$ to be the total number of prime factors of n congruent to a modulo q, and

$$\lambda(n; q, a) = (-1)^{\Omega(n;q,a)} \tag{1.4}$$

to be the completely multiplicative function that is -1 if n has an odd number of prime factors congruent to a modulo q, and 1 otherwise. They are related to the classical functions by

$$\lambda(n) = \prod_{a=0}^{q-1} \lambda(n; q, a), \quad \Omega(n) = \sum_{a=0}^{q-1} \Omega(n; q, a). \tag{1.5}$$

Using this we study the asymptotic behaviour instead of

$$L(x; q, a) = \sum_{n \le x} \lambda(n; q, a), \tag{1.6}$$

hence the distribution of the values of $\lambda(n; q, a)$. Also, we will be interested in r-fold products of $\lambda(n; q, a)$,

$$\lambda(n; q, a_1, \dots, a_r) = \prod_{i=1}^{r} \lambda(n; q, a_i) \tag{1.7}$$

where the a_i are residue classes modulo q, with $1 \le r \le q$, and define $\Omega(n; q, a_1, \dots, a_r)$ and $L(x; q, a_1, \dots, a_r)$ analogously.

1.2. **Prime factorizations.** Given a prime number p, we will call the *parity* of p in an integer n to be even or odd according to the exponent of p in the prime factorization of n. This includes the case where p is prime to n, in which case its exponent is zero and therefore having even parity.

Landau showed that the number of $n \le x$ containing an even (resp. odd) number of prime factors both tend to

$$\frac{1}{2}x + O(xe^{-c\sqrt{\log x}}) \tag{1.8}$$

with x tending to infinity, and c some positive constant. Soon after, Pólya asked whether $L(x)$ is nonpositive for all $x \ge 2$; this was shown to be false by Haselgrove [6], building on the work of Ingham [8], using the zeroes of $\zeta(s)$, and that in fact the sum must change sign infinitely often. Indeed, the first sign change was later computed to be around 9×10^8. A similar was problem posed by Turán on the positivity of partial sums of $\lambda(n)/n$, which was also shown to be false, with the first sign change taking place around 7×10^{13}. (See [12] for a discussion of these problems.)

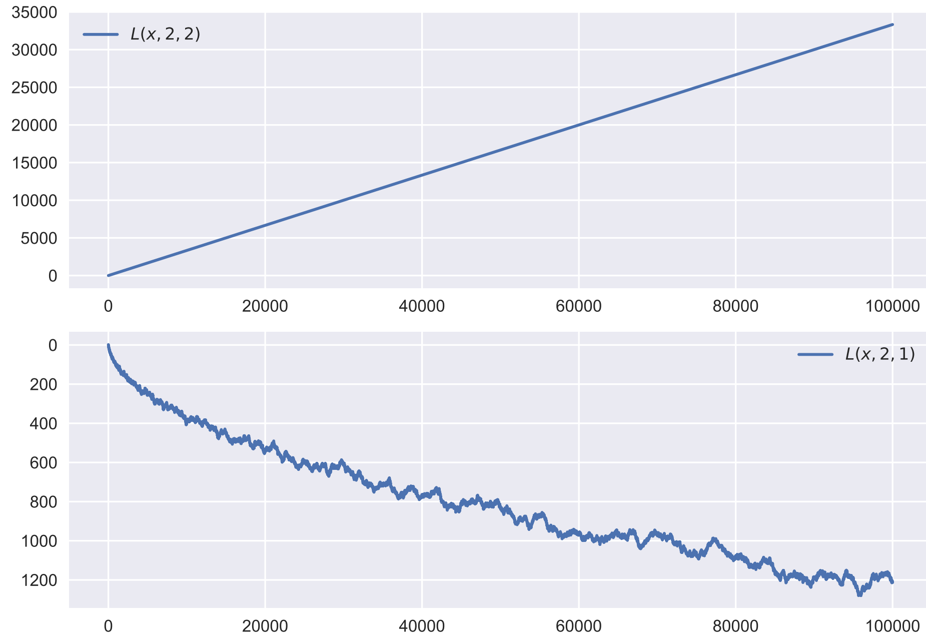

FIGURE 1. $L(x;2,2)$ and $L(x;2,1)$.

On the other hand, by the equidistribution of primes in arithmetic progressions, one might guess that the number of $n \leq x$ containing an even (resp. odd) number of prime factors $p \equiv a \pmod{q}$, for a fixed arithmetic progression would be evenly distributed over residue classes coprime to q. By our analysis of $\lambda(n;q,a)$, we find that this seems not to be the case.

For example, if we consider the parity of 2 and the parity of the odd primes separately, we find the behaviour of the partial sums as in Figure 1 above. Indeed, we prove that the sum $L(x;2,1)$ is $o(x)$, and $O_\epsilon(x^{1/2+\epsilon})$ on the Riemann hypothesis; whereas $L(x;2,2) \geq 0$ and tends to $\frac{1}{3}x$ unconditionally. Numerically, we find that $L(x;2,1) \leq 0$ at least up to $x \leq 10^9$. Interestingly, in spite of the expected square-root cancellation of $L(x;2,1)$, the graph suggests that $\lambda(x;2,1)$ favors -1 more than 1. That is, separating the primes dividing the modulus appears to 'tame' the randomness of $\lambda(n)$.

More generally, we have the following theorem.

Theorem 1.1. *Given any $q \geq 2$, let $a_1, \ldots, a_{\varphi(q)}$ be the residue classes modulo q such $(a_i, q) = 1$, and $b_1, \ldots, b_{q-\varphi(q)}$ the remaining residues classes. Then*

$$\sum_{n \leq x} \lambda(n;q,a_1,\ldots,a_{\varphi(q)}) = o(x), \tag{1.9}$$

for $x \geq 1$. Assuming the Riemann hypothesis, it is in fact $O_\epsilon(x^{1/2+\epsilon})$ for all $\epsilon > 0$.

On the other hand,

$$\sum_{n \leq x} \lambda(n;q,b_1,\ldots,b_k) = \left(\prod_{i=1}^{k} \prod_{p \mid b_i} \frac{p-1}{p+1}\right) x + o(x) \tag{1.10}$$

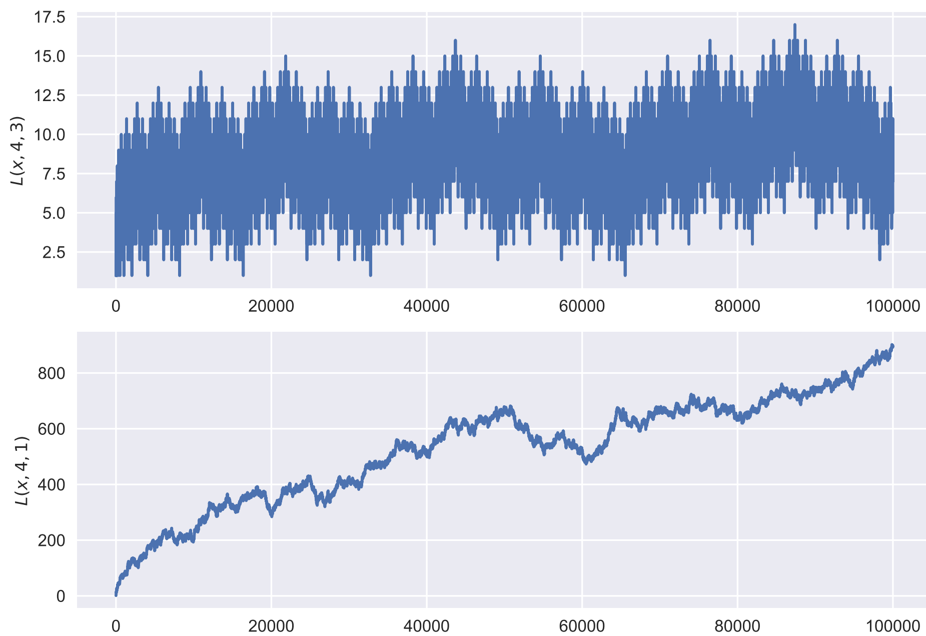

FIGURE 2. $L(x;4,3)$ and $L(x;4,1)$.

for any $1 \leq k \leq q - \varphi(q)$.

In fact, it is straightforward to show that the estimate $O_\epsilon(x^{1/2+\epsilon})$ is equivalent to the Riemann hypothesis, as in the classical Liouville function.

Having separated the odd primes, the natural thing to do next is to consider primes modulo 4. Surprisingly, we observe the behaviour as in Figure 2 above. In this case, the arithmetic function $\lambda(n;4,3)$ resembles the nonprincipal Dirichlet character modulo 4, and its partial sums are shown to be *positive*. On the other hand, the behaviour of $\lambda(n;4,1)$ turns out to be related to the classical $\lambda(n)$ restricted to arithmetic progressions modulo 4.

For general moduli, taking $r = \varphi(q)/2$, for certain choices of $a_1, \ldots, a_r$ coprime to q, the function $\lambda(n;q,a_1,\ldots,a_r)$ behaves similarly to a Dirichlet character, and in this sense is 'character-like,' in the sense of [2]. In this case, we can predict the behaviour of the function and its 'complement'; that is, when we are in the analogous setting to $\lambda(n;4,1)$ and $\lambda(n;4,3)$.

Theorem 1.2. *Let* $a_1, \ldots, a_{q-r}$ *and* $b_1, \ldots, b_r$ *be as above. Then*

$$\sum_{n \leq x} \lambda(n;q,a_1,\ldots,a_{q-r}) = O_\epsilon(x^{1/2+\epsilon}) \tag{1.11}$$

is equivalent to the generalised Riemann hypothesis for $L(s,\chi_q)$*; unconditionally, it is* $o(x)$*. On the other hand,*

$$\sum_{n \leq x} \lambda(n;q,b_1,\ldots,b_r) = O(\log x). \tag{1.12}$$

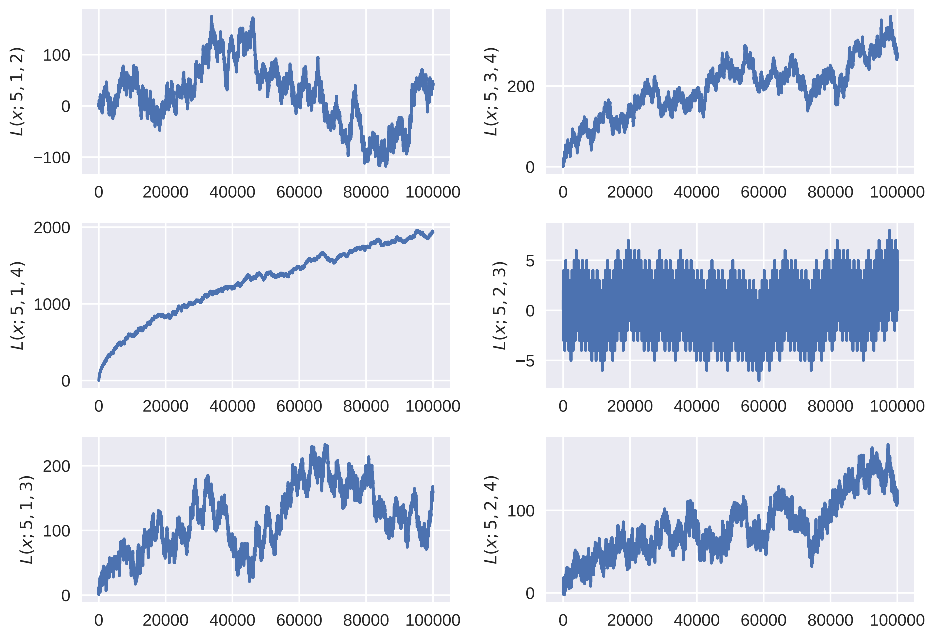

FIGURE 3. Combinations of $L(x; 5, *, *)$.

if there is only one nonprincipal real Dirichlet character modulo q; if there is more than one, then we have only $o(x^{1-\delta})$ for some $\delta > 0$.

Moreover, we show that when $r \neq \varphi(q)/2$, and $(a_i, q) = 1$, the behaviour of $\lambda(n; q, a_1, \ldots, a_r)$ is determined. Otherwise, the behaviour of $\lambda(n; q, a_1, \ldots, a_r)$ seems more difficult to describe precisely, and in this case it is interesting to ask the same question as Pólya did for $\lambda(n)$. For example, with modulus 5 we observe as in Figure 3, that except for the character-like function and its complement, the partial sums tend to fluctuate with a positive bias, except for $\lambda(n; 5, 1, 2)$, which already changes sign for small x. The remaining three remain positive up to $x \leq 10^7$, which leads us to ask whether they eventually change sign.

Theorem 1.3. *Let $a_1, \ldots, a_r$ be residue classes modulo q, coprime to q. Then for $r \neq \varphi(q)/2, \varphi(q)$,*

$$\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_r) = b_0 \frac{x}{(\log x)^{2 - \frac{2r}{\varphi(q)}}} + O\left(\frac{x}{(\log x)^{3 - \frac{2r}{\varphi(q)}}} \right), \tag{1.13}$$

where b_0 is an explicit constant such that $b_0 > 0$ if $2r < \varphi(q)$ and $b_0 < 0$ if $2r > \varphi(q)$.

If $r = \varphi(q)/2$ and $\lambda(n; q, a_1, \ldots, a_r)$ is not character like, we have again

$$\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{\varphi(q)}) = o(x), \tag{1.14}$$

for $x \geq 1$. Assuming the Riemann hypothesis, it is in fact $O_\epsilon(x^{1/2+\epsilon})$ for all $\epsilon > 0$.

The most intriguing aspect of our new family of Liouville-type functions, in light of the conjectures of Pólya, Turán, and even Mertens, is distinguishing when the partial sums of $\lambda(n;q,a_1,\dots,a_r)$ have any sign changes at all, or if a sign change occurs, then there must be infinitely many sign changes must follow. We give conditional answers to this question in Section 5.

We also prove some results on distribution of total number of primes in arithmetic progression; the analogues $\Omega(n;q,a)$ and $\omega(n;q,a)$ are more well-behaved, though we still observe some slight discrepancy in the implied constants in the growth of the partial sums, with respect to the residue class. For example, we show that

$$\sum_{n\le x}\omega(n;q,a)=\frac{1}{\varphi(q)}x\log\log x+g(q,a)x+o(x) \tag{1.15}$$

for some absolute constant $g(q,a)$ (see Proposition 4.1), and that $\omega(n;q,a)$ is distributed normally, as an application of the Erdős-Kac theorem.

Note that we have not considered the 'mixed' case, where $\lambda(n;a_1,\dots,a_r)$ contains both residue classes that are and are not coprime to q. Numerical experiments seem to suggest that they do affect the behaviour in small but observable ways, in particular, we observe that the adding several residue classes may cause a sum to fluctuate more. See Section 6 for a discussion and an example.

Remark 1.4. We mention the recent work of [2], in which the authors consider any subset A of prime numbers, and define $\Omega_A(n)$ to be the number of prime factors of n contained in n, counted with multiplicity. They then define a Liouville function for A to be

$$\lambda_A(n)=(-1)^{\Omega_A(n)}, \tag{1.16}$$

taking value -1 at primes in A and 1 at primes not in A, and show, for example, that $\lambda_A(n)$ is not eventually periodic in n. Our functions can be viewed as particular cases of $\lambda_A(n)$ where A is a set of primes in a given arithmetic progression.

1.3. **This paper.** This paper is organized as follows. In Section 2 we develop basic properties of the Liouville function for arithmetic progressions. In Section 3, we study the parity of the number of primes in arithmetic progressions appearing in a given prime factorization, while in Section 4 we study the distribution of the total number of such primes. In Section 5, we show a positive proportion of sign changes under certain hypotheses such as the generalized Riemann hypothesis, while in Section 6, we discuss numerical experiments and the observed behaviour of these Liouville functions.

Notation. Throughout the paper we fix the following conventions: ϵ will denote a positive real number and c an absolute constant, which will vary depending on the context. We also denote by $[x]$ the floor function, or the greatest integer less than x.

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2. First estimates

2.1. **Basic properties.** We develop some basic properties of the Liouville function for arithmetic progressions, analogous to the classical results. Using this, we prove a basic estimate for the distribution of $\lambda(n;q,a)$. Most of the statements in this section will be proven for $\lambda(n;q,a)$, and we leave to the reader the analogous statements for products $\lambda(n;q,a_1,\ldots,a_r)$.

We begin with a broad description.

Lemma 2.1. *$\lambda(n;q,a)$ is aperiodic and unbounded.*

Proof. The first statement follows as a special case of [2]. The second follows, for example, from the Erdős discrepancy problem,[1] but we will also prove this more directly below. $\square$

Recall that the classical Liouville function satisfies the identity

$$\sum_{d|n}\lambda(d)=\begin{cases}1 & \text{if } n \text{ is a perfect square,}\\ 0 & \text{otherwise.}\end{cases} \tag{2.1}$$

The following proposition gives the analogue of this identity.

Lemma 2.2. *Write $n=n_1n_2$ with n_1 not divisible by any prime $p\equiv a \pmod q$ dividing n. Then we have*

$$S(n;q,a):=\sum_{d|n}\lambda(d;q,a)=\begin{cases}\tau(n_1) & \textit{if } n_2 \textit{ is a perfect square,}\\ 0 & \textit{otherwise.}\end{cases} \tag{2.2}$$

where $\tau(n)$ is the divisor function.

Proof. By multiplicativity, it suffices to prove this when $n=p^r$ is a prime power. If $p\equiv a \pmod q$, then

$$S(p^r;q,a)=\sum_{i=0}^{r}\lambda(p^i;q,a)=\sum_{i=0}^{r}(-1)^i, \tag{2.3}$$

which is 1 is r is even, so that p^r is a perfect square, and is 0 otherwise. If $p\not\equiv a \pmod q$, on the other hand, then it is trivially true that $S(p^r;q,a)=r+1=\tau(p^r)$. $\square$

The trivial bound $|\lambda(n;q,a)|=1$ implies that

$$\left|\sum_{n\le x}\lambda(n;q,a)\right|\le x. \tag{2.4}$$

We will now do slightly better, at the expense of less elementary methods. We say a subset A of primes is said to have sifting density κ, if

$$\sum_{\substack{p\le x\\ p\in A}}\frac{\log p}{p}=\kappa\log x+O(1). \tag{2.5}$$

where $0\le\kappa\le 1$. In particular, we will take A to be the set of primes congruent to certain a modulo q. For example, we have $\kappa=0$ if $(a,q)>1$, and for $(a,q)=1$ with q odd, we have $0<\kappa\le\frac{1}{2}$ with equality only when $\varphi(q)=2$.

[1]It may seem like big hammer to invoke, but it is worth noting that amongst our $\lambda(n;q,a)$ are 'character-like' multiplicative functions considered in [2], whose $O(\log x)$ growth constitute near misses to the problem.

Proposition 2.3. *We have for $\kappa < \frac{1}{2}$, and $(a,q) = 1$,*

$$\sum_{n\le x} \lambda(n;q,a) = (1+o(1))\frac{C_\kappa x}{(\log x)^{2\kappa}} \tag{2.6}$$

where $C_\kappa > 0$ is an explicit constant depending on a, q and κ, and for $\kappa \ge \frac{1}{2}$,

$$\sum_{n\le x} \lambda(n;q,a) = o(x). \tag{2.7}$$

Proof. This follows from [2, Theorem 5], as an application of the Liouville function for A, choosing A to be a set of primes in arithmetic progression. □

Remark 2.4. More generally, if we take A to consist of several residue classes a_i modulo q, then κ will also vary accordingly according to the number of residue classes prime to q that are taken. In this case, we may replace the sum over $\lambda(n;q,a)$ by $\lambda(n;q,a_1,\dots,a_r)$ to obtain similar estimates.

We also record the following estimates for the classical Liouville function as a benchmark.

Lemma 2.5. *We have*

$$\sum_{n\le x} \lambda(n) = O(xe^{-c_1\sqrt{\log x}}). \tag{2.8}$$

and

$$\sum_{\substack{n\le x\\ n\in P}} \lambda(n) = -c_2\frac{x}{(\log x)^{\frac{r}{\varphi(q)}+1}} + O\left(\frac{x}{(\log x)^{\frac{r}{\varphi(q)}+2}}\right), \tag{2.9}$$

where P is a set of r residue classes coprime to $q > 2$ and $c_1, c_2 > 0$.

Proof. The first is well known. The second follows by the same method of proof of [4, Theorem 2]. □

In particular, we observe that when $\lambda(n)$ is restricted to arithmetic progressions (containing infinitely many primes), its partial sums tend to be negative. One can also show that its limiting distribution is negative using its relation to Lambert series; cf. [3, Theorem 1].

2.2. **Complete multiplicativity.** We now use some general facts about completely multiplicative functions to show what one might expect to hold in our case. Here we have not strived to provide the best bounds, but rather those simplest to state to illustrate the general picture.

The first shows the values of $\lambda(n)$ restricted to residue classes is equidistributed.

Proposition 2.6. *Let f be a completely multiplicative function, $A, Q \ge 1$, $x > Q$ and $(a,q) = 1$. Then*

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} f(n) = O(\frac{x}{q\sqrt{\log A}}) \tag{2.10}$$

for all $q > Q$, except possibly for multiples of one of at most two exceptional moduli.

Proof. See [2]. □

One may also consider a variant of the Chowla conjecture for the Liouville function: Fix a, q relatively prime. Given distinct integers $h_1, \ldots, h_k$, fix a sequence of signs $\epsilon_j = \pm 1$ for $1 \leq j \leq k$. Then one would like to know whether

$$\sum_{n \leq x} \prod_{i=1}^{k} \lambda(n + h_i; q, a) = o(x). \tag{2.11}$$

for all k. In particular, the number of $n \leq x$ such that $\lambda(n + j; q, a) = \epsilon_j$ for all$1 \leq j \leq k$ is

$$\left(\frac{1}{2^k} + o(1) \right) x. \tag{2.12}$$

Roughly, this tells us that $\lambda(n; q, a)$ takes the value 1 or -1 randomly.

Using known results, we have the following evidence towards the conjecture.

Proposition 2.7. *For every $h \geq 1$ there exists $\delta(h) > 0$ such that*

$$\frac{1}{x} \left| \sum_{n \leq x} \lambda(n; q, a) \lambda(n + 1; q, a) \right| \leq 1 - \delta(h) \tag{2.13}$$

for all sufficiently large x. Similarly,

$$\frac{1}{x} \left| \sum_{n \leq x} \lambda(n; q, a) \lambda(n + 1; q, a) \lambda(n + 2; q, a) \right| \leq 1 - \delta(h) \tag{2.14}$$

Proof. This follows as a special case of [10]. □

Remark 2.8. We also note that as an application of [10, Corollary 5] on sign changes of certain multiplicative functions, there exists a constant C such that every interval $[x, x + C\sqrt{x}]$ contains a number with an even number of prime factors in a fixed arithmetic progression a modulo q, and another one with an odd number of such prime factors.

3. Parity

We now turn to the average behaviour of our $\lambda(n; q, a)$. A refinement of Pólya's problem leads us to ask: consider the prime factorization of a composite number n. Do the primes in arithmetic progressions tend to appear an even or odd number of times? As described in the introduction, we show that one encounters biases, namely, that the answer depends strongly on the arithmetic progression chosen.

3.1. **Dirichlet series.**

Since $\lambda(n; q, a)$ is completely multiplicative, we can form the Dirichlet series generating function

$$D(s; q, a) := \sum_{n=1}^{\infty} \frac{\lambda(n; q, a)}{n^s} = \zeta(s) \prod_{p \equiv a \pmod{q}} \frac{1 - p^{-s}}{1 + p^{-s}} \tag{3.1}$$

using the Euler product in the case $(a, q) = 1$, and by the trivial bound converges absolutely for $\Re(s) > 1$. Taking the product over all such a, we obtain

$$\prod_{\substack{a \pmod{q} \\ (a,q)=1}} D(s; q, a) = \zeta(2s) \zeta(s)^{\varphi(q)-2} \prod_{p \mid q} \frac{1 + p^{-s}}{1 - p^{-s}}. \tag{3.2}$$

and we see that in the region $\Re(s) > 0$, the expression has a pole of order $\varphi(q)-2$ at $s = 1$ and a simple pole at $s = \frac{1}{2}$. Moreover, if we include residue classes a modulo q such that $(a, q) > 1$, for which $D(s; q, a)$ is equal to $\zeta(s)$ up a finite number of factors, we have

$$\prod_{a=0}^{q-1} D(s; q, a) = \zeta(2s)\zeta(s)^{q-2}, \tag{3.3}$$

generalizing the classical formula with $q = 1$.

Similarly, for products $\lambda(n; q, a_1, \ldots, a_r)$ with $(a_i, q) = 1$ for each i, we have

$$D(s; q, a_1, \ldots, a_r) = \zeta(s) \prod_{i=1}^{r} \prod_{p \equiv a_i \pmod q} \frac{1 - p^{-s}}{1 + p^{-s}} \tag{3.4}$$

and

$$D(s; q, a_1, \ldots, a_r) D(s; q, a'_1, \ldots, a'_{\varphi(q)-r}) = \zeta(2s) \prod_{p \mid q} \frac{1 + p^{-s}}{1 - p^{-s}}. \tag{3.5}$$

where $a'_1, \ldots, a'_{\varphi(q)-r}$ are the remaining residue classes coprime to q.

Remark 3.1. We note in passing that by the nonnegativity of the convolution $1 * \lambda(n; q, a_1, \ldots, a_r)$, together with Dirichlet's hyperbola method, it is a pleasant exercise to show that any such Dirichlet series $D(s; q, a_1, \ldots, a_r)$ is nonvanishing at $s = 1$, without appealing to the nonvanishing of $\zeta(s)$.

It is known that for $(a, q) = 1$, the Euler product

$$F_a(s) = \prod_{p \equiv a \pmod q} \frac{1}{1 - p^{-s}} \tag{3.6}$$

converges absolutely for $\Re(s) > 1$, and has analytic continuation to $\Re(s) \geq 1 - C/\log t$ for $|t| < T$, and $T \geq 10$ [9, p.212]. It can be expressed as

$$\prod_{p \equiv a \pmod q} \frac{1}{1 - p^{-s}} = \zeta(s)^{\frac{1}{\varphi(q)}} e^{G_a(s)}, \tag{3.7}$$

where $G_a(s)$ is given by

(3.8)

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \overline{\chi}(a) \left(\log L(s, \chi) + \sum_p \sum_{m=2}^{\infty} \frac{\chi(p) - \chi^m(p)}{m p^{ms}} \right) + \frac{1}{\varphi(q)} \sum_{p \mid q} \log(1 - p^{-s}).$$

Even though $\lambda(n; q, a)$ is not eventually periodic, we are still relate it at times to Dirichlet characters by the following identity.

Proposition 3.2. *Let $q > 2$. Given any nonprincipal real Dirichlet character χ_q modulo q, there is a combination of residue classes $a_1, \ldots, a_r$ such that*

$$D(s; q, a_1, \ldots, a_r) = L(s, \chi_q) \prod_{p \mid q} \frac{1}{1 - p^{-s}}. \tag{3.9}$$

Proof. Given χ_q, it is clear that we can choose a combination of of residue classes $a_1, \ldots, a_r$, coprime to q such that

$$\lambda(n; q, a_1, \ldots, a_r) = \chi_q(n) \tag{3.10}$$

for any n in $(\mathbf{Z}/q\mathbf{Z})^\times$. Then we may express the Dirichlet series as

$$D(s;q,a_1,\dots,a_r) = \prod_p \frac{1}{1-\lambda(p;q,a_1,\dots,a_r)p^{-s}} \tag{3.11}$$

$$= \prod_{p\nmid q} \frac{1}{1-\chi_q(p)p^{-s}} \prod_{p\mid q} \frac{1}{1-p^{-s}}, \tag{3.12}$$

and the result follows. □

Finally, note that by partial summation,

$$D(s;q,a) = s\int_1^\infty L(x;q,a)x^{-s-1}dx \tag{3.13}$$

for $\Re(s)$ sufficiently large. We will use this expression implicitly throughout.

3.2. **The odd primes and 2.** The first natural refinement is to ask what is the parity of (i) the odd primes and (ii) the prime 2 in prime factorizations. Indeed, we have:

Proposition 3.3. *Assuming the Riemann hypothesis,*

$$\sum_{n\le x} \lambda(n;2,1) = O_\epsilon(x^{1/2+\epsilon}) \tag{3.14}$$

for all $\epsilon > 0$, while unconditionally this sum is $o(x)$. On the other hand,

$$\sum_{n\le x} \lambda(n;2,2) = \frac{x}{3} + o(x) \tag{3.15}$$

and is nonnegative for all $x \ge 1$.

Proof. We first treat the simpler case $L(n;2,2)$. The Dirichlet series

$$D(s;2,2) = \zeta(s)\frac{1-2^{-s}}{1+2^{-s}} \tag{3.16}$$

has meromorphic continuation to $\Re(s) \ge 1$ with only a simple pole at $s = 1$ with residue $\frac{1}{3}$, and is holomorphic for $\Re(s) > 1$; hence we have

$$\sum_{n\le x} \lambda(n;2,2) = \frac{x}{3} + o(x) \tag{3.17}$$

by the Ikehara–Wiener theorem.

Now, notice that $\lambda(n;2,2)$ is always $1,-1,1$ when n is of the form $4k+1, 4k+2, 4k+3$ respectively. Only if it is of the form $4k$ can it take both 1 and -1 as values, in which case it is determined by the value $\lambda(k;2,2)$. Thus the first few summands of $L(x;2,2)$ are

$$1 - 1 + 1 + \lambda(4;2,2) + 1 - 1 + 1 + \lambda(8;2,2) + \dots \tag{3.18}$$

and continuing thus, we conclude that for $L(x;2,2) \ge 0$ for all $x \ge 1$.

Next, we treat the case $L(x;2,1)$. Notice that

$$D(s;2,1) = \frac{\zeta(2s)}{\zeta(s)}\frac{1+2^{-s}}{1-2^{-s}}, \tag{3.19}$$

where we recall from (1.2) that $\zeta(2s)/\zeta(s)$ is the Dirichlet series for the Liouville function. Since $\zeta(s)$ has a simple pole at $s = 1$ and is nonvanishing on $\Re(s) = 1$, it follows that $D(s;2,1)$ has analytic continuation to $\Re(s) \geq 1$ and has a zero at $s=1$. Thus we have

$$\sum_{n\leq x} \lambda(n;2,1) = o(x) \tag{3.20}$$

unconditionally. Assuming the Riemann hypothesis, $D(s;2,1)$ continues analytically to $\Re(s) > \frac{1}{2}$ and has a simple pole at $s=\frac{1}{2}$, and so $L(x;2,1) = O_\epsilon(x^{1/2+\epsilon})$. □

By a similar argument, we observe the following behaviour for general arithmetic progressions:

Theorem 3.4. *Given any $q \geq 2$, let $a_1,\ldots,a_{\varphi(q)}$ be the residue classes modulo q such $(a_i,q)=1$, and $b_1,\ldots,b_{q-\varphi(q)}$ be the remaining residues classes. Then assuming the Riemann hypothesis,*

$$\sum_{n\leq x} \lambda(n;q,a_1,\ldots,a_{\varphi(q)}) = O_\epsilon(x^{1/2+\epsilon}), \tag{3.21}$$

for all $\epsilon > 0$, while unconditionally this sum is $o(x)$.

On the other hand,

$$\sum_{n\leq x} \lambda(n;q,b_1,\ldots,b_k) = \left(\prod_{i=1}^{k}\prod_{p|b_i}\frac{p-1}{p+1}\right) x + o(x) \tag{3.22}$$

for any $1 \leq k \leq q-\varphi(q)$.

Proof. Simply observe that the Dirichlet series in this setting can be expressed as

$$D(s;q,a_1,\ldots,a_{\varphi(q)}) = \frac{\zeta(2s)}{\zeta(s)}\prod_{p|q}\frac{1+p^{-s}}{1-p^{-s}}, \tag{3.23}$$

and

$$D(s;q,b_1,\ldots,b_k) = \zeta(s)\prod_{i=1}^{k}\prod_{p|b_i}\frac{1-p^{-s}}{1+p^{-s}}, \tag{3.24}$$

and argue as in Proposition 3.3. □

Remark 3.5. Qualitatively, we note that since $(p-1)/(p+1) < 1$ for any prime p, the growth rate becomes slower and slower as more primes enter into the product.

The following corollary is proved in the same manner for the classical Liouville function, after the method of Landau.

Corollary 3.6. *With assumptions as in Theorem 3.4,*

$$\sum_{n\leq x} \lambda(n;q,a_1,\ldots,a_{\varphi(q)}) = o(x) \tag{3.25}$$

is equivalent to the prime number theorem.

By a similar reasoning, we shall see that one can also recover Dirichlet's theorem on primes in arithmetic progressions, but only in cases where there are no complex Dirichlet characters modulo q.

3.3. **A Chebyshev-type bias.** We show a result using the properties developed above, which can be interpreted as: the number of prime factors of the form $4k+1$ and $4k+3$ *both* tend to appear an even number of times, but the former having a much stronger bias. We first require the following formula.

Lemma 3.7. *Define the characteristic function*

$$c(x,k) = \begin{cases} 1 & [x] \equiv 2k \pmod{4k} \\ 0 & \textit{otherwise} \end{cases} \tag{3.26}$$

for any $x > 0$. Then

$$\sum_{n \le x} \lambda(n;4,3) = \sum_{k=1}^{\infty} c(x, 2^k), \tag{3.27}$$

the with finitely many terms on the right-hand side being nonzero.

Proof. To prove the formula, we repeatedly apply the elementary fact that if $n \equiv 1 \pmod 4$ (respectively $3 \pmod 4$), then the prime factors of n of the form 3 modulo 4 appear an even (respectively odd) number of times.

First, observe that $\lambda(n;3,4)$ is 1 or -1 if n is 1 or 3 modulo 4, thus the sum

$$\sum_{\substack{n \le x \\ n \equiv 1(2)}} \lambda(n;4,3) \tag{3.28}$$

is equal to $c(n,2)$. Then we move on to the even numbers, which, written as $2m, 2(m+1)$ and using $\lambda(2;4,3) = 1$, gives again the pattern 1 and -1 depending on whether m is 1 or 3 modulo 4. The even numbers of the form 4 and 6 modulo 8 contribute the term $c(n, 2^2)$.

Repeating this process we obtain the terms $c(n, 2^k)$ for all k, but certainly for k large enough this procedure will cover all $n \le x$, so only finitely many terms will be nonzero. ☐

Proposition 3.8. *Let χ_4 be the nonprincipal Dirichlet character modulo 4. Then*

$$\sum_{n \le x} \lambda(n;4,1) = O_\epsilon(x^{1/2+\epsilon}) \tag{3.29}$$

for any $\epsilon > 0$ assuming the generalised Riemann hypothesis for $L(s,\chi_4)$, while unconditionally this sum is $o(x)$. On the other hand,

$$\sum_{n \le x} \lambda(n;4,3) = O_\epsilon(x^{\epsilon}) \tag{3.30}$$

for any $\epsilon > 0$ and is nonnegative for $x \ge 1$.

Proof. We first prove the latter statement. The Dirichlet series of $\lambda(n;4,3)$ can be written as

$$D(s;4,3) = L(s,\chi_4)\frac{1}{1-2^{-s}}. \tag{3.31}$$

Thus we see that $D(s;4,3)$ has analytic continuation to the half-plane $\Re(s) > 0$.

From the explicit formula in Lemma 3.7 we also see immediately that $L(x;4,3)$ is nonnegative, and given any $C>0$ we can find x large enough so that $L(x;4,3)>C$. Hence $D(s;4,3)$ has a simple pole at $s=0$, and finally we conclude that

$$\sum_{n\leq x}\lambda(n;4,3)=O_\epsilon(x^\epsilon) \tag{3.32}$$

for any $\epsilon>0$.

On the other hand, from (3.2) we have that

$$D(s;4,1)D(s;4,3)=\zeta(2s)\frac{1+2^{-s}}{1-2^{-s}}, \tag{3.33}$$

which by (3.31) is

$$D(s;4,1)=\frac{\zeta(2s)}{L(s,\chi_4)}(1+2^{-s}) \tag{3.34}$$

Comparing both sides, we observe that $D(s;4,1)$ is analytic in the region $\Re(s)\geq 1$, giving $o(x)$ unconditionally by analytic continuation of $L(s,\chi_4)$. Moreover, assuming the generalised Riemann hypothesis for $L(s,\chi_4)$, we see that $D(s;4,1)$ in fact converges absolutely in $\Re(s)>\frac{1}{2}$, with only a simple pole at $s=\frac{1}{2}$, so that

$$\sum_{n\leq x}\lambda(n;4,1)=O_\epsilon(x^{1/2+\epsilon}), \tag{3.35}$$

as required. □

Remark 3.9. Indeed, we can numerically observe that the growth of $L(x;4,1)$ is extremely slow: for $x\leq 10^7$ the maximum value attained is 14, while for $x\leq 10^9$ the maximum value is 29.

The proposition above holds more generally for any $q\geq 2$, by the same method of proof, using the following observation: Let $r=\varphi(q)/2$. Then there is exactly one combination of residue classes, say $b_1,\dots,b_r$ such that

$$D(s;q,b_1,\dots,b_r)=L(s,\chi_q)\prod_{p|q}\frac{1}{1-p^{-s}} \tag{3.36}$$

where χ_q is a nonprincipal real Dirichlet character modulo q; whereas

$$D(s;q,a_1,\dots,a_{q-r})=\frac{\zeta(2s)}{L(s,\chi_q)}\prod_{p|q}(1-p^{-s}), \tag{3.37}$$

where $a_1,\dots,a_{q-r}$ are the remaining residue classes.

The following theorem is now immediate.

Theorem 3.10. *Let $a_1,\dots,a_{q-r}$ and $b_1,\dots,b_r$ be as above. Then*

$$\sum_{n\leq x}\lambda(n;q,a_1,\dots,a_{q-r})=O_\epsilon(x^{1/2+\epsilon}) \tag{3.38}$$

is equivalent to the Riemann hypothesis for $L(s,\chi_q)$, otherwise it is $o(x)$ unconditionally. On the other hand,

$$\sum_{n\leq x}\lambda(n;q,b_1,\dots,b_r)=O(\log x). \tag{3.39}$$

if there is only one nonprincipal real Dirichlet character modulo q, otherwise we have only $O_\epsilon(x^\epsilon)$.

Proof. The estimate (3.38) follows the same argument as before. For (3.39), we use the relation between χ_q and the Kronecker symbol, which follows by a modest extension of [2, Corollary 6] to composite q. ☐

Remark 3.11. We have not strived for the optimal unconditional bounds, that is, not assuming the generalised Riemann hypothesis. The relevant estimates can certainly be improved, for example, using the zero-free regions for the associated Dirichlet L-functions and $\zeta(s)$.

3.4. **General arithmetic progressions.** Now we turn to general arithmetic progressions. We now restrict to residue classes a coprime to q, which is the most interesting case. We will also assume moreover that $\varphi(q) > 2$.

Proposition 3.12. *Let $a_1, \ldots, a_r$ be residue classes modulo q, coprime to q, with $r \neq \varphi(q)/2, \varphi(q)$. Then*

$$\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_r) = b_0 \frac{x}{(\log x)^{2-\frac{2r}{\varphi(q)}}} + O\left(\frac{x}{(\log x)^{3-\frac{2r}{\varphi(q)}}}\right), \tag{3.40}$$

where b_0 is an explicit constant such that $b_0 > 0$ if $2r < \varphi(q)$ and $b_0 < 0$ if $2r > \varphi(q)$.

Proof. The proof is a slight modification of [9, Theorem 1]. We will give the main steps of the argument, leaving the details to the reader. We have that

$$D(s; q, a_1, \ldots, a_r) = \zeta(s) \prod_{i=1}^{r} \prod_{p \equiv a_i \pmod q} \frac{(1-p^{-s})^2}{1-p^{-2s}} \tag{3.41}$$

$$= \zeta(s)^{1-\frac{2r}{\varphi(q)}} \zeta(2s)^{\frac{r}{\varphi(q)}} \prod_{i=1}^{r} \exp(G_{a_i}(2s) - 2G_{a_i}(s)) \tag{3.42}$$

by (3.7). Applying Perron's formula, we have

$$\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_r) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} D(s; q, a_1, \ldots, a_r) \frac{x^s}{s} ds + O\left(\frac{x}{T}\right) \tag{3.43}$$

for $b > 1$ and $x, T \geq 2$.

We may analytically continue $D(s; q, a_1, \ldots, a_r)$ to the left of the line $\Re(s) = 1$, say to $\sigma \geq 1 - C/\log T$, so that the integral may be estimated by

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} D(s; q, a_1, \ldots, a_r) \frac{x^s}{s} ds = \frac{1}{2\pi i} \int_{\gamma} D(s; q, a_1, \ldots, a_r) \frac{x^s}{s} ds + O\left(\frac{x}{T}\right), \tag{3.44}$$

where γ is a closed loop around $s = 1$, with radius taken to be less than $1 - C/\log T$. In fact, we will choose T such that $\log T = C(\log x)^{1/2}$.

Now if we define the function $H(s)$ by the equation

$$\frac{D(s; q, a_1, \ldots, a_r)}{s} = (s-1)^{1-\frac{2r}{\varphi(q)}} H(s) \tag{3.45}$$

we can write

$$\frac{1}{2\pi i} \int_{\gamma} D(s; q, a_1, \ldots, a_r) \frac{x^s}{s} ds = x I(x) \tag{3.46}$$

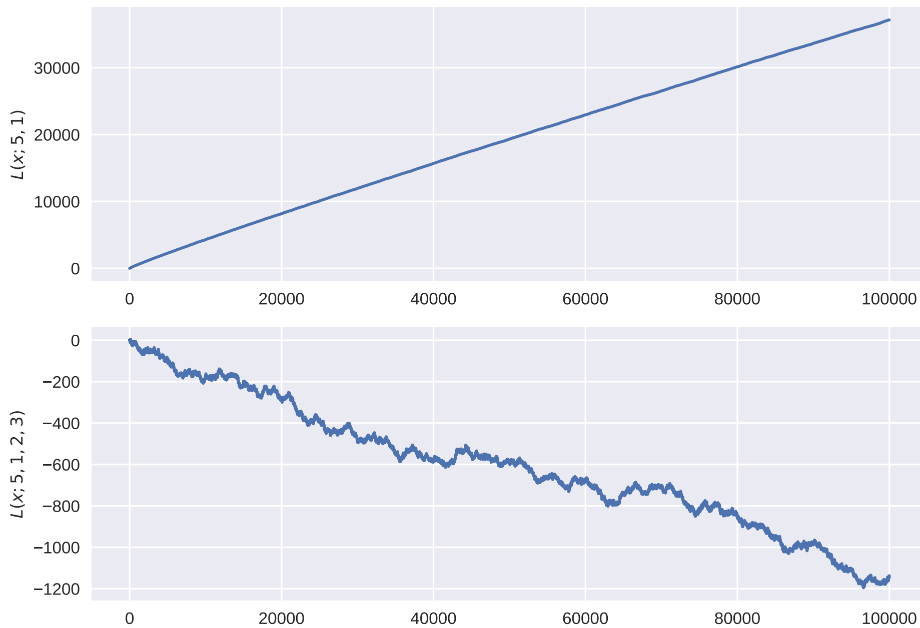

FIGURE 4. $L(x;5,1)$ and $L(x;5,1,2,3)$.

where

$$I(x) = \frac{1}{2\pi i}\int_{\gamma'} H(s+1)x^s s^{1-\frac{2r}{\varphi(q)}}\,ds \tag{3.47}$$

and γ' is the contour obtained by translating γ by $s \mapsto s+1$. Taking γ to have radius $1/\sqrt{\log x}$, the integral can be written as (cf. [9, p.216])

$$I(x) = \sum_{0\le j\le \sqrt{\log x}} \frac{B_j}{\Gamma\left(\frac{2r}{\varphi(q)} - j - 1\right)(\log x)^{2+j-\frac{2r}{\varphi(q)}}} + O(e^{-C\sqrt{\log x}}), \tag{3.48}$$

where B_j are the coefficients of the Taylor series expansion of $H(s)$ at $s=1$. In particular, $B_0 = H(1) > 0$. Set

$$b_0 = \frac{B_0}{\Gamma(\frac{2r}{\varphi(q)} - 1)}, \tag{3.49}$$

and notice that $\Gamma(\frac{2r}{\varphi(q)} - 1)$ is positive or negative depending on whether $2r/\varphi(q)$ is greater or lesser than 1, and is singular at $2r = \varphi(q)$. Since we may truncate the sum over j to one term with an error term of size $O((\log x)^{\frac{2r}{\varphi(q)}-3})$, the result follows. $\square$

Remark 3.13. The above proposition is illustrated in the case $q=5$ as in Figure 4. Indeed, from the graphs one would be led to ask if not only should the functions be asymptotically positive (resp. negative), but in fact positive (resp. negative) for all $x \ge 1$.

We now consider the most interesting case.

Proposition 3.14. *Let $a_1, \dots, a_r$ be residue classes modulo q, coprime to q, with $r = \varphi(q)/2$. Then for any $\lambda(n; q, a_1, \dots, a_r)$ that is neither character-like or the complement thereof, the sum*

$$\sum_{n \le x} \lambda(n; q, a_1, \dots, a_r) = O_\epsilon(x^{1/2+\epsilon}) \tag{3.50}$$

assuming the Generalized Riemann hypothesis.

Proof. The proof follows from a simple application of Ingham's analysis of $\lambda(n)$. Recall the Dirichlet series expression from (3.42), we have that

$$\sum_{n \le x} \lambda(n; q, a_1, \dots, a_r) = \zeta(2s)^{1/2} \prod_{i=1}^{r} \exp(2G_{a_i}(s) - G_{a_i}(2s)) \tag{3.51}$$

where we have used $r = \varphi(q)/2$. □

4. Distribution

In this section we provide a brief discussion on the distribution of the primes in arithmetic progressions in the number of prime factors.

4.1. The number of prime factors.

Recall the functions $\omega(n)$ counting the number of distinct prime factors of n, and $\Omega(n)$ counting the total number of prime factors of n. As in (1.5), we may define the analogous functions $\omega(n; q, a)$ and $\Omega(n; q, a)$ counting only primes congruent to a modulo q, so that

$$\omega(n) = \sum_{a=0}^{q-1} \omega(n; q, a), \quad \Omega(n) = \sum_{a=0}^{q-1} \Omega(n; q, a). \tag{4.1}$$

Proposition 4.1. *There exists an absolute constant $g(q, a)$ such that*

$$\sum_{n \le x} \omega(n; q, a) = \frac{1}{\varphi(q)} x \log\log x + g(q, a)x + o(x). \tag{4.2}$$

Proof. Write

$$\sum_{n \le x} \omega(n; q, a) = \sum_{n \le x} \sum_{\substack{p \mid n \\ p \equiv a \pmod q}} 1 = \sum_{\substack{p \le x \\ p \equiv a \pmod q}} \sum_{m \le x/p} 1 \tag{4.3}$$

which is

$$\sum_{\substack{p \le x \\ p \equiv a \pmod q}} \frac{x}{p} + O\left(\sum_{\substack{p \le x \\ p \equiv a \pmod q}} 1 \right) = \sum_{\substack{p \le x \\ p \equiv a \pmod q}} \frac{x}{p} + O\left(\frac{x}{\log x} \right) \tag{4.4}$$

by the prime number theorem. Now, using Mertens' theorem for primes in arithmetic progressions for $(a, q) = 1$ we have

$$\sum_{\substack{p \le x \\ p \equiv a \pmod q}} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + g(q, a) + o(1) \tag{4.5}$$

where $g(q, a)$ is an absolute constant. Applying this yields the proposition. □

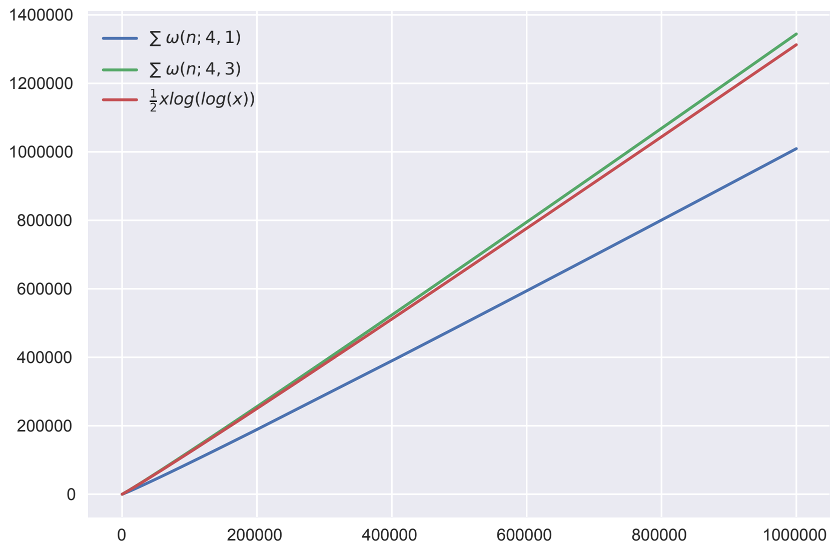

FIGURE 5. Distribution of $\omega(n;q,a)$.

Remark 4.2. We note that while the main term in Prop 4.1 is independent of a, the error term does appear to depend on the choice of residue class a (see Figure 5 below), but we do not study this here.

We may also consider moments, such as

$$\sum_{n\leq x}(\omega(n;q,a)-\frac{1}{\varphi(q)}\log\log x)^2=\frac{1}{\varphi(q)}x\log\log x+O(x) \tag{4.6}$$

by expanding the square and applying simple estimates.

Moreover, since $\omega(n;q,a)$ is completely additive, we may apply the Erdős–Kac theorem [5], which applies to strongly additive functions—additive functions f such that $f(mn)=f(m)+f(n)$ for all natural numbers m,n, and $|f(p)|\leq 1$ for all primes p—to immediately obtain the following statement.

Theorem 4.3 (Erdős–Kac). *Fix a modulus q and constants $A,B\in\mathbf{R}$. Then*

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n\leq x : A\leq\frac{\omega(n;q,a)-\frac{1}{\varphi(q)}\log\log x}{\sqrt{\frac{1}{\varphi(q)}\log\log x}}\leq B\right\}=\frac{1}{\sqrt{2\pi}}\int_A^B e^{-t^2/2}dt. \tag{4.7}$$

Hence $\omega(n;q,a)$ is also normally distributed.

5. Sign Changes and Biases

Let $r = \varphi(q)/2$, and let $a_1, \ldots, a_{q-r}$ denote the set of residue classes for which

$$D(s; q, a_1, \ldots, a_{q-r}) = \frac{\zeta(2s)}{L(s, \chi_q)} \prod_{p|q} (1 - p^{-s}), \tag{5.1}$$

so that $\lambda(n; q, a_1, \ldots, a_{q-r})$ is the complement to a character-like function. It is natural to ask whether $\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r})$ changes sign infinitely often; for $q = 1$, this is Pólya's conjecture. To this end, we have the following.

Proposition 5.1 (cf. [7, Theorem 2.6])**.** *Suppose that the generalised Riemann hypothesis for $L(s, \chi_q)$ is false, so that $\Theta := \sup\{\Re(\rho) \colon L(\rho, \chi_q) = 0\} > 1/2$. Then*

$$\liminf_{x \to \infty} \frac{\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r})}{x^{\Theta - \epsilon}} < 0, \qquad \limsup_{x \to \infty} \frac{\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_r)}{x^{\Theta - \epsilon}} > 0 \tag{5.2}$$

for every $\epsilon > 0$.

Proof. The proof is via the same method as [11, Theorem 15.2]; we omit the details. □

Proposition 5.2 (cf. [7, Theorem 2.7])**.** *Assume the generalised Riemann hypothesis for $L(s, \chi_q)$ and that $L(s, \chi_q)$ has a zero of order $m \geq 2$. Then*

$$\liminf_{x \to \infty} \frac{\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r})}{\sqrt{x} (\log x)^{m-1}} < 0, \qquad \limsup_{x \to \infty} \frac{\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r})}{\sqrt{x} (\log x)^{m-1}} > 0. \tag{5.3}$$

Proof. This is via the same method as [11, Theorem 15.3]. □

Of course, it is widely expected that $L(s, \chi_q)$ does indeed satisfy the generalised Riemann hypothesis and that all of its zeroes are simple. If we assume an additional widely believed conjecture on the behaviour of the zeroes of $L(s, \chi_q)$, then we can again show that $\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r})$ changes sign infinitely often.

Definition 5.3. We say that $L(s, \chi_q)$ satisfies the linear independence hypothesis if the set

$$\left\{\gamma \geq 0 \colon L\left(\frac{1}{2} + i\gamma, \chi_q\right) = 0\right\} \tag{5.4}$$

is linearly independent over the rationals.

In particular, the linear independence hypothesis implies that $L(1/2, \chi_q) \neq 0$.

Proposition 5.4 (cf. [7, Theorem 2.8])**.** *Assume the generalised Riemann hypothesis and the linear independence hypothesis for $L(s, \chi_q)$. Then*

$$\liminf_{x \to \infty} \frac{\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r})}{\sqrt{x}} < 0, \qquad \limsup_{x \to \infty} \frac{\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r})}{\sqrt{x}} > 0. \tag{5.5}$$

Proof. The proof follows the same method as [8, Theorem A]. □

Now we explain why there appears to be a positive bias in the limiting behaviour of $\sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r})$. This is for the same reason that $\sum_{n \leq x} \lambda(n)$ appears to have a heavy bias towards being negative; it is due to the fact that should it be the case that $L(1/2, \chi_q) > 0$, as is widely believed, then $D(s; q, a_1, \ldots, a_{q-r})$ has a pole at $s = 1/2$ with positive residue. To quantify this more precisely, we work with the limiting distribution of $e^{-y/2} \sum_{n \leq e^y} \lambda(n; q, a_1, \ldots, a_{q-r})$.

Theorem 5.5. *Assume the generalised Riemann hypotheses for $L(s, \chi_q)$, and that the bound*

$$\sum_{0<\gamma\leq T} \frac{1}{|L'(\rho, \chi_q)|^2} \ll T^\theta \tag{5.6}$$

holds for some $1 \leq \theta < 3 - \sqrt{3}$. Then

$$e^{-y/2} \sum_{n \leq e^y} \lambda(n; q, a_1, \ldots, a_r) \tag{5.7}$$

has a limiting distribution μ.

Suppose additionally that $L(s, \chi_q)$ satisfies the linear independence hypothesis. Then the Fourier transform $\widehat{\mu}$ of μ is given by

$$\widehat{\mu}(\xi) = e^{-ic\xi} \prod_{\gamma>0} J_0(|r(\gamma)\xi|), \tag{5.8}$$

where $J_0(z)$ is the Bessel function of the first kind,

$$c := \frac{1}{L(1/2, \chi_q)} \prod_{p|q} (1 - p^{-1/2}), \tag{5.9}$$

and

$$r(\gamma) := \frac{2\zeta(2\rho)}{\rho L'(\rho, \chi_q)} \prod_{p|q} (1 - p^{-\rho}). \tag{5.10}$$

The mean and median of μ are both equal to c, while the variance of μ is equal to $\frac{1}{2} \sum_{\gamma>0} |r(\gamma)|^2$.

Proof. This essentially follows from [1]. More precisely, [1, Lemmata 4.3 and 4.4] imply an explicit expression of the form

$$e^{-y/2} \sum_{n \leq e^y} \lambda(n; q, a_1, \ldots, a_{q-r}) = c + \Re\left(\sum_{0<\gamma\leq X} r(\gamma) e^{i\gamma y}\right) + \mathcal{E}(y, X) \tag{5.11}$$

for $y > 0$ and $X \geq 1$, where

$$\mathcal{E}(y, X) = O_\epsilon\left(\frac{ye^{y/2}}{X} + \frac{e^{y/2}}{yX^{1-\epsilon}} + (X^{\theta-2} \log X)^{1/2} + e^{-y(1/2-b)}\right) \tag{5.12}$$

for $0 < \epsilon < b < 1/4$, $x > 1$; cf. [1, Proof of Corollary 1.6]. With this in hand, the existence of μ follows from [1, Theorem 1.4], while the identity for $\widehat{\mu}$ is a consequence of [1, Theorem 1.9]. Finally, the proof of the identities for the mean and variance follow the same lines as [7, Proof of Corollary 6.3], while the proof of the identity for the median follows [7, Proof of Theorem 5.1]; cf. [1, Theorem 1.14]. ☐

Let

$$P := \left\{x \in [1, \infty) \colon \sum_{n \leq x} \lambda(n; q, a_1, \ldots, a_{q-r}) \geq 0\right\}, \tag{5.13}$$

and let

$$\delta(P) := \lim_{X\to\infty} \frac{1}{\log X} \int\limits_{P\cap[1,X]} \frac{dx}{x} \tag{5.14}$$

denote the logarithmic density of P, should it exist.

Theorem 5.6. *Assume the generalised Riemann hypotheses and linear independence hypothesis for $L(s,\chi_q)$, and that the bound*

$$\sum_{0<\gamma\leq T} \frac{1}{|L'(\rho,\chi_q)|^2} \ll T^{\theta} \tag{5.15}$$

holds for some $1 \leq \theta < 3-\sqrt{3}$. Then

$$\frac{1}{2} \leq \delta(P) < 1. \tag{5.16}$$

Proof. This is by the same method as [7, Proof of Theorem 1.5]. □

That is, the limiting logarithmic density of P is at least $1/2$ but strictly less than 1, so that 'most' of the time, $\sum_{n\leq x} \lambda(n;q,a_1,\ldots,a_{q-r})$ is nonnegative, but nevertheless it is negative a positive proportion of the time.

Finally, we mention that all of the results in this section are valid analogously for the case where $r=\varphi(q)$ and $a_1,\ldots,a_{\varphi(q)}$ is the set of residue classes coprime to q, so that

$$D(s;q,a_1,\ldots,a_{\varphi(q)}) = \frac{\zeta(2s)}{\zeta(s)} \prod_{p\mid q} \frac{1+p^{-s}}{1-p^{-s}}. \tag{5.17}$$

This has a pole at $s=1/2$ with negative residue; it is for this reason that there is a bias towards $\sum_{n\leq x} \lambda(n;q,a_1,\ldots,a_{\varphi(q)})$ being nonpositive.

Indeed, we can again conditionally prove the existence of a limiting distribution μ of $e^{-y/2}\sum_{n\leq e^y} \lambda(n;q,a_1,\ldots,a_{\varphi(q)})$, but now

$$c := \frac{1}{\zeta(1/2)} \prod_{p\mid q} \frac{1+p^{-1/2}}{1-p^{-1/2}} \tag{5.18}$$

and

$$r(\gamma) := \frac{2\zeta(2\rho)}{\rho\zeta'(\rho)} \prod_{p\mid q} \frac{1+p^{-\rho}}{1-p^{-\rho}}. \tag{5.19}$$

The mean and median of μ is c, which is negative; for this reason, the logarithmic density $\delta(P)$ of

$$P := \left\{ x \in [1,\infty) \colon \sum_{n\leq x} \lambda(n;q,a_1,\ldots,a_{\varphi(q)}) \leq 0 \right\} \tag{5.20}$$

is at least $1/2$ but strictly less than 1, so that $\sum_{n\leq x} \lambda(n;q,a_1,\ldots,a_{\varphi(q)})$ is nonpositive 'most' of the time, yet it is positive a positive proportion of the time.

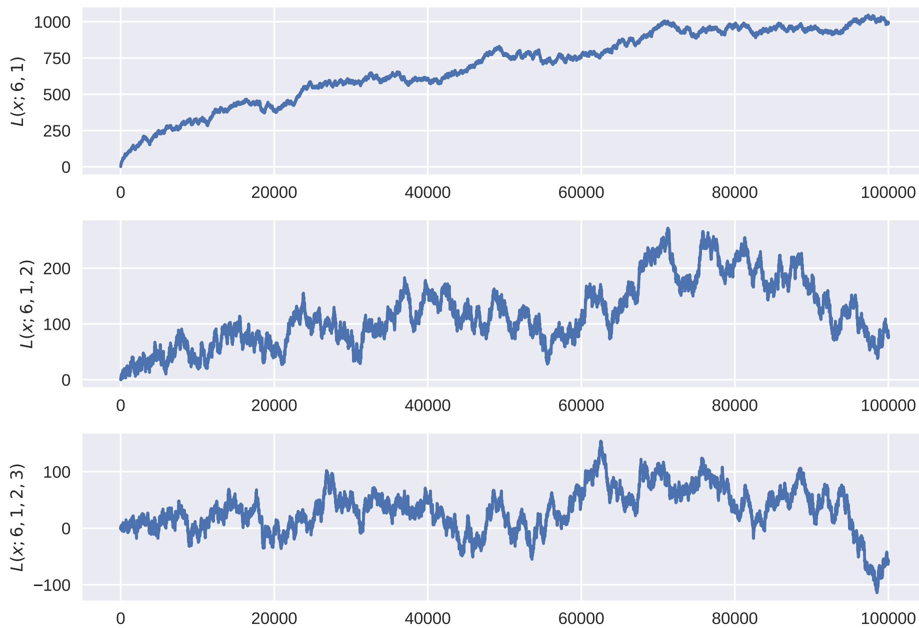

Figure 6. Mixing residue classes.

6. Numerical tests

6.1. **Numerical methods.** All computations in this paper were done on Fortran95 and Python3. The largest computation that we could carry out was that of $L(x;4,1)$, which we verified to be positive for all $1 < x \leq 10^{11}$. Most of the other calculations were carried out up to 10^8 or 10^9. Because of the limited amount of computer memory, we used an algorithm that calculates the parities in batches of size 10^8. Also, to avoid the problem of factorizing large integers, we used multiplication to build up the parities of numbers up to x. This results in a significant increase in the speed without any type of parallel computation. The codes for the computations of various combinations were primarily written in Python, which allows to easily construct all the required combinations.

6.1.1. *Mixing residue classes.* As exemplified in the Figure 6 above, we notice that the addition of residue classes 2 and 3, which divide 6, affect the fluctuations in the sum in a nontrivial manner. In fact, while we know that $L(x;6,1)$ is asymptotically positive, it appears that $L(x;6,1,2,3)$ should change sign infinitely often.

6.2. **Speculations on sieve parity.** We close with a vague speculation regarding sieve parity, following the heuristic of Tao [14, 3.10.2] regarding the parity problem, in terms of the Liouville function. Let A be a set we would like to sieve for. Then to get a lower bound on $|A|$ in say, $[x,2x]$ we set up the divisor sum lower bound

$$1_A(n) \geq \sum_{d|n} c_d \tag{6.1}$$

where the divisors are concentrated in some sieve level $d \leq R$. Summing over $n \leq x$ we get the form

$$|A| \geq \sum_{d|n} c_d \frac{x}{d} + \dots \tag{6.2}$$

If instead we multiply by the nonnegative weight $1 + \lambda(n; q, a)$ and sum, we obtain

$$2|A \backslash A_{a,q}| \geq \sum_{n \leq x} \sum_{d|n} c_d (1 + \lambda(n; q, a)) = \sum_{d|n} c_d \frac{x}{d} + L(x; q, a) + \dots \tag{6.3}$$

where $A_{a,q}$ denotes the set of $x \in A$ such that $x \equiv a \pmod q$. This suggests that one may be able to produce nontrivial lower bounds on the complement $|A \backslash A_{a,q}|$ by sieve theory, when $\varphi(q) > 2$.

In this light, we mention the result of [13] who prove, under certain conditions, the existence of a product of three primes below $x^{1/3}$, each congruent to a modulo q where $q \leq x^{1/16}$. In particular, they observe that their use of a sieve is not blocked by sieve parity. More importantly, in a future work we hope to relate $\lambda(n; q, a)$ to the Rosser–Iwaniec sieve, in which the sieve $S(A, z)$ is bounded by sets defined by the sign of $\lambda(n)$.

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