MODIFICATIONS OF THE STABLE TRACE FORMULA

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Abstract. We introduce a modification of the trace formula by means of certain maps of distribution, which are used to remove the contribution of weighted characters to the spectral side of the trace formula. As a result, we obtain a geometric expansion of the unitary part of the trace formula, as a first approximation to isolating the tempered, cuspidal spectrum. Additionally, we extend the results of Finis, Lapid, and Müller on the continuity of Arthur's noninvariant trace formula to the stable and endoscopic trace formulae, which allow for the use of noncompactly-supported test functions.

Contents

1. Introduction

Let G be a connected reductive group over a number field F, and let f be a suitable test function in the Hecke algebra of G. The noninvariant trace formula is a linear form $J(f)$ on G, that is made invariant by modifying the noninvariant distribution

$$I(f) = J(f) - \sum_{M \neq G} |W_0^M||W_0^G|^{-1} \hat{I}_M(\phi_M(f))$$

using certain maps ϕ_M. The resulting invariant linear form $I(f)$ then has geometric and spectral expansions parallel to those of $J(f)$,

$$\begin{aligned} I(f) &= \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma) I_M(\gamma, f) \\ &= \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M(\pi) I_M(\pi, f) d\pi \end{aligned}$$

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where γ are conjugacy classes and π are representations of Levi subgroups M of G [Art88b]. If G is quasiplit over F, the corresponding stabilisation

$$I(f) = \sum_{G'} \iota(G, G') \hat{S}^{G'}(f') \tag{1.1}$$

is a decomposition of $I(f)$ into a finite sum of stable distributions $S^{G'}$ on the elliptic endoscopic groups G' of G. For the case $G' = G^*$, the quasiplit inner form of G, the form $\hat{S}^{G^*}(f^*)$ is regarded as the stable part of $I(f)$. The resulting stable linear form $S(f)$ then comes with geometric and spectral expansions parallel to those of $I(f)$, and provide access to cases of functoriality for endoscopic groups. In the case that G is arbitrary, one obtains an endoscopic linear form $I^{\mathscr{E}}(f)$ instead. This foundation was laid down by Arthur [Art02, Art01, Art03], and extended to the twisted case by Moeglin and Waldspurger [MW16a, MW16b]. Of course, each of these results rests on the solution of the relevant Fundamental Lemmas.

In order to address functoriality beyond the endoscopic cases, one would like a further decomposition of $S(f)$ into primitive linear forms $P(f)$ as described by Arthur in [Art17, §2]. As Langlands first pointed out, this will require special treatment of the contribution of the tempered automorphic representations on the spectral side [Lan04, §1.6]. With this in mind, the first goal of this paper is to convert the contribution of weighted characters in the spectral expansion of $S(f)$,

$$\sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Phi(M,V,\zeta)} b^M(\phi) S_M(\phi, f) d\phi,$$

which are the terms associated to proper Levi subgroups $M \neq G$, into geometric distributions, and similarly for the endoscopic linear form $I^{\mathscr{E}}(f)$. This can be viewed as an important step in generalizing the analysis in [Won19b] for the case of GL(2) to general G. As we have described there, this method is in contrast to other recent work following ideas of [FLN10], whereby the geometric terms are modified instead. The remaining spectral terms corresponding to $M = G$, referred to as the 'purely unitary' part, consist of irreducible unitary characters

$$S_{\text{unit}}(f) = \int_{\Phi(G,V,\zeta)} b^G(\phi) f^G(\phi) d\phi$$

and with discrete part given by

$$S_{\text{disc}}(f) = \sum_{\phi \in \Phi_{\text{disc}}(G,V,\zeta)} b^G_{\text{disc}}(\phi) f^G(\phi).$$

Our results yield modified distributions $\tilde{I}^{\mathscr{E}}(f)$ and $\tilde{S}(f)$, whose spectral expansions consist solely of unweighted characters. We shall thus establish entirely geometric expansions for $I^{\mathscr{E}}_{\text{unit}}(f)$ and $S_{\text{unit}}(f)$, at the cost of the parallel structure that is found in the distributions $I^{\mathscr{E}}(f)$ and $S(f)$. This is the content of Corollary 5.3.

Theorem 1. *The unitary part of the stable trace formula has the geometric expansion*

$$S_{\text{unit}}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M,V,\zeta)} b^M(\delta) \tilde{S}_M(\delta, f)$$

where $\tilde{S}(\delta, f)$ is a family of modified linear forms on G, and similarly for $I^{\mathscr{E}}_{\text{unit}}(f)$.

In applications, it is the discrete part of the trace formula that is of primary interest, which is the invariant form

$$I_{\mathrm{disc}}(f) = \sum_M |W_0^M||W_0^G|^{-1} \sum_{s \in W(M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \mathrm{tr}(M_P(s,0))\mathscr{I}_P(0,f)),$$

with parallel stabiilization

$$I_{\mathrm{disc}}(f) = \sum_G \iota(G', G) S_{\mathrm{disc}}(f).$$

The modified stable linear form $\tilde{S}(f)$, or rather the geometric expansion that we have obtained for $S_{\mathrm{unit}}(f)$ represents a step in this direction. That our methods here are inspired by Arthur's work confirms the analogy described by Arthur between the endoscopic and beyond endoscopic settings, whereby obtaining a geometric expansion of $S_{\mathrm{cusp}}(f)$, the cuspidal tempered contribution to $S_{\mathrm{disc}}(f)$, should be analogous to making the trace formula invariant [Art17, §2].

We emphasize that it is $S_{\mathrm{cusp}}(f)$ that we are most interested in. In order to isolate it, it appears that what are now called basic functions will play a key role in establishing further refinements of the trace formula. Basic functions fall outside of the domain of test functions used in the trace formula, not being compactly supported; extending the noninvariant trace formula to include them is the subject of [FLM11, FL16, Par19]. The second result of this paper is the extension of these latter results to the invariant and stable trace formulas. Let us write $\mathscr{H}(G, V, \zeta)$ for the ζ^{-1}-equivariant Hecke algebra on $G_V = G(F_V)$, taken with respect to a finite set of valuations V of F, and an automorphic character ζ of a fixed induced torus Z of G. Also denote by $\mathscr{C}^\circ(G, V, \zeta)$ the larger space of functions that we have alluded to, containing $\mathscr{H}(G, V, \zeta)$. Then we have the following result, stated in Theorem 7.2.

Theorem 2. *The linear forms $I^{\mathscr{E}}(f)$ and $S(f)$ extend continuously from $\mathscr{H}(G, V, \zeta)$ to $\mathscr{C}^\circ(G, V, \zeta)$.*

In order to establish this, we have to first refine the noninvariant geometric and spectral expansions of $J(f)$ used in [FLM11, FL16], then pass to the invariant form $I(f)$, and finally stabilize it. The most difficult aspects of this passage concern the local behaviour of orbital integrals and characters, much of which have already been established by the fundamental works of Arthur, Moeglin, and Waldspurger. Most crucially, the Langlands-Shelstad transfer, which concerns smooth compactly supported functions and is a consequence of the Fundamental Lemma, extends naturally to the larger Schwartz space. It must be noted that to properly weight the stable trace formula by means of the basic function, the linear form $S(f)$ defined by Arthur is not in the right form. Rather, we have to revisit the noninvariant form $J(f)$ and rework the formula from there, which we shall do in a later paper [Won]. For now, these results taken together represent progress in the direction of a weighted stable trace formula, which we view as a precursor to the r-trace formula.

The proofs of these results draw heavily on Arthur's large body of work on the trace formula. The debt that this paper owes to it will be very clear, and also to the work of Moeglin and Waldspurger, explicating or superseding certain results of Arthur that remain unpublished.

We conclude with a brief outline of the paper. Section 2 introduces the basic notation, and the families of mappings $\iota_M^{\mathscr{E}}$ and τ_M modeled after Arthur's maps ϕ_M.

These maps are used in Section 3 to modify the distributions appearing on either side of the trace formula. The modified distributions satisfy properties similar to the original ones, and in particular they satisfy descent and splitting properties which are proved in Section 4. Having established the local theory, Section 5 provides the global expansions which lead to the modified linear forms $\tilde{I}^{\mathscr{E}}$ and $\tilde{S}$. In Sections 6 and 7 we refine the noninvariant trace formula as a linear form on $\mathscr{C}^\circ(G,V,\zeta)$ to the invariant and stable trace formulae respectively.

2. Maps of distributions

2.1. **Preliminaries.** Let G be a connected reductive group over a field F of characteristic zero. We denote by $\mathscr{L}(M)$ to be the collection of Levi subgroups of G containing M, $\mathscr{L}^0(M)$ the subset of proper Levi subgroups in $\mathscr{L}(M)$, and $\mathscr{P}(M)$ the collection of parabolic subgroups of G containing M. Let F be a global field, and V a finite set of places of F. We have the real vector space $\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbf{R})$, and the set

$$\mathfrak{a}_{M,V} = \{H_M(m) : m \in M(F_V)\}$$

is a subgroup of $\mathfrak{a}_M$, and $F_V = \prod_{v\in V} F_v$. It is equal to $\mathfrak{a}_M$ if V contains an archimedean place, and is a lattice in $\mathfrak{a}_M$ otherwise. The additive character group $\mathfrak{a}^*_{M,V} = \mathfrak{a}^*_M \backslash \mathfrak{a}^\vee_{M,V}$ equals $\mathfrak{a}^*_M$ in the first case, and is a compact quotient of $\mathfrak{a}^*_M$ in the second. Let A_M be the maximal split torus of a Levi subgroup M of G. We then identify the Weyl group of (G, A_M) with the quotient of the normaliser of M by M, thus

$$W^G(M) = \mathrm{Norm}_G(M)/M.$$

If M_0 is a minimal Levi subgroup of G, which we shall assume to be fixed, and denote $\mathscr{L} = \mathscr{L}(M_0), \mathscr{L}^0 = \mathscr{L}^0(M_0)$, and $W^G_0 = W^G(M_0)$.

Let Z be a central induced torus of G over F. Then following [Art99, §3] we define the pair (Z,ζ) where ζ is a character of $Z(F)$ if F is local, and an automorphic character of $Z(\mathbf{A})$ if F is global. Given a finite set of places V, we write $G_V = G(F_V)$ and write ζ_V for the restriction of ζ to the subgroup Z_V of $Z(\mathbf{A})$. We then write G^Z_V for the set of $x \in G_V$ such that $H_G(x)$ lies in the image of the canonical map from $\mathfrak{a}_Z$ to $\mathfrak{a}_G$. We shall assume that V contains the places over which G and ζ are ramified.

The stable trace formula requires that we work in fact with G a K-group as defined in [Art99, §1]. Thus

$$G = \coprod_\alpha G_\alpha \qquad \alpha \in \pi_0(G)$$

is a variety whose connected components G_α are reductive groups over F, equipped with an equivalence class of frames

$$(\psi, u) = \{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \alpha, \beta \in \pi_0(G)\}$$

satisfying natural compatibility conditions. Here $\psi_{\alpha\beta} : G_\alpha \to G_\beta$ in an isomorphism over $\bar{F}$, and $u_{\alpha\beta}$ is a locally constant function from $\Gamma = \mathrm{Gal}(\bar{F}/F)$ to the simply connected cover $G_{\alpha,\mathrm{sc}}$ of the derived group of G_α. Any connected reductive group is a component of an K-group that is unique up to weak isomorphism. It comes with a local product structure

$$G_V = \prod_{v\in V} \coprod_{\alpha_v \in \pi_0(G_v)} G_{v,\alpha_v}.$$

The introduction of K-groups is to steamline certain aspects of the study of connected groups, and the definitions for connected groups will extend to K-groups in a natural way. For example, a central induced torus Z of a K-group G will have central embeddings $Z \xrightarrow{\sim} Z_\alpha \subset Z(G_\alpha)$ for each α, and ζ determines a character ζ_α for each α. We shall call a K-group G quasisplit if it has a connected component that is quasisplit over F.

We shall have to pay special attention to the spaces of functions involved. We write $\mathscr{C}(G,V,\zeta) = \mathscr{C}(G_V^Z,\zeta_V)$ for the space of ζ^{-1}-equivariant Schwartz functions on G_V^Z, which contains the Hecke algebra $\mathscr{H}(G,V,\zeta) = \mathscr{H}(G_V^Z,\zeta_V)$ defined with respect to a choice of maximal compact subgroup K_∞ of G_{V_∞}, where V_∞ denotes the archimedean places in V. If F is a local field, we write $\mathscr{C}(G_v,\zeta_v)$ and $\mathscr{H}(G_v,\zeta_v)$ for the corresponding spaces. There are natural decompositions

$$\mathscr{C}(G_v,\zeta_v) = \bigoplus_{\alpha\in\pi_0(G)} \mathscr{C}(G_\alpha,\zeta_\alpha)$$

and

$$\mathscr{C}(G_V,\zeta_V) = \bigotimes_{v\in V} \mathscr{C}(G_v,\zeta_v),$$

and similarly for the Hecke algebra. We shall write $\mathscr{C}(G_v,\zeta_v) = \mathscr{C}(G,\zeta)$ and $\mathscr{H}(G_v,\zeta_v) = \mathscr{H}(G,\zeta)$ when the context is clear. Now, given $\gamma_V \in G_V^Z$, we have a continuous linear form $f \mapsto f_G$ on $\mathscr{C}(G_V^Z,\zeta_V)$, which can be defined as either the ζ-equivariant orbital integral at the conjugacy class γ_V of G_V^Z,

$$\int_{Z_V} \zeta_V(z) f_G(z\gamma_V)dz$$

where

$$f_G(\gamma_V) = |D(\gamma_V)|^{\frac{1}{2}} \int_{G_{\gamma_V}\cap G_V^Z\backslash G_V^Z} f(x^{-1}\gamma_V x)dx,$$

with $D(\gamma_v)$ the discriminant of γ_v, or as the character $f_G(\pi) = \mathrm{tr}(\pi(f))$ where π is an irreducible character of G_V^Z with central character ζ_V. The functions determine each other, so we may use either definition to form the invariant Schwartz space

$$I\mathscr{C}(G,V,\zeta) = I\mathscr{C}(G_V^Z,\zeta_V) = \{f_G : f \in \mathscr{C}(G_V^Z,\zeta)\}$$

and invariant Hecke space

$$\mathscr{I}(G,V,\zeta) = \mathscr{I}(G_V^Z,\zeta_V) = \{f_G : f \in \mathscr{H}(G_V^Z,\zeta)\},$$

equipped with topologies such that the surjective map $f \mapsto f_G$ is open and continuous. We shall write $I(f) = \hat{I}(f_G)$ for any invariant linear form I that lies in the image of the transpose of the map $f \to f_G$.

2.2. **Invariant characters.** We define $\mathscr{F}(G_V^Z,\zeta_V)$ to be the space of finite complex linear combinations of irreducible characters on G_V^Z with Z_V-central character equal to ζ_V, and identify an element $\pi \in \mathscr{F}(G_V^Z,\zeta_V)$ with the linear form

$$f \mapsto f_G(\pi) = \mathrm{tr}(\pi(f)) = J_G(\pi, f)$$

on $\mathscr{H}(G_V^Z,\zeta_V)$. The space $\mathscr{F}(G_V^Z,\zeta_V)$ has a canonical basis $\Pi(G_V^Z,\zeta_V)$ consisting of irreducible characters with Z_V-central character equal to ζ_V, and we can form a chain of subsets

$$\Pi_{\mathrm{disc}}(G_V^Z,\zeta_V) \subset \Pi_{\mathrm{unit}}(G_V^Z,\zeta_V) \subset \Pi_{\mathrm{temp}}(G_V^Z,\zeta_V) \subset \Pi(G_V^Z,\zeta_V)$$

consisting of characters that are discrete, unitary, and tempered respectively. We consider $\mathfrak{a}^*_{G,Z}$ as the subspace of linear forms on $\mathfrak{a}_G$ that are trivial on the image of $i\mathfrak{a}_Z$ in $\mathfrak{a}_G$. Then there is an action of $\mathfrak{a}^*_{G,Z}$ on $\Pi_{\text{unit}}(G_V, \zeta_V)$, given by $\lambda : \pi \to \pi_\lambda$, whose orbits can be identified with $\Pi_{\text{unit}}(G^Z_V, \zeta_V)$.

The Paley-Wiener space of functions on $\Pi_{\text{temp}}(G_V, \zeta_V) \times \mathfrak{a}_{M,V}$ is a subspace of $\mathscr{I}(G, V, \zeta)$. There is a continuous linear map from $\mathscr{H}(G, V, \zeta)$ to $\mathscr{I}(G, V, \zeta)$ given by $f \to f_G$ where

$$f_G(\pi, X) = \int_{i\mathfrak{a}^*_{G,V}} \text{tr}(\pi_\lambda(f))\text{e}^{\lambda(X)} d\lambda,$$

for $\pi \in \Pi_{\text{temp}}(G_V, \zeta_V)$ and $X \in \mathfrak{a}_{G,V}$. Let K be a maximal compact subgroup of $G)V$, Γ a finite subset of $\Pi(K)$, and N a positive integer. Also fix a positive function $||x||$ for $x \in G_V$ as in [Art89a, §11], and define $\mathscr{H}_N(G, V, \zeta)_\Gamma$ to be the space of smooth functions on G_V which are supported on the set of $x \in G_V$ such that $\log ||x|| \le N$. It has a topology given by the seminorms

$$||f||_D = \sup_{x \in G_V} |Df(x)|, \qquad f \in \mathscr{H}_N(G, V, \zeta)_\Gamma$$

where D is a differential operator on G_{V_∞}, and V_∞ is the subset of archimedean valuations in V. We then define the topological direct limits

$$\mathscr{H}(G, V, \zeta)_\Gamma = \varinjlim_N \mathscr{H}_N(G, V, \zeta)_\Gamma$$

$$\mathscr{I}(G, V, \zeta)_\Gamma = \varinjlim_N \mathscr{I}_N(G, V, \zeta)_\Gamma$$

with $\mathscr{I}_N(G, V, \zeta)_\Gamma$ defined analogously. We shall be interested in the larger spaces

$$\mathscr{H}_{\text{ac}}(G, V, \zeta) = \varinjlim_\Gamma \mathscr{H}_{\text{ac}}(G, V, \zeta)_\Gamma$$

$$\mathscr{I}_{\text{ac}}(G, V, \zeta) = \varinjlim_\Gamma \mathscr{I}_{\text{ac}}(G, V, \zeta)_\Gamma.$$

Here $\mathscr{H}_{\text{ac}}(G, V, \zeta)_\Gamma$ is the space of ζ^{-1}-equivariant functions f on G_V such that for every $b \in C^\infty_c(\mathfrak{a}_{G,V})$ the function

$$f^b(x) = f(x)b(H_G(x))$$

belongs to $\mathscr{H}(G, V, \zeta)_\Gamma$. We may also view it as the space of uniformly smooth ζ^{-1}-equivariant functions f on G_V such that for any $X \in \mathfrak{a}_{G,V}$, the restriction of f to the preimage of X in G_V is compactly supported. By uniformly smooth, we mean that the function f is bi-invarianat under an open compact subgroup of G_V. Similarly, $\mathscr{I}_{\text{ac}}(G, V, \zeta)_\Gamma$ is the space of ζ^{-1}-equivariant functions ϕ on $\Pi_{\text{temp}}(G_V, \zeta_V) \times \mathfrak{a}_{G,V}$ such that for every $b \in C^\infty_c(\mathfrak{a}_{G,V})$ the function

$$\phi^b(\pi, X) = \phi(\pi, X)b(X)$$

belongs to $\mathscr{I}(G, V, \zeta)_\Gamma$. We may also regard an element of $\mathscr{I}_{\text{ac}}(G, V, \zeta)$ as a function on the set of conjugacy classes of G_V by means of orbital integrals. The map $f \to f_G$ then extends to a continuous map from $\mathscr{H}_{\text{ac}}(G, V, \zeta)$ to $\mathscr{I}_{\text{ac}}(G, V, \zeta)$. More generally, there is a function

$$f_M(\pi, X) = \int_{i\mathfrak{a}^*_{M,S}} \text{tr}(\mathscr{I}_P(\pi_\lambda, f))\text{e}^{-\lambda(X)} d\lambda,$$

where $P \in \mathscr{P}(M), \pi \in \Pi_{\text{temp}}(M_V^Z, \zeta_V)$, and $X \in \mathfrak{a}_{M,V}$. Here $\mathscr{I}_P(\pi_\lambda)$ is the representation in $\Pi_{\text{temp}}(G_V, \zeta_V)$ induced from π_λ. Then $f \to f_M$ is a continuous linear map from $\mathscr{H}_{\text{ac}}(G, V, \zeta)$ to $\mathscr{I}_{\text{ac}}(M, V, \zeta)$.

2.3. **Invariant weighted characters.** We now want to define the distributions that occur in the spectral side of the trace formula. Recall the canonically normalised weighted character introduced in [Art98, §2],

$$J_M(\pi, f) = \operatorname{tr}(\mathscr{M}_M(\pi, P)\mathscr{I}_P(\pi, f))$$

where $\mathscr{I}_P(\pi)$ is the induced representation of G obtained from $\pi \in \Pi_{\text{unit}}(M_V^Z, \zeta_V)$ and $\mathscr{M}_M(\pi, P)$ is an operator constructed in a certain way from unnormalized intertwining operators, which we shall describe below. We can then define for any pair (π, X) in $\Pi(M_V^Z, \zeta_V) \times \mathfrak{a}_{M,V}$, the distribution

$$J_M(\pi, X, f) = \int_{i\mathfrak{a}_M^*} J_M(\pi_\lambda, f)\mathrm{e}^{-\lambda(X)} d\lambda, \qquad f \in \mathscr{H}(G, V, \zeta) \tag{2.1}$$

if $J_M(\pi_\lambda, f)$ is regular for $\lambda \in i\mathfrak{a}_M^*$, for example, if π is unitary. Whereas for more general representations $\pi \in \Pi(M_V^Z, \zeta_V)$ we define

$$J_M(\pi, X, f) = \sum_{P \in \ P(M)} \omega_P J_M(\pi_{\varepsilon_P}, X, f)\mathrm{e}^{-\varepsilon_P(X)}$$

where for each $P \in \mathscr{P}(M)$, ε_P is a small point in the positive chamber $(\mathfrak{a}_P^*)^+$ and

$$\omega_P = \operatorname{vol}(\mathfrak{a}_P^+ \cap B)\operatorname{vol}(B)^{-1},$$

where B is a ball in $\mathfrak{a}_M$ centered at the origin. The two definitions are compatible by a contour shift. More generally, we define the function

$$J_{M,\mu}(\pi, X, f) = J_M(\pi_\mu, X, f)\mathrm{e}^{-\mu(X)} = \int_{\mu + i\mathfrak{a}_M^*} J_M(\pi_\lambda, f)\mathrm{e}^{-\lambda(X)} d\lambda,$$

which is locally constant as a function of $\mu \in \mathfrak{a}_M^*$ on the complement of a finite set of affine hyperplanes.

The invariant weighted characters are then defined inductively by the relation

$$I_M(\pi, X, f) = J_M(\pi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{I}_M^L(\pi, X, \phi_L(f)),$$

where the map

$$\phi_M : \mathscr{H}_{\text{ac}}(G, V, \zeta) \to \mathscr{I}_{\text{ac}}(M, V, \zeta) \tag{2.2}$$

is based on the construction in [Art98, §2] using normalised weighted characters, which we briefly recall here. Suppose first that $\tilde{f}$ belongs to the Schwartz space $\mathscr{C}(G_V, \zeta_V)$. Then $\phi_M(\tilde{f})$ is defined to be the function on $\Pi_{\text{temp}}(M_V, \zeta_V)$ such that

$$\phi_M(\tilde{f}, \tilde{\pi}) = \operatorname{tr}(\mathscr{M}_M(\tilde{\pi}, P)\mathscr{I}_P(\tilde{\pi}, \tilde{f}))$$

for $P \in \mathscr{P}(M)$ and $\tilde{\pi} \in \Pi_{\text{temp}}(M_V, \zeta_V)$. The operator

$$\mathscr{M}_M(\tilde{\pi}, P) = \lim_{\Lambda \to 0} \sum_{Q \in \mathscr{P}(M)} (\otimes_{v \in V} \mathscr{M}_Q(\Lambda, \tilde{\pi}_v, P))\theta_Q(\Lambda)^{-1}, \tag{2.3}$$

with

$$\theta_Q(\Lambda) = \operatorname{vol}(\mathfrak{a}_M^G / Z(\Delta_Q^\vee))^{-1} \prod_{\alpha \in \Delta_Q} \Lambda(\alpha^\vee),$$

is defined as part of Arthur's theory of (G,M)-families, where the relevant (G,M)-family is a tensor product of (G,M)-families

$$\mathscr{M}_Q(\Lambda,\pi_v,P)=\mu_Q(\Lambda,\pi_v,P)\mathscr{I}_Q(\Lambda,\pi_v,P),\qquad Q\in\mathscr{P}(M),\ \Lambda\in i\mathfrak{a}_M^*$$

defined for π_v in general position. Here

$$\mu_Q(\Lambda,\pi_v,P)=\mu_{Q|P}(\pi)^{-1}\mu_{Q|P}(\pi_{\frac{1}{2}\Lambda}),$$

and the functions $\mu_{Q|P}(\pi_\Lambda)$ are Harish-Chandra's canonical family of μ-functions. If f and π are restrictions of $\tilde{f}$ and $\tilde{\pi}$ to G_V^Z and M_V^Z respectively, we set

$$\phi_M(f,\pi)=\int_{i\mathfrak{a}^*_{M,Z}}\phi_M(\tilde{f},\tilde{\pi}_\lambda)d\lambda.$$

From [Art02, §2] and [Art89a, §3] we know that ϕ_M maps $\mathscr{H}(G,V,\zeta)$ continuously to $\mathscr{I}(M,V,\zeta)$. If π is tempered, the inductive definition shows that

$$I_M(\pi,X,f)=\begin{cases}f_G(\pi,X), & M=G\\ 0 & M\neq G.\end{cases}\tag{2.4}$$

Moreover, if we consider only tempered π, then we may take f to be Schwartz, in which case we have that ϕ_M maps $\mathscr{C}(G_V^Z,\zeta_V)$ continuously to $I\mathscr{C}(M_V^Z,\zeta_V)$.

If π is nontempered, on the other hand, then the linear form $I_M(\pi,X,f)$ is a finite sum of residues of weighted characters in the complex domain. More generally, we set

$$I_{M,\mu}(\pi,X,f)=I_M(\pi_\mu,X,f)\mathrm{e}^{-\mu(X)},\qquad \mu\in\mathfrak{a}_M^*$$

satisfying

$$I_{M,\mu}=\sum_{P\in\mathscr{P}(M)}\omega_P I_{M,\mu+\varepsilon_P}(\pi,X,f).$$

It is again locally constant on the complement of a finite set of affine hyperplanes [Art88a, Lemma 3.2]. Let us also agree to write

$$J_M(\pi,f)=J_M(\pi_\mu,0,f)$$

and

$$I_M(\pi,f)=I_M(\pi_\mu,0,f)$$

for $f\in\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$.

2.4. **Stable weighted characters.** We say a distribution on G_V^Z is stable if it lies in the closed linear span of strongly regular, stable orbital integrals

$$f^G(\delta_V)=\sum_{\gamma_V}f_G(\gamma_V)$$

where δ_V is any strongly regular, stable conjugacy class in G_V^Z, and the sum is taken over the finite set of conjugacy classes in the stable conjugacy class δ_V containing γ_V. Let us write $S\mathscr{F}(G_V^Z,\zeta_V)$ for the subspace of ζ_V-equivariant stable distributions in $\mathscr{F}(G_V^Z,\zeta_V)$, and $\mathscr{S}(G_V^Z,\zeta_V)$ for the subspace of $\mathscr{I}(G_V^Z,\zeta)$ spanned by stable orbital integrals.

Referring to [Art02, §6], there is an endoscopic basis $\Phi^{\mathscr{E}}(G_V^Z,\zeta_V)$ of $\mathscr{F}(G_V^Z,\zeta_V)$, and a subset

$$\Phi(G_V^Z,\zeta_V)=\Phi^{\mathscr{E}}(G_V^Z,\zeta_V)\cap S\mathscr{F}(G_V^Z,\zeta_V)$$

that forms a basis of $S\mathscr{F}(G_V^Z,\zeta_V)$. They are defined in terms of abstract bases $\Phi_{\mathrm{ell}}(M_V^Z,\zeta_V)$ of the space $\mathscr{S}_{\mathrm{cusp}}(M_V^Z,\zeta_V)$ of stable orbital integrals of cuspidal

ζ_V-equivariant functions on M_V^Z, where a function is called cuspidal if it vanishes on any properly induced element. The basis $\Phi_{\mathrm{ell}}(M,\zeta)$ plays the formal role of cuspidal Langlands parameters. Let $i\mathfrak{a}_G^*$ be the imaginary dual space of $\mathfrak{a}$. The action $\phi \to \phi_\lambda$ of $i\mathfrak{a}_G^*$ then makes $\Phi_{\mathrm{ell}}(M_V^Z,\zeta_V)$ into a disjoint union of compact tori of the form $i\mathfrak{a}_{G,\phi}^* = i\mathfrak{a}_M^*/\mathfrak{a}_{M,\phi}^\vee$, where $\mathfrak{a}_{M,\phi}^\vee$ is the stabiliser of ϕ in $i\mathfrak{a}_M^*$. We note that $\mathscr{S}_{\mathrm{cusp}}(M_V^Z,\zeta_V)$ is the Paley-Wiener space on $\Phi_{\mathrm{ell}}(M_V^Z,\zeta_V)$, in the sense that its elements are supported on finitely many connected components, and on the component of any ϕ, pull back to a finite Fourier series on $i\mathfrak{a}_{M,\phi}^*$.

Let $\mathscr{E}(G)$ be the set of isomorphism classes of endoscopic data $(G',\mathcal{G}',s',\xi')$ for G over F that are locally relevant to G, which is to say that for every v, $G'(F_v)$ has a strongly G-regular element that is an image of some strongly G-regular conjugacy class in G_v. We shall write $\mathscr{E}(G,V)$ for the subset of elements $G' \in \mathscr{E}(G)$ that are unramified outside of V, and also $\mathscr{E}_{\mathrm{ell}}(G)$ and $\mathscr{E}_{\mathrm{ell}}(G,V)$ for the subsets of $\mathscr{E}(G)$ and $\mathscr{E}(G,V)$ respectively that are elliptic over F. Finally, if G' belongs to $\mathscr{E}_{\mathrm{ell}}(G)$, we fix a central extension $\tilde{G}'$ and an L-embedding $\tilde{\xi}' : \mathcal{G}' \to {}^L\tilde{G}'$ satisfying the conditions of [Art96, Lemma 2.1].

Now let M be a fixed Levi subgroup of the K-group G over a global field F, and let M' represent an elliptic endoscopic datum $(M',\mathcal{M}',s'_M,\xi'_M)$ of M [Art02, §6]. We shall assume that $\mathcal{M}'$ is an L-subgroup of ${}^LM$ and that ξ'_M is the identity embedding of $\mathcal{M}'$ into M'. We write $\mathscr{E}_{M'}(G)$ for the set of endoscopic data $(G',\mathcal{G}',s',\xi')$ for G over F, taken up to translation by s' in $Z(\hat{G})^\Gamma$, in which s' lies in $s'_M Z(\hat{M})^\Gamma$, $\hat{G}'$ is the connected centralizer of s' in $\hat{G}$, $\mathcal{G}'$ equals $\mathcal{M}'\hat{G}'$, and ξ' is the identity embedding of $\mathcal{G}'$ into ${}^LG$.

The invariant weighted characters are then stabilised as follows. Given a pair (ϕ,X) in $\Phi^{\mathscr{E}}(M_V^Z,\zeta_V)\times\mathfrak{a}_M^Z$, define the invariant linear form

$$I_M(\phi,X,f) = \sum_{\pi\in\Pi(M_V^Z,\zeta_V)} \Delta_M(\phi,\pi) I_M(\pi,X,f)$$

where $\Delta_M(\phi,\pi)$ is the spectral transfer factor defined in [Art02, §5], and for f in $\mathscr{H}(G,V,\zeta)$. If G is general, define for $\phi' \in \Phi((\tilde{M}'_V)^{\tilde{Z}'},\tilde{\zeta}'_V)$ the linear form

$$I_M^{\mathscr{E}}(\phi',X,f) = \sum_{G'\in\mathscr{E}^0_{M'}(G)} \iota_{M'}(G,G')\hat{S}_{\tilde{M}'}^{\tilde{G}'}(\phi',f') + \varepsilon(G) S_M^G(M',\phi',X,f) \tag{2.5}$$

with the requirement that

$$I_M^{\mathscr{E}}(\phi',X,f) = I_M(\phi,X,f)$$

in the case that G is quasisplit. Here $\varepsilon(G)$ is equal to 1 if G is quasisplit and 0 otherwise. If ϕ' and M' are locally relevant to M, Proposition 6.4 of [Art02] allows us to define the endoscopic form $I_M^{\mathscr{E}}(\pi,X,f)$ by inversion of the formula

$$I_M^{\mathscr{E}}(\phi',X,f) = \sum_{\pi\in\Pi(M_V^Z,\zeta_V)} \Delta_M(\phi',\pi) I_M^{\mathscr{E}}(\pi,X,f).$$

The distributions $I_M^{\mathscr{E}}(\pi,X,f)$ and $S_M^G(M',\phi',X,f)$ are then the main objects appearing in the spectral side of endoscopic and stable trace formulas respecitvely. In the case that G is quasisplit and $M' = M^*$,

$$S_M^G(\phi,X,f) = S_M^G(M^*,\phi^*,X,f).$$

These stable and endoscopic forms satisfy the usual descent and splitting formulas proved in [MW16b, X.4], parallel to those satisfied by the invariant forms [Art88a, §8–9]. Also, we shall set

$$I^{\mathscr{E}}_M(\pi, f) = I^{\mathscr{E}}_M(\pi_\mu, 0, f)$$

and

$$S_M(\phi, f) = S_M(\phi_\mu, 0, f)$$

as before.

2.5. **Maps of distributions.** We shall now introduce two new families of maps which will be endoscopic and stable analogues of ϕ_M. We first define the space

$$\mathscr{S}_{\mathrm{ac}}(G(F_V)) = \varinjlim_{\Gamma} \mathscr{S}_{\mathrm{ac}}(G(F_V))_\Gamma$$

parallel to the definition of $\mathscr{I}_{\mathrm{ac}}(G, V, \zeta)$ above by replacing $\Pi_{\mathrm{temp}}(G(F_V))$ with $\Phi_{\mathrm{temp}}(G(F_V))$. Then if G is arbitrary, we would like to define for each $M \in \mathscr{L}$ a map

$$\iota^{\mathscr{E}}_M : \mathscr{H}_{\mathrm{ac}}(G, V, \zeta) \to \mathscr{I}_{\mathrm{ac}}(M, V, \zeta)$$

such that

$$\iota^{\mathscr{E}}_M(f, \pi, X) = I^{\mathscr{E}}_M(\pi, X, f), \qquad \pi \in \Pi(M(F_V)), X \in \mathfrak{a}_{M,V},$$

and if G is quasisplit, we would like similarly a map

$$\tau_M : \mathscr{H}_{\mathrm{ac}}(G, V, \zeta) \to \mathscr{S}_{\mathrm{ac}}(M, V, \zeta)$$

such that

$$\tau_M(f, \phi, X) = S_M(\phi, X, f), \qquad \phi \in \Phi(M(F_V)), X \in \mathfrak{a}_{M,V}.$$

Recall that the map ϕ_M is defined initially for tempered representations π and then extended by analytic continuation. The definitions of the weighted characters for general elements π and ϕ above implicitly rely on this property.

We shall also define in this case the more general maps $\iota^{\mathscr{E}}_{M,\mu}$ and $\tau_{M,\mu}$ for $\mu \in \mathfrak{a}^*_M$ given by

$$\iota^{\mathscr{E}}_{M,\mu}(f, \pi, X) = I^{\mathscr{E}}_M(\pi_\mu, X, f)\mathrm{e}^{-\mu(X)}$$
$$\tau_{M,\mu}(f, \phi, X) = S_M(\phi_\mu, X, f)\mathrm{e}^{-\mu(X)}.$$

If $\mu = 0$, we shall omit the subscript μ and simply write $\phi_{M,0} = \phi_M$, for example. We shall provide inductive constructions of these maps using $\phi_{M,\mu}$ in the following proposition.

Proposition 2.1. *For each $\mu \in \mathfrak{a}^*_M$, there exist continuous linear maps $\iota^{\mathscr{E}}_{M,\mu}$ and $\tau_{M,\mu}$ from $\mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$ to $\mathscr{I}_{\mathrm{ac}}(M, V, \zeta)$ and $\mathscr{S}_{\mathrm{ac}}(M, V, \zeta)$ respectively.*

Proof. To study the properties of these maps, it will be useful for us to define an invariant form of the map ϕ_M,

$$\iota_M : \mathscr{H}_{\mathrm{ac}}(M, V, \zeta) \to \mathscr{I}_{\mathrm{ac}}(M, V, \zeta)$$

whose value at a point (π, X) in $\Pi(M(F_V)) \times \mathfrak{a}_{M,V}$ is given by

$$\iota_M(f, \pi, X) = I_M(\pi, X, f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G, V, \zeta),$$

and which can be equivalently defined by the inductive formula

$$\iota_M(f, \pi, X) = J_M(\pi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{I}^L_M(\pi, X, \phi_L(f)),$$

exactly as for the invariant weighted character. It follows that ι_M vanishes if π belongs to $\Pi_{\mathrm{temp}}(M(F_V))$ and $M \neq G$, and for nontempered representations it is defined by analytic continuation. Setting

$$\iota_{M,\mu}(f,\pi,X) = I_M(\pi_\mu, X, f)\mathrm{e}^{-\mu(X)}, \tag{2.6}$$

it follows from the inductive definition and properties of the transform ϕ_M that ι_M also maps $\mathscr{H}_{\mathrm{ac}}(M,V,\zeta)$ continuously to $\mathscr{I}_{\mathrm{ac}}(M,V,\zeta)$. Indeed, this can also be shown directly using the expansion of $I_{M,\mu}(\pi,X,f)$ in terms of residues of $J_M(\pi_\lambda,f)$ described in [Art89b, Theorem 4.1], and examining the continuity argument of $\phi_{M,\mu}$ in [Art89a, Theorem 12.1].

Now, given $\phi' \in \Phi((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$ we can then construct the map $\iota^{\mathscr{E}}_M$ by the inductive definition

$$\iota^{\mathscr{E}}_M(f,\phi',X) = \sum_{G' \in \mathscr{E}^0_{M'}(G)} \iota_{M'}(G,G')\hat{S}^{\tilde{G}'}_{\tilde{M}'}(f',\phi',X) + \varepsilon(G)\tau_M(M',f,\phi',X)$$

with the supplementary requirement that

$$\iota^{\mathscr{E}}_M(f,\phi',X) = \iota_M(f,\phi,X)$$

in the case that G is quasisplit. Certainly this agrees with our definition of $\iota^{\mathscr{E}}_M$ above. Also, if $G' = G^*$ we again set

$$\tau_M(f,\phi,X) = \tau_M(M^*,f,\phi^*,X).$$

If ϕ' and M' are locally relevant to M, we define the endoscopic form $I^{\mathscr{E}}_M(f,\pi,X)$ by the relation

$$\iota^{\mathscr{E}}_M(f,\phi',X) = \sum_{\pi \in \Pi(M^Z_V,\zeta_V)} \Delta_M(\phi',\pi)\iota^{\mathscr{E}}_M(f,\pi,X)$$

and the inversion formula

$$\sum_{\pi \in \Pi(M^Z_V,\zeta_V)} \Delta_M(\phi,\pi)\Delta_M(\pi,\phi_1) = \delta(\phi,\phi_1), \qquad \phi,\phi_1 \in \Phi^{\mathscr{E}}(M^Z_V,\zeta_V)$$

described in [Art02, §5], where $\delta(\phi,\phi_1) = 1$ if $\phi = \phi_1$ and 0 otherwise. It follows then from the inductive definitions that the maps $\iota^{\mathscr{E}}_{M,\mu}(f,\pi,X)$ and $\tau_{M,\mu}(f,\pi,X)$ are continuous. $\square$

As with ϕ_M, if we consider only tempered π and ϕ respectively, then we can allow for f belonging to the larger space $\mathscr{C}(G^Z_V,\zeta_V)$. In that case $\iota^{\mathscr{E}}_M$ would be a continuous map from $\mathscr{C}(G^Z_V,\zeta_V)$ to $I\mathscr{C}(G^Z_V,\zeta_V)$, and τ_M from $\mathscr{C}(G^Z_V,\zeta_V)$ to the stably invariant Schwarz space $S\mathscr{C}(G^Z_V,\zeta_V)$ generated by stable ζ-equivariant orbital integrals of functions $f \in \mathscr{C}(G^Z_V,\zeta_V)$.

3. Modified distributions

3.1. **Weighted characters.** The maps $\iota^{\mathscr{E}}_M$ and τ_M that we have just constructed allow us to modify the endoscopic and stable weighted characters in a manner similar to the invariant linear forms $I_M(\pi,X,f)$. Define inductively the modified linear forms

$$\tilde{I}^{\mathscr{E}}_M(\pi,X,f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$$

by setting

$$\tilde{I}^{\mathscr{E}}_M(\pi, X, f) = I^{\mathscr{E}}_M(\pi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L,\mathscr{E}}_M(\pi, X, \iota^{\mathscr{E}}_L(f)), \tag{3.1}$$

if G is arbitrary, and

$$\tilde{S}^G_M(\phi, X, f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$$

by setting

$$\tilde{S}^G_M(\phi, X, f) = S^G_M(\phi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{S}}^L_M(\phi, X, \tau_L(f)), \tag{3.2}$$

in the case that G is quasisplit. As usual, if $X = 0$ we shall simply write $\tilde{I}^{\mathscr{E}}_M(\pi, f)$ and $\tilde{S}^G_M(\phi, f)$. We shall also define more generally $\tilde{S}_M(M', \phi', X, \tau_L(f))$ in a similar manner, with

$$\tilde{S}^G_M(\phi, X, f) = \tilde{S}^G_M(M^*, \phi^*, X, \tau_L(f)).$$

Where the context is clear, we shall omit the superscript and write simply $\tilde{S}_M = \tilde{S}^G_M$.We then have the following analogue of (2.4).

Lemma 3.1. *We have that*

$$\tilde{I}^{\mathscr{E}}_M(\pi, X, f) = \begin{cases} f^G(\pi, X), & \text{if } M = G, \\ 0, & \text{if } M \neq G \end{cases}$$

if G is not quasiplit, and

$$\tilde{S}_M(\phi, X, f) = \begin{cases} f^G(\phi, X), & \text{if } M = G, \\ 0, & \text{if } M \neq G \end{cases}$$

if G is quasisplit.

Proof. First, suppose that G is arbitrary. If $M = G$, then $\tilde{I}^{\mathscr{E}}_M(\pi, X, f)$ reduces to $I^{\mathscr{E}}_G(\pi, X, f) = f^G(\pi, X)$ by definition. If $M \neq G$, the definitions imply that

$$I^{\mathscr{E}}_M(\pi, X, f) = \iota^{\mathscr{E}}_M(f, \pi, X) = \hat{\tilde{I}}^{\mathscr{E}}_M(\pi, X, f),$$

and the required idenitty follows inductively from (3.1).

Similarly, if G is quasiplit, then $\tilde{S}_M(\phi, X, f)$ reduces to $S_G(\phi, X, f) = f^G(\phi, X)$ if $M = G$. Otherwise, we have that

$$S_M(\phi, X, f) = \tau_M(f, \phi, X) = \hat{\tilde{S}}_M(\phi, X, f)$$

and the result then follows inductively from (3.2). □

This simple result will be the basis for our modification of the endoscopic and stable trace formulae, allowing us to isolate the unitary part of the spectral expansions. Recall that the analogous property for the invariant weighted characters $I_M(\pi, X, f)$ in (2.4) holds only for tempered representations π, whereas if π is not tempered they are defined by analytic continuation. In contrast, the lemma here for modified distributions above holds in the nontempered case also as implicitly we have depended on the definition of $I_M(\pi, X, f)$ for general π.

As with the stable and invariant weighted characters, the distributions $\tilde{I}^{\mathscr{E}}_M(\pi, X, f)$ and $\tilde{S}_M(\phi, X, f)$ do not assume too many values. Set

$$\tilde{I}^{\mathscr{E}}_{M,\mu}(\pi, X, f) = \tilde{I}^{\mathscr{E}}_M(\pi_\mu, X, f)\mathrm{e}^{-\mu(X)}$$

if G is arbitrary, and

$$\tilde{S}_{M,\mu}(\phi, X, f) = \tilde{S}_M(\phi_\mu, X, f)\mathrm{e}^{-\mu(X)}$$

if G is quasisplit, where $\mu \in \mathfrak{a}_M^*$.

Lemma 3.2. (a) *As functions of μ, $\tilde{I}^{\mathscr{E}}_{M,\mu}(\pi, X, f)$ and $\tilde{S}_{M,\mu}(\phi, X, f)$ are locally constant on the complement of a finite set of hyperplanes of the form $\mu(\alpha^\vee) = N$ for $N \in \mathbf{R}$ and α a root of (G, A_M).*

(b) *For each $P \in \mathscr{P}(M)$, given a small point ϵ_P in the chamber $(\mathfrak{a}_P^*)^+$, then*

$$\tilde{I}^{\mathscr{E}}_{M,\mu} = \sum_{P \in \mathscr{P}(M)} \omega_P \tilde{I}^{\mathscr{E}}_{M,\mu+\epsilon_P}(\pi, X, f)$$

and

$$\tilde{S}_{M,\mu} = \sum_{P \in \mathscr{P}(M)} \omega_P \tilde{S}_{M,\mu+\epsilon_P}(\phi, X, f).$$

Proof. Extending the definitions (3.1) and (3.2) to

$$\tilde{I}^{\mathscr{E}}_{M,\mu}(\pi, X, f) = I^{\mathscr{E}}_{M,\mu}(\pi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L,\mathscr{E}}_{M,\mu}(\pi, X, \iota^{\mathscr{E}}_L(f)) \tag{3.3}$$

and

$$\tilde{S}_{M,\mu}(\phi, X, f) = S_{M,\mu}(\phi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{S}}^{L}_{M,\mu}(\phi, X, \tau_L(f))$$

respectively, where

$$I^{\mathscr{E}}_{M,\mu}(\pi, X, f) = I^{\mathscr{E}}_M(\pi_\mu, X, f)\mathrm{e}^{-\mu(X)}$$

and

$$S_{M,\mu}(\phi, X, f) = S_M(\phi_\mu, X, f)\mathrm{e}^{-\mu(X)},$$

we see from the inductive definitions that the result will follow if we can establish the corresponding statement of (a) for $I^{\mathscr{E}}_{M,\mu}(\pi, X, f)$ and $S_{M,\mu}(\phi, X, f)$, which in turn follows from the inductive definition (2.5) and the corresponding statement for $I_{M,\mu}(\pi, X, f)$ (see also [MW16b, XI.6]).

On the other hand, assume inductively that (b) holds if G is replaced by any element $L \in \mathscr{L}^0(M)$. Then

$$\hat{\tilde{I}}^{L,\mathscr{E}}_{M,\mu}(\pi, X, \iota^{\mathscr{E}}_L(f)) = \sum_{R \in \mathscr{P}^L(M)} \omega_R \hat{\tilde{I}}^{L,\mathscr{E}}_{M,\mu+\epsilon_R}(\pi, X, \iota^{\mathscr{E}}_L(f))$$

and applying assertion (a) to L we see that this may be written as

$$\sum_{P \in \mathscr{P}(M)} \omega_P \hat{\tilde{I}}^{L,\mathscr{E}}_{M,\mu+\epsilon_P}(\pi, X, \iota^{\mathscr{E}}_L(f)).$$

The assertion then follows from the corresponding statement for $I^{\mathscr{E}}_M(\pi_\mu, X, f)$ and (3.3). The proof for $\tilde{S}_{M,\mu}$ follows by the same argument. $\square$

Let us also define inductively the modified invariant linear form

$$\tilde{I}_M(\pi, X, f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$$

by setting

$$\tilde{I}_M(\pi, X, f) = I_M(\pi, X, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L}_M(\pi, X, \iota_L(f)).$$

Then we have the following property as an immediate consequence of Local Theorem $2'$ of [Art02], which will be needed for the global expansion.

Lemma 3.3. *Let V be a finite set of valuations such that G and ζ are unramified outside of V.*

(a) *if G is arbitrary,*

$$\tilde{I}_M^{\mathscr{E}}(\pi, X, f) = \tilde{I}_M(\pi, X, f), \qquad \pi \in \Pi(M_V^Z, \zeta_V),\ f \in \mathscr{H}(G, V, \zeta)$$

(b)*Suppose that G is quasisplit, and that ϕ' belongs to $\Phi(\tilde{M}', \tilde{\zeta}')$ for some $M' \in \mathscr{E}_{\mathrm{ell}}(M)$. Then the linear form*

$$f \to \tilde{S}_M^G(M', \phi', X, f), \qquad f \in \mathscr{H}(G, V, \zeta)$$

vanishes unless $M' = M^*$, in which case it is stable.

3.2. **Supplementary maps.** In order to study these distributions, we shall have to define a family of supplementary maps. For any set of hyperplanes Φ in $\mathfrak{a}$, let $\mathscr{C}$ be the set of open connected components whose union is the complement of Φ in $\mathfrak{a}$. Then given $c \in \mathscr{C}$ and $X \in \mathfrak{a}$, we set

$$\omega(c, X) = \mathrm{vol}(c \cap B_X)\mathrm{vol}(B_X)^{-1}$$

where B_X is a small ball in $\mathfrak{a}$ centred at X. The function vanishes for any X outside the closure of c, and is locally constant on the strata of $\mathfrak{a}$ defined by the intersection of planes in Φ. We then define $\mathscr{H}^\Phi(G, V, \zeta)$ to be the space of functions f on G_V^Z such that

$$f(x) = \sum_{c \in \mathscr{C}} \omega(c, H_G(x)) f_c(x),$$

for $f_c \in \mathscr{H}(G, V, \zeta)$, and similarly $\mathscr{I}^\Phi(G, V, \zeta)$ to the space of functions g on $\Pi_{\mathrm{temp}}(G_V^Z, \zeta_V) \times \mathfrak{a}_G^Z$ of the form

$$g(\pi, X) = \sum_{c \in \mathscr{C}} \omega(c, X) g_c(\pi, X),$$

for $g_c \in \mathscr{I}(G, V, \zeta)$, and thirdly, $\mathscr{S}^\Phi(G, V, \zeta)$ to the space of functions h on $\Phi_{\mathrm{temp}}(G_V^Z, \zeta_V) \times \mathfrak{a}_G^Z$ of the form

$$h(\phi, X) = \sum_{c \in \mathscr{C}} \omega(c, X) h_c(\phi, X),$$

for $h_c \in \mathscr{S}(G, V, \zeta)$.

These spaces can be topologised in manner similar to the argument in [Art88a, §4] for the Hecke algebra. For example, we define for any positive integer N and finite subset Γ of $\Pi(K)$ the space $\mathscr{H}_N(G, V, \zeta)_\Gamma$ of smooth ζ-equivariant functions on $G(F_V)$ supported on the set

$$\{x \in G(F_V) : \log ||x|| \le N\}$$

and which transform on each side under K according to representations in Γ. Its topology is given by the seminorms

$$||f||_D = \sup_{x \in G(F_V)} |Df(x)|, \qquad f \in \mathscr{H}_N(G, V, \zeta)_\Gamma,$$

where D is a differential operator on $G(F_{V \cap V_\infty})$, with V_∞ is the set of archimedean valuations of F. Then $\mathscr{H}^\Phi(G, V, \zeta)$ can be obtained as the topological direct limit

$$\varinjlim_{\Gamma} \varinjlim_{N} \mathscr{H}_N^\Phi(G, V, \zeta)_\Gamma.$$

Here $\mathscr{H}_N^{\Phi}(G,V,\zeta)_\Gamma$ denotes the space of functions f such that f_c belongs to $\mathscr{H}_N(G,V,\zeta)_\Gamma$. Its topology is defined by the seminorms

$$||f||_D = \sup_{c\in\mathscr{C}} \sup_{\{x\in G(F_V):H_G(x)\in c\}} |Df_c(x)|.$$

Now, the collection of Φ being a partially ordered set, we can then define

$$\widetilde{\mathscr{H}}(G,V,\zeta) = \varinjlim_{\Phi} \mathscr{H}^{\Phi}(G,V,\zeta),$$

$$\widetilde{\mathscr{I}}(G,V,\zeta) = \varinjlim_{\Phi} \mathscr{I}^{\Phi}(G,V,\zeta),$$

and

$$\widetilde{\mathscr{S}}(G,V,\zeta) = \varinjlim_{\Phi} \mathscr{S}^{\Phi}(G,V,\zeta).$$

If V contains no archimedean valuations, then $\mathfrak{a}_{G,V}$ is a lattice in $\mathfrak{a}$, and the spaces $\widetilde{\mathscr{H}}(G,V,\zeta)$, $\widetilde{\mathscr{I}}(G,V,\zeta)$, and $\widetilde{\mathscr{S}}(G,V,\zeta)$ are equal to $\mathscr{H}(G,V,\zeta)$, $\mathscr{I}(G,V,\zeta)$, and $\mathscr{S}(G,V,\zeta)$ respectively. The extensions $\widetilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$, $\widetilde{\mathscr{I}}_{\mathrm{ac}}(G,V,\zeta)$, and $\widetilde{\mathscr{S}}_{\mathrm{ac}}(G,V,\zeta)$ are defined similarly.

Specifically, we shall consider the set of hyperplanes in $\mathfrak{a}_M$ given by the collection

$$\Phi = \{\mathfrak{a}_L : L \in \mathscr{L}(M), \dim(A_M/A_L) = 1\}.$$

The associated components are the usual chambers $\{\mathfrak{a}_P^+ : P \in \mathscr{P}(M)\}$. For each $P \in \mathscr{P}(M)$, we shall set $\omega_P(X) = \omega(\mathfrak{a}_P^+, X)$ so that $\omega_P(0)$ is the number ω_P, and let ν_P be the point in the associated chamber $(\mathfrak{a}_P^*)^+$ in $\mathfrak{a}_M^*$ whose distance from the walls is very large, so that $\iota^{\mathscr{E}}_{M,\nu_P}(f,\pi,X)$ and $\tau_{M,\nu_P}(f,\pi,X)$ are independent of ν_P. We define

$${}^c\iota^{\mathscr{E}}_M(f,\pi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\iota^{\mathscr{E}}_{M,\nu_P}(f,\pi,X)$$

and

$${}^c\tau_M(f,\phi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\tau_{M,\nu_P}(f,\phi,X)$$

for $f \in \mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$, $\pi \in \Pi_{\mathrm{temp}}(M_V^Z)$, $\phi \in \Phi_{\mathrm{temp}}(M_V^Z)$, and $X \in \mathfrak{a}_{M,V}$. One sees that $\iota^{\mathscr{E}}_{M,\nu_P}$ and τ_{M,ν_P} are continuous maps from $\widetilde{\mathscr{H}}(G,V,\zeta)$ to $\widetilde{\mathscr{I}}(G,V,\zeta)$ and $\widetilde{\mathscr{S}}(G,V,\zeta)$ respectively. It follows that ${}^c\iota^{\mathscr{E}}_M$ and ${}^c\tau_M$ also map $\widetilde{\mathscr{H}}(G,V,\zeta)$ continuously to $\widetilde{\mathscr{I}}(G,V,\zeta)$ and $\widetilde{\mathscr{S}}(G,V,\zeta)$ respectively. These definitions follow the supplementary map ${}^c\phi_M$, and accordingly send functions of compact support to functions of compact support.

Lemma 3.4. *The maps ${}^c\iota^{\mathscr{E}}_M(f,\pi,X)$ and ${}^c\tau_M(f,\phi,X)$ send $\widetilde{\mathscr{H}}(G,V,\zeta)$ continuously to $\widetilde{\mathscr{I}}(M,V,\zeta)$ and $\widetilde{\mathscr{S}}(M,V,\zeta)$ respectively.*

Proof. We would like to show that there is a positive integer N depending only on the support of $f \in \widetilde{\mathscr{H}}(G,V,\zeta)$ such that ${}^c\iota^{\mathscr{E}}_M(f,\pi,X)$ and ${}^c\tau_M(f,\phi,X)$ are supported on the ball in $\mathfrak{a}_{M,V}$ of radius N. From the definitions of ${}^c\iota^{\mathscr{E}}_M$ and ${}^c\tau_M(f,\pi,X)$, we see that it suffices to prove that for any $P \in \mathscr{P}(M)$ and X in the closure of $\mathfrak{a}_P^+ \cap \mathfrak{a}_{M,V}$, the functions ${}^c\iota^{\mathscr{E}}_{M,\nu_P}(f,\pi,X)$ and ${}^c\tau_{M,\nu_P}(f,\phi,X)$ are supported on the ball of radius N. Then from the decomposition

$$f(x) = \sum_{c\in\mathscr{C}} \omega(c, H_G(x)) f_c(x),$$

we may assume that each f_c are each supported on a set which depends only on the support of f. We may therefore assume that f itself belongs to $\mathscr{H}(G,V,\zeta)$, and therefore

$$^c\iota_M^{\mathscr{E}}(f,\pi,X) = \int_{\nu_P+i\mathfrak{a}^*_{M,V}} \mathrm{e}^{-\lambda(X)} I_M^{\mathscr{E}}(\pi_\lambda,f)d\lambda$$

and

$$^c\tau_M(f,\phi,X) = \int_{\nu_P+i\mathfrak{a}^*_{M,V}} \mathrm{e}^{-\lambda(X)} S_M(\phi_\lambda,f)d\lambda.$$

We then have to show that as a function of $X \in \mathfrak{a}_P^+$, the integrals are supported on a ball which depends only on the support of f. The proof now follows inductively from the same property of the map $^c\phi_M$ studied in [Art88a, §4] and the supplementary map defined by

$$^c\iota_M(f,\pi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\iota_{M,\nu_P}(f,\pi,X)$$

using (2.6). □

Our study of the maps $\iota_M^{\mathscr{E}}$ and τ_M leads us naturally to invariant maps $\theta_{\iota,M}^{\mathscr{E}}$ and $^c\theta_{\iota,M}^{\mathscr{E}}$ from $\widetilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$ to $\widetilde{\mathscr{I}}_{\mathrm{ac}}(M,V,\zeta)$ defined inductively by

$$\theta_{\iota,M}^{\mathscr{E}}(f) = {}^c\iota_M^{\mathscr{E}}(f) - \sum_{L\in\mathscr{L}^0(M)} \hat{\theta}_{\iota,M}^{L,\mathscr{E}}(\iota_L^{\mathscr{E}}(f)) \tag{3.4}$$

$$^c\theta_{\iota,M}^{\mathscr{E}}(f) = \iota_M^{\mathscr{E}}(f) - \sum_{L\in\mathscr{L}^0(M)} {}^c\hat{\theta}_{\iota,M}^{L,\mathscr{E}}({}^c\iota_L^{\mathscr{E}}(f)) \tag{3.5}$$

for any $f \in \widetilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$, and also stably invariant maps $\theta_{\tau,M}$ and $^c\theta_{\tau,M}$ from $\widetilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$ to $\widetilde{\mathscr{S}}_{\mathrm{ac}}(M,V,\zeta)$ inductively by setting

$$\theta_{\tau,M}(f) = {}^c\tau_M(f) - \sum_{L\in\mathscr{L}^0(M)} \hat{\theta}_{\tau,M}^{L}(\tau_L(f)) \tag{3.6}$$

$$^c\theta_{\tau,M}(f) = \tau_M(f) - \sum_{L\in\mathscr{L}^0(M)} {}^c\hat{\theta}_{\tau,M}^{L}({}^c\tau_L(f)). \tag{3.7}$$

We shall use these maps to study the modified distributions $\tilde{I}_M^{\mathscr{E}}(\pi,X,f)$ and $\tilde{S}_M(\phi,X,f)$ and their variants on $\tilde{\mathscr{H}}_{\mathrm{ac}}(G,V,\zeta)$ defined by

$$^c\tilde{I}_M^{\mathscr{E}}(\pi,X,f) = \tilde{I}_M^{\mathscr{E}}(\pi,X,f) - \sum_{L\in\mathscr{L}^0(M)} {}^c\hat{\tilde{I}}_M^{L,\mathscr{E}}(\pi,X,{}^c\iota_L^{\mathscr{E}}(f)) \tag{3.8}$$

and

$$^c\tilde{S}_M(\phi,X,f) = \tilde{S}_M(\phi,X,f) - \sum_{L\in\mathscr{L}^0(M)} {}^c\hat{\tilde{S}}_M^{L}(\phi,X,{}^c\tau_L(f)). \tag{3.9}$$

We then have the following basic relation between these maps and the modified characters.

Lemma 3.5. *Given $f \in \widetilde{\mathscr{H}}_{\mathrm{ac}}(G, V, \zeta)$, we have that*

$$\theta^{\mathscr{E}}_{\iota,M}(f,\pi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\tilde{I}_{M,\nu_P}(\pi,X,f),$$

$$^c\theta^{\mathscr{E}}_{\iota,M}(f,\pi,X) = {}^c\tilde{I}^{\mathscr{E}}_M(\pi,X,f)$$

for $\pi \in \Pi_{\mathrm{temp}}(M^Z_V, \zeta_V)$ and G non-quasisplit over F, and

$$\theta_{\tau,M}(f,\phi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\tilde{S}_{M,\nu_P}(\phi,X,f),$$

$$^c\theta_{\tau,M}(f,\phi,X) = {}^c\tilde{S}_M(\phi,X,f)$$

for $\phi \in \Phi_{\mathrm{temp}}(M^Z_V, \zeta_V)$ and G quasisplit over F.

Proof. The proof is a simple adaptation of the argument of [Art88a, Lemma 4.7], so we will be content with the general argument. By induction, we may assume that for any $L \in \mathscr{L}^0(M)$,

$$\hat{\theta}^{L,\mathscr{E}}_{\iota,M}(\iota^{\mathscr{E}}_L(f),\pi,X) = \sum_{R\in\mathscr{P}^L(M)} \omega_R(X)\hat{\tilde{I}}^{L,\mathscr{E}}_{M,\nu_R}(\pi,X,\iota^{\mathscr{E}}_L(f))$$

where $\mathscr{P}^L(M)$ denotes the set of parabolic subgroups of L containing M. The summands on the right are independent of the point ν_R as long as it remains highly regular in $(\mathfrak{a}^*_R)^+$. Now given any subset Φ' of hyperplanes Φ in $\mathfrak{a}$, with connected components $\mathscr{C}' \subset \mathscr{C}$, we have that

$$\omega(c',X) = \sum_{c\in\mathscr{C}:c\subset c'} \omega(c,X)$$

for any $c' \in \mathscr{C}'$. It follows then that

$$\hat{\theta}^{L,\mathscr{E}}_{\iota,M}(\iota^{\mathscr{E}}_L(f),\pi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\hat{\tilde{I}}^{L,\mathscr{E}}_{M,\nu_P}(\pi,X,\iota^{\mathscr{E}}_L(f)).$$

The definition (3.4) is equal to subtracting the sum over $L \in \mathscr{L}^0(M)$ of this from the function

$$^c\iota^{\mathscr{E}}_M(f,\pi,X) = \sum_{P\in\mathscr{P}(M)} \omega_P(X)\iota^{\mathscr{E}}_{M,\nu_P}(f,\pi,X).$$

The difference is equal to

$$\sum_{P\in\mathscr{P}(M)} \omega_P(X)\tilde{I}^{\mathscr{E}}_{M,\nu_P}(\pi,X,f),$$

since $\iota^{\mathscr{E}}_{M,\nu_P}(f,\pi,X) = \tilde{I}^{\mathscr{E}}_{M,\nu_P}(f,\pi,X)$, and the first identity follows. The second identity follows from a similar inductive argument using (3.5) and (3.8). The third and fourth identities are also proved in a similar fashion, using the parallel definitions (3.6), (3.7), and (3.9). □

3.3. **Weighted orbital integrals.** We would like to construct the linear forms that will occur in the modified geometric expansion, but first we shall have to recall the linear forms appearing in the geometric sides of the endoscopic and stable trace formulae. Let $\mathscr{D}(G_V^Z,\zeta_V)$ be the space of distributions on G_V^Z that are (i) invariant under G_V^Z-conjugation, (ii) ζ_V-equivariant under translation by Z_V, and (iii) supported on the preimage in G_V^Z of a finite union of conjugacy classes in $\bar{G}_V^Z = G_V^Z/Z_V$. It contains the invariant orbital integrals, and also derivatives of orbital integrals if V contain archimedean places. We shall write $\Gamma(G_V^Z,\zeta_V)$ for the fixed basis of this space as chosen in [Art02, §1].

Furthermore let $S\mathscr{D}(G_V^Z,\zeta_V)$ be the subspace of stable distributions in $\mathscr{D}(G_V^Z,\zeta)$. Recall that a distribution on G_V^Z is called stable if it lies in the closed linear span of strongly regular, stable orbital integrals. Following [Art02, §5], there is an endoscopic basis $\Delta^{\mathscr{E}}(G_V^Z,\zeta)$ of $\mathscr{D}(G_V^Z,\zeta_V)$ such that the intersection

$$\Delta(G_V^Z,\zeta_V) = \Delta^{\mathscr{E}}(G_V^Z,\zeta_V)\cap S\mathscr{D}(G_V^Z,\zeta_V)$$

forms a basis of $S\mathscr{D}(G_V^Z,\zeta_V)$. For any $f\in\mathscr{C}(G_V^Z,\zeta_V)$ and $\gamma\in\Gamma(M_V^Z,\zeta_V)$, we have the invariant linear form defined by the usual inductive formula

$$I_M(\gamma,f) = J_M(\gamma,f) - \sum_{L\in\mathscr{L}^0(M)} \hat{I}_M^L(\gamma,\phi_L(f)),$$

where $\phi_L(f)$ is the map (2.2), and $J_M(\gamma,f)$ is the noninvariant weighted orbital integral as in [Art02, §2].

We continue to assume that the endoscopic data M' and $\tilde{G}',\tilde{\xi}'$ are unramified away from the finite set V. Let $\delta'\in\Delta((\tilde{M}'_V)^{\tilde{Z}'},\tilde{\zeta}'_V)$. The corresponding endoscopic and stable linear forms on $\mathscr{S}(\tilde{G}'_V,\tilde{\zeta}'_V)$ are then defined inductively by the formula

$$I_M^{\mathscr{E}}(\delta',f) = \sum_{G\in\mathscr{E}^0_{M'}(G)} \iota_{M'}(G,G')\hat{S}_{M'}^{\tilde{G}'}(\delta',f') + \varepsilon(G)S_M^G(M',\delta',f)$$

together with the supplementary requirement that

$$I_M^{\mathscr{E}}(\delta',f) = I_M(\delta,f)$$

in the case that G is quasisplit and δ' maps to the element δ in $\Delta^{\mathscr{E}}(M_V^Z,\zeta_V)$. The coefficient $\iota_{M'}(G,G')$ vanishes unless G belongs to the finite set subset of elliptic elements in $\mathscr{E}_{M'}(G)$. The actual endoscopic distributions are in fact the family

$$I_M^{\mathscr{E}}(\gamma,f),\qquad \gamma\in\Gamma(M_V^Z,\zeta_V)$$

defined by the formula

$$I_M^{\mathscr{E}}(\delta',f) = \sum_{\gamma\in\Gamma(M_V^Z,\zeta_V)} \Delta_M(\delta',\gamma)I_M^{\mathscr{E}}(\gamma,f),$$

or equivalently,

$$I_M^{\mathscr{E}}(\gamma,f) = \sum_{\delta\in\Delta^{\mathscr{E}}(M_V^Z,\zeta_V)} \Delta_M(\gamma,\delta')I_M^{\mathscr{E}}(\delta,f),$$

which holds because $I_M^{\mathscr{E}}(\delta',f)$ depends only on the image δ of δ' in $\Delta^{\mathscr{E}}(M_V^Z,\zeta_V)$. Here $\Delta(\gamma,\delta)$ is the geometric transfer factor [Art02, §5], which satisfies the adjoint relations

$$\sum_{\delta\in\Delta^{\mathscr{E}}(G_V^Z,\zeta_V)} \Delta(\gamma,\delta)\Delta(\delta,\gamma_1) = \delta(\gamma,\gamma_1),\qquad \gamma,\gamma_1\in\Gamma(G_V^Z,\zeta_V) \tag{3.10}$$

and

$$\sum_{\gamma\in\Gamma(G^Z_V,\zeta_V)} \Delta(\delta,\gamma)\Delta(\gamma,\delta_1)=\delta(\delta,\delta_1),\qquad \delta,\delta_1\in\Delta^{\mathscr{E}}(G^Z_V,\zeta_V), \tag{3.11}$$

where $\delta(\gamma,\gamma_1)=1$ if $\gamma=\gamma_1$ and 0 otherwise, and similarly for $\delta(\gamma,\gamma_1)$. These distributions are the key objects on the geometric side of the stable trace formula.

We now would like to define our modification of these distributions using the maps we have introduced earlier. Define inductively the modified linear forms

$$\tilde{I}^{\mathscr{E}}_M(\gamma,f),\qquad f\in\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$$

by setting

$$\tilde{I}^{\mathscr{E}}_M(\gamma,f)=I^{\mathscr{E}}_M(\gamma,f)-\sum_{L\in\mathscr{L}^0(M)}\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma,\iota^{\mathscr{E}}_L(f)), \tag{3.12}$$

if G is arbitrary, and

$$\tilde{S}_M(\delta,f),\qquad f\in\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$$

by setting

$$\tilde{S}_M(\delta,f)=S_M(\delta,f)-\sum_{L\in\mathscr{L}^0(M)}\hat{\tilde{S}}^L_M(\delta,\tau_L(f)). \tag{3.13}$$

in the case that G is quasisplit. More generally, we define $\tilde{S}^G_M(M',\delta',f)$ analogously, whereby

$$\tilde{S}^G_M(\delta,f)=\tilde{S}^G_M(M^*,\delta^*,f)$$

and omitting the subscript G when the context is clear. The modified endoscopic distribution then satisfies the analogous inversion formula

$$\tilde{I}^{\mathscr{E}}_M(\delta',f)=\sum_{\gamma\in\Gamma(M^Z_V,\zeta_V)}\Delta_M(\delta',\gamma)\tilde{I}^{\mathscr{E}}_M(\gamma,f), \tag{3.14}$$

by the adjoint property of the transfer factors. We also define the variants

$${}^c\tilde{I}^{\mathscr{E}}_M(\gamma,f)=I^{\mathscr{E}}_M(\gamma,f)-\sum_{L\in\mathscr{L}^0(M)}{}^c\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma,{}^c\iota^{\mathscr{E}}_L(f)), \tag{3.15}$$

and

$${}^c\tilde{S}_M(\delta,f)=S_M(\delta,f)-\sum_{L\in\mathscr{L}^0(M)}{}^c\hat{\tilde{S}}^L_M(\delta,{}^c\tau_L(f)). \tag{3.16}$$

The supplementary maps that we have defined earlier allow us to describe the asymptotic behaviour of these distributions.

Proposition 3.6. *Let $f\in\widetilde{\mathscr{H}}(G,V,\zeta)$. Then for $\gamma\in\Gamma(M^Z_V,\zeta_V)$, we have*

$$\tilde{I}^{\mathscr{E}}_M(\gamma,f)={}^c\tilde{I}_M(\gamma,f)+\sum_{L\in\mathscr{L}^0(M)}{}^c\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma,\theta^{\mathscr{E}}_{\iota,L}(f)) \tag{3.17}$$

$${}^c\tilde{I}^{\mathscr{E}}_M(\gamma,f)=\tilde{I}_M(\gamma,f)+\sum_{L\in\mathscr{L}^0(M)}\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma,{}^c\theta^{\mathscr{E}}_{\iota,L}(f)) \tag{3.18}$$

if G is arbitrary, and for $\delta\in\Delta(M^Z_V,\zeta_V)$, we have

$$\tilde{S}_M(\delta,f)={}^c\tilde{I}_M(\delta,f)+\sum_{L\in\mathscr{L}^0(M)}{}^c\hat{\tilde{S}}_M(\delta,\theta_{\tau,L}(f)) \tag{3.19}$$

$$ {}^{c}\tilde{S}_M(\delta, f) = \tilde{I}_M(\delta, f) + \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{S}}_M(\delta, {}^{c}\theta_{\tau,L}(f)) \tag{3.20} $$

if G is quasisplit.

Proof. We assume inductively that each formula holds when G is replaced by a proper Levi subset. The formulas for G are then easily established from the definitions. We shall prove (3.17) here, and the rest will follow in the same way. Using the definitions (3.12) and (3.15) above, we have that

$$ \tilde{I}^{\mathscr{E}}_M(\gamma, f) - {}^{c}\tilde{I}^{\mathscr{E}}_M(\gamma, f) = \sum_{L \in \mathscr{L}^0(M)} {}^{c}\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, {}^{c}\iota^{\mathscr{E}}_L(f) - \sum_{L_1 \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L_1,\mathscr{E}}_M(\gamma, \iota^{\mathscr{E}}_{L_1}(f)). $$

By (3.5) the first sum is equal to

$$ \sum_{L \in \mathscr{L}^0(M)} {}^{c}\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, \theta^{\mathscr{E}}_{\iota,L}(f)) + \sum_{L_1 \in \mathscr{L}^0(M)} \sum_{L \in \mathscr{L}^{L_1}(M)} {}^{c}\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, \hat{\theta}^{L_1,\mathscr{E}}_{\iota,L}(\iota^{\mathscr{E}}_{L_1}(f))). $$

Applying (3.17) inductively to each $L_1 \in \mathscr{L}^0(M)$, we have that

$$ \sum_{L \in \mathscr{L}^{L_1}(M)} {}^{c}\hat{\tilde{I}}^{L,\mathscr{E}}_M(\gamma, \hat{\theta}^{L_1,\mathscr{E}}_{\iota,L}(\iota^{\mathscr{E}}_{L_1}(f))) = \hat{\tilde{I}}^{L_1,\mathscr{E}}_M(\gamma, \iota^{\mathscr{E}}_{L_1}(f)) $$

and the formula (3.17) then follows for G. □

Remark 3.7. By [Art88b, Corollary 5.3], we know that the distributions $I_M(\gamma, f)$ are supported on characters. Recall that a continuous linear map θ from $\mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$ to a topological vector space $\mathscr{V}$ is said to be supported on characters if it vanishes on the kernel of $\mathscr{I}(G, V, \zeta)$. That is, if $\theta(f) = 0$ for every function $f \in \mathscr{H}_{\mathrm{ac}}(G, V, \zeta)$ such that $f_G = 0$. Using this and a slight variation of the proof of [Art88a, Theorem 6.1], it follows inductively that the new distributions we have introduced, and in particular those occurring in the preceding proposition are also supported on characters. Since we do not have need of this property explicitly, we neglect to provide the details here.

Let us also define inductively the modified invariant linear form

$$ \tilde{I}_M(\delta, f), \qquad f \in \mathscr{H}_{\mathrm{ac}}(G, V, \zeta) $$

by setting

$$ \tilde{I}_M(\delta, f) = I_M(\delta, f) - \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}^L_M(\delta, \iota_L(f)). $$

Then we have the following property as an immediate consequence of Local Theorem $1'$ of [Art02], which will also be needed for the global expansion.

Lemma 3.8. *Let V be a finite set of valuations such that G and ζ are unramified outside of V.*

(a) *if G is arbitrary,*

$$ \tilde{I}^{\mathscr{E}}_M(\delta, f) = \tilde{I}_M(\delta, f), \qquad \pi \in \Gamma(M^Z_V, \zeta_V),\ f \in \mathscr{H}(G, V, \zeta) $$

(b)*Suppose that G is quasisplit, and that δ' belongs to $\Delta(\tilde{M}', \tilde{\zeta}')$ for some $M' \in \mathscr{E}_{\mathrm{ell}}(M)$. Then the linear form*

$$ f \to \tilde{S}^G_M(M', \delta', X, f), \qquad f \in \mathscr{H}(G, V, \zeta) $$

vanishes unless $M' = M^*$, in which case it is stable.

4. Descent and splitting formulas

We now want to establish descent and splitting formulas for our modified distributions, which will reduce the study of the compound distributions on the geometric side to the local setting.

4.1. **Descent.** For the descent formula, we shall prove it for a finite set of places V, which includes the special case that V contains a single place v. We shall take f to be a fixed function in $\mathscr{C}(G^Z_V, \zeta_V)$. Let R be a Levi subgroup of M, so that $\mathfrak{a}_M$ is a subspace of $\mathfrak{a}_R$ and whose orthogonal complement we denote by $\mathfrak{a}^M_R$. If L belongs to $\mathscr{L}(R)$, we then have a map

$$\mathfrak{a}^M_R \oplus \mathfrak{a}^L_R \to \mathfrak{a}^G_R.$$

As in the special case of [Art88a, §7], we define the coefficient $d^G_R(M,L)$ by setting it to be zero if the map is not an isomorphism, and otherwise we define $d^G_R(M,L)$ to be the volume in $\mathfrak{a}^G_R$ of the image of a unit cube in $\mathfrak{a}^M_R \oplus \mathfrak{a}^L_R$.

Let R be a proper Levi subgroup of M. Any Levi subgroup $L \in \mathscr{L}(R)$ comes with a dual Levi subgroup $\hat{L}$ of $\hat{G}$ that contains $\hat{R}$, from which we form the coefficient

$$\mathrm{e}^G_R(M,L) = d^G_R(M,L)|Z(\hat{M})^\Gamma \cap Z(\hat{L})^\Gamma / Z(\hat{G})^\Gamma|^{-1}$$

for each $L \in \mathscr{L}(R)$.

Proposition 4.1. (a) *Suppose that G is not quasiplit over F, and $\gamma \in \Gamma(M^Z_V, \zeta_V)$. Then for any $f \in \mathscr{H}(G,V,\zeta)$, we have*

$$\tilde{I}^{\mathscr{E}}_M(\gamma, f) = \sum_{L \in \mathscr{L}(R)} d^G_R(M,L) \hat{\tilde{I}}^{L,\mathscr{E}}_R(\gamma, f_L),$$

(b) *Suppose that G is quasiplit over F, and $\delta \in \Delta(M^Z_V, \zeta_V)$. Then for any $f \in \mathscr{H}(G,V,\zeta)$, we have*

$$\tilde{S}_M(\delta, f) = \sum_{L \in \mathscr{L}(R)} \mathrm{e}^G_R(M,L) \hat{\tilde{S}}^L_R(\delta, f_L). \tag{4.1}$$

Proof. We begin with the first equation. By the inversion formula (3.14) it is equivalent to establishing the analogue

$$\tilde{I}^{\mathscr{E}}_M(\delta', f) = \sum_{L \in \mathscr{L}(R)} d^G_R(M,L) \hat{\tilde{I}}^{L,\mathscr{E}}_R(\delta', f_L). \tag{4.2}$$

Recall that the endoscopic distribution also satisfies a descent formula

$$I^{\mathscr{E}}_M(\delta', f) = \sum_{L \in \mathscr{L}(R)} d^G_R(M,L) \hat{I}^{L,\mathscr{E}}_R(\delta', f_L) \tag{4.3}$$

in [Art99, Theorem 7.1] for strongly G-regular elements and [Won19a, Proposition 3.3] for general δ' (see also [MW16b, VI.4] for an alternate formulation). Then it follows from the definition (3.12) that $\tilde{I}^{\mathscr{E}}_M(\delta', f)$ can be expressed as the difference of the right-hand side of (4.3) and

$$\sum_{L_1 \in \mathscr{L}^0(M)} \hat{\tilde{I}}^{L_1,\mathscr{E}}_M(\delta', \iota^{\mathscr{E}}_{L_1}(f)). \tag{4.4}$$

We can assume inductively that (4.2) holds for each of the distributions $\hat{\tilde{I}}_M^{L_1,\mathscr{E}}(\delta')$ for $L_1 \in \mathscr{L}^0(M)$. Then the latter sum can be written as

$$\sum_{L_1\in\mathscr{L}^0(M)}\sum_{M_1\in\mathscr{L}^{L_1}(R)} d_R^{L_1}(M,M_1)\hat{\tilde{I}}_R^{M_1,\mathscr{E}}(\delta',\iota_{L_1}^{\mathscr{E}}(f)_{M_1}). \tag{4.5}$$

Now $\iota_{L_1}^{\mathscr{E}}(f)_{M_1}$ is a function in $\mathscr{I}_{\mathrm{ac}}(M_1,V,\zeta)$. By specializing the descent formula for endoscopic weighted characters $I_M^{\mathscr{E}}(\pi,X,f)$ in [MW16b, X.4.4] for twisted endoscopy, it follows that

$$I_{L_1}^{\mathscr{E}}(\pi,X,f) = \sum_{L\in\mathscr{L}(M_1)} d_{M_1}^G(L_1,L)I_{M_1}^L(\pi,X,f_L)$$

and hence the mapping $\iota_{L_1}^{\mathscr{E}}$ satisfies the descent property

$$\iota_{L_1}^{\mathscr{E}}(f)_{M_1} = \sum_{L\in\mathscr{L}(M_1)} d_{M_1}^G(L_1,L)\iota_{M_1}^{L,\mathscr{E}}(f)_L. \tag{4.6}$$

Let us agree to set $d_R^{L_1}(M,M_1)=0$ if L_1 does not contain both M and M_1. It follows from this that (4.5) is equal to

$$\sum_{L_1\in\mathscr{L}^0(M)}\sum_{M_1\in\mathscr{L}(R)}\sum_{L\in\mathscr{L}(M_1)} d_R^{L_1}(M,M_1)d_{M_1}^G(L_1,L)\hat{\tilde{I}}_R^{M_1,\mathscr{E}}(\delta',\iota_{M_1}^{L,\mathscr{E}}(f)_L). \tag{4.7}$$

We need only consider M_1 such that the coefficient $d_R^{L_1}(M,M_1)\neq 0$, so that $\mathfrak{a}_R^{L_1} = \mathfrak{a}_R^M \oplus \mathfrak{a}_R^{M_1}$. Also, if L is any element in $\mathscr{L}(M_1)$ such that $d_{M_1}^G(L_1,L)\neq 0$, then we have $\mathfrak{a}_{M_1}^G = \mathfrak{a}_{M_1}^{L_1}\oplus\mathfrak{a}_{M_1}^L$, and the Levi L in (4.3) and (4.7) can taken to be the same. The only part of the expression (4.7) depending on L_1 is

$$\sum_{L_1\in\mathscr{L}^0(M)} d_R^{L_1}(M,M_1)d_{M_1}^G(L_1,L). \tag{4.8}$$

If $L\neq M_1$, we can replace the sum over $\mathscr{L}^0(M)$ with $\mathscr{L}(M)$ since the term corresponding to $L_1=G$ vanishes. Then using [Art88a, (7.1)], or rather its extension to K-groups [Art99, Lemma 4.1], it follows that the sum is equal to $d_R^G(M,L)$. On the other hand, if $L=M_1$ then

$$d_{M_1}^G(L_1,L) = d_{M_1}^G(L_1,M_1) = 0$$

since $L_1\neq G$, so in this case all the summands are zero. It follows then that (4.7) is equal to

$$\sum_{L\in\mathscr{L}(R)}\sum_{\substack{M_1\in\mathscr{L}^L(M)\\ M_1\neq L}} d_R^G(M,L)\hat{\tilde{I}}_R^{M_1,\mathscr{E}}(\delta',\iota_{M_1}^{L,\mathscr{E}}(f)_L).$$

We can now combine this with (4.3). From the inductive definition of $\tilde{I}_M^{L,\mathscr{E}}(\delta')$ we see that the difference between (4.3) and (4.4) is equal to

$$\sum_{L\in\mathscr{L}(R)} d_R^G(M,L)\hat{\tilde{I}}_R^{L,\mathscr{E}}(\delta',f_L)$$

as desired.

The second equation is established in a similar manner, so we can afford to be brief, indicating the points of departure from the latter proof. Once again the stable linear forms $S_M(\delta, f)$ satisfy the descent formula

$$S_M(\delta, f) = \sum_{L \in \mathscr{L}(R)} \mathrm{e}_R^G(M, L)\hat{S}_R^L(\delta, f_L), \tag{4.9}$$

and it follows from the definition (3.13) that $\tilde{S}_M(\delta, f)$ can be expressed as the difference of the right hand side of (4.9) and

$$\sum_{L_1 \in \mathscr{L}^0(M)} \hat{\hat{S}}_M^{L_1}(\delta, \tau_{L_1}(f)).$$

Inductively applying the formula (4.1) to the distributions $\hat{\hat{S}}_M^{L_1}(\delta)$ for $L_1 \in \mathscr{L}^0(M)$, the latter sum can be written as

$$\sum_{L_1 \in \mathscr{L}^0(M)} \sum_{M_1 \in \mathscr{L}^{L_1}(R)} \mathrm{e}_R^{L_1}(M, M_1)\hat{\hat{S}}_R^{M_1}(\delta, \tau_{L_1}(f)_{M_1}). \tag{4.10}$$

Again using the descent formula for stable characters $S_M(\phi, X, f)$ in [MW16b, X.4.4],

$$S_{L_1}(\phi, X, f) = \sum_{L \in \mathscr{L}(M_1)} \mathrm{e}_{M_1}^G(L_1, L)S_{M_1}^L(\phi, X, f_L)$$

it follows that the mapping τ_{L_1} satisfies

$$\tau_{L_1}(f)_{M_1} = \sum_{L \in \mathscr{L}(M_1)} \mathrm{e}_{M_1}^G(L_1, L)\tau_{M_1}^L(f)_L. \tag{4.11}$$

It follows from this that (4.10) is equal to

$$\sum_{L_1 \in \mathscr{L}^0(M)} \sum_{M_1 \in \mathscr{L}(R)} \sum_{L \in \mathscr{L}(M_1)} \mathrm{e}_R^{L_1}(M, M_1)\mathrm{e}_{M_1}^G(L_1, L)\hat{\hat{S}}_R^{M_1}(\delta', \tau_{M_1}^L(f)_L). \tag{4.12}$$

We focus our attention on the product of the two coefficients, appealing to the definition to write

$$d_R^{L_1}(M, M_1)d_{M_1}^G(L_1, L)|Z(\hat{M})^\Gamma \cap Z(\hat{M}_1)^\Gamma/Z(\hat{L}_1)^\Gamma|^{-1}|Z(\hat{L}_1)^\Gamma \cap Z(\hat{L})^\Gamma/Z(\hat{G})^\Gamma|^{-1}.$$

Since we are assuming $d_{M_1}^G(L_1, L)$ is nonzero, it follows that the connected component $(Z(\hat{M}_1)^\Gamma)^0$ is equal to the product of the subgroups $(Z(\hat{L}_1)^\Gamma)^0$ and $(Z(\hat{L})^\Gamma)^0$. Since $Z(\hat{M}_1)^\Gamma$ is equal to the product of $(Z(\hat{M}_1)^\Gamma)^0$ and $Z(\hat{G})^\Gamma$, by [Art99, Lemma 1.1], we see then that

$$Z(\hat{M}_1)^\Gamma = Z(\hat{L}_1)^\Gamma Z(\hat{L})^\Gamma.$$

It follows then that the product of

$$\begin{aligned} |Z(\hat{L}_1)^\Gamma \cap Z(\hat{L})^\Gamma/Z(\hat{G})^\Gamma|^{-1} &= |Z(\hat{L}_1)^\Gamma/Z(\hat{L})^\Gamma \cap Z(\hat{L}_1)^\Gamma||Z(\hat{L}_1)^\Gamma/Z(\hat{G})^\Gamma|^{-1} \\ &= |Z(\hat{L}_1)^\Gamma Z(\hat{L})^\Gamma/Z(\hat{L})^\Gamma||Z(\hat{L}_1)^\Gamma/Z(\hat{G})^\Gamma|^{-1}, \end{aligned}$$

with the quantity $|Z(\hat{M})^\Gamma \cap Z(\hat{L}_1)^\Gamma Z(\hat{L})^\Gamma/Z(\hat{L}_1)^\Gamma|^{-1}$ is equal to

$$|Z(\hat{M})^\Gamma \cap Z(\hat{L})^\Gamma/Z(\hat{G})^\Gamma|^{-1}.$$

In particular, it is independent of L_1. Using the same identity for (4.8) to sum the product of $d_R^{L_1}(M,M_1)$ and $d_{M_1}^G(L_1,L)$, we have that

$$\sum_{L_1\in\mathscr{L}^0(M)} \mathrm{e}_R^{L_1}(M,M_1)\mathrm{e}_{M_1}^G(L_1,L)=\mathrm{e}_R^G(M,L).$$

And again taking into account the vanishing conditions for the coefficients, it follows that (4.12) is equal to

$$\sum_{L_1\in\mathscr{L}(R)}\sum_{\substack{M_1\in\mathscr{L}^L(M)\\ M_1\neq L}} \mathrm{e}_R^G(M,L)\hat{\tilde{S}}_R^{M_1}(\delta',\tau_{M_1}^L(f)_L). \tag{4.13}$$

Finally, using the inductive definition of $\tilde{S}_M^L(\delta)$ we see that the difference between (4.9) and (4.13) is equal to

$$\sum_{L\in\mathscr{L}(R)} \mathrm{e}_R^G(M,L)\hat{\tilde{S}}_R^L(\delta,f_L)$$

as desired. □

4.2. **Splitting.** For the splitting formula, we shall assume that V is a disjoint union of nonempty sets V_1 and V_2, and that the image of F_{V_i} in $\mathbf{R}$ under the absolute value is closed for each $i=1,2$. We shall also fix a function

$$f_V=f_{V_1}\times f_{V_2},\quad f_{V_i}\in\mathscr{C}(G_{V_i},\zeta_{V_i})$$

The splitting formulas will be expressed in terms of pairs of Levi subgroups $L_1,L_2\in\mathscr{L}(M)$. For any such pair, we define the coefficient

$$\mathrm{e}_M^G(L_1,L_2)=d_M^G(L_1,L_2)|Z(\hat{L}_1)^\Gamma\cap Z(\hat{L}_2)^\Gamma/Z(\hat{G})^\Gamma|^{-1}.$$

Note that if $d_M^G(L_1,L_2)$ is nonzero, then $\mathfrak{a}_{L_1}^*\cap\mathfrak{a}_{L_2}^*=\mathfrak{a}_G^*$ and the identity component of $Z(\hat{L}_1)^\Gamma\cap Z(\hat{L}_2)^\Gamma$ is the same as that of $Z(\hat{G})^\Gamma$. Therefore $\mathrm{e}_M^G(L_1,L_2)$ is also nonzero. We shall generally write L_i for the image of L_{i,V_i}.

Proposition 4.2. (a) *Suppose that G is arbitrary, and $\gamma_V=(\gamma_{V_1},\gamma_{V_2})\in\Gamma(G_V^Z,\zeta_V)$. Then for any $f\in\mathscr{H}(G,V,\zeta)$, we have*

$$\tilde{I}_M^{\mathscr{E}}(\gamma_V,f_V)=\sum_{L_1,L_2\in\mathscr{L}(M)} d_M^G(L_1,L_2)\hat{\tilde{I}}_M^{L_1,\mathscr{E}}(\gamma_{V_1},f_{V_1,L_1})\hat{\tilde{I}}_M^{L_2,\mathscr{E}}(\gamma_{V_2},f_{V_2,L_2}).$$

(b) *Suppose that G is quasisplit over F, and $\delta_V=(\delta_{V_1},\delta_{V_2})\in\Delta(G_V^Z,\zeta_V)$. Then for any $f\in\mathscr{H}(G,V,\zeta)$, we have*

$$\tilde{S}_M(\delta_V,f_V)=\sum_{L_1,L_2\in\mathscr{L}(M)} \mathrm{e}_M^G(L_1,L_2)\hat{\tilde{S}}_M^{L_1}(\delta_{V_1},f_{V_1,L_1})\hat{\tilde{S}}_M^{L_2}(\delta_{V_2},f_{V_2,L_2}).$$

Proof. The required formula in (a) has the analogue

$$\tilde{I}_M^{\mathscr{E}}(\delta'_V,f_V)=\sum_{L_1,L_2\in\mathscr{L}(M)} d_M^G(L_1,L_2)\hat{\tilde{I}}_M^{L_1,\mathscr{E}}(\delta'_{V_1},f_{V_1,L_1})\hat{\tilde{I}}_M^{L_2,\mathscr{E}}(\delta'_{V_2},f_{V_2,L_2}).$$

for $\delta'_V\in\Delta^{\mathscr{E}}((\tilde{M}')_V^{\tilde{Z}'},\tilde{\zeta}'_V)$. According to (3.14), the two formulas are equivalent, so it will be sufficient to establish the latter. The original endoscopic distribution satisfies the splitting formula

$$I_M^{\mathscr{E}}(\delta'_V,f_V)=\sum_{L_1,L_2\in\mathscr{L}(M)} d_M^G(L_1,L_2)\hat{I}_M^{L_1,\mathscr{E}}(\delta'_{V_1},f_{V_1,L_1})\hat{I}_M^{L_2,\mathscr{E}}(\delta'_{V_2},f_{V_2,L_2}). \tag{4.14}$$

as in [Won19a, Proposition 3.4] for general δ' and [Art99, Theorem 6.1] for strongly G-regular elements (see also [MW16b, VII.2] for an alternate formulation). Then it follows from the definition (3.12) that $\tilde{I}^{\mathscr{E}}_M(\delta'_V, f)$ can be expressed as the difference of the right-hand side of (4.14) and

$$\sum_{L\in\mathscr{L}^0(M)} \hat{\tilde{I}}^{L,\mathscr{E}}_M(\delta', \iota^{\mathscr{E}}_L(f)). \tag{4.15}$$

We can assume inductively that (4.14) holds for each of the distributions $\hat{\tilde{I}}^{L,\mathscr{E}}_M(\delta')$ for $L\in\mathscr{L}^0(M)$. Then the latter sum can be written as

$$\sum_{L\in\mathscr{L}^0(M)} \sum_{M_1,M_2\in\mathscr{L}^L(M)} d^L_M(M_1,M_2)\hat{\tilde{I}}^{M_1,\mathscr{E}}_M(\delta'_{V_1}, \iota^{\mathscr{E}}_L(f_{V_1})_{M_1})\hat{\tilde{I}}^{M_2,\mathscr{E}}_M(\delta'_{V_2}, \iota^{\mathscr{E}}_L(f_{V_2})_{M_2}). \tag{4.16}$$

We shall apply the descent formula (4.6) to each $\iota^{\mathscr{E}}_L(f_{V_i})_{M_i}$, that is,

$$\iota^{\mathscr{E}}_L(f_{V_i})_{M_i} = \sum_{L_i\in\mathscr{L}(M_i)} d^G_{M_i}(L,L_i)\iota^{L_i,\mathscr{E}}_{M_i}(f_{V_i})_{L_i}.$$

for $i=1.2$. It follows then that (4.16) can be written as the sum over $L\in\mathscr{L}^0(M)$, $M_1,M_2\in\mathscr{L}(M)$, $L_1\in\mathscr{L}(M_1)$, and $L_2\in\mathscr{L}(M_2)$ of

$$d^L_M(M_1,M_2)d^G_{M_1}(L,L_1)d^G_{M_2}(L,L_2)\hat{\tilde{I}}^{M_1,\mathscr{E}}_M(\delta'_{V_1}, \iota^{L_1,\mathscr{E}}_{M_1}(f_{V_1})_{L_1})\hat{\tilde{I}}^{M_2,\mathscr{E}}_M(\delta'_{V_2}, \iota^{L_2,\mathscr{E}}_{M_2}(f_{V_2})_{L_2}).$$

The only part that depends on L is again the sum

$$\sum_{L\in\mathscr{L}^0(M)} d^L_M(M_1,M_2)d^G_{M_1}(L,L_1)d^G_{M_2}(L,L_2).$$

We may assume that $d^L_M(M_1,M_2)$ and $d^G_{M_1}(L,L_1)$ are nonzero. This implies that $\mathfrak{a}^L_M = \mathfrak{a}^{M_1}_M \oplus \mathfrak{a}^{M_2}_M$ and $\mathfrak{a}^G_{M_1} = \mathfrak{a}^L_{M_1}\oplus\mathfrak{a}^{L_1}_{M_1}$, and it follows from this that $\mathfrak{a}^G_M$ is equal to

$$\mathfrak{a}^{M_1}_M\oplus\mathfrak{a}^G_{M_1} = \mathfrak{a}^{M_1}_M\oplus\mathfrak{a}^L_{M_1}\oplus\mathfrak{a}^{L_1}_{M_1} = \mathfrak{a}^{M_1}_M\oplus\mathfrak{a}^{M_2}_M\oplus\mathfrak{a}^{L_1}_{M_1} = \mathfrak{a}^{L_1}_M\oplus\mathfrak{a}^{M_2}_M,$$

hence $d^G_M(L_1,M_2)\neq 0$. We are also assuming that $d^G_{M_2}(L,L_2)$ is nonzero, and therefore $\mathfrak{a}^G_M = \mathfrak{a}^{M_1}_M\oplus\mathfrak{a}^{L_2}_M$, then by the same argument we have $d^G_M(M_1,L_2)\neq 0$. If $L_i\neq M_i$ and $L=G$ then $d^G_{M_i}(L,L_i)=0$ for each $i=1,2$, so in this case we may replace the sum over $\mathscr{L}^0(M)$ with $\mathscr{L}(M)$. It follows then from [MW16b, VI.4.2] that the sum is equal to $d^G_M(L_1,L_2)$.

On the other hand, the summands corresponding to $L_i = M_i$ and $L\neq G$ all vanish, so we may write

$$\sum_{L_1,L_2\in\mathscr{L}(M)} \sum_{\substack{M_i\in\mathscr{L}^{L_i}(M)\\ M_i\neq L_i, i=1,2}} d^G_M(L_1,L_2)\hat{\tilde{I}}^{M_1,\mathscr{E}}_M(\delta'_{V_1}, \iota^{L_1,\mathscr{E}}_{M_1}(f_{V_1})_{L_1})\hat{\tilde{I}}^{M_2,\mathscr{E}}_M(\delta'_{V_2}, \iota^{L_2,\mathscr{E}}_{M_2}(f_{V_2})_{L_2})$$

and then applying inductive definition to the form

$$\tilde{I}^{L_1\times L_2,\mathscr{E}}_{M\times M}(\delta'_V, f_{V,L_1\times L_2}),$$

which applies in particular to products of groups in a manner similar to [Art88a, §9], we arrive at

$$\sum_{L_1,L_2\in\mathscr{L}(M)} d^G_M(L_1,L_2)\hat{\tilde{I}}^{L_1,\mathscr{E}}_M(\delta'_{V_1}, f_{V_1,L_1})\hat{\tilde{I}}^{L_2,\mathscr{E}}_M(\delta'_{V_2}, f_{V_2,L_2})$$

as required.

The second required formula proceeds in a similar manner. The stable linear form also satisfies the splitting formula

$$S_M^G(\delta_V, f_V) = \sum_{L_1, L_2 \in \mathscr{L}(M)} \mathrm{e}_M^G(L_1, L_2) \hat{S}_M^{L_1}(\delta_{V_1}, f_{V_1, L_1}) \hat{S}_M^{L_2}(\delta_{V_2}, f_{V_2, L_2}) \tag{4.17}$$

as in [Won19a, Proposition 3.4] for general δ' and [Art99, Theorem 6.1] for strongly G-regular elements (see also [MW16b, VII.2] for an alternate formulation). Then it follows from the definition (3.13) that $\tilde{I}_M^{\mathscr{E}}(\delta'_V, f)$ can be expressed as the difference of the right-hand side of (4.17) and

$$\sum_{L \in \mathscr{L}^0(M)} \hat{\hat{S}}_M^L(\delta, \tau_L(f)). \tag{4.18}$$

We can assume inductively that (4.17) holds for each of the distributions $\hat{\hat{S}}_M^L(\delta)$ for $L \in \mathscr{L}^0(M)$. Then the latter sum can be written as

$$\sum_{L \in \mathscr{L}^0(M)} \sum_{M_1, M_2 \in \mathscr{L}^L(M)} \mathrm{e}_M^L(M_1, M_2) \hat{\hat{S}}_M^{M_1}(\delta_{V_1}, \tau_L(f_{V_1})_{M_1}) \hat{\hat{S}}_M^{M_2}(\delta_{V_2}, \tau_L(f_{V_2})_{M_2}). \tag{4.19}$$

We shall apply the descent formula (4.11) to each $\tau_L^{\mathscr{E}}(f_{V_i})_{M_i}$, that is,

$$\tau_L(f_{V_i})_{M_i} = \sum_{L_i \in \mathscr{L}(M_i)} \mathrm{e}_{M_i}^G(L, L_i) \tau_{M_i}^{L_i}(f_{V_i})_{L_i}.$$

for $i = 1.2$. It follows then that (4.19) can be written as the sum over $L \in \mathscr{L}^0(M)$, $M_1, M_2 \in \mathscr{L}(M)$, $L_1 \in \mathscr{L}(M_1)$, and $L_2 \in \mathscr{L}(M_2)$ of

$$\mathrm{e}_M^L(M_1, M_2) \mathrm{e}_{M_1}^G(L, L_1) \mathrm{e}_{M_2}^G(L, L_2) \hat{\hat{S}}_M^{M_1}(\delta_{V_1}, \tau_{M_1}^{L_1}(f_{V_1})_{L_1}) \hat{\hat{S}}_M^{M_2}(\delta_{V_2}, \tau_{M_2}^{L_2}(f_{V_2})_{L_2}).$$

The only part that depends on L is

$$\sum_{L \in \mathscr{L}^0(M)} \mathrm{e}_M^L(M_1, M_2) \mathrm{e}_{M_1}^G(L, L_1) \mathrm{e}_{M_2}^G(L, L_2). \tag{4.20}$$

It involves the product of the cardinalities

$$|Z(\hat{M}_1)^\Gamma \cap Z(\hat{M}_2)^\Gamma / Z(\hat{L})^\Gamma||Z(\hat{L})^\Gamma \cap Z(\hat{L}_1)^\Gamma / Z(\hat{G})^\Gamma||Z(\hat{L})^\Gamma \cap Z(\hat{L}_2)^\Gamma / Z(\hat{G})^\Gamma|.$$

It follows from [MW16b, VI.4.2] that this is equal to

$$|Z(\hat{L}_1)^\Gamma \cap Z(\hat{L}_2)^\Gamma / Z(\hat{G})^\Gamma|,$$

and hence (4.20) equals $\mathrm{e}_M^G(L_1, L_2)$, so that we may write (4.19) as

$$\sum_{L_1, L_2 \in \mathscr{L}(M)} \sum_{\substack{M_i \in \mathscr{L}^{L_i}(M) \\ M_i \neq L_i, i=1,2}} \mathrm{e}_M^G(L_1, L_2) \hat{\hat{S}}_M^{M_1}(\delta_{V_1}, \tau_{M_1}^{L_1}(f_{V_1})_{L_1}) \hat{\hat{S}}_M^{M_2}(\delta_{V_2}, \tau_{M_2}^{L_2}(f_{V_2})_{L_2}).$$

Then applying the inductive definition as before, we obtain

$$\sum_{L_1, L_2 \in \mathscr{L}(M)} \mathrm{e}_M^G(L_1, L_2) \hat{\hat{S}}_M^{L_1}(\delta_{V_1}, f_{V_1, L_1}) \hat{\hat{S}}_M^{L_2}(\delta_{V_2}, f_{V_2, L_2}).$$

as required. $\square$

5. Global expansions

We are now ready to apply the local study to the global endoscopic and stable trace formulas. We are assuming that G is a K-group over a global field F. Given $f \in \mathscr{H}(G,V,\zeta)$, define inductively

$$I^{\mathscr{E}}(f) = \sum_{G' \in \mathscr{E}^0(G,V)} \iota(G,G')\hat{S}'(f') + \varepsilon(G)S^G(f)$$

for linear forms $\hat{S}' = \hat{S}^{\tilde{G}'}$ on the spaces $\mathscr{S}(\tilde{G}',V,\tilde{\zeta}')$ defined inductively with the supplementary requirement that

$$I^{\mathscr{E}}(f) = I(f)$$

in the case that G is quasisplit. The distribution $S^G(f)$ is to be regarded as the stable part of $I(f)$, while the other terms can be considered as error terms. If G is quasisplit, in which case we take $G = G^*$, we write

$$S(f) = S^G(f) = I(f) - \sum_{G' \in \mathscr{E}^0(G,V)} \iota(G,G')\hat{S}'(f'),$$

which represents the stable trace formula.

We then define the modified linear forms on $\mathscr{H}(G,V,\zeta)$,

$$\tilde{I}^{\mathscr{E}}(f) = I^{\mathscr{E}}(f) - \sum_{L \in \mathscr{L}^0} |W_0^L||W_0^G|^{-1}\hat{\tilde{I}}^{L,\mathscr{E}}(\tau_L(f))$$

if G is arbitrary, and

$$\tilde{S}(f) = S(f) - \sum_{L \in \mathscr{L}^0} |W_0^L||W_0^G|^{-1}\hat{\tilde{S}}^L(\tau_L(f))$$

if G is quasisplit over F. Our present goal is to examine the spectral and geometric expansions of these new distributions.

5.1. **The spectral side.** To describe the global expansions further, we first have to recall the global coefficients. We shall describe them in some detail here, as we will also require them again later. We first consider families

$$c^V = \{c_v : v \notin V\}$$

of semisimple conjugacy classes c_v in the local L-group ${}^LG_v = G^\vee \rtimes W_{F_v}$, whose image in the local Weil group W_{F_v} is a Frobenius element. We let $C(G^V,\zeta^V)$ be the set of families c^V satisfying the requirement that each c_v is compatible with ζ_v in the sense that the image of c_v under the projection ${}^LG_v \to {}^LZ_v$ gives the unramified Langlands parameter of ζ_v. Moreover, we require that for any $G^\vee$ invariant polynomial A on ${}^LG$, we have that c satisfies the estimate

$$|A(c_v)| \leq q_v^{r_A}$$

for some $r_A > 0$ and for any $v \notin V$. By the Satake transform, any element $c \in \mathscr{C}(G^V,\zeta^V)$ can be identified with a K^V-unramified representation $\pi^V(c)$ in $\Pi(G^V,\zeta^V)$. Given $c \in \mathscr{C}(G^V,\zeta^V)$ and $\pi \in \Pi(G_V,\zeta_V)$, we write

$$\pi \times c = \pi \otimes \pi^V(c)$$

for the associated representation in $\Pi(G(\mathbf{A}),\zeta)$. If π belongs to $\Pi(G_V^Z,\zeta)$, then $\pi \times c$ belongs to the quotient $\Pi(G(\mathbf{A})^1,\zeta)$. We also define $C^V_{\mathrm{disc}}(G,\zeta)$ for the set of $c \in C(G^V,\zeta^V)$ such that $\pi \times c$ belongs to $\Pi_{\mathrm{disc}}(G,\zeta)$ for some $\pi \in \Pi_{\mathrm{disc}}(G,V,\zeta)$.

For $c \in C(G^V, \zeta^V)$ and a finite-dimensional representation ρ of ${}^LG$, we can form the Euler product

$$L(s, c, \rho) = \prod_{v \notin V} \det(1 - \rho(c_v) q_v^{-s})^{-1},$$

where $s \in \mathscr{C}$ and q_v is the cardinality of the residue field of F_v. There is a natural action of $\lambda \in \mathfrak{a}^*_{G,Z}$ on $c \to c_\lambda$, which we write as $c \mapsto c_\lambda$. Let $M \in \mathscr{L}$ and $M^\vee \subset G^\vee$ be a dual Levi subgroup. Then there is a bijection $P \mapsto P^\vee$ from $\mathscr{P}(M)$ to $\mathscr{P}(M^\vee)$, the set of Γ-stable parabolic subgroups of $G^\vee$ with Levi component $M^\vee$. Given $P, Q \in \mathscr{P}(M)$, let $\rho_{Q|P}$ denote the adjoint representation of ${}^LM$ the Lie algebra of the intersection of the unipotent radicals of $P^\vee$ and opposite $\bar{Q}^\vee$. We can then define the normalizing factors

$$r_{Q|P}(c_\lambda) = L(0, c_\lambda, \rho_{Q|P}) L(1, c_\lambda, \rho_{Q|P})^{-1}$$

and form the (G, M)-family

$$r_Q(\Lambda, c_\lambda) = r_{Q|\bar{Q}}(c_\lambda)^{-1} r_{Q|\bar{Q}}(c_{\lambda+\Lambda/2})$$

for $Q \in \mathscr{P}(M)$ and $\Lambda \in i\mathfrak{a}^*_M$.

To define the global spectral coefficient $a^G(\pi)$ on $\Pi(G^Z_V, \zeta_V)$, we set

$$a^G(\pi) = \sum_{L \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{c \in C^V_{\text{disc}}(M,\zeta)} a^M_{\text{disc}}(\pi_M \times c) r^G_M(c)$$

where for each c, the product $\pi_M \times c$ represents a finite sum of representations $\dot{\pi}$ in $\Pi_{\text{unit}}(M(\mathbf{A}), \zeta)$ and $a^M_{\text{disc}}(\pi_M \times c)$ is the corresponding sum of spectral coefficients $a^M_{\text{disc}}(\dot{\pi})$ defined in [Art88b, §4]. It follows from the definition that $a^G(\pi)$ is supported on the subset $\Pi(G, V, \zeta)$ of $\Pi(G^Z_V, \zeta)$. We fix a Borel measure $d\pi$ on $\Pi(G, V, \zeta)$ by requiring that

$$\int_{\Pi(G,V,\zeta)} h(\pi) d\pi = \sum_{M \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\rho \in \Pi_{\text{disc}}(M, V\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\rho^G_\lambda) d\lambda$$

for any $h \in C_c(\Pi(G, V, \zeta)$. If π lies in $\Pi^{\mathscr{E}}(G^Z_V, \zeta_V)$, we set

$$a^{G,\mathscr{E}}(\pi) = \sum_{G'} \sum_{\phi'} \iota(G, G') b^{\tilde{G}'}(\phi') \Delta_G(\phi', \pi) + \varepsilon(G) \sum_{\phi} b^G(\phi) \Delta_G(\phi, \pi),$$

with G', ϕ', and ϕ summed over $\mathscr{E}^0_{\text{ell}}(G, V)$, $\Phi((\tilde{G}'_V, \tilde{\zeta}'_V)$ and $\Phi^{\mathscr{E}}(G^Z_V, \zeta_V)$ respectively, and the coefficients $b^{\tilde{G}'}(\phi)$ defined inductively with the requirement that

$$a^{G,\mathscr{E}}(\pi) = a^G(\pi)$$

in the case that G is quasisplit. The coefficients $a^{G,\mathscr{E}}(\pi)$ and $b^G(\phi)$ are in fact supported on the discrete subsets $\Pi^{\mathscr{E}}(G, V, \zeta)$ and $\Phi(G, V, \zeta)$ respectively. These spaces come with corresponding Borel measures $d\pi$ and $d\phi$, given by

$$\int_{\Pi^{\mathscr{E}}(G,V,\zeta)} h(\pi) d\pi = \sum_{M \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\rho \in \Pi^{\mathscr{E}}_{\text{disc}}(M, V\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\rho^G_\lambda) d\lambda$$

for any $h \in C_c(\Pi^{\mathscr{E}}(G, V, \zeta)$, and

$$\int_{\Phi(G,V,\zeta)} h(\phi) d\phi = \sum_{M \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\chi \in \Phi_{\text{disc}}(M, V\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\chi^G_\lambda) d\lambda$$

for any $h \in C_c(\Phi(G, V, \zeta)$ respectively.

We then have for any f in $\mathscr{H}(G,V,\zeta)$ the spectral expansions of the endoscopic and stable linear forms

$$I^{\mathscr{E}}(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi(M,V,\zeta)}a^{M,\mathscr{E}}(\pi)I_M^{\mathscr{E}}(\pi,f)d\pi \tag{5.1}$$

and

$$S(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Phi(M,V,\zeta)}b^{M}(\phi)S_M(\phi,f)d\phi. \tag{5.2}$$

Due to problems of absolute convergence, in [Art02] these forms are further decomposed into a sum of linear forms

$$I^{\mathscr{E}}(f)=\sum_{t\geq 0}I_t^{\mathscr{E}}(f)$$

and similarly $S_t(f)$, where $t\geq 0$ is the norm of the imaginary part of the archimedean infinitesimal parameter associated to a representation. This is no longer strictly necessary due to the results of [FLM11].

Theorem 5.1. (a) *If G is arbitrary, then $\tilde{I}^{\mathscr{E}}(f)$ has a spectral expansion*

$$\tilde{I}^{\mathscr{E}}(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\pi)\tilde{I}_M^{\mathscr{E}}(\pi,f)d\pi. \tag{5.3}$$

(b) *If G is quasisplit, then $\tilde{S}(f)$ has a spectral expansion*

$$\tilde{S}(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Phi(M,V,\zeta)}b^{M}(\phi)\tilde{S}_M(\phi,f)d\phi \tag{5.4}$$

for any $f\in\mathscr{H}(G,V,\zeta)$.

Proof. The proofs of the two statements are the same, so we will be content with proving (a). Using the spectral expansion (5.1) of $I^{\mathscr{E}}(f)$, it follows from the definition that $\tilde{I}^{\mathscr{E}}(f)$ is equal to the difference of

$$I^{\mathscr{E}}(f)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi(M,V,\zeta)}a^{M,\mathscr{E}}(\pi)I_M^{\mathscr{E}}(\pi,f)d\pi$$

and

$$\sum_{L\in\mathscr{L}}|W_0^L||W_0^G|^{-1}\hat{\tilde{I}}^{L,\mathscr{E}}(\iota_L^{\mathscr{E}}(f)).$$

Now assume inductively that (5.3) holds for $L\in\mathscr{L}^0(M)$, so that we have

$$\tilde{I}^{L,\mathscr{E}}(g)=\sum_{M\in\mathscr{L}^L}|W_0^M||W_0^L|^{-1}\int_{\Pi^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\pi)\tilde{I}_M^{L,\mathscr{E}}(\pi,g)d\pi$$

for any $g\in\mathscr{H}(L,V,\zeta_L)$ and ζ_L the restriction of ζ to L. The sums in the expression

$$\sum_{L\in\mathscr{L}}|W_0^L||W_0^G|^{-1}\sum_{M\in\mathscr{L}^L}|W_0^M||W_0^L|^{-1}\int_{\Pi^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\pi)\tilde{I}_M^{L,\mathscr{E}}(\pi,\iota_L^{\mathscr{E}}(f))d\pi$$

can be rewritten as

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{L\in\mathscr{L}^0(M)}\int_{\Pi^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\pi)\tilde{I}_M^{L,\mathscr{E}}(\pi,\iota_L^{\mathscr{E}}(f))d\pi.$$

It follows that $\tilde{I}^{\mathscr{E}}(f)$ can be expressed as

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\pi)\left(I^{\mathscr{E}}(\pi,f)-\sum_{L\in\mathscr{L}^0(M)}\hat{\tilde{I}}_M^{L,\mathscr{E}}(\pi,\iota_L^{\mathscr{E}}(f))\right)d\pi,$$

which by (3.1) is equal to

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Pi^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\pi)\tilde{I}_M(\pi,f)d\pi,$$

which proves (5.3).

The proof of (b) follows from the spectral expansion (5.2) of $S(f)$, assuming (5.4) inductively and the definition (3.2). □

5.2. **Geometric side.** We now turn to the geometric expansion. Recall that two elements $\dot{\gamma}$ and $\dot{\gamma}_1$ in $G(F_S)$ with standard Jordan decompositions $\dot{\gamma}=c\dot{\alpha}$ and $\dot{\gamma}_1=c_1\dot{\alpha}_1$ are said to be (G,S)-equivalent if there is an element $\dot{\delta}\in G(F)$ such that $\dot{\delta}^{-1}c_1\delta=c$ and $\dot{\delta}^{-1}\alpha_1\delta$ is conjugate to α in $G_c(F_S)$. Beginning with the global geometric coefficient $a^G(S,\dot{\gamma})$ for the (G,S)-equivalence class $\dot{\gamma}\in(G(F))_{G,S}$ in [Art86, (8.1)], we define the geometric coefficient

$$a^G_{\mathrm{ell}}(\dot{\gamma})=\sum_{\{\dot{\gamma}\}}|Z(F,\dot{\gamma})|^{-1}a^G(S,\dot{\gamma})(\dot{\gamma}/\dot{\gamma}_S)$$

for any admissible element $\dot{\gamma}\in\Gamma(G_S,\zeta_S)$ in the sense of [Art02, §1]. Here $\{\dot{\gamma}\}$ runs over $Z(F)\cap Z_SZ(\mathfrak{o}^S)$-orbits in $(G(F))_{G,S}$ that map to $\dot{\gamma}_S$, and such that the $G(\mathbf{A}^S)$-conjugacy class of $\dot{\gamma}$ in $G(\mathbf{A}^S)$ meets K^S,

$$Z(F,\dot{\gamma})=\{z\in Z(F):z\dot{\gamma}=\dot{\gamma}\},$$

and $(\dot{\gamma}/\dot{\gamma}_S)$ is the ratio of the invariant measure on $\dot{\gamma}$ and the signed measure on $\dot{\gamma}$ that comes with $\dot{\gamma}_S$. The coefficient vanishes on the complement of the subset of orbital integrals $\Gamma_{\mathrm{orb}}(G_S^Z,\zeta_S)$ in $\Gamma(G_S,\zeta_S)$.

Now let $\mathcal{K}(\bar{G}_S^V)$ be the set of conjugacy classes in $\bar{G}_S^V=G_S^V/Z_S^V$ that are bounded. Any element $k\in\mathcal{K}(\bar{G}_S^V)$ induces a distribution $\gamma_S^V(k)$ in the subset $\Gamma_{\mathrm{orb}}(G_S^V,\zeta_S^V)$. Given $k\in\mathcal{K}(\bar{G}_S^V)$ and $\gamma\in\Gamma(G_V^Z,\zeta_V)$, we write

$$\gamma\times k=\gamma\times\gamma_S^V(k)$$

for the associated element in $\Gamma(G_S^Z,\zeta_S)$. We then define the unramified weighted orbital integrals

$$r_M^G(k)=J_M(\gamma_S^V(k),u_S^V)$$

as functions on $\mathcal{K}(\bar{G}_S^V)$. To define the global geometric coefficient $a^G(\gamma)$ on $\Gamma(G_V^Z,\zeta_V)$, we set

$$a^G(\gamma)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{k\in\mathcal{K}^V_{\mathrm{ell}}(\bar{M},S)}a^M_{\mathrm{ell}}(\gamma_M\times k)r_M^G(k)\tag{5.5}$$

where S is is any finite set of valuations containing V and such that the set $\gamma\times K^V$ is S-admissible. Here $\gamma\times k$ is viewed as a finite linear combination of elements $\dot{\gamma}\in\Gamma(M_S^Z,\zeta_S)$ and $a^M_{\mathrm{ell}}(\gamma_M\times k)$ is the corresponding finite linear combination of values $a^M_{\mathrm{ell}}(\dot{\gamma}_S)$.

If γ lies in $\Gamma(G_V^Z, \zeta_V)$, we set

$$a^{G,\mathscr{E}}(\gamma) = \sum_{G'} \sum_{\delta'} \iota(G, G') b^{\tilde{G}'}(\delta') \Delta_G(\delta', \gamma) + \varepsilon(G) \sum_{\delta} b^G(\delta) \Delta_G(\delta, \gamma)$$

with G', δ', and δ summed over $\mathscr{E}^0_{\text{ell}}(G, V), \Delta((\tilde{G}'_V, \tilde{\zeta}'_V)$ and $\Delta^{\mathscr{E}}(G_V^Z, \zeta_V)$ respectively, and the coefficients $b^{\tilde{G}'}(\delta)$ defined inductively with the requirement that

$$a^{G,\mathscr{E}}(\gamma) = a^G(\gamma)$$

in the case that G is quasisplit. The coefficients $a^{G,\mathscr{E}}(\gamma)$ and $b^G(\delta)$ are in fact supported on the discrete subsets $\Gamma^{\mathscr{E}}(G, V, \zeta)$ and $\Delta^{\mathscr{E}}(G, V, \zeta)$ respectively.

We have for any f in $\mathscr{H}(G, V, \zeta)$ the geometric expansions of the endoscopic and stable linear forms

$$I^{\mathscr{E}}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M, V, \zeta)} a^{M,\mathscr{E}}(\gamma) I_M^{\mathscr{E}}(\gamma, f) \tag{5.6}$$

in the case that G is arbitrary, and

$$S(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M, V, \zeta)} b^M(\delta) S_M(\delta, f) \tag{5.7}$$

in the case that G is quasisplit.

Theorem 5.2. (a) *If G is arbitrary, then $\tilde{I}^{\mathscr{E}}(f)$ has a geometric expansion*

$$\tilde{I}^{\mathscr{E}}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M, V, \zeta)} a^{M,\mathscr{E}}(\gamma) \tilde{I}_M^{\mathscr{E}}(\gamma, f). \tag{5.8}$$

(b) *If G is quasisplit, then $\tilde{S}(f)$ has a geometric expansion*

$$\tilde{S}(f) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta \in \Delta(M, V, \zeta)} b^M(\delta) \tilde{S}_M(\delta, f) \tag{5.9}$$

for any $f \in \mathscr{H}(G, V, \zeta)$.

Proof. The proofs of the two statements are the same and also parallel to Theorem 5.1, so we will be content with proving (a). Assume inductively that (5.8) holds for $L \in \mathscr{L}^0(M)$, so that we have

$$\tilde{I}^{L,\mathscr{E}}(f) = \sum_{M \in \mathscr{L}^L} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M, V, \zeta)} a^{M,\mathscr{E}}(\gamma) \tilde{I}_M^{\mathscr{E}}(\gamma, f)$$

for any $g \in \mathscr{H}(L, V, \zeta_L)$ and ζ_L the restriction of ζ to L. Using the geometric expansion (5.6) of $I^{\mathscr{E}}(f)$, it follows from the definition that $\tilde{I}^{\mathscr{E}}(f)$ is equal to the difference between

$$\sum_{M \in \mathscr{L}} |W_0^L||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M, V, \zeta)} a^{M,\mathscr{E}}(\gamma) I_M^{\mathscr{E}}(\gamma, f)$$

and

$$\sum_{L \in \mathscr{L}^0} |W_0^L||W_0^G|^{-1} \sum_{M \in \mathscr{L}^L} |W_0^M||W_0^L|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M, V, \zeta)} a^{M,\mathscr{E}}(\gamma) \hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma, \iota_L^{\mathscr{E}}(f)).$$

The second expression can be written as

$$\sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M, V, \zeta)} a^{M,\mathscr{E}}(\gamma) \sum_{L \in \mathscr{L}^0(M)} \hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma, \iota_L^{\mathscr{E}}(f)),$$

and it follows that $\tilde{I}(f)$ equals

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\gamma)\left(I_M(\gamma,f)-\sum_{L\in\mathscr{L}^0(M)}\hat{\tilde{I}}_M^{L,\mathscr{E}}(\gamma,\iota_L^{\mathscr{E}}(f))\right),$$

which by (3.12) is equal to

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma^{\mathscr{E}}(M,V,\zeta)}a^{M,\mathscr{E}}(\gamma)\tilde{I}_M(\gamma,f),$$

thus proving (5.8).

The proof of (b) follows from the geometric expansion (5.7) of $S(f)$, assuming (5.9) inductively and the definition (3.13). □

We can now put these together to obtain our first main result.

Corollary 5.3. *The linear forms $I^{\mathscr{E}}_{\mathrm{unit}}(f)$ and $S_{\mathrm{unit}}(f)$ have geometric expansions given by the geometric expansions of $\tilde{I}^{\mathscr{E}}(f)$ and $\tilde{S}(f)$ in (5.8) and (5.9) respectively.*

Proof. We first observe that the modified distributions $\tilde{I}^{\mathscr{E}}(f)$ and $\tilde{S}(f)$ differ from the original distributions $I^{\mathscr{E}}(f)$ and $S(f)$ in the contribution of the proper Levi subgroups $L\in\mathscr{L}^0$. On the spectral side, it follows from Lemma 3.1 that the latter contribution vanishes, so we have only the terms corresponding the $M=G$, namely

$$\tilde{I}^{\mathscr{E}}(f)=I^{\mathscr{E}}_{\mathrm{unit}}(f)$$

and

$$\tilde{S}(f)=S_{\mathrm{unit}}(f)$$

on the one hand as a result of Theorem 5.1, and on the hand the geometric expansions given in Theorem 5.2(a) and 5.2(b) respectively. □

Remark 5.4. We observe that the proofs of the global expansions are more or less formal, following the inductive definitions of the modified linear forms. In particular, once we have established the validity of the forms $I^{\mathscr{E}}$ and S for the larger space of non-compactly supported test functions, we shall also have the modified global expansion as above.

6. Continuity of the invariant trace formula

6.1. **The refined expansion.** Let now F be a number field, and let $\mathscr{H}(G)$ be the Hecke algebra on $G(\mathbf{A})^1$. We first recall the noninvariant linear form $J(f)$ on $\mathscr{H}(G,V,\zeta)$ established in [Art02, §2] from the original linear form on $\mathscr{H}(G)$. The noninvariant trace formula is a continuous, $Z(F)$-invariant linear form on $\mathscr{H}(G)$ consisting of two different expansions

$$J(\dot{f}^1)=\sum_{\mathfrak{o}\in\mathscr{O}}J_{\mathfrak{o}}(\dot{f}^1)=\sum_{\chi\in\mathfrak{X}}J_\chi(\dot{f}^1)$$

for any $\dot{f}^1\in\mathscr{H}(G)$. Here $\mathscr{O}$ is the set of $\mathscr{O}$-equivalence classes of element in $G(F)$, whereby two elements are equivalent if their semisimple parts are $G(F)$-conjugate, and $\mathfrak{X}$ is the set of cuspidal automorphic data $\chi=\{(P,\sigma)\}$, where P is a standard parabolic subgroup of G with Levi subgroup M_P and σ is an irreducible representation of $M_P(\mathbf{A})^1$, up to a certain equivalence relation as described in [Art82]. There is a natural projection

$$\dot{f}^1 \to \dot{f}^\zeta$$

from $\mathscr{H}(G)$ onto the space $\mathscr{H}(G,\zeta) = \mathscr{H}(G(\mathbf{A})^Z,\zeta)$ given by

$$\dot{f}^\zeta(x) = \int_{Z(\mathbf{A})^x} \dot{f}^1(zx)\zeta(zx)dz \tag{6.1}$$

where $x \in G(\mathbf{A})^Z$ and $Z(\mathbf{A})^x$ is the set of $z \in Z(\mathbf{A})$ such that $H_G(zx) = 0$. We can then define a linear form on $\mathscr{H}(G,\zeta)$ by

$$J(\dot{f}^\zeta) = J^\zeta(\dot{f}^1) = \int_{Z(F)\backslash Z(\mathbf{A})^1} J(\dot{f}^1_z)\zeta(z)dz \tag{6.2}$$

where $\dot{f}^1_z$ denotes the translation of $\dot{f}^1$ by a point $z \in Z(\mathbf{A})^1$. The integral is readily seen to be convergent, and depends only on the image $\dot{f}^\zeta$ of $\dot{f}^1$ in $\mathscr{H}(G,\zeta)$. Next, given a function $f \in \mathscr{H}(G,V,\zeta)$, we can also define a linear form on $\mathscr{H}(G,V,\zeta)$ by setting

$$J(f) = J(\dot{f})$$

where $\dot{f} = f \times u^V$ is the function in $\mathscr{H}(G,\zeta)$ that is defined as follows. We shall denote by G^V the product of groups $G(F_v)$ over places v outside of the finite set V. Then let u^V be the function on G^V with support equal to $K^V Z^V$ such that

$$u^V(kz) = \zeta(z)^{-1}, \qquad k \in K^V, z \in Z^V,$$

so that $f \to \dot{f}$ gives a map from $\mathscr{H}(G,V,\zeta)$ to $\mathscr{H}(G,\zeta)$. We then have the noninvariant linear form on $\mathscr{H}(G,V,\zeta)$ given by

$$J(f) = J(\dot{f}) = J^\zeta(\dot{f}^1)$$

where $\dot{f}$ is any function in $\mathscr{H}(G)$ whose projection $\dot{f}^\zeta$ onto $\mathscr{H}(G,\zeta)$ equals $\dot{f} = f \times u^V$. It follows from this that $J(f)$ has the parallel expansions

$$J(f) = \sum_{\mathfrak{o}\in\mathscr{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} J_\chi(f)$$

which we would like extend to a larger family of noncompactly supported test functions.

We first recall the space of functions constructed in [FLM11] extending the usual space of test functions $C_c^\infty(G) = C_c^\infty(G(\mathbf{A})^1)$. For any compact open subgroup K of $G(\mathbf{A}_f)$ the space $G(\mathbf{A})^1/K$ is a differentiable manifold. Any element $X \in \mathcal{U}(\mathfrak{g}^1)$, the universal enveloping algebra of the Lie algebra $\mathfrak{g}^1$ of $G(\mathbf{R})^1 = G(\mathbf{R}) \cap G(\mathbf{A})^1$ defines a left-invariant differentiable operator $f * X$ on $G(\mathbf{A})^1/K$. Let $\mathscr{C}^\circ(G,K)$ be the space of smooth, right-K-invariant functions on $G(\mathbf{A})^1$ which belong to $L^1(G(\mathbf{A})^1)$ together with all their derivatives. It is a Fréchet space under the seminorms

$$||f * X||_{L^1}, \qquad X \in \mathcal{U}(\mathfrak{g}^1).$$

Denote by $\mathscr{C}^\circ(G)$ the union of $\mathscr{C}^\circ(G,K)$ as K varies over open compact subgroups of $G(\mathbf{A}_f)^1$, and endow $\mathscr{C}^\circ(G)$ with the inductive limit topology. The main results of [FLM11, FL16] together imply that the (fine) spectral and (coarse) geometric expansions of $J(f)$ converge absolutely as sums over $\mathfrak{X}$ and $\mathscr{O}$ respectively, and extend from $C_c^\infty(G)$ to continuous linear forms on $\mathscr{C}^\circ(G)$.

As with the Hecke algebra, we shall also define the corresponding spaces $\mathscr{C}^\circ(G,\zeta)$ and $\mathscr{C}^\circ(G,V,\zeta)$ obtained from the spaces of ζ^{-1}-equivariant functions on $G(\mathbf{A})^Z$ and G_V^Z respectively, in a manner parallel to $\mathscr{C}^\circ(G)$. The resulting spaces are natural subspaces of the Schwartz spaces $\mathscr{C}(G), \mathscr{C}(G,\zeta)$, and $\mathscr{C}(G,V,\zeta)$ respectively. Moreover, we will again take G to be a K-group, so that

$$\mathscr{C}^\circ(G) = \bigoplus_{\alpha\in\pi_0(G)} \mathscr{C}^\circ(G_\alpha)$$

and similarly with the spaces $\mathscr{C}^\circ(G,\zeta)$ and $\mathscr{C}^\circ(G,V,\zeta)$.

Lemma 6.1. *The linear form J on $\mathscr{H}(G,V,\zeta)$ extends to a continuous linear form on $\mathscr{C}^\circ(G,V,\zeta)$.*

Proof. We follow the passage of J from $\mathscr{H}(G)$ to $\mathscr{H}(G,V,\zeta)$. There is a natural projection

$$\dot f^1 \to \dot f^\zeta$$

from $\mathscr{C}^\circ(G)$ to $\mathscr{C}^\circ(G,\zeta)$ given by the formula (6.1). Given the linear form J on $\mathscr{C}^\circ(G)$, we define a linear form on $\mathscr{C}^\circ(G,\zeta)$ by

$$J(\dot f^\zeta) = J^\zeta(\dot f^1)$$

where the right-hand side is defined as in (6.2).

Now let f be a function in $\mathscr{C}^\circ(G,V,\zeta)$. Given any function $\dot f$ in $\mathscr{C}^\circ(G)$ whose projection $\dot f^\zeta$ onto $\mathscr{C}^\circ(G,\zeta)$ equals $\dot f = f \times u^V$, we have the noninvariant linear form on $\mathscr{C}^\circ(G,V,\zeta)$ given by

$$J(f) = J(\dot f) = J^\zeta(\dot f^1)$$

as before, with both spectral and geometric sides converging absolutely. By the construction of the linear forms on each space, it follows that the form $J(f)$ on $\mathscr{C}^\circ(G,V,\zeta)$ is the continuous extension of the corresponding linear form on $\mathscr{H}(G,V,\zeta)$. □

In order to pass to the invariant trace formula, we first have to refine the expansion of the noninvariant trace formula. In particular, we need to express both sides in terms of the basic distributions $J_M(\gamma,f)$ and $J_M(\pi,f)$.

Proposition 6.2. *Let $f\in\mathscr{C}^\circ(G,V,\zeta)$. Then the linear form $J(f)$ has a geometric expansion*

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma) J_M(\gamma,f). \tag{6.3}$$

Proof. The linear form $J(f)$ obtained in Lemma 6.1 has the coarse geometric expansion

$$J(f) = \sum_{\mathfrak{o}\in\mathscr{O}} J_{\mathfrak{o}}(f)$$

with the sums converging absolutely. Let G^0 be the connected component of the identity in G, and G_c the identity component of the centralizer of a semisimple element c in $G(F)$. Then the equivalence class $\mathfrak{o}$ consists of elements in $G(F)$ whose semisimple Jordan components belong in the same $G^0(F)$ orbit. There is another equivalence relation, which depends on a finite set of places S, which we shall assume contains V. The (G,S)-equivalence classes are defined to be the sets

$$G(F)\cap(\sigma U)^{G^0(F)} = \{g^{-1}\sigma u g : g\in G^0(F), u\in U\cap G^0(F)\}$$

where σ is a semisimple element of $G^0(F)$, and U is a unipotent conjugacy class in $G_\sigma(F)$. Any class $\mathfrak{o} \in \mathscr{O}$ breaks up into a finite set $(\mathfrak{o})_{G,S}$ of (G,S)-equivalence classes.

Let $\dot{f}^1$ be any function in $\mathscr{C}^\circ(G)$ whose projection $\dot{f}^\zeta$ onto $\mathscr{C}^\circ(G,\zeta)$ equals the function $\dot{f} = f \times u^V$. Suppose moreover that

$$\dot{f}^1 = \dot{f}^1_S \times u^{S,1}, \qquad \dot{f}^1_S \in \mathscr{C}^\circ(G(F_S)^1).$$

for $S \supset V$ large enough. The space $\mathscr{C}^\circ(G(F_S)^1)$ naturally embeds in $\mathscr{C}^\circ(G)$.It follows from [Art86, Theorem 8.1] that there is an expansion

$$J_{\mathfrak{o}}(\dot{f}^1) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma} \in (M(F) \cap \mathfrak{o})_{M,S}} a^M(S,\dot{\gamma}) J_M(\dot{\gamma}, \dot{f}^1_S) \tag{6.4}$$

for any $\mathfrak{o} \in \mathscr{O}$, $\dot{f}^1_S \in C_c^\infty(G(F_S)^1)$, and S containing a finite set $S_{\mathfrak{o}}$ of valuations of F including the archimedean places. Here $J_M(\dot{\gamma}, \dot{f}^1_S)$ is the weighted orbital integral of $\dot{f}^1_S$ over the conjugacy class of $\dot{\gamma}$ in G_S, and is a tempered distribution by [Art94]. The derivation of this formula relies on a combinatorial argument and descent to unipotent weighted orbital integrals, and in particular remains valid so long as the distribution $J_{\mathfrak{o}}(\dot{f}^1)$ is absolutely convergent, and thus for $\dot{f}^1$ belonging to the larger space $\mathscr{C}^\circ(G)$.

In order to sum over the classes $\mathfrak{o} \in \mathscr{O}$, we have to modify the proof of [Art86, Theorem 9.2] and appeal to [FL16, Theorem 7.1] instead for the convergence of the sum since $\dot{f}^1$ no longer has compact support. Let

$$\mathrm{ad}(G^0(\mathbf{A}))_{\mathfrak{o}} = \{x^{-1}\gamma x : x \in G^0(\mathbf{A}), \gamma \in \mathfrak{o}\},$$

and write $\mathscr{O}_\Delta$ for the set of classes $\mathfrak{o}$ such that $\mathrm{ad}(G^0(\mathbf{A}))_{\mathfrak{o}}$ meets the support of $\dot{f}^1_S$. Since $J_{\mathfrak{o}}$ annihilates any function which vanishes on $\mathrm{ad}(G^0(\mathbf{A}))_{\mathfrak{o}}$, we obtain therefore

$$\sum_{\mathfrak{o} \in \mathscr{O}} J_{\mathfrak{o}}(\dot{f}^1) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\mathfrak{o} \in \mathscr{O}_\Delta} \sum_{\dot{\gamma} \in (M(F) \cap \mathfrak{o})_{M,S}} a^M(S,\dot{\gamma}) J_M(\dot{\gamma}, \dot{f}^1_S).$$

Now suppose that $\dot{\gamma}$ is any element of $(M(F))_{F,S}$. Then $\dot{\gamma}$ is contained in a unique class $\mathfrak{o} \in \mathscr{O}$, and it follows from [Art88c, Theorem 5.2] that $J_M(\dot{\gamma}, \dot{f}^1_S)$ vanishes if $\dot{f}^1_S$ vanishes on $\mathrm{ad}(G^0(\mathbf{A}))_{\mathfrak{o}}$, hence $J_M(\dot{\gamma}, \dot{f}^1_S)$ vanishes unless $\mathfrak{o}$ belongs to $\mathscr{O}_\Delta$. From this we have that

$$J(\dot{f}^1) = \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma} \in (M(F_S))_{F,S}} a^M(\dot{\gamma}) J_M(\dot{\gamma}, \dot{f}^1_S).$$

The rest of the argument is similar to the proof of [Art02, Proposition 2.2], so we can be brief. For a fixed set of valuations S, the linear form $J(\dot{f}^1)$ is K^S-invariant, we may then write

$$J(f) = \int_{Z(F)Z(\mathfrak{o}^S)\backslash Z(\mathbf{A})^1} J(\dot{f}^1_z)\zeta(z)dz$$

as

$$\sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma} \in (M(F))_{M,S}} a^M(S,\dot{\gamma}) \int_{Z_{S,\mathfrak{o}} \backslash Z_S^1} J_M(z\dot{\gamma}, \dot{f}^1_S)\zeta(z)dz$$

since $Z(\mathbf{A}) = Z(F)Z_S Z(\mathfrak{o}^S)$ and $J_M(\dot{\gamma}, \dot{f}^1_{S,z}) = J_M(z\dot{\gamma}, \dot{f}^1_S)$ for any $z \in Z_S$. Then using the definition of the coefficient $a^M(\gamma)$ in (5.5) it follows that the geometric expansion of $J(f)$ can be written as

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)J_M(\gamma,f)$$

as required. □

Remark 6.3. We note that we have not obtained the absolute convergence of this refined geometric expansion. For semisimple elements γ, this follows from [FL11, Theorem 1], which proves the absolute convergence of the semisimple contribution to (6.4), and by the argument above one deduces the absolute convergence of the semisimple contribution to the refined geometric expansion (6.3). As the authors point out, the absolute convergence of the unipotent contribution would require a uniform bound on the global geometric coefficients, which at present are known only for $\mathrm{GL}(n)$ [Mat15, Theorem 1.1]. Fortunately, this is not needed for the applications that we are interested in, which is the comparison of trace formulae.

We next refine the spectral expansion.

Proposition 6.4. *Let $f \in \mathscr{C}^\circ(G,V,\zeta)$. Then the linear form $J(f)$ has a spectral expansion*

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M(\pi)J_M(\pi,f)d\pi, \tag{6.5}$$

with the integrals converging absolutely.

Proof. The linear form $J(f)$ obtained in Lemma 6.1 has the fine spectral expansion

$$J(f) = \sum_{\chi\in\mathfrak{X}} J_\chi(f)$$

which converges absolutely, and where $J_\chi(f)$ is equal to the sum over $M \in \mathscr{L}$ of the product of

$$|W_0^M||W_0^G|^{-1}|\det(s-1)_{\mathfrak{a}_M^G}|^{-1}$$

with

$$\sum_{\pi\in\Pi_{\mathrm{unit}}(M,\zeta)} \sum_{L\in\mathscr{L}(M)} \sum_{s\in W^L(M)_{\mathrm{reg}}} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \mathrm{tr}(\mathscr{J}_L(P,\lambda)J_P(s,0)\mathscr{I}_{P,\chi,\pi}(\lambda,f))d\lambda,$$

as stated in [Art82, Theorem 8.2]. Here

$$\mathscr{J}_L(P,\lambda) = \lim_{\Lambda\to 0} \sum_{Q\in\mathscr{P}(M)} \mathscr{J}_Q(P,\lambda,\Lambda)\theta_Q(\Lambda)^{-1},$$

for $\Lambda \in i\mathfrak{a}_M^*$ near to 0, is the limit of (G,M)-families

$$\mathscr{J}_Q(P,\lambda,\Lambda) = J_{P|Q}(\lambda)J_{Q|P}(\lambda+\Lambda)$$

and $J_{Q|P}(\lambda)$ is the global unnormalized operator intertwining the actions of the induced representations $\mathscr{I}_P(\pi_\lambda)$ and $\mathscr{I}_Q(\pi_\lambda)$. Also

$$J(P,s) = J_{P|P}(s,0).$$

It is a consequence of [FLM11, Corollary 1] that the sums are finite and the integrals are absolutely convergent with respect to the trace norm, and define distributions on $\mathscr{C}^\circ(G)$. We note that the absolute convergence is proved for an expansion slightly different from the above, but is shown to be equivalent in [FLM11, §5.3]. Importantly, the sum over π does not occur in the latter, but the necessary estimate for this sum, which is not necessarily finite, is contained in [FLM11, §5.1]. (See also [Par19, Theorem 7.2] for the twisted case.)

Beginning with

$$J(f) = J^{\zeta}(\dot{f}^1) = \int_{Z(F)\backslash Z(\mathbf{A})^1} J(\dot{f}^1_z)\zeta(z)dz,$$

where $\dot{f}^1$ is any function in $\mathscr{C}^\circ(G)$ whose projection onto $\mathscr{C}^\circ(G,\zeta)$ equals $\dot{f} = f \times u^V$, it follows from the argument of [Art88b, Theorem 4.4] and the definition of $a^G_{\mathrm{disc}}(\dot{\pi})$ that $J(f)$ has an expansion

$$\int_{Z(F)\backslash Z(\mathbf{A})^1} \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\pi}\in\Pi_{\mathrm{disc}}(M)} \int_{i\mathfrak{a}^*_{M,Z}\backslash i\mathfrak{a}^*_{G,Z}} a^M_{\mathrm{disc}}(\dot{\pi}_\lambda) J_M(\dot{\pi}_\lambda, \dot{f}^1_z)\zeta(z)d\lambda dz$$

where

$$J_M(\dot{\pi}_\lambda, \dot{f}^1_z) = \mathrm{tr}(\mathscr{J}_M(\dot{\pi}_\lambda, P)\mathscr{I}_P(\dot{\pi}_\lambda, \dot{f}^1_z))$$

is the global unnormalized weighted character on $\mathscr{C}^\circ(G)$. It is a consequence of [FLM11, §5.1] that the inner integral converges absolutely. On the other hand, the integral over $Z(F)\backslash Z(\mathbf{A})^1$ annihilates the contribution of $\dot{\pi}$ coming from the complement of $\Pi_{\mathrm{disc}}(M,\zeta)$ in $\Pi_{\mathrm{disc}}(M)$, hence $J(f)$ equals

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\pi}\in\Pi_{\mathrm{disc}}(M,\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} a^M_{\mathrm{disc}}(\dot{\pi}_\lambda) J_M(\dot{\pi}_\lambda, \dot{f})d\lambda. \tag{6.6}$$

Then arguing as in [Art02, Proposition 3.3], it follows from the definition of $a^M(\pi)$ that the spectral expansion (6.6) equals

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M(\pi) J_M(\pi, f)d\pi$$

where

$$J_M(\pi_\lambda, f) = \mathrm{tr}(\mathscr{M}_M(\pi_\lambda, P)\mathscr{I}_P(\pi_\lambda, f)), \qquad L\in\mathscr{L}(M), P\in\mathscr{P}(L)$$

is the local normalized weighted character. It is related to the global unnormalized character by the formula

$$J_M(\dot{\pi}_\lambda, \dot{f}) = \sum_{L\in\mathscr{L}(M)} r^L_M(c_\lambda) J_L(\pi^L_\lambda, f),$$

and hence is defined for f belonging to $\mathscr{C}^\circ(G,V,\zeta)$. Also, the operator $\mathscr{J}_Q(\Lambda,\dot{\pi}_\lambda,P)$ is a scalar multiple of $\mathscr{M}_Q(\Lambda,\pi_\lambda,P)$, that is,

$$\mathscr{J}_Q(\Lambda,\dot{\pi}_\lambda,P) = r_Q(\Lambda,c_\lambda,P)\mu_Q(\Lambda,c_\lambda,P)\mathscr{M}_Q(\Lambda,\pi_\lambda,P),$$

where the coefficient $r_Q(\Lambda,c_\lambda,P)$ is defined in [Art98, §2], and it follows then that the integral over $\Pi(M,V,\zeta)$ converges absolutely. □

6.2. **The invariant expansion.** Given the noninvariant linear form J on $\mathscr{H}(G,V,\zeta)$, we have already discussed the invariant linear form I also on $\mathscr{H}(G,V,\zeta)$ obtained by setting inductively

$$I(f) = J(f) - \sum_{M\neq G} |W_0^M||W_0^G|^{-1}\hat{I}_M(\phi_M(f)) \tag{6.7}$$

for the maps ϕ_M described in (2.2).

Proposition 6.5. *The invariant linear form I on $\mathscr{H}(G,V,\zeta)$ extends to a continuous linear form on $\mathscr{C}^\circ(G,V,\zeta)$. It has the spectral and geometric expansions given by*

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1}\int_{\Pi(M,V,\zeta)} a^M(\pi)I_M(\pi,f)d\pi$$
$$= \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)I_M(\gamma,f).$$

Proof. We recall that for any $\tilde{f}\in\mathscr{C}(G_V,\zeta_V)$, the function $\phi_M(\tilde{f})$ is defined to be the function on $\Pi_{\mathrm{temp}}(M_V^Z,\zeta_V)$ whose value at $\tilde{\pi}$ is the tempered distribution $J_M(\tilde{\pi},\tilde{f})$ [Art94, §2], and

$$\phi_M(f,\pi) = \int_{i\mathfrak{a}^*_{M,Z}} \phi_M(\tilde{f},\tilde{\pi})d\lambda$$

where f and π are the restrictions of $\tilde{f}$ and $\tilde{\pi}$ to G_V^Z and M_V^Z respectively. We also define

$$\phi_M(\tilde{f},\tilde{\pi},X) = J_M(\tilde{f},\tilde{\pi},X), \qquad X\in\mathfrak{a}_{M,V}$$

and $\phi_M(f,\pi,X)$ using the definition (2.1). In this case, it follows from [Art98, Lemma 3.1] that ϕ_M maps $\mathscr{C}(G_V^Z,\zeta_V)$ continuously to $I\mathscr{C}(G_V^Z,\zeta_V)$. For general π in $\Pi(M,V,\zeta)$, it follows from the proof of Proposition 6.4 that $J_M(\pi,f)$ is well-defined for $f\in\mathscr{C}^\circ(G,V,\zeta)$, and moreover the integral

$$J_M(\tilde{f},\tilde{\pi},X) = \int_{i\mathfrak{a}^*_{M,V}/i\mathfrak{a}^*_{G,V}} J_M(\tilde{\pi}_\lambda,\tilde{f}^Z)e^{-\lambda(X)}d\lambda$$

converges absolutely. Here Z is the image in $\mathfrak{a}_{G,V}$ of X.

On the other hand, the weighted orbital integrals $J_M(\gamma,f)$ are tempered distributions on $\mathscr{C}(G,V,\zeta)$ as a consequence of [Art94, Theorem 4.1]. Altogether, it follows that the invariant distributions defined inductively by

$$I_M(\pi,f) = J_M(\pi,f) - \sum_{L\in\mathscr{L}^0(M)} \hat{I}_M^L(\pi,\phi_L(f))$$

and

$$I_M(\gamma,f) = J_M(\gamma,f) - \sum_{L\in\mathscr{L}^0(M)} \hat{I}_M^L(\gamma,\phi_L(f))$$

on either side of the invariant trace formula hold for functions f in $\mathscr{C}^\circ(G,V,\zeta)$.

Beginning with the linear form J on $\mathscr{C}^\circ(G,V,\zeta)$, we define the invariant linear form I as in (6.7). We can see that the absolute value of $I(f)$ extends to a continuous linear form on $\mathscr{C}^\circ(G,V,\zeta)$, by assuming inductively that the statement holds for $L\in\mathscr{L}^0$ then applying the continuity of the map ϕ_M on $\mathscr{C}(G_V^Z,\zeta_V)$ and the linear form J. But we shall also arrive at the same conclusion once we have obtained the desired expansions. Let us first show that $I(f)$ has the geometric expansion

$$I(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1}\sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma)I_M(\gamma,f).$$

Assume inductively that the required expansion holds if G is replaced by any group $L\in\mathscr{L}^0$. Combining this with the geometric expansion (6.3) for J, we see then that $I(f)$ equals

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(M,V,\zeta)} a^M(\gamma) \left(J_M(\gamma,f) - \sum_{L\in\mathscr{L}^0(M)} \hat{I}_M^L(\gamma,f) \right),$$

and by definition of $I_M(\gamma,f)$ this yields the required geometric expansion for $I(f)$. On the other hand, the spectral expansion

$$I(f) = \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M(\pi) I_M(\pi,f) d\pi$$

follows in a similar manner. That is, assuming inductively that the required identity holds for $L\in\mathscr{L}^0$, and using the spectral expansion (6.5) for J it follows that $I(f)$ equals

$$\sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M,V,\zeta)} a^M(\gamma) \left(J_M(\pi,f) - \sum_{L\in\mathscr{L}^0(M)} \hat{I}_M^L(\pi,f) \right) d\pi.$$

Then by definition of $I_M(\pi,f)$ this yields the required spectral expansion for $I(f)$. □

7. Continuity of the stable trace formula

7.1. **Transfer.** We now turn to the stabilization. The Langlands-Shelstad transfer conjecture states that for any $G'_v\in\mathscr{E}(G_v)$, the map that sends $f\in\mathscr{H}(G_v,\zeta_v)$ to the function

$$f'(\delta') = f^{G'}(\delta') = \sum_{\gamma\in\Gamma_{\mathrm{reg}}(G_v,\zeta_v)} \Delta(\delta',\gamma) f_G(\gamma)$$

on $\Delta_{\mathrm{reg}}(\tilde{G}'_v,\tilde{\zeta}'_v)$ exists, and maps $\mathscr{H}(G_v,\zeta_v)$ continuously to the space $\mathscr{S}(\tilde{G}'_v,\tilde{\zeta}'_v)$. In the nonarchimedean case, the Langlands-Shelstad transfer was proved for smooth functions $f\in C_c^\infty(G_v)$ as a consequence of [Wal97] and the solution of the Fundamental Lemma [Ngô10], and thus holds also for ζ^{-1}-equivariant space $C_c^\infty(G_v,\zeta_v)$. Moreover, since the orbital integrals are tempered distributions, it makes sense to formulate the smooth transfer for the larger Schwartz space $\mathscr{C}(G_v,\zeta_v)$, in which case the transfer would lie in the corresponding space of stable orbital integrals $S\mathscr{C}(\tilde{G}'_v,\tilde{\zeta}'_v)$ [Art99, §3]. Recall that we are taking G to be a K-group, so if f equals $\oplus_\alpha f_\alpha$, then

$$f' = \sum_{\alpha\in\pi_0(G)} f'_\alpha.$$

The Langlands-Shelstad transfer for Schwartz functions is then a simple consequence of the smooth transfer. We note that in the archimedean case, the result follows from work of Shelstad (c.f. [She08]).

Lemma 7.1. *Let F_v be a nonarchimedean local field. Then for $f\in\mathscr{C}(G_v,\zeta_v)$, the map from f to the function*

$$f'(\delta') = f^{G'}(\delta') = \sum_{\gamma\in\Gamma_{\mathrm{reg}}(G_v,\zeta_v)} \Delta(\delta',\gamma) f_G(\gamma)$$

on $\Delta_{reg}(\tilde{G}'_v,\tilde{\zeta}'_v)$ exists, and maps $\mathscr{C}(G_v,\zeta_v)$ continuously to $S\mathscr{C}(G_v,\zeta_v)$.

Proof. The proof relies on the fact that $C_c^\infty(G_v,\zeta_v)$ is a dense subspace of $\mathscr{C}(G_v,\zeta_v)$. Given f in $\mathscr{C}(G_v,\zeta_v)$, we may choose a sequence (f_n) in $C_c^\infty(G_v,\zeta_v)$ converging to f as n tends to infinity. Applying the Langlands-Shelstad transfer, it follows then that there is a family of transfers (f_n') in $\mathscr{C}(\tilde{G}_v',\tilde{\zeta}_v')$ such that for any $\delta'\in\Delta_{\mathrm{reg}}(\tilde{G}_v',\tilde{\zeta}_v')$, we have

$$f_n'(\delta')=\sum_{\gamma\in\Gamma_{\mathrm{reg}}(G_v,\zeta_v)}\Delta(\delta',\gamma)f_{n,G}(\gamma)$$

in $\mathscr{S}(\tilde{G}_v',\tilde{\zeta}_v')$.

Estimating then the difference

$$|f_n'(\delta')-f_{n+1}'(\delta')|\le\sum_{\gamma\in\Gamma_{\mathrm{reg}}(G_v,\zeta_v)}|\Delta(\delta',\gamma)||f_{n,G}(\gamma)-f_{n+1,G}(\gamma)|$$

for any fixed δ', where we note that the sums are finite since the orbital integral of f_n is compactly supported on the regular set for any n, we see that the difference

$$|f_{n,G}(\gamma)-f_{n+1,G}(\gamma)|$$

converges in the space of orbital integrals of functions in $C_c^\infty(G_v,\zeta_v)$. It follows that $f_n(\delta')$ converges in $\mathscr{S}(\tilde{G}_v',\tilde{\zeta}_v')$, and by continuity in $S\mathscr{C}(\tilde{G}_v',\tilde{\zeta}_v')$. By completeness, we denote by f' the function in $\mathscr{C}(\tilde{G}_v',\tilde{\zeta}_v')$ such that f_n' converges to f'. We note that the choice of f' is unique only up to stable conjugacy, and satisfies the identity

$$f'(\delta')=\sum_{\gamma\in\Gamma_{\mathrm{reg}}(G_v,\zeta_v)}\Delta(\delta',\gamma)f_G(\gamma).$$

as required. □

The stabilization of the trace formula relies on the local results of Arthur on orbital integrals such as in [Art96, Art99, Art06, Art08, Art16]. In order to stabilize the invariant linear form $I(f)$ for $f\in\mathscr{C}^\circ(G,V,\zeta)$, we note that these results hold for general Schwartz functions $f\in\mathscr{C}(G,V,\zeta)$ either as explicitly stated, or otherwise can be shown using the fact that the linear forms $I_M(\gamma,f)$ extend to tempered distributions on G. These local transfer mappings are required to construct the stable basis $\Delta(G_V^Z,\zeta_V)$ that is used to index the geometric side.

We have to show that this construction holds in our case also. While Arthur's stabilization is carried out for functions f belonging to $\mathscr{H}(G,V,\zeta)$, his construction of these spaces holds generally for functions in $\mathscr{C}(G,V,\zeta)$, as long as the transfer mappings exist. Note also that the twisted stabilization of Moeglin and Waldspurger is proved for functions in $C_c^\infty(G)$, given the existence of smooth transfer. We shall summarize Athur's construction here, extended to the slightly more general setting of $\mathscr{C}^\circ(G,V,\zeta)$.

7.2. **Geometric transfer factors.** As usual, if S' is a stable, tempered $\tilde{\zeta}'$-equivariant distribution on $\tilde{G}'(F_v)$, then we write $\hat{S}'$ for the corresponding continuous linear form on $S\mathscr{C}(\tilde{G}_v',\tilde{\zeta}_v')$. Applying the transfer to each of the components G_{α_v} of G_v, we have a mapping

$$f_v\to f_v'=f_v^{\tilde{G}'}$$

from $\mathscr{C}(G_v,\zeta_v)$ to $S\mathscr{C}(\tilde{G}_v',\tilde{\zeta}_v')$, which can be identified with a mapping

$$a_v\to a_v'$$

from $I\mathscr{C}(G_v, \zeta_v)$ to $S\mathscr{C}(\tilde{G}'_v, \tilde{\zeta}'_v)$. It follows that the product mapping from $\prod_v a_v$ to $\prod_v a'_v$ gives a linear mapping from $I\mathscr{C}(G_V, \zeta_V)$ to $S\mathscr{C}(\tilde{G}'_V, \tilde{\zeta}'_V)$. This mapping is attached to the product G'_V of the data G'_v, which we can think of as the endoscopic data of G over F_V. Letting G'_V vary, we obtain a mapping

$$I\mathscr{C}(G_V, \zeta_V) \to \prod_{G'_V} S\mathscr{C}(\tilde{G}'_V, \tilde{\zeta}'_V)$$

by putting together the individual images of a'. The image $I\mathscr{C}^{\mathscr{E}}(G_V, \zeta_V)$ of $I\mathscr{C}(G_V, \zeta_V)$ fits into a sequence of inclusions

$$I\mathscr{C}^{\mathscr{E}}(G_V, \zeta_V) \subset \bigoplus_{\{G'_V\}} I\mathscr{C}^{\mathscr{E}}(G'_V, G_V, \zeta_V) \subset \prod_{\Delta_V} S\mathscr{C}(\tilde{G}'_V, \tilde{\zeta}'_V) \tag{7.1}$$

in which the summand $I\mathscr{C}^{\mathscr{E}}(G'_V, G_V, \zeta_V)$ is a vector space of families of functions on $\tilde{G}'$ parametrized by transfer factors for G and $\tilde{G}'$, depending only on the F_V-isomorphism class of G'_V.

The mappings of functions have dual analogues for distributions. Given G'_V with auxiliary data $\tilde{G}'_V$ and $\tilde{\xi}'_V$, assume that δ' belongs to the space of stable distributions $S\mathscr{D}((\tilde{G}'_V)^{\tilde{Z}'_V}, \tilde{\zeta}'_V)$. By Lemma 7.1, we may evaluate the transfer f' of any function f in $\mathscr{C}^\circ(G, V, \zeta)$ at δ'. Since the distribution $f \to f'(\delta')$ belongs to $\mathscr{D}(G_V^Z, \zeta_V)$, we can construct the extended geometric transfer factors at each local place

$$\Delta(\delta', \gamma), \qquad G' \in \mathscr{E}(G), \delta' \in \Delta(\tilde{G}', \tilde{\zeta}'), \gamma \in \Gamma(G, \zeta).$$

defined for fixed bases $\Delta(\tilde{G}', \tilde{\zeta}')$ of the spaces $S\mathscr{D}(\tilde{G}', \tilde{\zeta}')$ such that

$$f'(\delta') = \sum_{\gamma \in \Gamma(G,\zeta)} \Delta(\delta', \gamma) f_G(\gamma) \tag{7.2}$$

holds for $\delta' \in \Delta(\tilde{G}', \tilde{\zeta}')$ and $f \in \mathscr{C}(G, \zeta)$. We can see that the extended local transfer factor, as a function on $\Delta(\tilde{G}', \tilde{\zeta}') \times \Gamma(G, \zeta)$ is defined in the exact same manner as [Art02, §4], and depends linearly on δ'. We can then define the global transfer factor as the corresponding product

$$\Delta(\delta, \gamma) = \prod_{v \in V} \Delta(\delta_v, \gamma_v)$$

for $\delta \in \Delta^{\mathscr{E}}(G_V, \zeta_V)$ and $\gamma \in \Gamma(G_V, \zeta_V)$. The sequence of inclusions (7.1) is dual to a sequence of surjective linear mappings

$$\prod_{G'_V} S\mathscr{D}((\tilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V) \to \bigoplus_{\{G'_V\}} \mathscr{D}^{\mathscr{E}}(G'_V, G_V^Z, \zeta_V) \to \mathscr{D}^{\mathscr{E}}(G_V^Z, \zeta_V) \tag{7.3}$$

between spaces of distributions. Since f' is the image of the function f_G in $I\mathscr{C}(G, V, \zeta)$, it follows that $f'(\delta')$ depends only on the image δ of δ' in $\mathscr{D}^{\mathscr{E}}(G_V^Z, \zeta_V)$. In other words,

$$f'(\delta') = f_G^{\mathscr{E}}(\delta),$$

where $f_G^{\mathscr{E}}$ is the image of f_G in $I\mathscr{C}^{\mathscr{E}}(G, V, \zeta)$, so that by the adjoint relations (3.10) and (3.11) the map $f_G \to f_G^{\mathscr{E}}$ is an isomorphism. The same is true therefore of the coefficients $\Delta_G(\delta', \gamma)$, so we may write

$$\Delta_G(\delta, \gamma) = \Delta_G(\delta', \gamma)$$

for $\gamma \in \Gamma(G_V^Z, \zeta_V)$ and complex numbers $\Delta_G(\delta, \gamma)$ that depend linearly on $\delta \in \mathscr{D}^{\mathscr{E}}(G_V^Z, \zeta_V)$. The image in $\Delta^{\mathscr{E}}(G_V^Z, \zeta_V)$ of the subspace

$$S\mathscr{D}((G_V^*)^{Z_V^*}, \zeta_V^*) \simeq S\mathscr{D}(G_V^*, G_V^Z, \zeta_V)$$

can be identified with the subspace $S\mathscr{D}(G_V^Z, \zeta_V)$ of stable distributions in $\mathscr{D}(G_V^Z, \zeta_V)$.

The coefficients in the geometric expansion should really be regarded as elements in the appropriate completion of $\mathscr{D}(M_V^Z, \zeta_V)$ and $S\mathscr{D}(M_V^Z, \zeta_V)$, which we shall identify with the dual space of $\mathscr{D}(M_V^Z, \zeta_V)$ by fixing suitable bases $\Gamma(M_V^Z, \zeta_V)$ and $\Delta(M_V^Z, \zeta_V)$ of the relevant spaces of distributions. In particular, we shall fix a basis $\Delta((\tilde{G}'_V)^{\tilde{Z}'_V}, \tilde{\zeta}'_V)$ of $S\mathscr{D}((\tilde{G}'_V)^{\tilde{Z}'_V}, \tilde{\zeta}'_V)$ for any F_V-endoscopic datum G'_V with auxiliary data $\tilde{G}'_V$ and $\tilde{\xi}'_V$. We also fix a basis $\Delta^{\mathscr{E}}(G_V^Z, \zeta_V)$ of $\mathscr{D}^{\mathscr{E}}(G_V^Z, \zeta_V)$ such that

$$\Delta(G_V^Z, \zeta_V) = \Delta^{\mathscr{E}}(G_V^Z, \zeta_V) \cap S\mathscr{D}(G_V^Z, \zeta_V)$$

forms a basis of $S\mathscr{D}(G_V^Z, \zeta_V)$, and in the case that G is quasisplit, that $\Delta(G_V^Z, \zeta_V)$ is isomorphic to the image of the basis $\Delta((G_V^*)^{Z^*}, \zeta_V)$.

7.3. **Spectral transfer factors.** The construction on the spectral side is parallel. In place of the spaces of distributions described by (7.3), we have the spectral analogue $\mathscr{F}(G_V^Z, \zeta_V)$ of $\mathscr{D}(G_V^Z, \zeta_V)$, and the sequence of maps

$$\prod_{G'_V} S\mathscr{F}((\tilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V) \to \bigoplus_{\{G'_V\}} \mathscr{F}^{\mathscr{E}}(G'_V, G_V^Z, \zeta_V) \to \mathscr{F}^{\mathscr{E}}(G_V^Z, \zeta_V).$$

In place of the basis $\Gamma(G_V^Z, \zeta_V)$ of $\mathscr{D}(G_V^Z, \zeta_V)$, we have the basis

$$\Pi(G_V^Z, \zeta_V) = \coprod_{t \geq 0} \Pi_t(G_V^Z, \zeta_V)$$

of $\mathscr{F}(G_V^Z, \zeta_V)$ consisting of irreducible characters. If ϕ' belongs to $S\mathscr{F}((\tilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$, then the distribution $f \to f'(\phi')$ belongs to $\mathscr{F}(G_V^Z, \zeta_V)$, we can construct the spectral transfer factors at each local place

$$\Delta(\phi', \pi), \qquad G' \in \mathscr{E}(G), \phi' \in \Phi(\tilde{G}', \tilde{\zeta}'), \pi \in \Pi(G, \zeta)$$

defined for fixed bases $\Phi(\tilde{G}', \tilde{\zeta}')$ of the spaces $S\mathscr{F}(\tilde{G}', \tilde{\zeta}')$ such that

$$f'(\phi') = \sum_{\pi \in \Pi(G_v, \zeta_v)} \Delta(\phi', \pi) f_G(\pi)$$

holds for $\phi' \in \Phi(\tilde{G}', \tilde{\zeta}')$ and $f \in \mathscr{C}(G, \zeta)$, parallel to (7.2). We also define the corresponding product

$$\Delta(\phi, \pi) = \prod_{v \in V} \Delta(\phi_v, \pi_v)$$

for $\phi \in \Phi^{\mathscr{E}}(G_V, \zeta_V)$ and $\pi \in \Pi(G_V, \zeta_V)$. Given an element ϕ' in $S\mathscr{F}((\tilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$, We have that $f'(\phi')$ depends only on the image ϕ of ϕ' in $\mathscr{F}^{\mathscr{E}}(G_V^Z, \zeta_V)$, that is,

$$f'(\phi') = f_G^{\mathscr{E}}(\phi),$$

and the spectral coefficients satisfy the relation

$$\Delta_G(\phi, \pi) = \Delta_G(\phi', \pi)$$

for $\pi \in \Pi(G_V^Z, \zeta_V)$ and complex numbers $\Delta_G(\phi, \pi)$ that depend linearly on $\phi \in \mathscr{F}^{\mathscr{E}}(G_V^Z, \zeta_V)$. They satisfy adjoint relations parallel to the geometric transfer factors. Here as in [Art02, §5] we shall fix an endoscopic basis $\Phi^{\mathscr{E}}(G_V^Z, \zeta_V)$ of $\mathscr{F}(G_V^Z, \zeta_V)$, and a subset

$$\Phi(G_V^Z, \zeta_V) = \Phi^{\mathscr{E}}(G_V^Z, \zeta_V) \cap S\mathscr{F}(G_V^Z, \zeta_V)$$

that forms a basis of $S\mathscr{F}(G_V^Z, \zeta_V)$, and in the case that G is quasisplit, such that $\Phi(G_V^Z, \zeta_V)$ is isomorphic to the image of the basis $\Phi((G_V^*)^{Z^*}, \zeta_V)$. If v is archimedean, we can identify $\Phi(\tilde{G}'_v, \tilde{\zeta}'_v)$ with the relevant set of Langlands parameters. If v is nonarchimedean, we construct $\Phi(\tilde{G}'_v, \tilde{\zeta}'_v)$ in terms of abstract bases $\Phi_{\mathrm{ell}}(G_v, \zeta_v)$ of the cuspidal subspaces $\mathscr{S}_{\mathrm{cusp}}(M_v, \zeta_v)$, and similar objects for endoscopic groups M' of M, where we observe that the relevant constructions of [Art96] extend readily to $\mathscr{C}(G_v, \zeta_v)$ (see [Art03, p.825]).

7.4. **The stable and endoscopic expansions.** Having defined the relevant objects, we now turn to the continuity of the stable trace formula. As before, our attention will be on extending the arguments in [Art02, Art01, Art03], which will essentially follow from properly constructing the natural generalizations of the required objects. As the stabilization of the trace formula involves a much more intricate argument than that needed for the invariant trace formula, we are forced to follow the same path here. We note that a similar argument is provided in [MW16a, MW16b] for the stabilization of the twisted trace formula.

Theorem 7.2. *The linear forms $I^{\mathscr{E}}$ and S extend continuously from $\mathscr{H}(G, V, \zeta)$ to $\mathscr{C}^\circ(G, V, \zeta)$.*

Proof. We first observe that Global Theorem $1'$ in [Art02, §7] states that the global geometric coefficients satisfy

$$a^{G,\mathscr{E}}(\gamma) = a^G(\gamma), \qquad \gamma \in \Gamma^{\mathscr{E}}(G, V, \zeta)$$

for any G, and that

$$b^G(\delta), \qquad \delta \in \Delta^{\mathscr{E}}(G, V, \zeta)$$

vanishes on the complement of $\Delta(G, V, \zeta)$ if G is quasisplit. Notice that $\Gamma^{\mathscr{E}}(G, V, \zeta)$ and $\Delta(G, V, \zeta)$ are constructed as subsets of bases $\Gamma(G_V^Z, \zeta_V)$ and $\Delta(G_V^Z, \zeta_V)$ of the spaces $\mathscr{D}(G_V^Z, \zeta_V)$ and $S\mathscr{D}(G_V^Z, \zeta_V)$ respectively. In particular, we see that this space contains the orbital integrals, and also derivatives of orbital integrals in the archimedean cases, of functions f in $\mathscr{C}^\circ(G, V, \zeta)$. Similarly, Global Theorem $2'$ states that the global geometric coefficients satisfy

$$a^{G,\mathscr{E}}(\pi) = a^G(\pi), \qquad \pi \in \Pi_t^{\mathscr{E}}(G, V, \zeta)$$

for any G, and that

$$b^G(\phi), \qquad \delta \in \Phi_t^{\mathscr{E}}(G, V, \zeta)$$

vanishes on the complement of $\Phi_t(G, V, \zeta)$ if G is quasisplit. Here the spaces

$$\Pi_t^{\mathscr{E}}(G, V, \zeta), \quad \Phi_t^{\mathscr{E}}(G, V, \zeta), \quad \Phi_t(G, V, \zeta)$$

are the subset of elements in

$$\Pi^{\mathscr{E}}(G, V, \zeta), \quad \Phi^{\mathscr{E}}(G, V, \zeta), \quad \Phi(G, V, \zeta)$$

respectively whose archimedean infinitesimal characters ν have norms $t = ||\mathrm{Im}(\nu)||$. Notice that $\Phi_t(G, V, \zeta)$ and $\Pi_t^{\mathscr{E}}(G, V, \zeta)$ are constructed as discrete subsets of the bases $\Pi_t^{\mathscr{E}}(G_V^Z, \zeta_V)$ and $\Phi_t(G_V^Z, \zeta_V)$ of the spaces $\mathscr{F}(G_V^Z, \zeta_V)$ and $S\mathscr{F}(G_V^Z, \zeta_V)$ respectively. As we have indicated above, in both the geometric and spectral cases, the construction of the endoscopic spaces implicitly rely on the Langlands-Shelstad transfer, hence by Lemma 7.1 these spaces exist unconditionally.

Let S be a finite set of valuations containing V. There is a natural map

$$f \mapsto \dot{f}_S = f \times u_S^V$$

from $\mathscr{C}^\circ(G,V,\zeta)$ to $\mathscr{C}^\circ(G,S,\zeta)$. We shall define an admissible subspace $\mathscr{C}^\circ_{\mathrm{adm}}(G,S,\zeta)$ of $\mathscr{C}^\circ(G,S,\zeta)$, using the same notion of admissibility in [Art02, §1]. The polynomial $\det(1+t-\mathrm{Ad}(x)) = \sum_k D_k(x)t^k$ for $x \in G$ defines a morphism

$$\mathcal{D} = (D_0,\dots,D_d) : G \to \mathbf{G}_a^{d+1}$$

where $d = \dim G$. If X is a nonzero point in $\mathbf{G}_a^{d+1}$, we shall denote $X_{\min} = X_k$ where k is the smallest integer such that X_k is nonzero. Let $\mathscr{O}^S$ be product of all v not in S of $\mathscr{O}_v$, the ring of integers of F_v. We call a subset C_S of $F_S^{d+1}\backslash\{0\}$ admissible if any point X in the intersection

$$F^{d+1} \cap (C_S \times (\mathscr{O}^S)^{d+1})$$

satisfies $|X_{\min}|_v = 1$ for all $v \notin S$. Assume moreover that S contains the places over which G and ζ are ramified, and that $Z(A) = Z(F)Z_S Z(\mathscr{O}^S)$. Then we call a subset Δ_S of G_S admissible if $\mathcal{D}(\Delta_S)$ is admissible in F_S^{d+1}. This implies that

$$|D(\dot{\gamma})|_v = 1$$

for all $\gamma \in G(F) \cap (\Delta_S \times K^S)$ and $v \notin S$. Also, Δ_S is admissible if and only if its projection onto $\bar{G}_S = G_S/Z_S$ is admissible. Finally, we define $\mathscr{C}^\circ_{\mathrm{adm}}(G,S,\zeta)$ to be the subspace of functions in $\mathscr{C}^\circ(G,S,\zeta)$ whose support is admissible. Also, we shall say a subset Δ of $G(\mathbf{A})$ is S-admissible if for some finite set S there is an admissible subset C_S of F_S^{d+1} such that $\mathcal{D}(\Delta)$ is contained in $C_S \times (\mathscr{O}^S)^{d+1}$. We note that it is this condition of admissibility and S-admissibility that the reductions of [Art02, Art01, Art03] are based upon, rather than the compact support of the test functions f.

Having made these preliminary remarks, we now proceed as follows. Let I be the invariant linear form on $\mathscr{C}^\circ(G,V,\zeta)$ obtained in Proposition 6.5. If G is arbitrary, we define an endoscopic linear form inductively by setting

$$I^{\mathscr{E}}(f) = \sum_{G' \in \mathscr{E}_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}'(f')$$

for stable linear forms $\hat{S}' = \hat{S}^{\tilde{G}'}$ on $S\mathscr{C}^\circ(G,V,\zeta)$. In the case that G is quasisplit, we define a linear form

$$S^G(f) = I(f) - \sum_{G' \in \mathscr{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}'(f')$$

and also the endoscopic linear form by the trivial relation

$$I^{\mathscr{E}}(f) = I(f).$$

We assume inductively that if G is replaced by a quasisplit inner K-form of $\tilde{G}'$, the corresponding analogue of S^G is defined and stable. At this stage, the reductions of [Art02, Art01] can now be applied without difficulty. In particular, if on the geometric side, we define $I^{\mathscr{E}}_{\mathrm{orb}}(f)$ and $S^G_{\mathrm{orb}}(f)$ to be the summands corresponding to

$M = G$ in $I^{\mathscr{E}}(f)$ and $S^G(f)$ respectively, we see from the proof of [Art02, Theorem 10.1] that if G is arbitrary,

$$I^{\mathscr{E}}(f) - I^{\mathscr{E}}_{\text{orb}}(f) = \sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) I^{\mathscr{E}}_M(\gamma, f)$$

and if G is quasisplit, we have that $S^G(f) - S^G_{\text{orb}}(f)$ is equal to

$$\sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathscr{E}'_{\text{ell}}(M,V)} \iota(M, M') \sum_{\delta' \in \Delta(\tilde{M}', V, \tilde{\zeta}')} b^{\tilde{M}'}(\delta') S^G_M(M', \delta', f).$$

While on the spectral side, we define $I^{\mathscr{E}}_{t,\text{unit}}(f)$ and $S^G_{t,\text{unit}}$ using the decomposition according to the norm of the archimedean infinitesimal character,

$$I^{\mathscr{E}}(f) = \sum_{t \geq 0} I^{\mathscr{E}}_t(f)$$

and

$$S^G(f) = \sum_{t \geq 0} S^G_t(f).$$

It follows then from the proof of [Art02, Theorem 10.6] that if G is arbitrary,

$$I^{\mathscr{E}}_t(f) - I^{\mathscr{E}}_{t,\text{unit}}(f) = \sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathscr{E}}_t(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) I^{\mathscr{E}}_M(\pi, f) d\pi$$

and if G is quasisplit, we have that $S^G_t(f) - S^G_{t,\text{unit}}(f)$ is equal to

$$\sum_{M \in \mathscr{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathscr{E}'_{\text{ell}}(M,V)} \iota(M, M') \int_{\Phi_{t'}(\tilde{M}', V, \tilde{\zeta}')} b^{\tilde{M}'}(\phi') S^G_M(M', \phi', f) d\phi'.$$

These identities reduce the study of the global geometric coefficients $a^{G,\mathscr{E}}(\gamma), b^G(\delta)$ and global spectral coefficients $a^{G,\mathscr{E}}(\pi), b^G(\phi)$ to the terms $M = G$ in their expansion, namely $a^{G,\mathscr{E}}_{\text{ell}}(\gamma), b^G_{\text{ell}}(\delta)$ and $a^{G,\mathscr{E}}_{\text{disc}}(\pi), b^G_{\text{disc}}(\phi)$ respectively by the arguments of Propositions 10.3 and 10.7 of [Art02]. Moreover, the global descent formula of [Art01, Corollary 2.2] further reduces the study of the global geometric coefficients to unipotent elements. (Our extension of the notion of admissibility is crucial here for the extension of this result, which is a long but straightforward verification.) More precisely, given an admissible element $\dot{\gamma}_S$ in $\Gamma^{\mathscr{E}}_{\text{ell}}(G, S, \zeta)$ with Jordan decomposition $\dot{\gamma}_S = c_S \dot{\alpha}_S$, we have

$$a^{G,\mathscr{E}}_{\text{ell}}(\dot{\gamma}_S) = \sum_c \sum_{\dot{\alpha}} i^{\bar{G}}(S, c) |\bar{G}_{c,+}(F)/\bar{G}_c(F)|^{-1} a^{G_c,\mathscr{E}}_{\text{ell}}(\dot{\alpha})$$

and if G is quasisplit, given an admissible element $\dot{\delta}_S$ in $\Delta_{\text{ell}}(G, S, \zeta)$ with Jordan decomposition $\dot{\delta}_S = d_S \dot{\beta}_S$, we have

$$b^G_{\text{ell}}(\dot{\delta}_S) = \sum_d \sum_{\dot{\beta}} j^{\bar{G}^*}(S, d) |\bar{G}^*_{d,+}(F)/\bar{G}^*_d(F)|^{-1} b^{G^*_d}_{\text{ell}}(\dot{\beta}),$$

where $G_{c,+}$ denotes the centralizer of c in G, and G_c is the identity component of $G_{c,+}$. We refer the reader to [Art01] for complete definitions of these expressions.

Turning to the local setting, the analogues of Local Theorems 1 and 2 of [Art02, §6] for $f \in \mathscr{C}^\circ(G, V, \zeta)$ follow from the analogues of Local Theorems 1′ and 2′, which concern the compound linear forms $I^{\mathscr{E}}_M(\gamma, f)$, $S^G_M(M', \delta', f)$ and $I^{\mathscr{E}}_M(\pi, f)$, $S^G_M(M', \phi', f)$ as a consequence of the geometric and spectral splitting and descent formulae respectively [Art02, Propositions 6.1 and 6.3]. We recall that the required geometric formulae are given in [Won19a, §3], whereas the spectral formulae can be deduced from [MW16b, X.4].

To apply the arguments of [Art03], we require analogous constructions of various subspaces of the Hecke space $\mathscr{H}(G,V,\zeta)$ used therein. If G is quasisplit, we define the unstable subspace

$$\mathscr{C}^{\circ,\mathrm{uns}}(G,V,\zeta)$$

of functions $f\in\mathscr{C}^\circ(G,V,\zeta)$ such that $f^G=0$. It is spanned by functions $f=\prod_{v\in V}f_v$ such that for some $v\in V$, f_v satisfies the property that $f_v^G=0$. We shall also define the subspace

$$\mathscr{C}^\circ_M(G,V,\zeta)$$

of functions $f\in\mathscr{C}^\circ(G,V,\zeta)$ such that f_v is M-cuspidal at two places $v\in V$. Recall that $f_v\in\mathscr{C}(G_v,\zeta_v)$ is said to be M-cuspidal if $f_{v,L_v}=0$ for any element $L_v\in\mathscr{L}_v$ that does not contain a G_v-conjugate of M_v. If v is a nonarchimedean place, we define

$$\mathscr{C}^\circ(G_v,\zeta_v)^{00}$$

to be the subspace of functions $f\in\mathscr{C}^\circ(G_v,\zeta_v)$ such that $f_{v,G}(z_v\alpha_v)=0$ for any z_v in the center of $\bar{G}_v=G_v/Z_v$ and α_v in the basis $R_{\mathrm{unip}}(G_v,\zeta_v)$ of unipotent orbital integrals in [Art03, §3]. We lastly define

$$\mathscr{C}^\circ(G_v,\zeta_v)^0$$

analogously, with α_v ranging over the parabolic subset $R_{\mathrm{unip,par}}(G_v,\zeta_v)$. We also write $\mathscr{C}^\circ(G.V,\zeta)^0$ for the product of functions $f_v\in\mathscr{C}^\circ(G_v,\zeta_v)^0$ for $v\in V$, and similarly for $\mathscr{C}^\circ(G.V,\zeta)^{00}$. We shall denote by the intersections of these various spaces by using overlapping notation, for example, we write $\mathscr{C}^\circ_M(G_v,\zeta_v)^0=\mathscr{C}^\circ_M(G_v,\zeta_v)\cap\mathscr{C}^\circ(G_v,\zeta)^0$.

The remainder of the proof proceeds by a double induction on integers r_{der} and d_{der} such that

$$0<r_{\mathrm{der}}<d_{\mathrm{der}}.$$

Namely, we assume inductively that Local Theorem 1 holds if $\dim(G_{\mathrm{der}})<d_{\mathrm{der}}$ and if

$$\dim(G_{\mathrm{der}})=d_{\mathrm{der}},\quad \dim(A_M\cap G_{\mathrm{der}})<d_{\mathrm{der}}$$

for a local non-archimedean field; the archimedean transfer for $f\in\mathscr{C}(G,\zeta)$ follows from [Art08, Theorem 1.1]. We assume inductively that Global Theorems 1 and 2 hold if $\dim(G_{\mathrm{der}})<r_{\mathrm{der}}$. In both local and global cases, we assume that if G is not quasisplit and $\dim(G_{\mathrm{der}})=d_{\mathrm{der}}$, the relevant theorems hold for the quasisplit inner K-form of G.

We will be content with recapitulating the broad strokes of the arguments in [Art03]. We shall use the subscripts 'unip' to denote the unipotent variant of objects with subscript 'ell,' and 'par' with the objects corresponding to terms $M\neq G$. For example, we write

$$I_{\mathrm{unip}}(f,S)=\sum_{\alpha\in\Gamma_{\mathrm{unip}}(G,V,\zeta)}a^G_{\mathrm{unip}}(\alpha,S)f_G(\alpha)$$

where

$$a^G_{\mathrm{unip}}(\alpha,S)=\sum_{k\in\mathcal{K}^V_{\mathrm{unip}}((\bar{G},S)}a^G_{\mathrm{ell}}(\alpha\times k)r_G(k)$$

for $\alpha \in \Gamma_{\mathrm{unip}}(G, V, \zeta)$. By the inductive definitions we obtain $I^{\mathscr{E}}_{\mathrm{unip}}$ and S^G_{unip} analogously. The global induction hypothesis then implies that

$$I^{\mathscr{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = \sum_t \left(I^{\mathscr{E}}_{t,\mathrm{disc}}(f) - I_{t,\mathrm{disc}}(f)\right) - \sum_z \left(I^{\mathscr{E}}_{t,\mathrm{unip}}(f, S) - I_{t,\mathrm{unip}}(f, S)\right)$$

and if G is quasisplit and f belongs to $\mathscr{C}^{\circ,\mathrm{uns}}(G, V, \zeta)$, then

$$S^G_{\mathrm{par}}(f) = \sum_t S^G_{t,\mathrm{disc}}(f) - \sum_z S^G_{z,\mathrm{unip}}(f, S),$$

where z belongs to the quotient $Z(G)_{V,\mathfrak{o}} Z_V / Z_V$, and $Z(G)_{V,\mathfrak{o}}$ is the subgroup of elements in $(Z(G))(F)$ such that for every $v \notin V$, the element z_v is bounded in $(Z(G))(F_v)$. The induction hypotheses further lead to a cancellation of p-adic singularities, allowing us to express

$$I^{\mathscr{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = |W(M)|^{-1} \hat{I}^M(\varepsilon_M(f))$$

for f in the intersection $\mathscr{C}^{\circ}_M(G, V, \zeta)^0$. and if G is quasisplit,

$$S^G_{\mathrm{par}}(f) = |W(M)|^{-1} \sum_{M' \in \mathscr{E}_{\mathrm{ell}}(M,v)} \iota(M, M') \hat{S}^{\tilde{M}'}(\varepsilon^{M'}(f))$$

for f in the intersection $\mathscr{C}^{\circ,\mathrm{uns}}_M(G, V, \zeta)^0$ as in [Art03, Corollary 3.3]. Here ε_M is a map from $\mathscr{C}^{\circ}(G_v, \zeta_v)^0$ to the subspace of cuspidal functions in $\mathscr{I}_{\mathrm{ac}}(M_v, \zeta_v)$ such that

$$\varepsilon_M(f_v, \gamma_v) = I^{\mathscr{E}}_M(\gamma_v, f_v) - I_M(\gamma_v, f_v)$$

for any $\gamma_v \in \Gamma(M_v, \zeta_v)$, and in the case that G_v is quasisplit, ε^M is a map from $\mathscr{C}^{\circ,\mathrm{uns}}(G_v, \zeta_v)^0$ to $\mathscr{S}_{\mathrm{ac}}(M_v, \zeta_v)$ such that

$$\varepsilon^M(f_v, \delta_v) = S^G_M(\delta_v, f_v)$$

for any $\delta_v \in \Delta(M_v, \zeta_v)$. These maps are given in [Art03, Proposition 3.1], also studied in Chapters VIII and IX of [MW16b], and can be seen as generalizing the mapping ϕ_M in a direction different from the maps $\iota_M, \iota^{\mathscr{E}}_M$, and τ_M that we have constructed earlier.

The separation of the spectral sides according to infinitesimal character follows from [Art03, §4–5] and the properties of the function spaces we have defined, but is not strictly necessary given the absolute convergence of the spectral side. On the other hand, the stabilization of the invariant local trace formula in [Art02, §10] and [Art03, §6] extends to our setting following Lemma 7.1, and together with the global results above lead to the proof of Local Theorem 1 in the nonarchimedean case, again using the local and global induction hypotheses. We note that this implies Local Theorem 2 according to an unpublished work of Arthur, and we may also refer to sections X.5 and X.7 of [MW16b] for a variant argument.

To complete the global theorems, we apply the local theorems to conclude that

$$I^{\mathscr{E}}_{\mathrm{par}}(f) - I_{\mathrm{par}}(f) = \sum_{M \in \mathscr{L}^0} |W^M_0||W^G_0|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma)(I^{\mathscr{E}}_M(\gamma, f) - I_M(\gamma, f))$$

vanishes for $\mathscr{C}^{\circ}(G, V, \zeta)$, and if G is quasisplit, that

$$S^G_{\mathrm{par}}(f) = \sum_{M \in \mathscr{L}^0} |W^M_0||W^G_0|^{-1} \sum_{\delta^* \in \Delta(M^*,V,\zeta^*)} b^{M^*}(\delta^*) S^G_M(M^*, \delta^*, f)$$

vanishes for $f \in \mathscr{C}^{\circ,\mathrm{uns}}(G,V,\zeta)$. The induction argument on r_{der} implies that the terms $I^{\mathscr{E}}_{t,\mathrm{disc}}(f) - I_{t,\mathrm{disc}}(f)$ and $S^G_{t,\mathrm{disc}}(f)$ vanish for f in $\mathscr{C}^\circ(G,V,\zeta)$ and $\mathscr{C}^{\circ,\mathrm{uns}}(G,V,\zeta)$ respectively, so that

$$\sum_z \left(I^{\mathscr{E}}_{t,\mathrm{unip}}(f,S) - I_{t,\mathrm{unip}}(f,S)\right) = 0, \qquad f \in \mathscr{C}^\circ(G,V,\zeta)$$

and

$$\sum_z S^G_{z,\mathrm{unip}}(f,S) = 0, \qquad f \in \mathscr{C}^{\circ,\mathrm{uns}}(G,V,\zeta)$$

in the case that G is quasisplit. Choosing $V = S$, and using the property that the linear forms

$$\dot{f}_S \to \dot{f}_{S,G}(z\dot{\alpha}_S), \qquad z \in Z(\bar{G})_{S,\mathfrak{o}},\ \dot{\alpha}_S \in \Gamma^{\mathscr{E}}_{\mathrm{unip}}(G,S,\zeta)$$

on the subspace of admissible functions in $\mathscr{C}^\circ(G,S,\zeta)$ are linearly independent, we conclude from the definitions of $I_{t,\mathrm{unip}}$ and $I^{\mathscr{E}}_{t,\mathrm{unip}}$ that

$$a^{G,\mathscr{E}}_{\mathrm{ell}}(\dot{\alpha}_S) - a^G_{\mathrm{ell}}(\dot{\alpha}_S) = 0$$

for $\dot{\alpha}_S \in \Gamma^{\mathscr{E}}_{\mathrm{unip}}(G,S,\zeta)$, and similarly

$$\dot{f}_S \to \dot{f}^{\mathscr{E}}_{S,G}(z\dot{\beta}_S), \qquad z \in Z(\bar{G})_{S,\mathfrak{o}},\ \dot{\beta}_S \in \Delta^{\mathscr{E}}_{\mathrm{unip}}(G,S,\zeta)\backslash\Delta_{\mathrm{unip}}(G,S,\zeta)$$

on the subspace of admissible functions in $\mathscr{C}^{\circ,\mathrm{uns}}(G,S,\zeta)$ are linearly independent, whence we conclude that

$$b^G_{\mathrm{ell}}(\dot{\beta}_S) = 0$$

for $\dot{\alpha}_S$ in the complement of $\Delta_{\mathrm{unip}}(G,S,\zeta)$ in $\Delta^{\mathscr{E}}_{\mathrm{unip}}(G,S,\zeta)$. Applying the global descent formula to the coefficients then yields the geometric Global Theorem 1. The spectral Global Theorem 2 follows similarly, using the vanishing of

$$\sum_t \left(I^{\mathscr{E}}_{t,\mathrm{disc}}(f) - I_{t,\mathrm{disc}}(f)\right) = 0, \qquad \dot{f} \in \mathscr{C}^\circ(G,\zeta)$$

and

$$\sum_t S^G_{t,\mathrm{disc}}(f) = 0, \qquad \dot{f} \in \mathscr{C}^{\circ,\mathrm{uns}}(G,\zeta).$$

Arguing as in the geometric case we conclude that

$$a^{G,\mathscr{E}}_{\mathrm{disc}}(\dot{\pi}) - a^{G,\mathscr{E}}_{\mathrm{disc}}(\dot{\pi}) = 0$$

for any $\dot{\pi} \in \Pi_{t,\mathrm{disc}}(G,\zeta)$, and in the case that G is quasisplit,

$$b^G_{\mathrm{disc}}(\dot{\phi}) = 0$$

for any $\dot{\phi}$ in the complement of $\Phi_{t,\mathrm{disc}}(G,\zeta)$ in $\Phi^{\mathscr{E}}_{t,\mathrm{disc}}(G,\zeta)$, and the desired result follows.

Finally, we can conclude from these general remarks the extension of the endoscopic and stable trace formulae, with the required expansions

$$\begin{aligned} I^{\mathscr{E}}(f) &= \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathscr{E}}(M,V,\zeta)} a^{M,\mathscr{E}}(\gamma) I^{\mathscr{E}}_M(\gamma,f) \\ &= \sum_t \sum_{M \in \mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi^{\mathscr{E}}_t(M,V,\zeta)} a^{M,\mathscr{E}}(\pi) I^{\mathscr{E}}_M(\pi,f) d\pi \end{aligned}$$

and

$$\begin{aligned} S(f) &= \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \sum_{\delta\in\Delta(M,V,\zeta)} b^M(\delta)S_M(\delta,f) \\ &= \sum_t \sum_{M\in\mathscr{L}} |W_0^M||W_0^G|^{-1} \int_{\Phi_t(M,V,\zeta)} b^M(\phi)S_M(\phi,f)d\phi \end{aligned}$$

for $f \in \mathscr{C}^\circ(G,V,\zeta)$. □

Our earlier modifications of the endoscopic and stable linear forms then applies to this setting also.

Corollary 7.3. *The modified distributions $\tilde{I}^{\mathscr{E}}$ and $\tilde{S}$ extend to continuous linear forms on $\mathscr{C}^\circ(G,V,\zeta)$.*

Proof. This follows from the fact that the maps τ_M and $\iota_M^{\mathscr{E}}$ used to construct the modified distributions are defined for functions in $\mathscr{C}(G,V,\zeta)$ (though we only have need of the smaller space $\mathscr{H}_{\mathrm{ac}}(G,V,\zeta)$ in the application), whence we may apply Theorem 7.2 and argue as in the first part of the proof of Proposition 6.5. □

As we have alluded to in the beginning, the extension of the stable linear form S to noncompactly-supported test functions in $\mathscr{C}^\circ(G,V,\zeta)$ does not allow for proper use of the basic function, which belongs to $\mathscr{C}^\circ(G)$ and is nontrivial at almost all places v. To correct for this, we have to reconsider the passage from $\mathscr{C}^\circ(G,\zeta)$ to $\mathscr{C}^\circ(G,V,\zeta)$, which requires, among other things, a reconsideration of the global geometric coefficients that depend on the finite set S in a complicated way.

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