ON THE BALANCED VORONOÏ FORMULA FOR GL_N

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Abstract. Miller and F. Zhou have proved a balanced Voronoï summation formula for GL_N over $\mathbf{Q}$, which allows one to control the dimensions of the Kloosterman sums appearing on either side of the Voronoï formula. In this note, we prove a balanced Voronoï formula over an arbitrary number field, starting with the Voronoï summation formula of A. Ichino and N. Templier over number fields, allowing one to extend recent results on spectral reciprocity laws to number fields, in special cases.

1. Introduction

1.1. The Voronoï summation formula is an equality between a weighted sum of Fourier coefficients of an automorphic form twisted by an additive character, and dual weighted sum of Fourier coefficients of the dual form twisted by a Kloosterman sum. The Voronoï formula for GL_2 is a basic tool in the study of automorphic forms, while more general applications have followed with the more general formulas for GL_N proved by Goldfeld-Li [GL08] and Miller-Schmid [MS11] over $\mathbf{Q}$, and Ichino-Templier [IT13] over number field F which, importantly, removes any ramification assumptions in the previous cases.

A balanced formula on GL_N was first obtained by Zhou [Zho16] under certain restrictions, then later in general by Miller-Zhou [MZ17], in which the lengths of the hyper-Kloosterman sums on either side of the formula can be chosen in a 'balanced' manner. This was applied in the recent work of Blomer-Li-Miller [BLM17] to prove a spectral reciprocity law via a so-called 'Kuznetsov-Voronoï-Kuznetsov triad' for a spectral sum of automorphic L-functions on $\mathrm{GL}_4 \times \mathrm{GL}_2$ as follows: a Kuznetsov trace formula on GL_2 is applied, and then the balanced Voronoï formula for GL_4 is used on the geometric side, and the Kuznetsov formula is applied again to the dual geometric side, to give a dual spectral sum. As an application, the authors prove a non-vanishing result for automorphic L-functions on $\mathrm{GL}_4 \times \mathrm{GL}_2$. A modified version has also been developed in Blomer-Khan [BK17], and is used to bound moments of twisted L-functions on GL_4. The mechanics of the spectral reciprocity law suggest that a general formula may exist for $\mathrm{GL}_{2N} \times \mathrm{GL}_N$. Unfortunately, even for $N = 3$ one finds that the Kuznetsov formula involves Kloosterman sums of varying lengths, which prevents a direct application of the balanced Voronoï formula as in the $N = 2$ case.

1.2. In this paper, we generalise the balanced Voronoï to a general number field. Besides allowing for extensions of the results on spectral reciprocity laws to number fields in special cases, another key motivation in our work is that rather than

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Kloosterman sums, more general Kloosterman integrals appear on either side of the balanced formula, which allows for the possibility of a more flexible relative trace formula, which involves Kloosterman integrals, to be used in place of the Kuznetsov trace formula.

A second motivation for our study comes from a somewhat different source. Recent developments with regards to the conjectures of Braverman and Khazdan [BK00] such as [Ngô14, Ngô16, BNS16] developing geometric methods to generalise the theory of Godement and Jacquet [GJ72], which proves the functional equation of standard automorphic L-functions on GL_N using Poisson summation. In particular, [Ngô16] proposes a construction of the conjectural ρ-Fourier transform $\mathscr{F}^\rho$, which generalizes the Hankel transform that occurs in the Voronoï formula for GL_2. The existence of balanced Voronoï formulas then suggests that the ρ-Poisson summation formula of the form

$$\sum_{\gamma\in G(F)}\phi(\gamma)=\sum_{\gamma\in G(F)}\mathscr{F}^\rho(\phi)(\gamma),$$

where ϕ belongs to a certain ρ-Schwartz space $\mathscr{S}^\rho(G(\mathbf{A}_F))$, can be again 'balanced' in a similar manner, and it would be interesting to explore potential applications to the analytic theory of L-functions.

1.3. **Main result.** Our method essentially follows that of Miller-Zhou, where instead of starting with the Voronoï formula of Miller-Schmid [MS11] over $\mathbf{Q}$ we use the more general formula of Ichino-Templier [IT13], and avoid the use of multiple Dirichlet series. The key observation is that the proof of the balanced Voronoï formula reduces to the usual Voronoï formula through a series of character sums, parallel to the repeated use of the crucial identity [MZ17, Lemma 3.2].

Let $N+2=L+M$. Let T be the maximal torus of diagonal matrices in GL_N, and T^L,T^M disjoint sub-tori of dimensions $L-1$ and $M-1$ respectively, so that $T\simeq T^L\times T^M$. The case $L=1$ then reduces to the ordinary Voronoï summation formula. Then referring to Section 2 below for the definitions and notations, our balanced Voronoï formula is as follows.

Theorem 1. *Let $\pi=\otimes_v\pi_v$ be an irreducible cuspidal automorphic representation of $GL_N(\mathbf{A}_F)$, and let S be the set of places of F over which π_v is ramified. For any $\zeta\in\mathbf{A}_F^S$ and $\omega_S\in C_c^\infty(F_S^\times)$, we have*

$$\begin{aligned}(1.1)\qquad &\sum_{t\in T^M_\zeta/T^M_\circ}\sum_{\gamma\in F^\times}Kl_M(\gamma\zeta,t)W^S_\circ\Big(\begin{pmatrix}\gamma & \\ & 1_{N-1}\end{pmatrix}\Big)w_S(\gamma)\\ &=\sum_{s\in T^L_\zeta/T^L_\circ}c_s\sum_{\gamma\in F^\times}Kl_L(\gamma\zeta^{-1},s)\tilde{W}^S_\circ\Big(\begin{pmatrix}\gamma & \\ & 1_{N-1}\end{pmatrix}a(s)\Big)\tilde{w}_S(\gamma),\end{aligned}$$

where c_s is a constant depending on s.

We have unfortunately not made the constant c_s explicit, though it can be easily estimated by our computations below. In principle, one should nonetheless be able to apply this formula to obtain a generalization the spectral reciprocity formula of [BLM17, Theorem 3] in the case $N=2$ to totally real number fields, using the relevant Kuznetsov formula of Bruggeman and Miatello.

Remark 2. We briefly describe how the notation in [MZ17, Theorem 1.1] can be compared to ours. First, for the same N, their parameters are chosen such that

$M' + L' + 2 = N$. Our choice of $M + L - 2 = N$ here differs from theirs due to the convention on hyper-Kloosterman sums used in [IT13]. Their balanced Voronoï formula takes the form:

$$\sum_{\mathbf{D}|\mathbf{Q}} D_1^M \dots D_M^1 \sum_{n=1}^{\infty} Kl_M(\bar{a}, n, c; \mathbf{Q}, \mathbf{D}) A(\mathbf{q}, \mathbf{D}, n) \omega\Big(\frac{nD_1^{M+1} \dots D_M^2}{q_1^L \dots q_L^1}\Big)$$
$$= \sum_{\mathbf{d}|\mathbf{q}} \frac{d_1^L \dots d_L^1}{c^{L+1}} \sum_{n=1}^{\infty} \sum_{\epsilon=\pm} Kl_L(a, \epsilon n, c; \mathbf{q}, \mathbf{d}) \tilde{A}(\mathbf{Q}, \mathbf{d}, n) \Omega\Big(\frac{(-1)^M \epsilon n d_1^{L+1} \dots d_L^2}{c^N Q_1^M \dots Q_M^1}\Big).$$

Letting $F = \mathbf{Q}$, we specialise $\psi(x_v)$ to be $e^{-2\pi i x_\infty}$ for $x_\infty \in \mathbf{R}$, and $e^{2\pi i x_p}$ for $x_p \in \mathbf{Q}_p$. Our ζ corresponds to $\frac{\bar{a}}{c}$, and the set of places R are the prime divisors of c. Our $\gamma \in F^\times$ correspond to the arguments of ω and Ω above. Our $t \in T_\zeta^M / T_\circ^M$ corresponds to their sequence of positive integers $d_1, \dots, d_M$, where up to units we have $(t_2, \dots, t_{M-2}, t_{M-1})$ equal to $\frac{1}{c}(d_1 d_2 \dots d_M, \dots, d_1 d_2, d_1)$, and similarly $s \in T_\zeta^L / T_\circ^L$ corresponds to $D_1, \dots, D_L$. Their hyper-Kloosterman sum $Kl_N(a, n, c; \mathbf{q}, \mathbf{d})$ corresponds to $Kl_N(\gamma\zeta^{-1}, t)$ as outlined in [IT13, p.72]. Finally, the Fourier coefficients A correspond to $W_{\circ f}$ and $\tilde{W}_{\circ f}$ up to normalisation as in (2.1), and our functions $\omega, \tilde{\omega}$ correspond to their ω, Ω respectively, though their test function ω is compactly supported on $(0, \infty)$.

2. The Voronoï formula of Ichino-Templier

2.1. Let F be a number field, and $\mathbf{A} = \mathbf{A}_F$ the ring of adeles. Also let F_v be a completion of F at a prime v, with ring of integers $\mathcal{O}_v$. Fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of $F \backslash \mathbf{A}$. Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $GL_N(\mathbf{A}_F)$, $n \geq 2$, and let S be the set of places of F over which π_v is ramified. Let $\mathbf{A}^S$ be the adeles with trivial component above S. Define the unramified Whittaker function of π^S to be $W_\circ^S = \prod_{v \notin S} W_{\circ v}$, and similarly for the contragredient representation $\tilde{\pi}^S$ we write $\tilde{W}_\circ^S$, where

$$\tilde{W}_\circ^S(g) = W_\circ^S(w^t g^{-1})$$

for all $g \in GL_N(\mathbf{A}^S)$, and w is the long Weyl element of GL_N. Over $\mathbf{Q}$, they are related to the Fourier coefficients $A(m_1, m_2, \dots, m_{N-1})$ of π by the following relation:

$$\prod_{p<\infty} W_p(\Delta_m) = \prod_{i=1}^{N-1} |m_i|^{-i(n-i)/2} A(m_1, m_2, \dots, m_{N-1}), \tag{2.1}$$

where

$$\Delta_m = \mathrm{diag}(m_1 \dots m_{N-1}, m_2 \dots m_{N-1}, \dots, m_{N-1}, 1)$$

is a diagonal matrix in $GL_N(\mathbf{Q})$.

2.1.1. *Measures.* Throughout, we make the following choices of measures: The measure dx_v on the local field F_v is chosen to be self-dual with respect to the fixed additive character ψ_v. Fix a non-zero differential form ω in $\mathrm{Hom}_F(\wedge^{\mathrm{top}}\mathrm{Lie}(U), F)$ and also for Y, so that ω_v and dx_v determine a measure on $\mathrm{Lie}(U)(F_v)$, hence an invariant measure on $U(F_v)$. The product of these measures gives the Tamagawa measure.

2.1.2. *Generalised Bessel transforms.* Define for each $w_v \in C_c^\infty(F_v^\times)$ a dual function $\tilde{\omega}_v$ such that

$$\int_{F_v^\times} \tilde{\omega}_v(y)\chi(y)^{-1}|y|^{s-\frac{N-1}{2}}dy$$
$$= \chi(-1)^{N-1}\gamma(1-s,\pi_v\times\chi,\psi_v)\int_{F_v^\times} w_v(y)\chi(y)|y|^{1-s-\frac{N-1}{2}}dy$$

for all $\mathrm{Re}(s)$ large enough and any unitary character χ of $F_v^\times$. This defines $\tilde{\omega}_v$ uniquely in terms of π_v, ψ_v, and ω_v, independent of the choice of Haar measure dy. Note that $\tilde{\omega}_v(x)$ is smooth of rapid decay, but not necessarily compactly supported, as $|x| \to \infty$, which is important for the convergence of the dual sum.

2.1.3. *Kloosterman integrals.* Define for any $\gamma_v, \zeta_v \in F_v^\times$, the hyper-Koosterman integral,

$$K_v(\gamma_v,\zeta_v,\tilde{W}_{\circ v}) := |\zeta_v|^{N-2}\int_{U_\tau^-(F_v)} \overline{\psi}_v(u_{N-2,N-1})\tilde{W}_{\circ v}(\tau u)du$$

where

$$\tau = \begin{pmatrix} & 1 & \\ 1_{N-2} & & \\ & & 1\end{pmatrix}\begin{pmatrix}1_{N-2} & & \\ & -\gamma_v\zeta_v^{-1} & \\ & & -\zeta\end{pmatrix},$$

and set

$$K_R(\gamma,\zeta,\tilde{W}_{\circ R}) = \prod_{v\in R} K_v(\gamma_v,\zeta_v,\tilde{W}_{\circ v})$$

for $\gamma, \zeta \in \mathbf{A}_R^\times$. It relates to hyper-Kloosterman sums as follows: Let T be the maximal torus of diagonal matrices in GL_N, then

$$K_v(\gamma_v,\zeta_v,\tilde{W}_{\circ v}) = |\zeta_v|^{N-2}\sum_{T(F)^+/T(\mathcal{O}_v)} \tilde{W}(t)Kl_N(\gamma_v\zeta_v^{-1},t)$$

where the sum is taken over elements $t = (t_1,\dots,t_N)$ in $T(F_v)^+/T(\mathcal{O}_v)$ such that

$$1 \le |t_2| \le \cdots \le |t_N| = |\zeta_v|, \text{ and } |t_1t_2\dots t_{N-1}| = |\zeta_v|.$$

Here $Kl_N(\gamma\zeta^{-1},t)$ is the hyper-Kloosterman sum of dimension $N-1$, can be expressed as

$$\sum_{v_{N-1}\in t_{N-1}\mathcal{O}_v^\times/\mathcal{O}_v}\cdots\sum_{v_2\in t_2\mathcal{O}_v^\times/\mathcal{O}_v}\psi(v_{N-1}+\cdots+v_2)\psi((-1)^n\gamma\zeta_v^{-1}v_2^{-1}\dots v_{N-1}^{-1})$$

by [IT13, Corollary 6.7].

2.1.4. *Voronoï formula.* We can now state the main result of Ichino and Templier [IT13, Theorem 1], which will be the basis for our balanced Voronoï formula.

Theorem 3 (Ichino-Templier)**.** *Let $\zeta \in \mathbf{A}_F^S$, and R the set of places v such that $|\zeta_v| > 1$. Then with notation as above, we have*

$$\sum_{\gamma\in F^\times}\psi(\gamma\zeta)W_\circ^S\Big(\begin{pmatrix}\gamma & \\ & 1_{N-1}\end{pmatrix}\Big)w_S(\gamma)$$
$$= \sum_{\gamma\in F^\times}K_R(\gamma,\zeta,\tilde{W}_{\circ R})\tilde{W}_\circ^{R\cup S}\Big(\begin{pmatrix}\gamma & \\ & 1_{N-1}\end{pmatrix}\Big)\tilde{w}_S(\gamma),$$

for any $\omega_S \in C_c^\infty(F_S^\times)$.

From the preceding discussion, we can expand the right-hand side along the maximal torus T to obtain an expression in terms of Kloosterman sums:

$$\sum_{t\in T_\zeta/T_\circ}\sum_{\gamma\in F^\times} Kl_N(\gamma\zeta^{-1},t)\tilde{W}_\circ^S\Big(\begin{pmatrix}\gamma & \\ & 1_{N-1}\end{pmatrix}a(t)\Big)\tilde{w}_S(\gamma),$$

here T_ζ denotes the set of $(t_2,\dots,t_{N-1})$ in F_R^{N-2} such that

$$1\le |t_2|_v\le\cdots\le |t_{N-1}|_v\le|\zeta|_v$$

for all $v\in R$. Here $T_\circ=(\mathcal{O}_R^\times)^{N-2}$ and $a(t)$ is the diagonal matrix $(t_1,\dots,t_N)$ in $T(\mathbf{A}_R)/T(\mathcal{O}_R)$ uniquely completed such that $|t_N|_v=|\zeta|_v$ and $|t_1\cdots t_N|_v=1$ for all $v\in R$. Taking $F=\mathbf{Q}$, and π to be unramified at every finite prime, this recovers the main result of [MS11] (see [IT13, Theorem 2]).

3. Proof of Theorem 1

We are now ready to prove a balanced Voronoï formula over an arbitrary number field, which specialises to Theorem 3 at $M=0$. First, we open up the hyper-Kloosterman sum on the left-hand side of (1.1), and then bring in the γ sum,

$$\sum_{t\in T_\zeta^M/T_\circ^M}\ \sum_{\substack{v_{M-1}\in t_{M-1}\mathcal{O}_R^\times/\mathcal{O}_R\\ \cdots\\ v_2\in t_2\mathcal{O}_R^\times/\mathcal{O}_R}}\psi(v_{M-1}+\cdots+v_2)\ \times$$
$$\sum_{\gamma\in F^\times}\psi((-1)^M\gamma\zeta^{-1}v_2^{-1}\dots v_{M-1}^{-1})W_\circ^S\Big(\begin{pmatrix}\gamma & \\ & 1_{N-1}\end{pmatrix}\Big)w_S(\gamma). \tag{3.1}$$

Note that interchanging the summation is justified by the compact support of the test function ω_S. Then applying the Voronoï summation of Theorem 3 to the inner sum, we obtain the dual expression

$$\sum_{t\in T_\zeta^M/T_\circ^M}\ \sum_{\substack{v_{M-1}\in t_{M-1}\mathcal{O}_R^\times/\mathcal{O}_R\\ \cdots\\ v_2\in t_2\mathcal{O}_R^\times/\mathcal{O}_R}}\psi(v_{M-1}+\cdots+v_2)\ \times$$
$$\sum_{s\in T_\zeta/T_\circ}\sum_{\gamma\in F^\times}Kl_N(\gamma\breve{\zeta}^{-1},s)\tilde{W}_\circ^S\Big(\begin{pmatrix}\gamma & \\ & 1_{N-1}\end{pmatrix}a(s)\Big)\tilde{w}_S(\gamma), \tag{3.2}$$

where we have denoted $\breve{\zeta}:=(-1)^M\zeta^{-1}v_2^{-1}\dots v_{M-1}^{-1}$. Recall that here T is the maximal split torus in G, so that relabelling indices if necessary, we may decompose any $s\in T_\zeta/T_\circ$ into $s=s_1s_2$ where

$$\begin{aligned}s_1&=(t_1,\dots,t_{L-1})\in T_\zeta^L/T_\circ^L,\\ s_2&=(t_L,\dots,t_{N-1})\in T_\zeta^M/T_\circ^M,\end{aligned}$$

such that

$$\begin{aligned}1\le|t_1|_v&\le\cdots\le|t_{L-1}|_v\le|\breve{\zeta}|_v,\\ 1\le|t_L|_v&\le\cdots\le|t_{N-1}|_v\le|\breve{\zeta}|_v\end{aligned}$$

for all $v\in R$. Note that s_2 is an $(M-2)$-tuple.

Now on the dual side, opening up the $(N-1)$-dimensional hyper-Kloosterman sum along t_2, down to $(L-1)$ dimension, we have

$$Kl_N(\gamma\check{\zeta}^{-1}, s_1s_2) = \sum_{u_{N-1}\in t_{N-1}\mathcal{O}_R^\times/\mathcal{O}_R} \cdots \sum_{u_L\in t_L\mathcal{O}_R^\times/\mathcal{O}_R} \psi(u_{N-1}+\cdots+u_L) Kl_L(\gamma\zeta_L^{-1}, s_1)$$

where

$$\zeta_L = (-1)^{N-L}\check{\zeta}u_L\dots u_{N-1} = \zeta^{-1}v_2^{-1}\dots v_{M-1}^{-1}u_L\dots u_{N-1}.$$

Only the innermost sum over γ is infinite, so we may rearrange the order of summation by pairing the v_{M-1} sum with the u_L sum, the v_{M-2} sum with the u_{L+1} sum, and so on. Separating the s_1 and s_2 sums, we write (3.2) as

$$\sum_{t\in T_\zeta^M/T_\circ^M} \sum_{\substack{v_{M-1}\in t_{M-1}\mathcal{O}_R^\times/\mathcal{O}_R\\ \cdots\\ v_2\in t_2\mathcal{O}_R^\times/\mathcal{O}_R}} \sum_{s_2\in T_\zeta^M/T_\circ^M} \sum_{\substack{u_{N-1}\in t_{N-1}\mathcal{O}_R^\times/\mathcal{O}_R\\ \cdots\\ u_L\in t_L\mathcal{O}_R^\times/\mathcal{O}_R}} \psi(v_{M-1}+u_L)\dots\psi(v_2+u_{N-1})$$

$$\times \sum_{s_1\in T_\zeta^L/T_\circ^L} \sum_{\gamma\in F^\times} Kl_L(\gamma\zeta_L^{-1}, s_1)\tilde{W}_\circ^S\Big(\begin{pmatrix}\gamma & \\ & 1_{N-1}\end{pmatrix}a(s_1)\Big)\tilde{w}_S(\gamma), \tag{3.3}$$

where the third sum is over $s_2 = (t_L, \dots, t_{N-1})$ as above. It remains then to evaluate the first line, noting that it is independent of the second line except for ζ_L. To treat the first four sums, we separate also the sum on t in $T_\zeta^M/T_\circ^M$ into its $(M-2)$ components $(t_2, \dots, t_{M-1})$ such that $1 \le |t_1|_v \le \dots \le |t_{M-1}|_v \le |\zeta|_v$ for all $v \in R$. So the first two sums of (3.3) then reads for every fixed $s_2, u_L, u_{L+1}, \dots, u_{N-1}$ as follows:

$$\sum_{\substack{t_{M-1}\in F_R\\ |t_{M-1}|\le|\zeta_R|}} \sum_{v_{M-1}\in t_{M-1}\mathcal{O}_R^\times/\mathcal{O}_R} \psi(v_{M-1}+u_L)\cdots \sum_{\substack{t_2\in F_R\\ |t_2|\le|t_3|}} \sum_{v_2\in t_2\mathcal{O}_R^\times/\mathcal{O}_R} \psi(v_2+u_{N-1}). \tag{3.4}$$

Consider then the first pair. We observe that for each fixed $s_2 \in T_\zeta^M/T_\circ^M$ and $u_L \in t_L\mathcal{O}_R^\times/\mathcal{O}_R$, the sum:

$$\sum_{|t_{M-1}|\le|\zeta_R|} \sum_{v_{M-1}\in t_{M-1}\mathcal{O}_R^\times/\mathcal{O}_R} \psi(v_{M-1}+u_L)$$

is nonzero only if $t_{M-1} = t_L, u_L \equiv -v_{M-1} \pmod{\mathcal{O}_R}$. To see this, simply observe that

$$\sum_{v_{M-1}\in t_{M-1}\mathcal{O}_R^\times/\mathcal{O}_R} \psi(v_{M-1}+u_L) = |t_{M-1}|$$

if $t_{M-1} = t_L$ and $u_L \equiv -v_{M-1}$ modulo $\mathcal{O}_R$, and is zero otherwise. We note that this is the analogue of Lemma 3.2 of [MZ17]. This implies that $v_{M-1}^{-1}u_L \equiv -1 \bmod \mathcal{O}_R$ in ζ_L. Moving on the second pair, for fixed t_{M-1} and u_{L+1},

$$\sum_{|t_{M-2}|\le|t_{M-1}|} \sum_{v_{M-2}\in t_{M-2}\mathcal{O}_R^\times/\mathcal{O}_R} \psi(v_{M-2}+u_{L+1})$$

we see that the sum is again nonzero only if $t_{M-2} = t_{L+1}$ and $u_{L+1} \equiv -v_{M-2} \bmod \mathcal{O}_R$, and zero otherwise. Applying this $M-2$ times, we collect the evaluated sums

(3.3) into a constant c_{s_1}, and finally the sum reduces to

$$\sum_{s_1 \in T^L_\zeta / T^L_\circ} c_{s_1} \sum_{\gamma \in F^\times} Kl_L(\gamma\zeta_L^{-1}, s_1) \tilde{W}^S_\circ\Big(\begin{pmatrix} \gamma & \\ & 1_{N-1} \end{pmatrix} a(s_1)\Big) \tilde{w}_S(\gamma)$$

as desired.

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