# ENDOSCOPIC POINTS ON SIEGEL EIGENVARIETIES

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Abstract. Generalizing to GSp(4), a recent result of Ludwig relating the geometry of the eigenvariety to the endoscopy theory of automorphic forms on SL(2), we prove the existence of certain non-classical $p$-adic automorphic forms on GSp(4), arising from endoscopic automorphic representations on GSp(4) of paramodular level constructed from Yoshida lifts.

Contents



## 1. Introduction

Given reductive groups $G$ and $H$ over a global field $F$, and a functorial transfer (when they exist) of automorphic representations

$$\pi_H \mapsto \pi_G \tag{1.1}$$

associated to an $L$-homomorphism ${}^LH$ to ${}^LG$, we may interpolate this transfer to obtain a canonical finite morphism of eigenvarieties

$$\mathscr{X}_H \to \mathscr{X}_G$$

as was studied in various cases, including non-endoscopic cases, and in general by [JN]. This is generally referred to as $p$-adic Langlands functoriality, and the functorial transfer is more properly a transfer of packets of automorphic representations indexed by Langlands parameters $\phi$.

We shall be interested in the case where $H$ is an elliptic endoscopic group of $G$. Let $\pi_G$ be an endoscopic representation of $G$, in the sense that it is the functorial image of some automorphic representation $\pi_H$ on $H$. Then $\pi_G$ gives rise to a classical point on the eigenvariety of $G$, which we shall call an endoscopic point. For $G = \mathrm{SL}(2)$, Ludwig [Lud18] showed that there exist endoscopic points on the eigenvariety of SL(2), such that non-classical $p$-adic automorphic forms occur in the fibre of this endoscopic point.

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In this paper, we prove the analogous result in the case of $G = \mathrm{GSp}(4)$ with paramodular level, which we shall take as a simplifying assumption. This group has only one elliptic endoscopic subgroup, namely $H = \mathrm{GSO}(2,2)$, and the corresponding endoscopic representations are Yoshida lifts from a pair of classical modular forms. Given the eigenvariety $\mathscr{X}_G$ of $G$, we construct a middle-degree eigenvariety $\mathscr{X}_{\mathrm{mid}}$ associated to $\mathscr{X}_G$, imitating a construction of Bergdall–Hansen for Hilbert modular forms, whose points support only cohomology in middle-degree, where the automorphic forms we are interested in are concentrated [BH17]. Choosing certain endoscopic representations $\pi$, we construct a classical point $x$ on $\mathscr{X}_{\mathrm{mid}}(\overline{\mathbb{Q}}_p)$ with critical slope, and consider the stalk of the sheaf of automorphic forms $\mathscr{M}_x$ where

$$\mathscr{M}_x = H^3_c(S(K_f), \mathscr{D}_\lambda \otimes k_x)_{m_x} \xrightarrow{\iota_x} H^3_c(S(K_f), \mathscr{L}_\lambda(k_x))_{m_x}$$

as in Definition 5.2, and denote by $\mathscr{M}_x^{\mathrm{cl}}$ the image of a fixed section of this map.

**Theorem 1.1.** *Let $G = \mathrm{GSp}(4)$ be defined over $\mathbb{Q}$, and $S$ a finite set of finite places of $\mathbb{Q}$. Let $\Pi_\phi$ be a cohomological cuspidal endoscopic global $L$-packet of $G$ of paramodular level at primes in $S$. Moreover, assume that*

*(1) for all $v \notin S$, $\Pi_{\phi,v} = \{\pi_v\}$, and $\pi_v$ is an unramified representation of $G(\mathfrak{o}_v)$*
*(2) for all $v \in S$, $\Pi_{\phi,v}$ is supercuspidal and has cardinality two.*

*Then there exists an automorphic representation $\pi \in \Pi_\phi$ whose $p$-refinement gives rise to classical point $x$ on $\mathscr{X}_{\mathrm{mid}}(\overline{\mathbb{Q}}_p)$ such that there exists non-classical $p$-adic automorphic forms in the fibre of $\mathscr{M}$ at $x$, in the sense that*

$$\mathscr{M}_x/\mathscr{M}_x^{\mathrm{cl}}$$

*is nontrivial, or in other words, $\iota_x$ is not an isomorphism.*

This is in contrast to the situation at any stable point $z$ in a neighborhood of $x$, which is obtained from a non-endoscopic representation of $G$, where $\mathscr{M}_z = \mathscr{M}_z^{\mathrm{cl}}$.

While the result is proved over $\mathbb{Q}$, in principle it extends to totally real fields without any serious difficulty. Our method of proof largely follows that of Ludwig for SL(2). As a side result, we also prove a rigidity theorem for $L$-packets in connected components of the eigenvariety, generalizing an intermediate result of Ludwig [Lud18], in the SL(2) case. In particular, we show in Proposition 3.1 that the cardinality of $L$-packets are constant on connected components of $\mathscr{X}_{\mathrm{mid}}$. We include this result as it may be of interest in connection to the inertial local Langlands correspondence [AMS18] discussed in Section 3.2, which leads us to ask whether the latter may be connected to the geometry of the eigenvariety.

The endoscopic automorphic forms which we consider are transfers from the unique elliptic endoscopic group GSO(2, 2), and can be viewed as Siegel modular forms that arise as Yoshida lifts. On the other hand, we mention the recent result of Berger and Betina which studies endoscopic representations that arise as Saito-Kurokawa lifts, thus transfers from a non-elliptic endoscopic group, and deduce certain (non)smoothness properties of the Siegel eigenvariety at those points [BB19]. Their method follows that of Bellaïche [Bel08] for U(3), where the non-smooth points are again constructed from endoscopic representations.

This paper is organized as follows: In §2, we recall the basic notions of $L$-packets on GSp(4) and their relevant properties. In §3, we introduce the eigenvariety for GSp(4) following the construction of Urban and the middle-degree eigenvariety following Bergdall and Hansen. In §4, we study the points on the eigenvariety corresponding to $p$-stabilizations of Yoshida lifts. In §5, we prove the main theorem, that is, the existence of 'endoscopic' $p$-adic automorphic forms on $G$.

**Notation.** We lay out some notations that will be used throughout this article. We denote by $G$ the algebraic group GSp(4) over $\mathbb{Q}$ realised as the set of matrices $g$ in GL(4) that satisfy ${}^t gJg = \mu(g)J$ for a unique $\mu(g) \in \mathbb{Q}^\times$, where

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Let $B$ be the Borel subgroup consisting of upper triangular matrices. Let $T$ be the maximal torus inside $B$. It consists of elements of the form

$$t(a_1, a_2; a_0) := \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & a_0 a_2^{-1} & \\ & & & a_0 a_1^{-1} \end{bmatrix}.$$

We choose the following basis for the character lattice. Define $e_i(t(a_1, a_2; a_0)) = a_i$ for $i = 0, 1, 2$. The group of algebraic characters $X(T) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_0$. Choose the simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2 - e_0$, also the co-roots $\alpha_1^\vee = (u \mapsto t(u, u^{-1}; 1))$ and $\alpha_2^\vee = (u \mapsto t(1, u; 1))$. The corresponding fundamental weights $\lambda_{\alpha_1} = e_1$ and $\lambda_{\alpha_2} = e_1 + e_2$, which are uniquely determined upto $\mathbb{Z}e_0$. We will denote the weight $\mu = \mu_1 e_1 + \mu_2 e_2 + \mu_0 e_0$ by $(\mu_1, \mu_2, \mu_0)$. A weight $\mu$ is dominant if it is a non-negative combination of the fundamental weights; that is, $\mu$ is dominant if and only if $\mu_1 \geq \mu_2 \geq 0$.

Let $\delta = \delta_B$ denote the modulus character for the Borel $B$. For our choice of $B$, this is explicitly given by

$$\delta(b) = |(a_1^2 a_2)^2 a_0^{-3}|,$$

where we write $b = tn$ and $t = t(a_1, a_2; a_0)$.

Let $W = N_G(T)/T$ denote the Weyl group of the torus $T$ in $G$. The Weyl group acts on the torus by $w \cdot t = wtw^{-1}$. In our case, $W$ is isomorphic to $S_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2$. We identify $(\mathbb{Z}/2\mathbb{Z})^2$ with functions $\varepsilon : \{1, 2\} \to \{\pm 1\}$. An element $w = (\sigma, \varepsilon) \in W$ of the Weyl group act on the torus as follows:

$$w \cdot t(a_1, a_2; a_0) = t(a'_{\sigma(1)}, a'_{\sigma(2)}; a_0),$$

where $a'_i = a_i$ if $\varepsilon(i) = 1$ and $a'_i = a_0 a_i^{-1}$ if $\varepsilon(i) = -1$.

## 2. $L$-packets

2.1. **Global $L$-packets.** Let $\mathbb{A}$ the adele ring of $\mathbb{Q}$, and consider the discrete $L^2$-spectrum $L^2_{\text{disc}}(G_F \backslash G_{\mathbb{A}}, \omega)$ relative to a central character $\omega$ of $G(\mathbb{A})$. Let $\pi$ be an irreducible constituent occurring with multiplicity $m(\pi)$ in this space. For each admissible $L$-homomorphism

$$\phi : L_{\mathbb{Q}} \to {}^L G,$$

one associates a global $L$-packet $\Pi_\phi$, a finite set of irreducible automorphic representations $\pi$. Here $L_{\mathbb{Q}}$ is the conjectural automorphic Langlands group whose existence is not known, but in the case of GSp($2n$) we have an unconditional replacement provided by [Xu18], though we do not need it explicitly. The packet $\Pi_\phi$ will be called stable if the parameter $\phi$ does not factor through a map

$$\phi : L_{\mathbb{Q}} \to {}^L H$$

for any endoscopic group $H$ of $G$. We then call $\Pi_\phi$ endoscopic if it is not a stable packet. The multiplicity $m(\pi)$ for $\pi \in \Pi_\phi$ is in general nonconstant as one varies over $\pi \in \Pi_\phi$ if $\Pi_\phi$ is endoscopic. We shall call an automorphic representation $\pi$ stable if it belongs to a stable packet, or equivalently, if it is not the image of a transfer from an endoscopic group as in (1.1). The discrete spectrum of $G(\mathbb{A})$ can then be decomposed into stable and endoscopic parts, we refer to [CG15, §3] for a precise description of the endoscopic discrete spectrum.

2.2. **Endoscopic classification.** For the remainder of this section, we shall take $F$ to be a non-archimedean local field of characteristic zero. We briefly describe the local Langlands correspondence over $F$ as in [GT11]. Let $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil–Deligne group of $F$, and $\Phi(G)$ the set of equivalence classes of admissible homomorphisms

$$\phi : WD_F \to G^\vee,$$

where $G^\vee$ is the Langlands dual group of $G$, and the equivalence is taken up to $G^\vee$-conjugacy. Note that $\mathrm{GSp}(4)^\vee = \mathrm{GSp}(4)$. Let $\Phi(G)$ denote the set of equivalence classes of admissible irreducible representations of $G$. Then the main theorem of [GT11] proves the existence of a finite-to-one and onto map

$$L : \Pi(G) \to \Phi(G),$$

satisfying various compatibility conditions which uniquely determine the map $L$. One then has a partition of the set $\Pi(G)$ into finite fibres of the map $L$, called the local $L$-packets. For a $\phi \in \Phi(G)$, let $\Pi_\phi$ denote the associated local $L$-packet.

The group $G$ has a unique proper elliptic endoscopic subgroup $H$ corresponding to the endoscopic transfer (1.1),

$$\mathrm{GSO}(2,2) \simeq (\mathrm{GL}_2 \times \mathrm{GL}_2)/\{(z, z^{-1}) \mid z \in \mathbb{G}_m\},$$

with dual group $\mathrm{GSpin}(4,\mathbb{C})$; where by elliptic we mean that $H^\vee$ is not contained in any Levi subgroup of $G^\vee$. An $L$-parameter $\phi$ is called endoscopic if it factors through $\mathrm{GSpin}(4,\mathbb{C})$, the dual group of $\mathrm{GSO}(2,2)$. As a 4-dimensional Weil–Deligne representation, such a parameter has the form $\phi = \phi_1 \oplus \phi_2$, where $\phi_1, \phi_2$ are 2-dimensional representations of $W_F$ with $\det \phi_1 = \det \phi_2$. The associated $L$-packet has size two if and only if $\phi_1$ and $\phi_2$ are irreducible (but possibly equivalent). When $\phi$ is not endoscopic, the associated $L$-packet is singleton.

Suppose that $\phi_i$ above are unramified $L$-parameters, corresponding to principal series representations $\tau_i$. Then $\phi = \phi_1 \oplus \phi_2$ is also unramified and corresponds to a principal series representation. This is the Yoshida lift of $\tau_1$ and $\tau_2$ and is denoted by $Y(\tau_1, \tau_2)$.

Now let $\phi$ be an $L$-parameter with the corresponding $L$-packet $\Pi_\phi$ consisting entirely of supercuspidal representations, in which case we shall call $\Pi_\phi$ a supercuspidal $L$-packet. We then have the following two possibilities for $\phi$:

(1) $\phi = \phi_1 \oplus \phi_2$, where $\phi_1, \phi_2$ are distinct irreducible 2-dimensional representations of $W_F$ with $\det \phi_1 = \det \phi_2$. In this case, the $L$-packet is of size two.
(2) $\phi$ is a 4-dimensional representation of $W_F$, in which case it belongs to a singleton $L$-packet.

More precisely, the endoscopic $L$-packet (the first case) consists of representations

$$Y(\tau_1, \tau_2) := \theta(\tau_1 \boxtimes \tau_2), \quad Y^D(\tau_1, \tau_2) := \theta(\tau_1^D \boxtimes \tau_2^D);$$

where $\tau_1$ and $\tau_2$ are irreducible, square-integrable, admissible representations of $\mathrm{GL}(2, F)$ with common central character, and $\tau_i^D$ is the Jacquet-Langlands transfer of $\tau_i$ to $D^\times$. Here $D$ is the unique quaternion algebra over $F$ and $\theta$ is the theta correspondence from $\mathrm{GSO}(D)$ described in [GT11, Theorem 5.6]. If $\tau_1 \not\simeq \tau_2$, then $Y^D(\tau_1, \tau_2)$ is non-generic supercuspidal, and $Y(\tau_1, \tau_2)$ is generic and also supercuspidal if and only if $\tau_1$ and $\tau_2$ are both supercuspidal [CG15, §8.2].

In the second case, for a quadratic extension $K/F$, we write

$$\phi = \mathrm{Ind}_{W_K}^{W_F}(\phi_\rho) \tag{2.1}$$

where $\phi_\rho$ is an irreducible 2-dimensional representation of $W_K$ which does not extend to $W_F$, but $\det \phi_\rho$ does. The supercuspidal representation $\rho$ of $\mathrm{GL}_2(K)$ associated to $\phi_\rho$ is not obtained via base change from $\mathrm{GL}_2(F)$. The singleton supercsupidal $L$-packet consists of the theta lift of $\Pi_\rho \boxtimes \mu$ from $\mathrm{GL}_4(K) \times \mathrm{GL}_1(F)$ to $\mathrm{GSp}_4(F)$, where $\Pi_\rho$ is the automorphic induction of $\mathrm{GL}_2(K)$ to $\mathrm{GL}_4(F)$ described in [GT11, p. 1877].

2.3. **Paramodular level structure.** Let $F$ be a non-archimedean local field, with valuation ring $\mathcal{O}$ and uniformizer $\varpi$. The paramodular level structures are subgroups of matrices in $G$ of the form

$$K(n) = \{g \in G(F) : \mu(g) \in \mathcal{O}^\times\} \cap \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & (\varpi^{-n}) \\ (\varpi^n) & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ (\varpi^n) & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ (\varpi^n) & (\varpi^n) & (\varpi^n) & \mathcal{O} \end{pmatrix}.$$

We say that an irreducible admissible representation of $G(F)$, with trivial central character is paramodular of level $n$, if it admits a nonzero fixed vector under the action of $K(n)$ for some $n \geq 0$.

Globally, the endoscopic lift, say $\pi$, from GSO(2,2) is known classically as a Yoshida lift. From the multiplicity formula for the global $L$-packet in the discrete spectrum of $G$, one knows that the local constituent $\pi_v$ is non-generic for a finite set of valuations $v$, whose cardinality is even. On the other hand, our requirement that $\pi$ be paramodular implies that $\pi_v$ is paramodular and therefore generic at every finite place $v$ [RS07]. Over $\mathbb{Q}$, the parity condition then forces $\pi_v$ to also be generic at the archimedean place, and hence can only come from non-holomorphic Siegel modular forms.

Our endoscopic representation will be taken to be a non-holomorphic Yoshida lift of paramodular level. It follows from work of Schmidt [Sch18] that the newforms on $G$ of paramodular level satisfy a strong multiplicity one theorem; see also [Rob01] for the corresponding multiplicity one statement for Yoshida lifts.

2.4. **Langlands parameter for unramified principal series.** Returning to the local setting, let $\pi$ be an unramified principal series for $G$ over $F$. These representations are obtained as parabolic inductions from characters. More precisely, let $\chi_i : F^\times \to \mathbb{C}^\times$ be characters and let $\chi : T(F) \to \mathbb{C}^\times$ denote the map

$$t(a_1, a_2; a_0) \mapsto \chi_1(a_1)\chi_2(a_2)\chi_0(a_0)$$

constructed from the $\chi_i$. We inflate $\chi$ to a character of $B$ via the quotient map $B \to T$. Then $\pi$ is given by the action of $G(F)$ on the space

$$\mathrm{Ind}_B^G \chi = \left\{ f : G(F) \to \mathbb{C} \;\middle|\; \begin{array}{c} f \text{ is smooth, and } f(bg) = \delta^{1/2}(b)\chi(b)f(g), \\ \forall b \in B(F), g \in G(F) \end{array} \right\}$$

by right translations. We denote these representations by $\pi(\chi)$.

The Langlands paramter $\varphi : W_F \to \mathrm{GSp}_4(\mathbb{C})$ of $\pi(\chi)$ is given by

$$w \mapsto \begin{bmatrix} \chi_1\chi_2\chi_0(w) & & & \\ & \chi_1\chi_0(w) & & \\ & & \chi_2\chi_0(w) & \\ & & & \chi_0(w) \end{bmatrix}.$$

Note that this is an unramified parameter (depends only on the image of the Frobenius), since the $\chi_i$ are unramified characters.

Now suppose that $\pi(\chi)$ is also a Yoshida lift, that is,

$$\pi = Y(\tau_1, \tau_2), \tag{2.2}$$

where $\tau_1$ and $\tau_2$ are unramified representations for GL(2). Suppose that $\tau_1 = \pi(\eta_{11}, \eta_{12})$ and $\tau_2 = \pi(\eta_{21}, \eta_{22})$. By assumption, we also know that $\eta_{11}\eta_{12} = \eta_{21}\eta_{22}$. Then the Langlands parameter attached to the Yoshida lift is given by

$$w \mapsto \begin{bmatrix} \eta_{11}(w) & & & \\ & \eta_{21}(w) & & \\ & & \eta_{22}(w) & \\ & & & \eta_{12}(w) \end{bmatrix}.$$

This implies that

$$\chi_0 = \eta_{12}, \quad \chi_1 = \eta_{21}\eta_{12}^{-1} = \eta_{11}\eta_{22}^{-1}, \quad \chi_2 = \eta_{11}\eta_{21}^{-1} = \eta_{22}\eta_{12}^{-1}. \tag{2.3}$$

2.5. **An adjoint lift.** We continue to take $F$ to be a non-archimedean local field and $G = \mathrm{GSp}(4)/F$. We embed the dual group into the general linear group, and compose with the adjoint representation:

$$\mathrm{GSp}(4, \mathbb{C}) \longrightarrow \mathrm{GL}(4, \mathbb{C}) \xrightarrow{\mathrm{Ad}^0} \mathrm{GL}(15, \mathbb{C}). \tag{2.4}$$

This induces a transfer of representations $\pi$ of $G$, which we shall refer to as the adjoint lift, and denote it simply as Ad. We record the following property regarding the image of this transfer, which will be used later to study the relation of $L$-packets to the eigenvariety.

**Proposition 2.1.** *Let $\pi$ be a supercuspidal representation of $G(F)$, and $\pi'$ a smooth irreducible representation of $G(F)$. If the adjoint lifts of $\pi$ and $\pi'$ to* GL(15) *lie in the same Bernstein component, then $\pi'$ is also supercuspidal. Moreover, the $L$-packets $\Pi_\phi$ and $\Pi_{\phi'}$ containing $\pi$ and $\pi'$ respectively have the same cardinality.*

*Proof.* Using the notation of Asgari and Schmidt [AS08, §2], we compute first the image of non-supercuspidal parameters $(\rho, N)$, where $\rho$ is a continuous homomorphism from $W_F$ to $\mathrm{GL}_4(\mathbb{C})$ and $N$ is a nilpotent matrix in $\mathrm{GL}_4(\mathbb{C})$ such that

$$\rho(w)N\rho(w)^{-1} = \nu(w)N$$

for all $w \in W_F$, where $\nu$ is the normalized absolute value on $F$. When $N$ is trivial, the parameters lie in the Bernstein component associated to the minimal parabolic $(1, \ldots, 1)$. There are five nontrivial $N$:

$$N_1 = \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix} \quad N_2 = \begin{bmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \quad N_3 = \begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$N_4 = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix} \quad N_5 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}.$$

By direct computation, we find that under the adjoint representation $\mathrm{Ad}(N_i)$ belongs to Bernstein blocks indexed by the following partitions of 15:

$$\begin{aligned} &(2,1,\ldots,1), && i=1, \\ &(2,1,\ldots,1), && i=2, \\ &(2,2,2,2,1,\ldots,1), && i=3, \\ &(2,2,2,2,1,\ldots,1), && i=4, \\ &(4,3,3,2,2,1), && i=5. \end{aligned}$$

Now we examine the supercuspidal representations. In the first case, we have

$$\phi = \phi_1 \oplus \phi_2 = \mathrm{Ind}_{W_K}^{W_F}(\chi_1) \oplus \mathrm{Ind}_{W_K}^{W_F}(\chi_2),$$

where $K/F$ is an extension of degree 2 and $\chi_1, \chi_2$ are characters of $W_K$. Moreover, $\phi_1 \ncong \phi_2$. Let $\sigma$ represent the non-trivial class in $W_F/W_K$. Denote by ${}^{\sigma}\chi_i$ the conjugate of $\chi_i$ under the nontrivial Galois action. Then the contragredient representations are given by

$$\tilde{\phi}_i = \mathrm{Ind}_{W_K}^{W_F}(\chi_i^{-1}), \quad \tilde{\phi}_i\big|_{W_K} = \chi_i^{-1} \oplus {}^{\sigma}\chi_i^{-1}.$$

The adjoint representation decomposes as

$$\phi \otimes \tilde{\phi} = \phi_1 \otimes \tilde{\phi}_1 \oplus \phi_2 \otimes \tilde{\phi}_2 \oplus \phi_1 \otimes \tilde{\phi}_2 \oplus \tilde{\phi}_1 \otimes \phi_2,$$

and in particular the last two summands are irreducible representations of dimension 4, since $\phi_i$ are not isomorphic, whereas the first two summands decompose into $1 \oplus \mathrm{Ad}(\phi_i)$. Since, $\phi_i$ are induced representations from index 2 subgroups, the representations $\mathrm{Ad}(\phi_i)$ break up as the non-trivial quadratic character on $W_F$ corresponding to $K$ and irreducible 2-dimensional representations. This thus corresponds to the partition $(4,4,2,2,1,1,1)$, hence lies in a Bernstein component different from those above.

In the second supercuspidal case, the parameter given by (2.1) corresponds to the partition $(8,6,1)$ determined by $\mathrm{Ad}(\phi)$, and the computation is left to the reader. □

## 3. Eigenvarieties

In this section, we recall Urban's construction of eigenvarieties for reductive groups that admit discrete series representations at infinity, which obviously includes GSp(4). Our main reference for this section will be [Urb11].

3.1. **Eigenvarieties for reductive groups.** The eigenvariety is constructed from an eigenvariety datum using Buzzard's Eigenvariety Machine, where an eigenvariety datum is a tuple $\mathfrak{D} = (\mathscr{W}, \mathscr{Z}, \mathscr{M}, \mathbb{T}, \psi)$ where

(1) the weight space $\mathscr{W}$ is a separated, reduced, equidimensional, relatively factorial rigid analytic space;
(2) the spectral variety $\mathscr{Z} \subset \mathscr{W} \times \mathbb{A}^1$ is a Fredholm hypersurface, parametrizing eigenvalues of a distinguished operator $U \in \mathbb{T}$ on a graded module $M^*$ of $p$-adic automorphic forms;
(3) the Hecke algebra $\mathbb{T}$, a commutative $\mathbb{Q}_p$-algebra;

(4) $\mathscr{M}$ a natural spreading out of $M^*$ to a coherent analytic sheaf over $\mathscr{Z}$;
(5) and, an action $\psi : \mathbb{T} \to \mathrm{End}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{M})$.

Given an eigenvariety datum as above, there exists a rigid analytic space $\mathscr{X}$ with a finite morphism $\pi : \mathscr{X} \to \mathscr{Z}$, a weight morphism $w : \mathscr{X} \to \mathscr{W}$, an algebra homomorphism $\phi_{\mathscr{X}} : \mathbb{T} \to \mathscr{O}(\mathscr{X})$, and a coherent sheaf $\mathscr{M}^\dagger$ on $\mathscr{X}$ together with a canonical isomorphism $\mathscr{M} \simeq \pi_* \mathscr{M}^\dagger$ compatible with the actions of $\mathbb{T}$ on $\mathscr{M}$ and $\mathscr{M}^\dagger$ via $\psi$ and $\phi_{\mathscr{X}}$ respectively. The points of $\mathscr{X}$ lying over $z \in \mathscr{Z}$ are in bijection with the generalized eigenspaces of the action of $\mathbb{T}$ on the stalk $\mathscr{M}_z$. We call $\mathscr{X}$ the eigenvariety associated to this datum.

Given a general reductive group $G$, let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup and let $K^p$ and $K_p$ denote the away from $p$ and $p$-part of $K_f$ respectively. The eigenvariety $\mathscr{X}_{G,K^p}$ with 'tame level' $K^p$ is constructed by making specific choices for the eigenvariety datum. In our context, these choices are made in [Urb11]. Note also that, since our reductive group $G$ and tame level $K^p$ are fixed we suppress them from our notations and denote the eigenvariety only by $\mathscr{X}$. Let $\mathscr{W}$ denote the rigid analytic space whose $\overline{\mathbb{Q}}_p$-points are given by

$$\mathscr{W}(\overline{\mathbb{Q}}_p) = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times). \tag{3.1}$$

We take the weight space $\mathscr{W}_{K^p}$ to be the rigid subspace $\mathscr{W}_{K^p} \subset \mathscr{W}$ cut out by weights $\lambda$ that are trivial on $Z_G(\mathbb{Q}) \cap K^p I \subset T(\mathbb{Z}_p)$.

Given any point $\lambda \in \mathscr{W}(\overline{\mathbb{Q}}_p)$, the image of $\lambda$ generates a subfield $k_\lambda$ of $\overline{\mathbb{Q}}_p$ finite over $\mathbb{Q}_p$. Then, define a Fréchet $k_\lambda$-module $\mathscr{D}_\lambda$ of locally analytic distributions equipped with a continuous $k_\lambda$-linear action of $I$, the Iwahori subgroup of $G(\mathbb{Z}_p)$ associated to $B$. We have the locally symmetric space

$$S(K_f) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty^\circ K_f, \tag{3.2}$$

where $K_\infty^\circ$ is the identity component of a subgroup $K_\infty \subset G(\mathbb{R})$ that is maximal compact modulo the center. The quotient

$$(G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty^\circ K_f^p \times \mathscr{D}_\lambda)/I \to S(K^p I) \tag{3.3}$$

defines a local system on $S(K^p I)$, which we again denote by $\mathscr{D}_\lambda$. The local system is non-trivial if and only if $\lambda \in \mathscr{W}_{K^p}$.

A key ingredient in the eigenvariety datum is the sheaf $\mathscr{M}$. This sheaf is constructed via the overconvergent cohomology groups $H_c^*(S(K^p I), \mathscr{D}_\lambda)$, or more generally $H_c^*(S(K^p I), \mathscr{D}_\Omega)$ for affinoid open sets $\Omega \subset \mathscr{W}_{K^p}$. Here, $\mathscr{D}_\Omega$ is a Fréchet $\mathscr{O}(\Omega)$-module, with a continuous action of $I$, such that $\mathscr{D}_\Omega \otimes_{\mathscr{O}(\Omega)} k_\lambda \cong \mathscr{D}_\lambda$ for any $\lambda \in \Omega$. These are used in the construction of the sheaf $\mathscr{M}$. The Hecke action on these cohomology groups is used in the construction of $\psi$. Note that, by taking the compactly supported cohomology here, we are actually constructing the cuspidal eigenvariety.

An algebraic $B$-dominant weight $\lambda$ can naturally be viewed as an element in $\mathscr{W}$. These weights in $\mathscr{W}$ are called algebraic weights in $\mathscr{W}$, and they form a Zariski dense subset of $\mathscr{W}$. Given an algebraic weight $\lambda \in \mathscr{W}_{K^p}$, let $\mathscr{L}_\lambda$ denote the associated algebraic irreducible representation. There is a surjective $I$-equivariant map $i_\lambda : \mathscr{D}_\lambda \to \mathscr{L}_\lambda$ inducing a Hecke-equivariant and degree-preserving map

$$\iota_\lambda : H_c^*(S(K_f), \mathscr{D}_\lambda) \to H_c^*(S(K_f), \mathscr{L}_\lambda),$$

where we take $K_f = K^p I$. The right hand side can be identified as a Hecke-module with automorphic representations of 'cohomological type.' In fact, after base change to $\mathbb{C}$, there is a splitting

$$H^*(S(K_f),\mathscr{L}_\lambda)_{\mathbb{C}} \simeq H^*_{\text{cusp}}(S(K_f),\mathscr{L}_\lambda)_{\mathbb{C}} \oplus H^*_{\text{Eis}}(S(K_f),\mathscr{L}_\lambda)_{\mathbb{C}};$$

and, a canonical isomorphism of graded Hecke-modules

$$H^*_{\text{cusp}}(S(K_f),\mathscr{L}_\lambda)_{\mathbb{C}} \simeq \bigoplus_{\pi\in L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}}))} m(\pi)H^*(\mathfrak{g},K_\infty;\pi_\infty\otimes\mathscr{L}_\lambda)\otimes\pi_f^{K_f},$$

where $m(\pi)$ is the multiplicity of $\pi$. A representation $\pi$ that contributes non-trivially in this decomposition is called cuspidal and cohomological of weight $\lambda$. Note that this splitting already holds for sufficiently large finite extensions of $\mathbb{Q}$. Note also that, since $G_\infty$ in our case satisfies the Harish-Chandra condition, the cuspidal cohomology matches with the inner cohomology (image of compactly supported cohomology), i.e., $H^*_{\text{cusp}} = H^*_!$.

Following Ash and Stevens, a finite-slope eigenpacket of weight $\lambda$ is an algebra homomorphism $\phi : \mathbb{T} \to \overline{\mathbb{Q}}_p$ such that $H^*(S(K_f),\mathscr{D}_\lambda)\otimes_{k_\lambda}\overline{\mathbb{Q}}_p$ has a common eigenvector with eigenvalue $\phi(T)$ for all $T\in\mathbb{T}$ and $\phi(U)\neq 0$, for the controlling operator $U$. Given an algebraic weight $\lambda$, a finite-slope eigenpacket $\phi$ is called classical if $H^*(S(K_f),\mathscr{L}_\lambda)$ is non-zero after localization at $\ker\phi$. Furthermore, $\phi$ is called non-critical if the map $\iota_\lambda$ is an isomorphism after localization at $\ker\phi$. We say a point $x\in\mathscr{X}(\overline{\mathbb{Q}}_p)$ is classical if the weight $\lambda_x$ is algebraic, and the associated eigenpacket $\phi_x$ is classical.

We also recall the control theorem due to Ash and Stevens, and Urban [Urb11, Proposition 4.3.10]: Let $t_0\in T(\mathbb{Q}_p)$ be the torus element used to define the controlling operator in the construction of the eigenvariety. Any $h\in\mathbb{Q}$ is called a small slope for $\lambda$ if

$$h < \inf_{w\in W\backslash\{1\}} v_p(w\cdot\lambda(t_0)) - v_p(\lambda(t_0)),$$

where $W$ is the Weyl group of $T$. Then the control theorem states that $\iota_\lambda$ induces a natural isomorphism,

$$H^*(S(K_f),\mathscr{D}_\lambda)_{\le h} \cong H^*(S(K_f),\mathscr{L}_\lambda)_{\le h},$$

as Hecke modules.

Using the work Bergdall and Hansen on Hilbert modular forms, we shall also define a certain middle-degree eigenvariety associated to $G$ as follows. Consider the sheaf

$$\mathscr{M}^{\text{BM}}_* = \bigoplus_j \mathscr{M}^{\text{BM}}_j,$$

defined to be the homology sheaf of the complex giving rise to Borel-Moore homology, viewed as complexes of coherent $\mathscr{O}_{\mathscr{X}}$-modules. Similarly, we have a decomposition

$$\mathscr{M}^* = \bigoplus_j \mathscr{M}^j,$$

for the cohomology sheaf of the complex giving rise to compactly supported cohomology. Let $\mathscr{X}$ (the cuspidal eigenvariety) be the $\mathbb{Q}_p$-rigid analytic space associated to the eigenvariety datum with compactly-supported cohomology sheaf $\mathscr{M}^*$, and $\mathscr{X}^{\text{BM}}$ associated to the eigenvariety datum with Borel-Moore homology sheaf $\mathscr{M}^{\text{BM}}_*$. Let $\mathscr{X}^{\text{red}}$ be the nilreduction of $\mathscr{X}$. Then there is a canonical morphism $\tau : \mathscr{X}^{\text{red}} \to \mathscr{X}^{\text{BM}}$ as in [BH17, Prop 6.4.1], and a natural closed immersion $i : \mathscr{X}^{\text{red}} \to \mathscr{X}$.

Given a sheaf $\mathscr{M}$, we shall denote by $\mathrm{supp}(\mathscr{M})$ its support. Then we define the middle-degree eigenvariety as:

$$\mathscr{X}_{\mathrm{mid}} = \mathscr{X} - \left[ \Big( \bigcup_{j=4}^{6} \mathrm{supp}(\mathscr{M}^j) \Big) \cup \Big( \bigcup_{j=0}^{2} \mathrm{supp}(\iota_* \tau^* \mathscr{M}_j^{\mathrm{BM}}) \Big) \right]. \tag{3.4}$$

As an application of [BH17, Prop 6.4.3], it follows then that the classical points $x$ in $\mathscr{X}_{\mathrm{mid}}(\overline{\mathbb{Q}}_p)$ support cohomology only in middle degree. Also, $\mathscr{X}_{\mathrm{mid}}$ is Zariski-open in $\mathscr{X}$, in particular if $x \in \mathscr{X}_{\mathrm{mid}}$ then any sufficiently small good neighborhood of $x$ in $\mathscr{X}$ is actually contained in $\mathscr{X}_{\mathrm{mid}}$.

3.2. **Relation to $L$-packets.** Let now $z$ be a classical point on $\mathscr{X}_{\mathrm{mid},K^p}(\overline{\mathbb{Q}}_p)$ arising from an automorphic representation $\pi$, and let $\Pi_z$ be the global $L$-packet containing $\pi$. Following [Lud18], we call the point $z$ stable if the packet $\Pi_z$ is stable, and otherwise we shall call it an endoscopic point. The following proposition relates supercuspidal $L$-packets to the geometry of the eigenvariety.

**Proposition 3.1.** *Let $x, y \in \mathscr{X}_{\mathrm{mid}}(\overline{\mathbb{Q}}_p)$ be classical points belonging to the same connected component. Let $\pi_x$ and $\pi_y$ be automorphic representations giving rise to $x$ and $y$ respectively, belonging to the packets $\Pi_x$ and $\Pi_y$. If for some $l \neq p$, $\Pi_{x,l}$ is supercuspidal, then $\Pi_{y,l}$ is also supercuspidal and $|\Pi_{x,l}| = |\Pi_{y,l}|$.*

*Proof.* Let $\rho_x$ and $\rho_y$ be the 4-dimensional Galois representations corresponding to $\pi_x$ and $\pi_y$ respectively by [Wei05]. Composing with the adjoint lift (2.4) we obtain 15-dimensional Galois representations, so that applying [BC09, Proposition 7.1.1] to the Zariski-dense subset of classical points we obtain a 15-dimensional pseudorepresentation on $\mathscr{X}_{\mathrm{mid}}$. It then follows from [BC09, Lemma 7.8.17] that we have an equality of pseudorepresentations

$$\mathrm{trace}(\mathrm{Ad}(\rho_x))|_{I_l} = \mathrm{trace}(\mathrm{Ad}(\rho_y))|_{I_l},$$

where $I_l$ is the inertia group of $\mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$. By the inertial Langlands correspondence, the lifts of $\pi_{x,l}$ and $\pi_{y,l}$ to $\mathrm{GL}(15)$ lie in the same Bernstein component, and it follows from Proposition 2.1 that $\Pi_{y,l}$ is also supercuspidal and of the same cardinality as $\Pi_{x,l}$. $\square$

The local Langlands correspondence for $G = \mathrm{GSp}_4$ is known to be compatible with normalized parabolic induction in the following sense: Let $P \subset G$ be a parabolic subgroup with Levi factor $M$. Let $\sigma$ and $\pi$ be irreducible representations of $M$ and $G$ respectively, and assume that $\pi$ is an irreducible subquotient of $\mathrm{Ind}_P^G(\sigma)$. Then the infinitesimal characters

$$\phi_\pi|_{W_F} : WD_F \to {}^LG, \text{ and } \phi_\sigma|_{W_F} : WD_F \to {}^LM \hookrightarrow {}^LG$$

are $G^\vee$-conjugate. Here an infinitesimal character is defined as a $G^\vee$-conjugacy class of an admissible homomorphism of $W_F \to {}^LG$. It follows from this that the restriction $\phi_\pi|_{W_F}$ depends only on the cuspidal support of $\pi$, that is, the $G$-conjugacy class $(M, \sigma)$ associated to $\pi$. By [Hai14, Remark 5.2.3], this is known for $\mathrm{GL}(n)$ and $\mathrm{GSp}(4)$.

Recall the (weak) Bernstein decomposition of the category of irreducible admissible representations of $G(F)$, with $F$ a nonarchimedean field,

$$\mathrm{Irr}(G(F)) = \bigsqcup_{s \in \mathfrak{B}(G(F))} \mathrm{Irr}(G(F))_s.$$

Here $\mathfrak{B}(G(F))$ are the Bernstein components of $G(F)$, consisting of equivalence classes of inertial pairs $s = [M, \sigma]_{G(F)}$ where $M$ is a Levi subgroup of $G(F)$ and $\sigma$ a cuspidal representation of $L$. Two such pairs $s_1, s_2$ are said to be (inertially) equivalent if there exists an unramified character $\chi_1$ of $M_1$ and an element $g \in G(F)$ such that $gM_2g^{-1} = M_1$ and $h\cdot\sigma_2 = \sigma_1 \otimes \chi_1$. Then the set $\mathrm{Irr}(G(F))_s$ consists of irreducible smooth $G(F)$-representations $\pi$ whose cuspidal support lies in $s$.

Referring to [AMS18, §8] for details, there is a similar decomposition of the set of (enhanced) $L$-parameters

$$\Phi_e(G(F)) = \bigsqcup_{s^\vee \in \mathfrak{B}(G(F))^\vee} \Phi_e(G(F))_{s^\vee}.$$

where the enhanced $L$-parameters $(\phi, \rho) \in \Phi_e(G(F))_{s^\vee}$ have cuspidal support contained in the inertial equivalence class $s^\vee = ({}^L M, \phi, q\epsilon)$. Two such classes $s_1^\vee, s_2^\vee$ are said to be (inertially) equivalent if there exists $g \in G^\vee$ and $z$ in

$$Z((G^\vee \rtimes I_{F_v}) \cap {}^L M)^\circ_{\mathrm{Frob}}$$

such that $g\,{}^L M_2 g^{-1} = {}^L M_1$ and $(z\phi_1, q\epsilon) = (g\phi_2 g^{-1}, g \cdot q\epsilon)$. The action $z\phi$ is given by $(z\phi)|_{I_{F_v} \times SL_2(\mathbb{C})} = \phi|_{I_{F_v} \times SL_2(\mathbb{C})}$ and $(z\phi)(\mathrm{Frob}) = z\phi(\mathrm{Frob})$. Note that this generalizes the inertial equivalence defined in [Hai14, Definition 5.3.3] for usual $L$-parameters.

Then using the refinement of the local Langlands correspondence by [AMS18], we have a bijection $s \leftrightarrow s^\vee$ giving a correspondence

$$\mathrm{Irr}(G)_s \leftrightarrow \Phi_e(G)_{s^\vee}$$

where $\Phi_e(G)$ is the set of extended Langlands parameters, again decomposed into a disjoint union of sets indexed by $s^\vee \in \mathfrak{B}(G)^\vee$ obtained from a generalized Springer correspondence. The following is then an immediate consequence of the preceding definitions and Proposition 3.1.

**Corollary 3.2.** *Given supercuspidal $L$-packets $\Pi_\phi$ and $\Pi_{\phi'}$ as in Proposition 3.1, the associated enhanced $L$-parameters belong to same component $s^\vee$.*

## 4. $p$-stabilisation of Yoshida lifts

In the section, we study the points on the Siegel eigenvariety corresponding to Yoshida lifts. Throughout this section, we fix $\pi = Y(\tau_1, \tau_2)$ to be a Yoshida lift as in (2.2). We also assume that $\pi$ is unramified at $p$.

4.1. **Cohomological weights.** We now describe the weights $\mu \in X(T)$, that support a Yoshida lift $\pi$ in cohomology.

Let $W_\mathbb{R} = \mathbb{C} \sqcup j\mathbb{C}$ be the Weil group of $\mathbb{R}$. Let $\varphi_{i,\infty} : W_\mathbb{R} \to \mathrm{GL}_2(\mathbb{C})$ be the Langlands parameter associated to $\tau_i$, for $i = 1, 2$, respectively. Then

$$\varphi_{i,\infty}(re^{i\theta}) = \begin{bmatrix} r^{2t_i}e^{il_i\theta} & \\ & r^{2t_i}e^{-il_i\theta} \end{bmatrix}, \qquad \varphi_{i,\infty}(j) = \begin{bmatrix} & (-1)^{l_i} \\ 1 & \end{bmatrix},$$

for $i = 1, 2$; where $l_i \in \mathbb{Z}_{\geq 1}$ and $t_i \in \frac{1}{2}\mathbb{Z}$. In particular, we know that $\tau_{i,\infty} = |\det(\cdot)|^{t_i} \otimes D_{l_i}$, where $D_l$ is the discrete series representation with Blattner parameter $l$ and central character $\mathrm{sgn}^l$. Note that $t_1 = t_2 =: t$ and $l_1$ and $l_2$ have the same parity since $\tau_{1,\infty}$ and $\tau_{2,\infty}$ have the same central character. The Langlands parameter for $\pi_\infty$ is, $\varphi_\infty : W_\mathbb{R} \to \mathrm{GSp}_4(\mathbb{C})$ given by

$$\varphi_\infty(re^{i\theta}) = r^{2t} \begin{bmatrix} e^{il_1\theta} & & & \\ & e^{il_2\theta} & & \\ & & e^{-il_2\theta} & \\ & & & e^{-il_1\theta} \end{bmatrix}, \quad \varphi_\infty(j) = \begin{bmatrix} & & & (-1)^{l_1} \\ & & (-1)^{l_2} & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Let $\mathfrak{gl}_2$ denote the complex Lie algebra of $\mathrm{GL}_2(\mathbb{R})$. For the representations $\tau_{i,\infty}$, we know that the weights for which the relative Lie algebra cohomology

$$H^*(\mathfrak{gl}_2, \mathbb{R}_{>0}\mathrm{SO}_2(\mathbb{R}); \tau_{i,\infty} \otimes V_{\lambda_i}^\vee) \neq 0$$

are given by

$$\lambda_i = \left(t + \frac{l_i - 1}{2}, t - \frac{l_i - 1}{2}\right).$$

Similarly, for $G_\infty = \mathrm{GSp}_4(\mathbb{R})$ and $\pi_\infty$, we know that the weight for which the relative Lie algebra cohomology

$$H^*(\mathfrak{g}, K_\infty^\circ; \pi_\infty \otimes V_\mu^\vee) \neq 0$$

is given by

$$\mu = \left(\frac{l_1 + l_2}{2} - 2\right) e_1 + \left(\frac{l_1 - l_2}{2} - 1\right) e_2 + \left(\frac{2t - l_1 + 3}{2}\right) e_0. \tag{4.1}$$

Note that since $2t$ and $l_i - 3$ have the same parity, all the coefficients above are integers. Since $l_2 \geq 1$ and $l_1 > l_2$, the weight $\mu$ is also dominant.

4.2. **$p$-stabilization.** Assume, as above, that $\pi_p$ is unramified. A $p$-stabilization $\nu$ of $\pi$ is an irreducible constituent of $\pi_p^I$ as a $T(\mathbb{Q}_p)$-module. The tuple $(\pi, \nu)$ contributes to a point on the eigenvariety.

We now need to understand the possible $p$-stabilizations for the representation $\pi$. Let $\mathcal{H}(G, I)$ denote the Iwahori-Hecke algebra and let $\mathcal{H}(T, T^\circ)$ denote the unramified Hecke algebra of the torus, where $T^\circ = T(\mathbb{Z}_p)$. Since $T^\circ \subset I$, we are interested in the structure of $\pi^I$ as a $\mathcal{H}(T, T^\circ)$-module.

Let $S_I : \mathcal{H}(G, I) \longrightarrow \mathcal{H}(T, T^\circ)$ denote the twisted Satake transform given by

$$S_I(\phi)(m) = \int_N \phi(mn)\, dn.$$

For any character $\chi : T \to \mathbb{C}^\times$, we also define a left action of the Weyl group on $\chi$ by the formula

$${}^w\chi(m) = \chi(w^{-1} \cdot m),$$

where $\delta$ is the modulus character for $B$. We extend any character $\chi : T \to \mathbb{C}^\times$ to an algebra homomorphism $\chi : \mathcal{H}(T, T^\circ) \to \mathbb{C}$ in the obvious way.

The dimension of $\pi^I$ is 8, with a basis vector $e_w$ for each $w \in W$. The action of any operator $U_t = [ItI]$, for a dominant $t = \mathrm{diag}(t_{11}, t_{22}, t_{33}, t_{44})$, with respect to this basis is of the form

$$U_t(e_w) = \sum_{w' \leq w} c(U_t, \chi, w, w') e_{w'},$$

where $\leq$ is with respect to the Bruhat ordering and

$$c(U_t, \chi, w, w) = \delta(t)^{1/2}\, {}^w\chi(S_I(U_t)).$$

So, the elements of the Weyl group are in bijection with the set of $p$-stabilizations, and this leads us to an explicit computation of $S_I(U_t)$ in order to determine the slope of the corresponding eigenpacket.

This is a standard computation similar to that of the spherical Satake transform. We write the double coset $ItI = \sqcup_i x_i I$. Write each of these as $x_i = t_i n_i$ for $t_i \in T$ and $n_i \in N$. It can be shown that all the $t_i = t$ in the Iwahori case. Then

$$\begin{aligned} S_I(\mathbb{1}_{ItI}) = m \mapsto \int_N \mathbb{1}_{ItI}(mn)dn &= \sum_i \int_N \mathbb{1}_{x_i I}(mn)dn \\ &= \sum_i \mathbb{1}_{t_i T^\circ} \\ &= \#X \mathbb{1}_{tT^\circ}, \end{aligned}$$

where $X = \{x_i\}$. We can find an explicit set of coset representatives $X$, as in [Hid95, §2]. The set $X$ consists of matrices $\eta t$, where $\eta \in N(\mathbb{Z}_p)$ and $\eta_{ij}$ ranges over representatives in $\mathbb{Z}_p$ for $\frac{\mathbb{Z}_p}{t_{jj}^{-1} t_{ii} \mathbb{Z}_p}$, for $j > i$. Any matrix in $N(\mathbb{Z}_p)$ is of the form

$$\begin{bmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & x_4 & x_2 - x_1 x_4 \\ & & 1 & -x_1 \\ & & & 1 \end{bmatrix}.$$

Fix $t_0 = \mathrm{diag}(p^3, p^2, p, 1)$ and $u_0 = [It_0I]$. For these choices, we compute that $\#X = p^7$ and

$$S_I(u_0) = p^7 \mathbb{1}_{t_0}.$$

The eigenvalue of $u_0$ acting on $e_w \in (\pi^I)^{ss}$ is

$$\delta^{1/2}(t_0)^w \chi(p^7 t_0) = p^7 \delta^{1/2}(t_0) \chi(w^{-1} \cdot t_0).$$

Since, we have an explicit formula

$$\delta^{1/2}(t(p^{a_1}, p^{a_2}; p^{a_0})) = |p|^{2a_1 + a_2 - \frac{3}{2}a_0} = p^{\frac{3}{2}a_0 - 2a_1 - a_2},$$

for the modulus character, we get

$$p^{\frac{7}{2}} \chi(w^{-1} \cdot t_0)$$

as the eigenvalue for $e_w$ for the action of $U_{t_0}$.

To summarise, the $p$-stabilizations $\nu$ of $\pi_p$ correspond to elements of the Weyl group with corresponding eigenvalues $\alpha_\nu = p^{\frac{7}{2}} \chi(w^{-1} \cdot t_0)$.

We compute these eigenvalues for the 8 choices of $p$-stabilizations and enumerate them in the third column of Table 1. This calculation works for any unramified principal series. When $\pi$ is a Yoshida lift, we use the relations in (2.3) to write these eigenvalues in terms of the Satake parameters of $\tau_{i,p}$; these values are listed in the fourth column of Table 1. We fix the following notation

$$\alpha_1 = \eta_{11}(p), \quad \beta_1 = \eta_{12}, \quad \alpha_2 = \eta_{21}, \quad \text{and } \beta_2 = \eta_{22};$$

for the Satake parameters of $\tau_i$ at $p$.

4.3. **Slopes of $p$-stabilizations of Yoshida lifts.** In this section, we compute the slopes of the $p$-stabilizations of Yoshida lifts. We also discuss the notion of points on the eigenvariety with noncritical slopes.

The Hecke operator $U_t$ arising from $t \in T(\mathbb{Q}_p)$ is called a controlling operator if $|\alpha(t)|_p < 1$ for all positive roots $\alpha$. Note that it is sufficient to verify this condition for positive simple

TABLE 1. $p$-stabilizations of $\pi_p$

| $\nu$ | $w^{-1}$ | Eigenvalue $\alpha_\nu$ | $\alpha_\nu$ when $\pi$ is Yoshida lift |
|---|---|---|---|
| $\nu_1$ | $1 \rtimes (1,1)$ | $p^{\frac{7}{2}}\chi_1(p^3)\chi_2(p^2)\chi_0(p^3)$ | $p^{\frac{7}{2}}\alpha_1^2\alpha_2$ |
| $\nu_2$ | $1 \rtimes (1,-1)$ | $p^{\frac{7}{2}}\chi_1(p^3)\chi_2(p)\chi_0(p^{-3})$ | $p^{\frac{7}{2}}\alpha_1\alpha_2^2$ |
| $\nu_3$ | $1 \rtimes (-1,1)$ | $p^{\frac{7}{2}}\chi_2(p^2)\chi_0(p^3)$ | $p^{\frac{7}{2}}\beta_1\beta_2^2$ |
| $\nu_4$ | $1 \rtimes (-1,-1)$ | $p^{\frac{7}{2}}\chi_2(p)\chi_0(p^3)$ | $p^{\frac{7}{2}}\beta_1^2\beta_2$ |
| $\nu_5$ | $(12) \rtimes (1,1)$ | $p^{\frac{7}{2}}\chi_1(p^2)\chi_2(p^3)\chi_0(p^3)$ | $p^{\frac{7}{2}}\alpha_1^2\beta_2$ |
| $\nu_6$ | $(12) \rtimes (1,-1)$ | $p^{\frac{7}{2}}\chi_1(p^2)\chi_0(p^3)$ | $p^{\frac{7}{2}}\beta_1\alpha_2^2$ |
| $\nu_7$ | $(12) \rtimes (-1,1)$ | $p^{\frac{7}{2}}\chi_1(p)\chi_2(p^3)\chi_0(p^3)$ | $p^{\frac{7}{2}}\alpha_1\beta_2^2$ |
| $\nu_8$ | $(12) \rtimes (-1,-1)$ | $p^{\frac{7}{2}}\chi_1(p)\chi_0(p^3)$ | $p^{\frac{7}{2}}\beta_1^2\alpha_2$ |

roots. Applying this to the simple roots $\alpha_1$ and $\alpha_2$, an element $t = t(p^{a_1}, p^{a_2}; p^{a_0})$ will give rise to a controlling operator $U_t$, if

$$a_1 - a_2 > 0 \qquad \text{and} \qquad 2a_2 - a_0 > 0.$$

These conditions are satisfied by the operator $u_0$ corresponding to

$$t_0 = t(p^3, p^2; p^3) = \mathrm{diag}(p^3, p^2, p, 1).$$

We will take this to be our default controlling operator.

From the eigenvariety construction of Urban, the eigenvariety $\mathscr{X}$ is equipped with a map $\phi : \mathbb{T} \to \mathscr{O}(\mathscr{X})$ which interpolates the normalized eigenvalues of the Hecke operator at classical points. Let $x \in \mathscr{X}$ correspond to a $p$-stabilization $(\pi, \nu)$. We denote the cohomological weight of $\pi$ by $\mu = \omega(x)$. Let $\alpha_\nu(u_0)$ denote the $u_0$-eigenvalue for the $p$-stabilization $\nu$. Then

$$\phi(u_0)_x = |\mu(t_0)|_p^{-1}\alpha_\nu(u_0) = p^{3\mu_1+2\mu_2+3\mu_0}\alpha_\nu(u_0),$$

where $\mu = \mu_1 e_1 + \mu_2 e_2 + \mu_0 e_0$. The valuation $h_x := v_p(\phi(u_0)_x)$ is called the slope of the point $x \in \mathscr{X}$.

The slope $h_x$ at a classical point on the eigenvariety is given by

$$h_x = 3\mu_1 + 2\mu_2 + 3\mu_0 + v_p(\alpha_\nu(u_0))$$

Now suppose that $\pi$ is a Yoshida lift. Then, the $\alpha_\nu(u_0)$ are the eigenvalues listed in the third column of Table 1 and the weight $\mu$ is of the form (4.1). Then

$$h_x = -\frac{7}{2} + 3t + l_1 + \frac{l_2}{2} + v_p(\alpha_\nu(u_0)).$$

We write $h_1 = v_p(\alpha_1)$ and $h_2 = v_p(\alpha_2)$; and since $\alpha_1\beta_1 = \alpha_2\beta_2 = |\det(pI)|^t = p^{-2t}$, we have $v_p(\beta_1) = -2t - h_1$ and $v_p(\beta_2) = -2t - h_2$.

For a classical modular form $f$, let $\tau(f)$ denote the associated unitary automorphic representation. We may write $\tau_i = \tau(f_i) \otimes |\det|^t$, for classical eigenforms $f_i$ of weight $l_i + 1$. Suppose that $a_p(f_i)$ is the $p$th Fourier coefficient of $f_i$, we let $\alpha_i^\circ$ and $\beta_i^\circ$ denote the roots of the Hecke polynomial $X^2 - a_p(f_i)X + p^{l_i}$. We then have

$$\alpha_i = p^{-\left(t+\frac{l_i}{2}\right)}\alpha_i^\circ \qquad \text{and} \qquad \beta_i = p^{-\left(t+\frac{l_i}{2}\right)}\beta_i^\circ.$$

In particular,

$$h_i = -t - \frac{l_i}{2} + h_i^\circ,$$

where $h_i^\circ = v_p(\alpha_i^\circ)$. Substituting these values in the fourth column of Table 1, we get the slopes of the $p$-stabilizations of Yoshida lifts in terms of the slopes of its constituent $\mathrm{GL}_2$ representations. These are listed in the second column of Table 2. Without loss of generality, we arrange the $\alpha_i$ and $\beta_i$ such that $v_p(\alpha_i) \leq v_p(\beta_i)$. Since $\alpha_i^\circ \beta_i^\circ = p^{l_i}$, we also have $0 \leq h_i^\circ \leq l_i/2$.

TABLE 2. Slopes of $p$-stabilizations of Yoshida lifts

| $\nu$ | $h_x$ | Min value | Max value | when $f_i$ are ordinary |
|---|---|---|---|---|
| $\nu_1$ | $2h_1^\circ + h_2^\circ$ | $0$ | $l_1 + \frac{l_2}{2}$ | $0$ |
| $\nu_2$ | $\frac{l_1}{2} - \frac{l_2}{2} + h_1^\circ + 2h_2^\circ$ | $\frac{l_1 - l_2}{2}$ | $l_1 + \frac{l_2}{2}$ | $\frac{l_1 - l_2}{2}$ |
| $\nu_3$ | $\frac{3l_1}{2} + \frac{3l_2}{2} - h_1^\circ - 2h_2^\circ$ | $l_1 + \frac{l_2}{2}$ | $\frac{3(l_1+l_2)}{2}$ | $\frac{3(l_1+l_2)}{2}$ |
| $\nu_4$ | $2l_1 + l_2 - 2h_1^\circ - h_2^\circ$ | $l_1 + \frac{l_2}{2}$ | $2l_1 + l_2$ | $2l_1 + l_2$ |
| $\nu_5$ | $l_2 + 2h_1^\circ - h_2^\circ$ | $\frac{l_2}{2}$ | $l_1 + l_2$ | $l_2$ |
| $\nu_6$ | $\frac{3l_1}{2} - \frac{l_2}{2} - h_1^\circ + 2h_2^\circ$ | $l_1 - \frac{l_2}{2}$ | $\frac{3l_1+l_2}{2}$ | $\frac{3l_1-l_2}{2}$ |
| $\nu_7$ | $\frac{l_1}{2} + \frac{3l_2}{2} + h_1^\circ - 2h_2^\circ$ | $\frac{l_1+l_2}{2}$ | $\frac{l_1+3l_2}{2}$ | $\frac{l_1+3l_2}{2}$ |
| $\nu_8$ | $2l_1 - 2h_1^\circ + h_2^\circ$ | $l_1$ | $2l_1 + \frac{l_2}{2}$ | $2l_1$ |

**Definition 4.1.** The point $x \in \mathscr{X}$ with slope $h_x$ and weight $\mu$ is said to have noncritical slope if

$$h_x < [\mu(\alpha^\vee) + 1]v_p(\alpha(t_0)),$$

for each simple positive root $\alpha$.

Applying this definition to the two simple positive roots $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2 - e_0$ in our case, the noncritical slope condition becomes

$$h_x < \min\{\mu_1 - \mu_2 + 1, \mu_2 + 1\}.$$

If $x$ is a classical endoscopic point, then the weight $\mu$ is given by (4.1), and

$$h_x < \min\left\{l_2, \frac{l_1 - l_2}{2}\right\}$$

becomes noncritical slope condition. In particular, any classical endoscopic point has at least one refinement with critical slope.

4.4. **Endoscopic points with critical slope.** We now fix $f_1$ and $f_2$ that are ordinary at $p$. That is, $h_i^\circ = 0$ and $f_i$ have an ordinary $p$-stabilization and a critical $p$-stabilization. Let $\tau_i = \tau(f_i) \otimes |\det|^t$. Let $\pi$ be the Yoshida lift of $\tau_1$ and $\tau_2$. We now prove the following useful proposition.

**Proposition 4.2.** *Let $x \in \mathscr{X}$ be a classical endoscopic point corresponding to $(\pi, \nu)$ with critical slope. Then there exists a rigid analytic neighbourhood $U$ of $x$ in $\mathscr{X}$, and a dense subset $U^{\mathrm{cl}} \subset U$ of classical points such that any $z \in U^{\mathrm{cl}}$ comes from a stable automorphic representation $\pi$.*

*Proof.* We work with the $p$-stabilization $\nu_4$; the other cases are similar. The $p$-stabilization $\nu_4$ has slope $2l_1 + l_2$, which is the maximum possible and always a critical slope. Moreover, this $p$-stabilization corresponds to choosing the critical $p$-stabilizations for $f_i$ and taking its image under a $p$-adic Yoshida lifting.

We start with a neighbourhood $U$ of $x$ such that the slope of $\phi(u_0)$ is constant on $U$. This slope is $2l_1 + l_2$. Suppose that $x' \in U$ is a classical endoscopic point. Then, $x'$ is obtained as a $p$-adic Yoshida lift of points on the eigenvariety for $\mathrm{GL}(2)$ corresponding to $p$-stabilizations of representations $\tau_i'$. Suppose that $\tau_1'$ and $\tau_2'$ arise from classical modular forms $f_1'$ and $f_2'$. Shrinking $U$ if necessary, we assume that the normalized slope of the $p$-stabilizations of $\tau_i'$ are $l_i$.

Let $\tau_i'$ have cohomological weight

$$\lambda_i' = \left(t' + \frac{l_i' - 1}{2}, t' - \frac{l_i' - 1}{2}\right).$$

Then the normalized slopes of the $p$-stabilizations of $\pi_i'$ are $t' + \frac{l_i'}{2} + v_p(\alpha_i')$ and $t' + \frac{l_i'}{2} + v_p(\beta_i')$. These values are equal to ${h_i'}^{\circ}$ and $l_i' - {h_i'}^{\circ}$, where without loss of generality we assume that ${h_i'}^{\circ} \leq l_i' - {h_i'}^{\circ}$. We then have

$$l_i = l_i' - {h_i'}^{\circ} \quad \text{or} \quad l_i = {h_i'}^{\circ}.$$

Hence, we get two possible values for each of ${h_1'}^{\circ}$ and ${h_2'}^{\circ}$; and hence for each $p$-stabilization $\nu_i'$, there are four possible values for $h_{x'}$. However, since $h_{x'} = 2l_1 + l_2$ is fixed, this gives a relation among $(l_1', l_2', t')$. Hence, the classical endoscopic points in $U$ are codimension 1, and the set $U^{\mathrm{cl}}$ consisting of all classical points in $U$ that are not endoscopic is a dense subset of $U$. □

## 5. Proof of main theorem

Before proving the main theorem, we show the existence of $L$-packets satisfying the required assumptions.

**Proposition 5.1.** *The $L$-packets satisfying the assumptions in Theorem 1.1 exist.*

*Proof.* Take two automorphic representations $\tau_1$ and $\tau_2$ of $\mathrm{GL}_2$, such that $\tau_1 \ncong \tau_2$, but they have the same central character. Moreover, assume that $\tau_{i,\ell}$ are unramified for all $\ell \notin S$, and supercuspidal for all $\ell \in S$. The Yoshida lift $Y(\tau_1, \tau_2)$ will satisfy all the assumptions in 1.1. □

Let $\phi : \mathbb{T} \to \mathscr{O}(\mathscr{X})$ be as before. For any point $x \in \mathscr{X}_{\mathrm{mid}}$, let $\phi_x : \mathbb{T} \to k_x$ denote the composition

$$(\mathscr{O}(\mathscr{X}) \to \mathscr{O}(\mathscr{X})_x \to k_x) \circ \phi,$$

where $k_x$ is the residue field of $x$. Let $m_x$ denote the kernel of $\phi_x$. By construction,

$$\mathscr{M}_x = H_c^3(S(K_f), \mathscr{D}_\lambda \otimes k_x)_{m_x},$$

since $x$ is in the middle degree eigenvariety.

**Definition 5.2.** Let $x$ be a classical point on $\mathscr{X}(\overline{\mathbb{Q}}_p)$. We have the homomorphism

$$\iota_x : H^3_c(S(K_f), \mathscr{D}_\lambda \otimes k_x)_{m_x} \to H^3_c(S(K_f), \mathscr{L}_\lambda(k_x))_{m_x}.$$

We let $\mathscr{M}_x^{\mathrm{cl}}$ denote the subspace of $\mathscr{M}_x$ spanned by the image of a section of this map, when $\phi_{\mathscr{X},x}$ comes from a cuspidal automorphic representation. Note that if $x$ is a point with non-critical slope, then $\iota_x$ is an isomorphism and $\mathscr{M}_x = \mathscr{M}_x^{\mathrm{cl}}$.

Note that there is no canonical definition of a classical subspace in Urban's eigenvariety construction, in contrast to [Lud18] which uses the eigenvariety construction of Emerton via completed cohomology, in which case the subspace classical automorphic forms is well-defined.

We now prove the main theorem.

*Proof of Theorem 1.1.* Choose the tame level $K^p$ to be of paramodular level such that $\pi_\ell^{K_\ell}$ is one dimensional for all $\ell \in S$, and $K_\ell$ is the maximal compact subgroup for $\ell \neq p$ not in $S$.

Let $x \in \mathscr{X}_{\mathrm{mid}}$ be a classical point as in Proposition 4.2, and $z$ a stable point in a neighborhood of $x$. Recall that we have $\mathscr{M} \simeq \pi_* \mathscr{M}^\dagger$, by abuse of notation we shall identify these two spaces. We shall show that the quotient $\mathscr{M}_x / \mathscr{M}_x^{\mathrm{cl}}$ is nontrivial by showing that the dimension

$$\dim_{k_x} \mathscr{M}_x \geq \dim_{k_z} \mathscr{M}_z = \dim_{k_z} \mathscr{M}_z^{\mathrm{cl}} > \dim_{k_x} \mathscr{M}_x^{\mathrm{cl}}, \tag{5.1}$$

where the first equality follows from the semicontinuity of the fibre rank, and that the sheaf $\mathscr{M}$ is coherent.

To that end, let $\pi_z$ denote the automorphic representation associated to the classical point $z$. Let $X(z)$ denote the set of all automorphic representations $\pi$ such that $\pi_\ell$ is unramified for all finite places $\ell \notin S$, and $\pi_\ell$ is in the same $L$-packet as $\pi_{z,\ell}$ for $\ell \in S \cup \{\infty\}$.

We then have,

$$\mathscr{M}_z^{\mathrm{cl}} = H^3_c(S(K_f), \mathscr{L}_\lambda)_{m_z} = \bigoplus_{\pi \in X(z)} m(\pi) H^3(\mathfrak{g}, K_\infty^\circ; \pi_\infty \otimes \mathscr{L}_\lambda) \otimes (\pi_f^p)^{K^p} \otimes (\pi_p^I)_z,$$

and similarly for $\mathscr{M}_x^{\mathrm{cl}}$. Since $\pi_z$ is stable, $m(\pi) \geq 1$ for all $\pi \in X(z)$. Moreover, for each $\ell \in S$, there are two possibilities for $\pi_\ell$: a generic supercuspidal representation or a non-generic supercuspidal representation. Since our tame level $K^p$ is paramodular, we have $(\pi_f^p)^{K^p} \neq 0$ only when each $\pi_\ell$ is the generic supercuspidal representation in its $L$-packet. Hence,

$$\dim \mathscr{M}_z^{\mathrm{cl}} \geq 2.$$

At the endoscopic point $x$ arising from a Yoshida lift, for any $\pi \in X(x)$, Arthur's multiplicity formula gives us that $m(\pi) \neq 0$ if and only if

$$\#\{\ell \in S \cup \{\infty\} \mid \pi_\ell \text{ is non-generic}\}$$

is even. Since $(\pi_f^p)^{K^p} = 0$ if any of the $\pi_\ell$ is non-generic, the only contribution to $\mathscr{M}_x^{\mathrm{cl}}$ comes when all of the $\pi_\ell$ are generic. Hence

$$\dim \mathscr{M}_x^{\mathrm{cl}} = 1,$$

as desired. ∎

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