Analysis of some Chirp related Models

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Abstract

Chirp signals play a very crucial role in various signal processing applications, ranging from radar and sonar systems to biomedical imaging and communication systems. Here we study a brief analysis of the Elementary Chirp model and Chirp model. We first provide an overview of both models, discussing their mathematical formulations and underlying assumptions. We discuss how to estimate the unknown parameters of the model using our usual Least Squares method and Approximate Least squares method. Also we perform some numerical experiments to study the finite sample performance of the two estimation methods. Additionally, we analyse two Sonar data using both the models.

1 Introduction

In this project we are going to study some models which are commonly used in signal processing and communication systems. There are many such models but we are going to focus on the Elementary Chirp model and Chirp model.

The multicomponent Elementary Chirp model is described as follows:

$$y_t = \sum_{k=1}^{p} \left[A_k \cos(\beta_k t^2) + B_k \sin(\beta_k t^2)\right] + X_t; \quad t = 1, 2, \ldots, n. \tag{1}$$

Here y_t has p number of components, the k-th component is corresponding to the chirp rate β_k. The linear parameters $A_k,\ B_k,\ k = 1, 2 \ldots, p$ are unknown amplitudes attached to the chirp rate parameters β_k; β_k is also known as frequency rate parameter. The additive noise $\{X_t\}$ is a sequence of stationary random variables with mean zero and finite variance. The number of components p is assumed to be known. The task is to estimate the unknown parameters, that is A_k, B_k and β_k, $k = 1, 2, \ldots, p$ based on a sample of size n, $\{y_t; t = 1, \ldots, n\}$.

Now we describe the multicomponent Chirp model as follows:

$$y_t = \sum_{k=1}^{p} \left[A_k \cos(\alpha_k t + \beta_k t^2) + B_k \sin(\alpha_k t + \beta_k t^2)\right] + X_t; \quad t = 1, 2, \ldots, n. \tag{2}$$

Here α_k are the non-zero unknown frequencies, β_k are the unknown chirp rates whereas A_k, B_k, $k = 1, 2, \ldots, p$ are the unknown amplitudes; $\{X_t\}$ is same as defined earlier.

The Elementary chirp model is a special case of Chirp model. In Chirp model each component has a non-zero frequency and it changes with a rate, different for each component. Whereas the Elementary Chirp model does not have a frequency parameter, that is, basically the frequency is zero. This zero frequency changes with a rate β_k for the k-th component.

The multicomponent Elementary Chirp model as well as Chirp model has received considerable attention in recent times. Some of the examples include radar pulse reconstruction, sonar pulse detection etc. One can see Mboup and Adali [2] and references cited therein. In this project we plan to analyze some Sonar data using Elementary Chirp and Chirp models.

Our main goal is to estimate the non-linear parameters that is the frequencies and the chirp rates and as a consequence the amplitudes by the least squares and the approximate least squares method.
We will also focus on the asymptotic distribution of the parameter estimates for the Elementary Chirp model and Chirp model under certain assumptions and look at different scenarios like how does the Mean Square Error (MSE) of the estimates of the parameters changes with varying sample size and error variance.
We consider two Sonar datasets, a Sonar-mines data and a Sonar-rocks data. We analyse these dataset using both the models considered in this work. The project is organised as follows: in Section 2 we discuss the least square estimation method for the Elementary Chirp model and in Section 3 for the Chirp model. Numerical experiments are reported in Section 4 and two real life datasets are analysed in Section 5. We conclude our work in Section 6 and some of the theoretical results and associated lemmas are given in Section 7.

2 Least Squares Estimation (LSE) Method for Elementary Chirp Model

Least squares estimation is a valid estimation for estimating the unknown parameters of a linear or a non-linear model. In this section we shall discuss how to estimate the unknown parameters in an Elementary Chirp model with one component that is model (1) with $p = 1$.
The model under consideration is:

$$y_t = A^0 \cos(\beta^0 t^2) + B^0 \sin(\beta^0 t^2) + X_t \; ; \quad t = 1, 2, \ldots, n. \tag{3}$$

- $(y_1, \ldots, y_n)$ is the data observed at n time points.
- A^0 and B^0 are the true values of the unknown amplitudes and β^0 is the true unknown chirp rate.
- We denote $\boldsymbol{\theta^0} = (A^0, B^0, \beta^0)^T$ and $\boldsymbol{\theta} = (A, B, \beta)^T$.

Assumption 1:- The sequence of additive error $\{X_t\}$ has the following linear structure

$$X_t = \sum_{k=-\infty}^{\infty} a_k e_{t-k}, \tag{4}$$

where $\{e_t\}$ is a sequence of independent and identically distributed (i.i.d) random variables with mean zero and finite variance σ^2. The arbitrary real valued sequence $\{a_k\}$ satisfies the following condition of absolute summability;

$$\sum_{k=-\infty}^{\infty} |a_k| < \infty. \tag{5}$$

Note that any stationary process like AR, MA or ARMA can be expressed as (4).
We write the Elementary Chirp model with one component in the usual matrix notation:

$$\mathbf{y} = \mathbf{X}(\beta^0)\boldsymbol{\lambda}^{\mathbf{0}} + \boldsymbol{\epsilon}, \tag{6}$$

where $\mathbf{y}$=$(y_1, \ldots, y_n)^T$, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)^T$, and $\boldsymbol{\lambda}^{\mathbf{0}} = (A^0, B^0)^T$ is the vector of linear parameters and

$$\mathbf{X}(\beta^0) = \begin{bmatrix} \cos(\beta^0) & \sin(\beta^0) \\ \cos(\beta^0 2^2) & \sin(\beta^0 2^2) \\ \vdots & \vdots \\ \cos(\beta^0 n^2) & \sin(\beta^0 n^2) \end{bmatrix}.$$

Let $\boldsymbol{\lambda} = (A, B)^T$, the Residual Sum of Squares (RSS) is then defined as

$$Q(A, B, \beta) = Q(\boldsymbol{\theta}) = \sum_{t=1}^{n} \left[y_t - A\cos(\beta t^2) - B\sin(\beta t^2)\right]^2, \tag{7}$$

and in matrix notation this is same as

$$Q(\boldsymbol{\theta}) = (\mathbf{y} - \mathbf{X}(\beta)\boldsymbol{\lambda})^T(\mathbf{y} - \mathbf{X}(\beta)\boldsymbol{\lambda}). \tag{8}$$

For a given β we minimize $Q(\boldsymbol{\theta})$ with respect to $\boldsymbol{\lambda}$ and get

$$\hat{\boldsymbol{\lambda}}(\beta) = (\hat{A}(\beta), \hat{B}(\beta))^T = (\mathbf{X}(\beta)^T\mathbf{X}(\beta))^{-1}\mathbf{X}(\beta)^T\mathbf{y}. \tag{9}$$

Replacing $\boldsymbol{\lambda}$ by $\hat{\boldsymbol{\lambda}}(\beta)$ in $Q(\boldsymbol{\theta})$, we get

$$Q(A(\beta), B(\beta), \beta) = \mathbf{y}^T(I - P_{\mathbf{X}(\beta)})\mathbf{y} = \mathbf{y}^T\mathbf{y} - \mathbf{y}^T P_{\mathbf{X}(\beta)}\mathbf{y}. \tag{10}$$

So minimizing (10) with respect to β is equivalent to maximizing $R(\beta) = \mathbf{y}^T P_{\mathbf{X}(\beta)}\mathbf{y}$, where $P_{\mathbf{X}(\beta)} = \mathbf{X}(\beta)(\mathbf{X}(\beta)^T\mathbf{X}(\beta))^{-1}\mathbf{X}(\beta)^T$ is the projection matrix on the column space of $\mathbf{X}(\beta)$.
Thus by minimizing $Q(\beta)$ or maximizing $R(\beta)$ we obtain the LSE of β say $\hat{\beta}$ and using this LSE of β we obtain the estimates of the linear parameters $\hat{A}(\hat{\beta})$ and $\hat{B}(\hat{\beta})$ by (9).

2.1 Approximate Least Squares Estimation (ALSE) Method

In signal processing we extensively use the notion of Periodogram function which is an usual estimate of the spectral density of a signal. In fact the smooth Periodogram function is used to estimate the spectral density. The Periodogram function is defined as follows:

$$I(\omega) = \frac{1}{n}\left|\sum_{t=1}^{n} y_t e^{-i\,(\omega t)}\right|^2. \tag{11}$$

Now for the Elementary Chirp model the Periodogram function is modified as follows:

$$I(\beta) = \frac{1}{n}\left|\sum_{t=1}^{n} y_t e^{-i\,\beta t^2}\right|^2. \tag{12}$$

The Periodogram function in (12) can be written as :

$$I(\beta) = \frac{1}{n}\left[\left(\sum_{t=1}^{n} y_t \cos(\beta t^2)\right)^2 + \left(\sum_{t=1}^{n} y_t \sin(\beta t^2)\right)^2\right]. \tag{13}$$

The ALSE of β is obtained by maximizing (13) with respect to β. Let the ALSE of β be $\tilde{\beta}$, so once we get the estimate of the non-linear parameter we estimate our linear parameters by

$$\tilde{A} = \frac{2}{n}\sum_{t=1}^{n} y_t \cos(\tilde{\beta} t^2), \quad \tilde{B} = \frac{2}{n}\sum_{t=1}^{n} y_t \sin(\tilde{\beta} t^2). \tag{14}$$

Implementation

- In model (3) we have β as the non-linear parameter and A and B as the linear parameters and there is no closed form solution for the LSE of β. We need to use some iterative method to find the LSE of β we need an initial estimate, close enough to the true parameter value. So we try to find the initial estimate of β by maximizing the Periodogram function $I(\beta)$ with respect to β over a grid of $\{\frac{\pi k}{n^2},\ k = 1, \ldots, n^2 - 1\}$ by a grid search method.

- We obtain the initial value of β by maximizing I(β) in (13). Using that initial value of β we calculate the initial estimates of A and B.

- The linear parameters are replaced by their initial estimates in the RSS in (7) and the latter becomes a function of β then
$$Q(\beta) = \sum_{t=1}^{n}\left[y_t - \bar{A}\cos\left(\beta t^2\right) - \bar{B}\sin\left(\beta t^2\right)\right]^2. \tag{15}$$

- $Q(\beta)$ is minimized with respect to β to get the LSE of β. So we finally get the least squares estimates as $\hat{\boldsymbol{\theta}} = (\hat{A}, \hat{B}, \hat{\beta})^T$.

- In this way we get the LSE of the non-linear and linear parameters using the least squares method which uses the maximization of the Periodogram function for getting the initial estimate of the chirp rate parameter.

The following assumption is also needed to develop the theoretical properties of the estimators.

Assumption 2:- A^0 and B^0 are not simultaneously equal to zero.

2.2 Multicomponent Elementary Chirp Model

Here we consider the model given in equation (1). Let us denote the parameter vector corresponding to the k-th component as $\boldsymbol{\theta_k} = (A_k, B_k, \beta_k)^T$ and $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \ldots, \boldsymbol{\theta}_p^T)^T$ is the parameter vector of order $3p$. Then the LSE of $\boldsymbol{\theta}$ minimizes

$$Q(\boldsymbol{\theta}) = \sum_{t=1}^{n} \left[y_t - \sum_{k=1}^{p} (A_k \cos(\beta_k t^2) + B_k \sin(\beta_k t^2)) \right]^2, \tag{16}$$

with respect to A_k, B_k and β_k, $k = 1, \ldots, p$. We write $\mathbf{y} = (y_1, \ldots, y_n)^T$ and

$$\boldsymbol{X(\beta)} = [\mathbf{X}_1(\beta_1), \ldots, \mathbf{X}_p(\beta_p)],$$

$$\boldsymbol{X_k}(\beta_k) = \begin{bmatrix} \cos(\beta_k) & \sin(\beta_k) \\ \cos(\beta_k 2^2) & \sin(\beta_k 2^2) \\ \vdots & \vdots \\ \cos(\beta_k n^2) & \sin(\beta_k n^2) \end{bmatrix}, \tag{17}$$

and $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^T, \ldots, \boldsymbol{\lambda}_p^T)^T$, $\boldsymbol{\lambda}_k = (A_k, B_k)^T$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^T$, $k = 1, \ldots, p$. Then $Q(\boldsymbol{\theta})$ can be written as

$$Q(\boldsymbol{\theta}) = (\mathbf{y} - \boldsymbol{X(\beta)\lambda})^T (\mathbf{y} - \boldsymbol{X(\beta)\lambda}). \tag{18}$$

For a given $\boldsymbol{\beta}$, the LSEs of $\boldsymbol{\lambda}$ is $\hat{\boldsymbol{\lambda}}(\boldsymbol{\beta}) = \boldsymbol{(X(\beta)^T X(\beta))}^{-1} \boldsymbol{X(\beta)^T y}$. Replacing this in $Q(\boldsymbol{\theta})$, we have

$$Q(\hat{\boldsymbol{\lambda}}(\boldsymbol{\beta}), \boldsymbol{\beta}) = \boldsymbol{y}^T (\boldsymbol{I} - \boldsymbol{P_{X(\beta)}}) \boldsymbol{y},$$

where $\boldsymbol{P_{X(\beta)}} = \boldsymbol{X(\beta)(X(\beta)^T X(\beta))}^{-1} \boldsymbol{X(\beta)}^T$, is the projection matrix on the column space of $\boldsymbol{X(\beta)}$. Then minimizing $Q(\hat{\boldsymbol{\lambda}}(\boldsymbol{\beta}), \boldsymbol{\beta})$ with respect to $\boldsymbol{\beta}$ boils down to maximizing $R(\boldsymbol{\beta})$ with respect to $\boldsymbol{\beta}$, where

$$R(\boldsymbol{\beta}) = \boldsymbol{y}^T \boldsymbol{X(\beta)(X(\beta)^T X(\beta))}^{-1} \boldsymbol{X(\beta)}^T \boldsymbol{y}. \tag{19}$$

For large n we have the following simplification as :

$$\hat{\boldsymbol{\lambda}}_k(\beta_k) = (\boldsymbol{X}_k(\beta_k)^T \boldsymbol{X}_k(\beta_k))^{-1} \boldsymbol{X}_k(\beta_k)^T \boldsymbol{y}, \tag{20}$$

$$R(\boldsymbol{\beta}) = \sum_{k=1}^{p} \boldsymbol{y}^T \boldsymbol{X}_k(\beta_k) (\boldsymbol{X}_k(\beta_k)^T \boldsymbol{X}_k(\beta_k))^{-1} \boldsymbol{X}_k(\beta_k)^T \boldsymbol{y}. \tag{21}$$

Let $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \ldots, \hat{\beta}_p)^T$ maximizes $R(\boldsymbol{\beta})$ and then the linear parameters are obtained as $\hat{\boldsymbol{\lambda}}_k(\hat{\beta}_k)$, $k = 1, \ldots, p$. For the multicomponent Elementary Chirp model to estimate the unknown parameters we use a sequential procedure to find the ALSEs. The sequential method is described as follows:

- Compute $\tilde{\beta_1}$ by maximizing the Periodogram-like function

$$I_1(\beta) = \frac{1}{n} \left[\left(\sum_{t=1}^{n} y_t \cos(\beta t^2) \right)^2 + \left(\sum_{t=1}^{n} y_t \sin(\beta t^2) \right)^2 \right]. \tag{22}$$

  Then the linear parameter estimates are obtained by substituting $\tilde{\beta_1}$ in (14). Thus

$$\tilde{A}_1 = \frac{2}{n} \sum_{t=1}^{n} y_t \cos(\tilde{\beta_1} t^2), \quad \tilde{B}_1 = \frac{2}{n} \sum_{t=1}^{n} y_t \sin(\tilde{\beta_1} t^2). \tag{23}$$

- Now we have the estimates of the parameters of the first component of the observed signal. We subtract the contribution of the first component from the original signal y_t to remove the effect of the first component and obtain new data, say

$$y_t^{(1)} = y_t - \tilde{A}_1 \cos(\tilde{\beta}_1 t^2) - \tilde{B}_1 \sin(\tilde{\beta}_1 t^2), \quad t = 1, \ldots, n.$$

- Next we compute $\tilde{\beta}_2$ by maximizing $I_2(\beta)$ which is obtained by replacing the original data vector by the new data vector in equation (22) and $\tilde{A}_2$ and $\tilde{B}_2$ by substituting $\tilde{\beta}_2$ in (23).

- We continue this process upto p steps to get all the estimates. Asymptotically the ALSEs obtained in this method and the LSEs are equivalent.

3 Least Squares Estimation Method for Chirp Model

In this section we discuss the estimation of the parameters in a Chirp Model in (2). Firstly we consider for $p = 1$ i.e one component. The one component Chirp Model is described as follows:

$$y_t = A^0 \cos\left(\alpha^0 t + \beta^0 t^2\right) + B^0 \sin\left(\alpha^0 t + \beta^0 t^2\right) + X_t; \quad t = 1, 2, \ldots, n. \tag{24}$$

- $\{y_1, \ldots, y_n\}$ is the data observed at n time points.

- A^0 and B^0 are the true values of the unknown amplitudes, α^0 is the true non-zero frequency and β^0 is the true unknown frequency/chirp rate.

- We denote $\boldsymbol{\theta^0} = (A^0, B^0, \alpha^0, \beta^0)^T$ and $\boldsymbol{\theta} = (A, B, \alpha, \beta)^T$.

- $\{X_t\}$ satisfies Assumption 1.

We write the model in matrix notation as

$$\mathbf{y} = \mathbf{X}(\alpha^0, \beta^0)\boldsymbol{\lambda^0} + \boldsymbol{\epsilon}, \tag{25}$$

where $\mathbf{y} = (y_1, \ldots, y_n)^T$, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)^T$, and $\boldsymbol{\lambda^0} = (A^0, B^0)$ is the vector of linear parameters and

$$\mathbf{X}(\alpha^0, \beta^0) = \begin{bmatrix} \cos(\alpha^0 + \beta^0) & \sin(\alpha^0 + \beta^0) \\ \cos(\alpha^0 2 + \beta^0 2^2) & \sin(\alpha^0 2 + \beta^0 2^2) \\ \vdots & \vdots \\ \cos(\alpha^0 n + \beta^0 n^2) & \sin(\alpha^0 n + \beta^0 n^2) \end{bmatrix}.$$

Let $\boldsymbol{\lambda} = (A, B)^T$, the RSS is then defined as

$$Q(A, B, \alpha, \beta) = Q(\boldsymbol{\theta}) = \sum_{t=1}^{n} \left[y_t - A\cos\left(\alpha t + \beta t^2\right) - B \sin\left(\alpha t + \beta t^2\right)\right]^2. \tag{26}$$

In matrix notation this is same as

$$Q(\boldsymbol{\theta}) = (\mathbf{y} - \mathbf{X}(\alpha,\beta)\boldsymbol{\lambda})^T(\mathbf{y} - \mathbf{X}(\alpha,\beta)\boldsymbol{\lambda}), \tag{27}$$

for a given (α, β) we minimize $Q(\boldsymbol{\theta})$ with respect to $\boldsymbol{\lambda}$ and get

$$\hat{\boldsymbol{\lambda}}(\alpha,\beta) = (\hat{A}(\alpha,\beta), \hat{B}(\alpha,\beta))^T = (\mathbf{X}(\alpha,\beta)^T\mathbf{X}(\alpha,\beta))^{-1}\mathbf{X}(\alpha,\beta)^T\mathbf{y}. \tag{28}$$

Replacing $\boldsymbol{\lambda}$ by $\hat{\boldsymbol{\lambda}}(\alpha,\beta)$ in $Q(\theta)$ we get

$$Q(A(\alpha,\beta), B(\alpha,\beta), \alpha, \beta) = \mathbf{y}^T(I - P_{\mathbf{X}(\alpha,\beta)})\mathbf{y} = \mathbf{y}^T\mathbf{y} - \mathbf{y}^T P_{X(\alpha,\beta)}\mathbf{y}. \tag{29}$$

So minimizing (29) with respect to (α, β) is equivalent to maximizing $R(\alpha,\beta) = \mathbf{y}^T P_{X(\alpha,\beta)}\mathbf{y}$, where $P_{\mathbf{X}(\alpha,\beta)}$ is the projection matrix. Thus minimizing $Q(\alpha,\beta)$ or maximizing $R(\alpha,\beta)$ we obtain the LSE of (α,β) which is $(\hat{\alpha},\hat{\beta})$ and using these LSEs we obtain the estimates of the linear parameters $\hat{A}(\hat{\alpha},\hat{\beta})$ and $\hat{B}(\hat{\alpha},\hat{\beta})$ by (28). Now since the least squares method as discussed earlier is an iterative method which needs a good initial estimate of the parameters. These are obtained by maximizing the Periodogram function at the fourier frequencies $\{(\frac{\pi j}{n}, \frac{\pi k}{n^2});\ \ j = 1, \ldots, n-1,\ k = 1, \ldots, n^2 - 1\}$. The Periodogram function for the Chirp model looks like

$$I(\alpha,\beta) = \frac{1}{n}\left|\sum_{t=1}^{n} y_t\, e^{-i(\alpha t + \beta t^2)}\right|^2. \tag{30}$$

The ALSEs of α and β are obtained by maximizing $I(\alpha,\beta)$ with respect to α and β simultaneously. Once we estimate the non-linear parameters we can use them to estimate A and B by the following, let $\tilde{\alpha}$ and $\tilde{\beta}$ be the ALSEs of α and β respectively then we have:

$$\tilde{A} = \frac{2}{n}\sum_{t=1}^{n} y_t \cos(\tilde{\alpha} t + \tilde{\beta} t^2), \quad \tilde{B} = \frac{2}{n}\sum_{t=1}^{n} y_t \sin(\tilde{\alpha} t + \tilde{\beta} t^2). \tag{31}$$

The least squares estimators of the parameters for a multicomponent Chirp model in (2) are obtained by minimizing

$$Q(\boldsymbol{\theta}) = \sum_{t=1}^{n}\left[y_t - \sum_{k=1}^{p}(A_k\cos(\alpha_k t + \beta_k t^2) + B_k \sin(\alpha_k t + \beta_k t^2))\right]^2, \tag{32}$$

where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \ldots, \boldsymbol{\theta}_p^T)^T$ where the parameter vector of the k-th component is $\boldsymbol{\theta_k} = (A_k, B_k, \alpha_k, \beta_k)^T$. Simplifications similar to Elementary Chirp model can also be done for implementation.

The ALSEs are again obtained by using the sequential method discussed in the earlier section for the Elementary Chirp model.

4 Numerical Experiments

In this section we study how the least squares and the approximate least squares estimation methods behave for finite samples for the following three models;

1. One component Elementary Chirp model:

$$y_t = A\cos(\beta t^2) + B\sin(\beta t^2) + X_t; \quad t = 1, \ldots, n. \tag{33}$$

2. One component Chirp model:

$$y_t = A\cos(\alpha t + \beta t^2) + B\sin(\alpha t + \beta t^2) + X_t; \quad t = 1, \ldots, n. \tag{34}$$

3. Two component Elementary Chirp model:

$$y_t = A_1\cos(\beta_1 t^2) + B_1\sin(\beta_1 t^2) + A_2\cos(\beta_2 t^2) + B_2\sin(\beta_2 t^2) + X_t; \quad t = 1, \ldots, n. \tag{35}$$

where $\{X_t\}$ is the error process, and we consider two cases here

- **Case-I** $\{X_t\}$ is a sequence of i.i.d $N(0, \sigma^2)$ random variables with $\sigma^2 < \infty$.
- **Case-II** $\{X_t\}$ is a sequence of stationary process. Firstly we consider $\{X_t\}$ to come from a $MA(1)$ process which is given by;

$$X_t = \epsilon_t + \rho\, \epsilon_{t-1}, \quad t = 1, \ldots, n. \tag{36}$$

  where $\epsilon_t \sim iid\ N(0, \sigma_0^2), \forall\, t = 1, \ldots, n$ and $\rho = 0.5$.

We report these numerical experiments results conducted for different model parameters, different sample sizes and different error variances. The sample sizes (n) which we consider through out this section are $100, 200, 300, 500, 600$. Now regarding the model variance σ^2 we note that:

- If we consider case-I, then the error variance σ^2 is same as the model variance σ^2.
- If we consider case-II, then the error variance σ_0^2 is not same as the model variance, in this case the model variance $\sigma^2 = \sigma_0^2(1 + \rho^2)$, so when we say we are taking different values for the model variance for e.g, say 0.05 we mean that we are considering the value of $\sigma^2 = \sigma_0^2(1 + \rho^2) = 0.05$ which means the error variance is $\sigma_0^2 = \frac{0.05}{1+\rho^2}$.

The values of σ^2 considered here are $0.001, 0.01, 0.05, 0.1, 1.0, 2.0$, respectively. Note that the number of replications used in these experiments is 2000. Firstly we consider the one component Elementary Chirp model with the following parameter values;

$$\beta^0 = 0.7, \quad A^0 = 2, \quad B^0 = 3. \tag{37}$$

We generate the data and estimate the parameters using both the estimation methods (Least squares and Approximate Least squares) for a given sample size and model variance σ^2 for each case, i.i.d set up and stationary set up. We replicate the process 2000 times and calculate the average estimates of β, A, B along with their MSEs for both the estimation methods for different sample sizes with a fixed value of $\sigma^2 = 1$ and observe its behaviour. We repeat a similar experiment by fixing the sample size at 100 for different values of σ^2. The plots are only reported for the non linear parameters. We also report how the observed values of the signals are able to estimate the true values by both the methods for a fixed sample size 100 and $\sigma^2 = 1$.

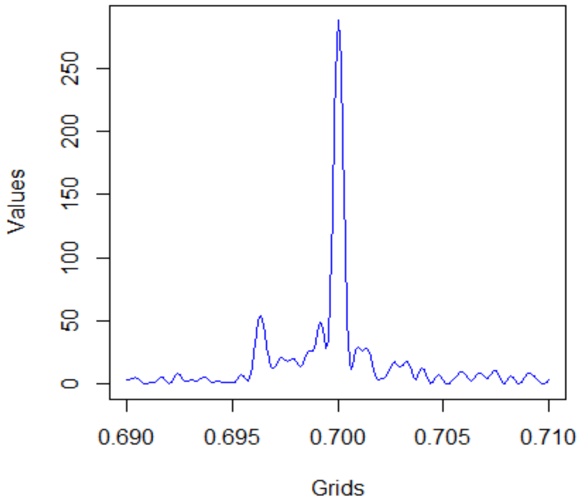


Figure 1: Periodogram function for one component Elementary Chirp model under stationary error.

Now we report the average estimate of β and its MSE for a fixed value of $\sigma^2 = 1$ and varying sample sizes .

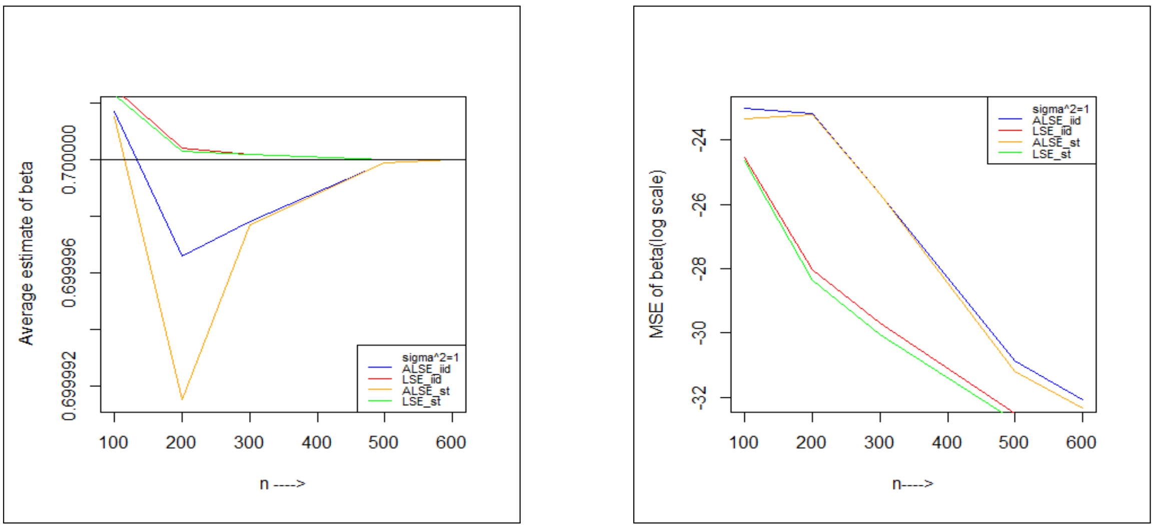
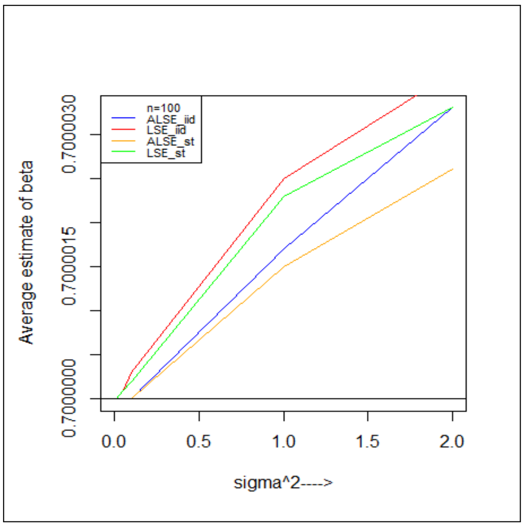


Figure 2: Left panel: Average estimate of β. Right panel: MSE of β.
blue: ALSE under i.i.d error, red: LSE under i.i.d error, orange: ALSE under stationary error, green: LSE under stationary error.

For fixed $n = 100$ and varying σ^2 we report the average estimate of β and its MSE;



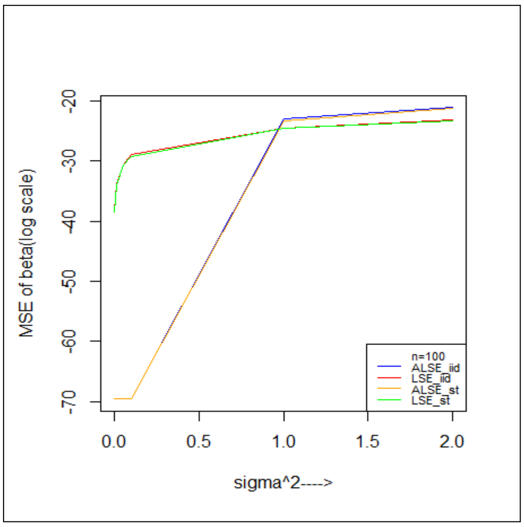
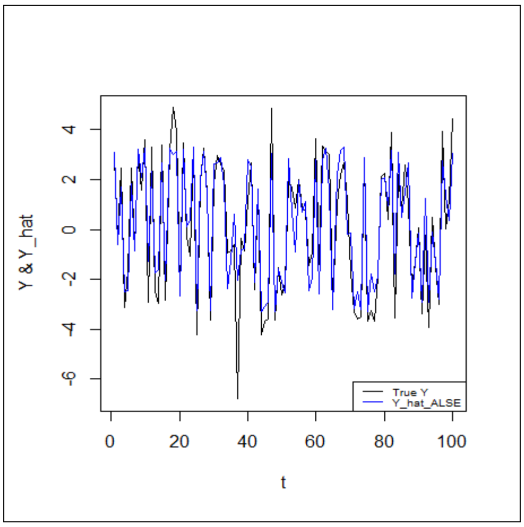


Figure 3: Left panel: Average estimate of β, Right panel: MSE of β.
blue: ALSE under i.i.d error, red: LSE under i.i.d error, orange: ALSE under stationary error, green: LSE under stationary error.

The fitted signals by both the estimation methods under the i.i.d and stationary set up are given below;



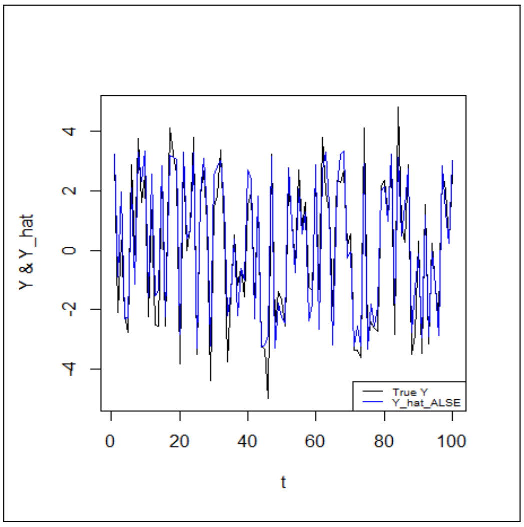


Figure 4: Left panel: Fitted signals by ALSE method under i.i.d error. Right panel: under stationary error.

The fitted signals (blue) obtained by the ALSE method matches quite well with the true signal (black) under both the cases i.i.d and stationary error.

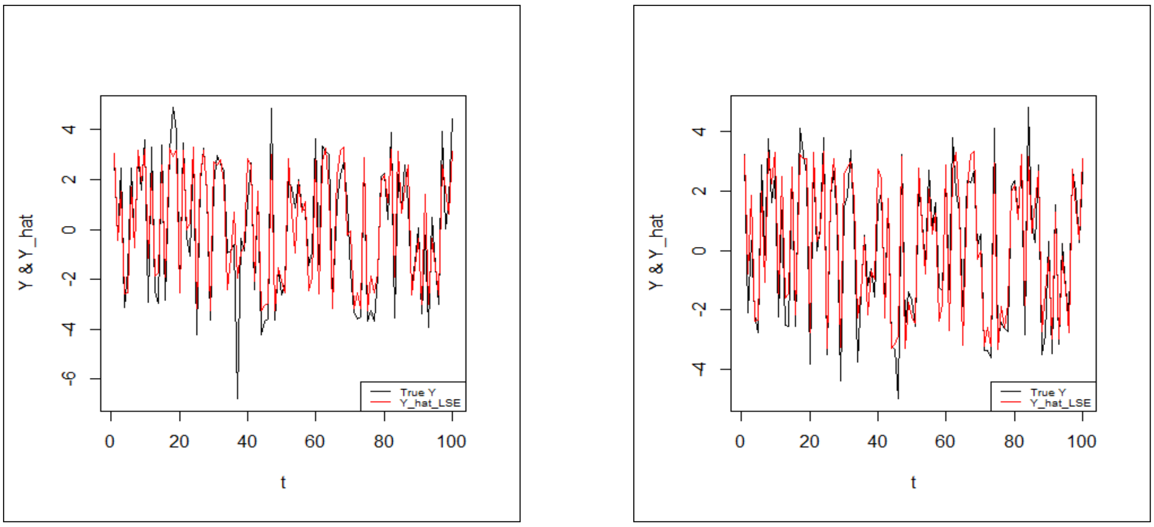


Figure 5: Left panel: Fitted signals by LSE method under i.i.d error. Right panel: under stationary error.

The fitted signals (red) obtained by the LSE method matches quite well with the true signal (black) under both the cases i.i.d and stationary error.

Now we consider a one component Chirp model with the true parameter values

$$\alpha^0 = 0.8, \quad \beta^0 = 0.6, \quad A^0 = 1, \quad B^0 = 2. \tag{38}$$

We first fix the error variance $\sigma^2 = 1$ and report the estimates of the non linear parameters α and β and also their MSE's for different values of sample sizes.

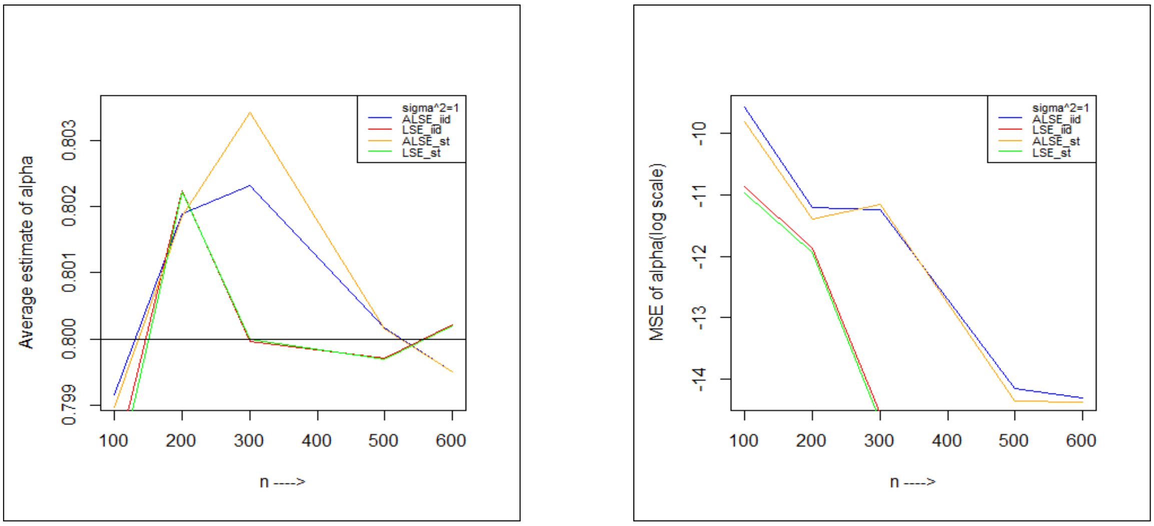
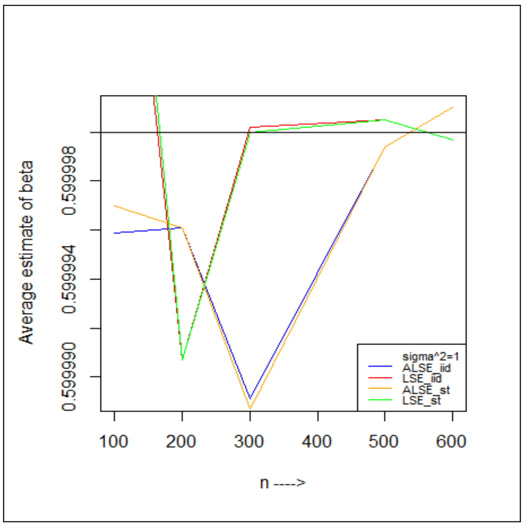


Figure 6: Left panel: Average estimate of α. Right panel: MSE of α.

blue: ALSE under i.i.d error, red: LSE under i.i.d error, orange: ALSE under stationary error, green: LSE under stationary error.



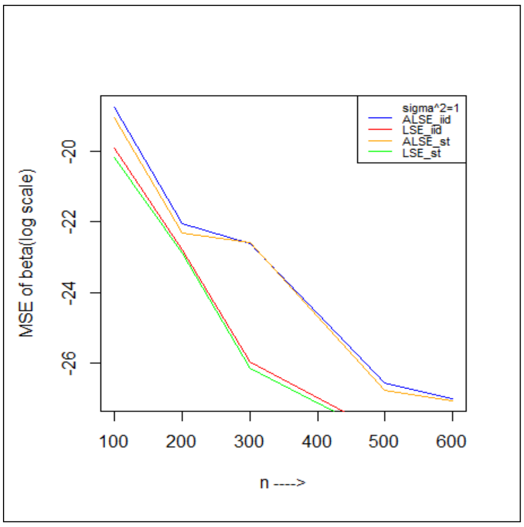
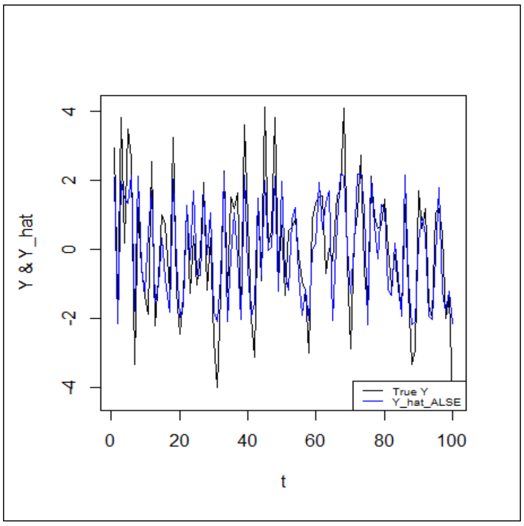


Figure 7: Left panel: Average estimate of β. Right panel: MSE of β.
blue: ALSE under i.i.d error, red: LSE under i.i.d error, orange: ALSE under stationary error, green: LSE under stationary error.

In simulation studies reported here, we observe that average biases and MSEs for LSE are marginally smaller than those of ALSE in case of both the models. Though LSE and ALSEs are asymptotically equivalent, LSEs are marginally better for finite samples.
For the Chirp model also we report how the observed values of the signals are able to estimate the true values by both the methods for a fixed sample size 100 and $\sigma^2 = 1$ for both i.i.d and stationary error are given below;



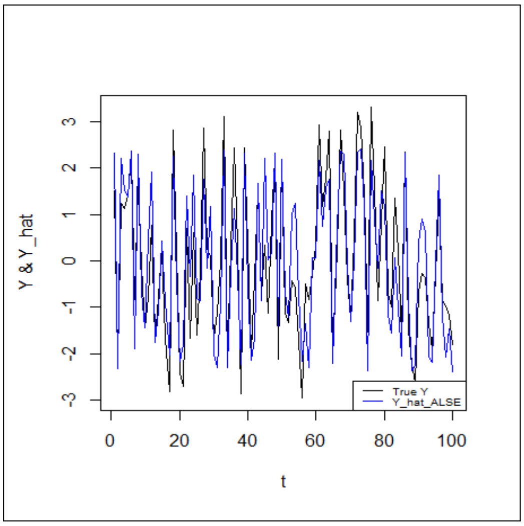


Figure 8: Left panel: Fitted signals by ALSE method under i.i.d error. Right panel: under stationary error.

The fitted signals (blue) obtained by the ALSE method matches quite well with the true signal (black) under both the cases i.i.d and stationary error for the Chirp model.

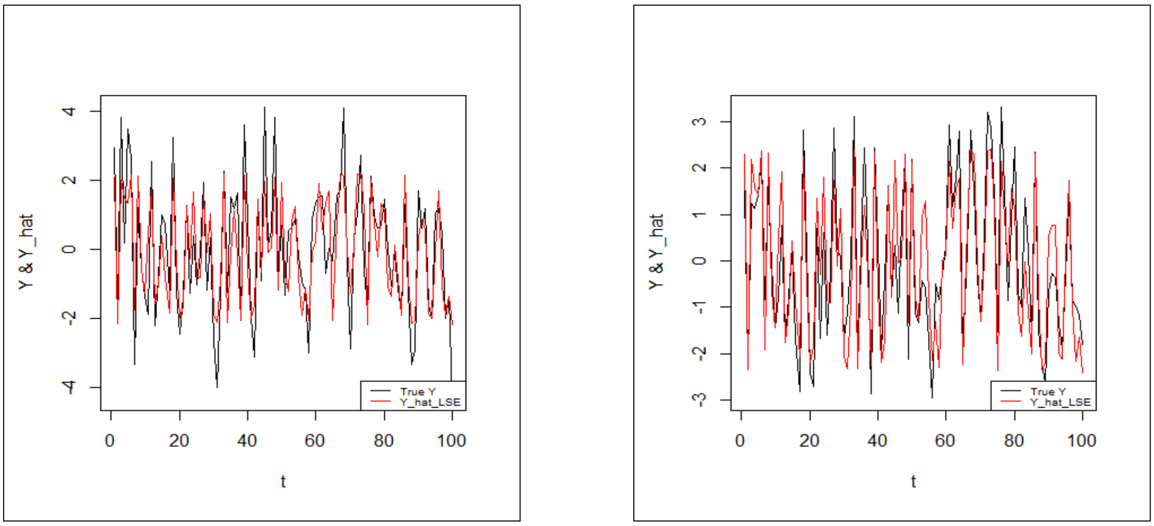


Figure 9: Left panel: Fitted signals by LSE method under i.i.d error. Right panel: under stationary error.

The fitted signals (red) obtained by the LSE method matches quite well with the true signal (black) under both the cases i.i.d and stationary error for the Chirp model.

We have visualised the change of MSE, and average estimates of the parameters under the stationary $MA(1)$ process for both Elementary Chirp model and Chirp model, now we try to emphasize on another widely used stationary process the $AR(1)$ process for simulating a 2 component Elementary Chirp model. Here referring (35) we have $\{X_t\}$ as;

$$X_t = \phi X_{t-1} + \epsilon_t, \quad t = 1, \ldots, n, \tag{39}$$

where ϵ_t are i.i.d random variables coming from $N(0, \sigma^2)$ $\forall\, t = 1, \ldots, n$ with $\sigma^2 < \infty$. For simulation purpose we consider the true value of $\phi = 0.5$ and that of $\sigma^2 = 1$. For the two component Elementary Chirp model the true values of the parameter which we consider are

$$\beta_1^0 = 0.3, \quad \beta_2^0 = 1, \quad A_1^0 = 3, \quad A_2^0 = 1, \quad B_1^0 = 4, \quad B_2^0 = 2. \tag{40}$$

Now here just from a single replication we obtain the fitted signals, and try to check whether the residuals obtained are stationary, if so then fit a suitable stationary process to the residuals.

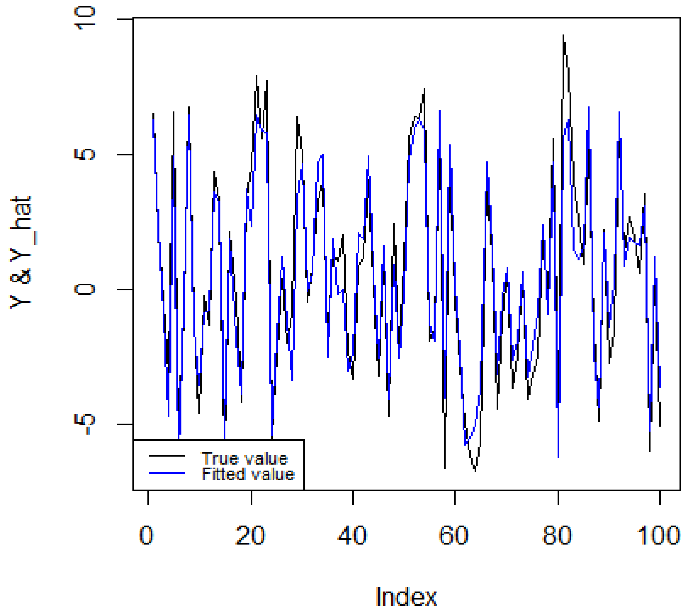


Figure 10: Fitted signals for a 2 component Elementary Chirp model.

We observe that the fitted signals (blue) and the true signal (black) matches quite well.

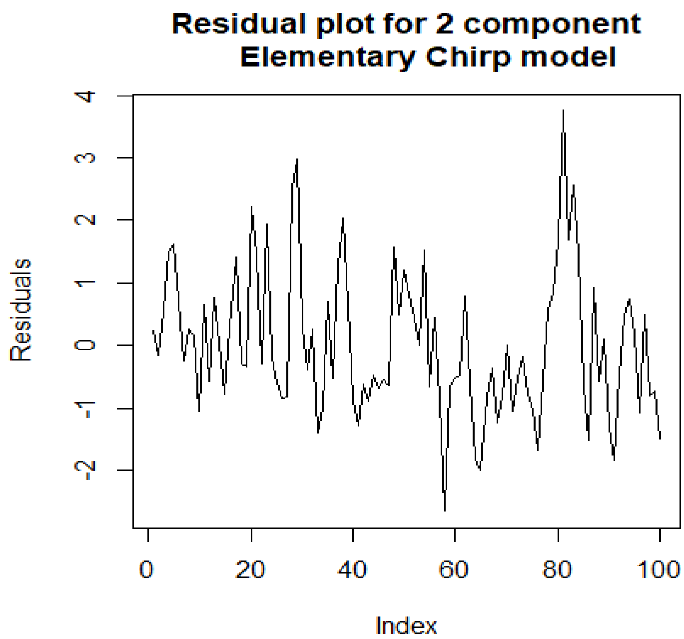


Figure 11: Residuals obtained from a 2 component Elementary Chirp model.

We observe from the above plot that the residuals look stationary, still we perform the **Augmented Dickey Fuller** test for conformity. The p value of the test comes out to be 0.012 which leads us to reject the null hypothesis and accept the fact that the residuals are stationary at 5% level of significance. Now to have an idea about the most suitable stationary process that will fit the residuals well, we focus on the Partial Auto-correlation function (PACF) and the Auto-correlation function (ACF).

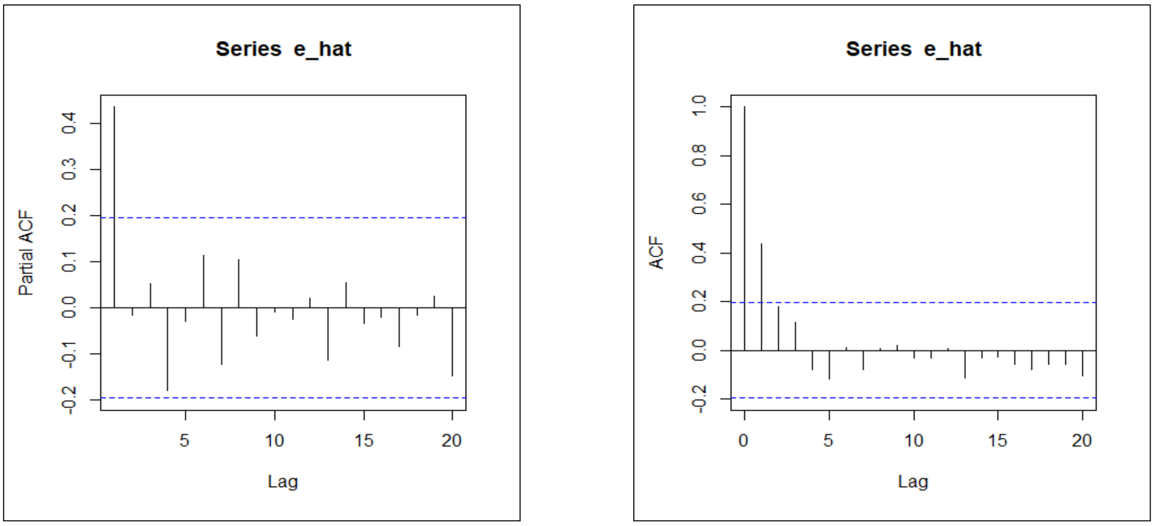


Figure 12: Left panel: PACF of the residuals. Right panel: ACF.

We compare different models based on the Bayesian Information Criterion (BIC) values.

AR(1)	AR(2)	AR(3)	AR(4)
299.6904	304.2745	308.5638	309.6804

MA(1)	MA(2)	MA(3)	MA(4)
301.3622	305.2609	306.841	311.1455

ARMA(1,1)	ARMA(1,2)	ARMA(2,1)	ARMA(2,2)
304.2658	308.6445	308.4666	312.7794

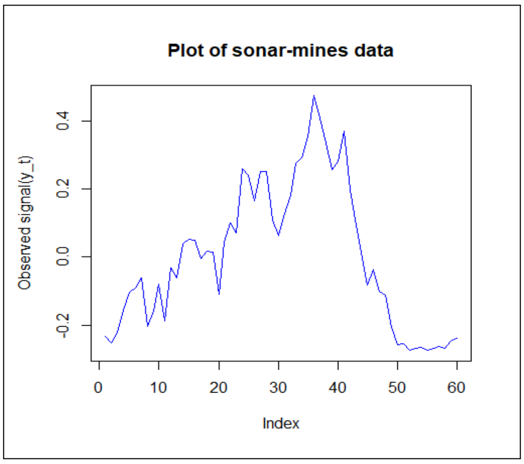
Comparing all the values we see the BIC value turns out to be the least for $AR(1)$ process, hence we fit an $AR(1)$ process to the residuals and the estimated coefficient comes out to be $\hat{\phi} = 0.4699$ which is quite close to 0.5.

All the plots reported for the Chirp and Elementary Chirp model are in support of our classical theory of Statistics i.e whether the error is stationary or i.i.d the average estimates obtained by the ALSE and LSE methods are consistent as well as the MSE decreases as the sample size increases. Besides this as we increase the model variance the average estimates moves away from the true parameters and the MSE also increases considerably.

5 Data Analysis

5.1 Introduction

The Sonar-mines and the Sonar-rocks datasets are used for the purpose of data analysis. The data were collected from the UCI Irvine Machine Learning Repository. The data "Sonar.mines" contains a particular pattern obtained by bouncing sonar signals off a metal cylinder at a particular angle. There are other such datasets collected at various angles and under various conditions. The data "Sonar.rocks" contains a pattern obtained from rocks under a similar condition. Both the datasets consists of the response values over time. There are 60 observations in each dataset. We denote the observed data by y_t. Note that from here onwards we center the data and consider it throughout the discussion.



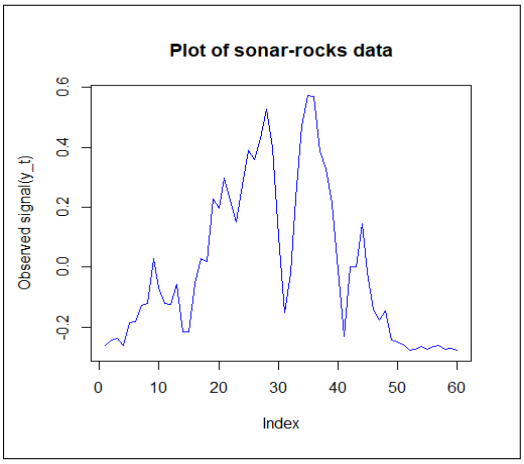


Figure 13: Left panel: Sonar-mines data, Right panel:Sonar-rocks data.

5.2 Analysis

We try to fit both the model multicomponent Elementary Chirp model and Chirp model on each of the dataset. To fit these models on the datasets we need to know p, the number of components before going for estimating the model parameters. The procedure to estimate the number of components which is followed in this study is given below:

1. Here we are handling a real life dataset hence we do not know the number of components (p) which will best fit the data so need to estimate it by some method.

2. Let us first consider the fitting of a multicomponent Elementary Chirp model. We adopt a sequential procedure to estimate p and also the corresponding model parameters.

3. We first start with a one component model where we have A_1, B_1 as the linear parameters and β_1 as the non-linear parameter.

4. We perform a gridsearch method over the grid $\{k\frac{\pi}{n^2}, \quad k = 1, \ldots, (n-1)^2\}$, where n is the number of observations. We take that point to be the initial choice of β_1 where the periodogram function is maximised.

5. Using the initial choice of β_1 we maximise the Periodogram function until we get the final estimate of β_1. After we get the estimate of the non-linear parameter we get the estimates of the linear parameters given by the equation (23).

6. Next we remove the effect of the first component from the data and compute

$$y_t^{(1)} = \left(y_t - \hat{A}_1 \cos(\hat{\beta}_1 t^2) - \hat{B}_1 \sin(\hat{\beta}_1 t^2)\right), \quad t = 1, \ldots, n,$$

then we use this $y_t^{(1)}$ as the current observed data and again perform a grid search over the grid $\{k\frac{\pi}{n^2}, \ k = 1, \ldots, (n-1)^2\}$ to get an initial estimate of β_2 and then we continue the same process for k number of times.

7. For each case we compute the residuals i.e $y_t - \hat{y}_t$ and check for the stationarity of the error, and try to fit a stationary process to the error. The stage at which these residuals becomes stationary and the fit is also reasonable we take that model to be the one with which we try to fit the data.

8. For the multicomponent Chirp model the entire procedure remains the same only that here instead of having one non-linear parameter we have two non-linear parameters α, β, the frequency and the chirp rate respectively. So here we have to perform a simultaneous grid search over $\{(\frac{\pi j}{n}, \frac{\pi k}{n^2}), \quad j = 1, \ldots, n-1, \ \ k = 1, \ldots, n^2-1\}$. Then the same steps are followed to estimate the number of components p and then we fit a multicomponent chirp model and use the sequential procedure to estimate the model parameters.

5.3 Visualization and fitting of a Sonar-mines data

In step 5 of Section 5.2 we said that the initial choice of β_1 is obtained by maximizing the Periodogram function given by (12) over the grid, so here we have a glimpse of the Periodogram function for the Sonar-mines data where y_t in (12) is the observed signal.

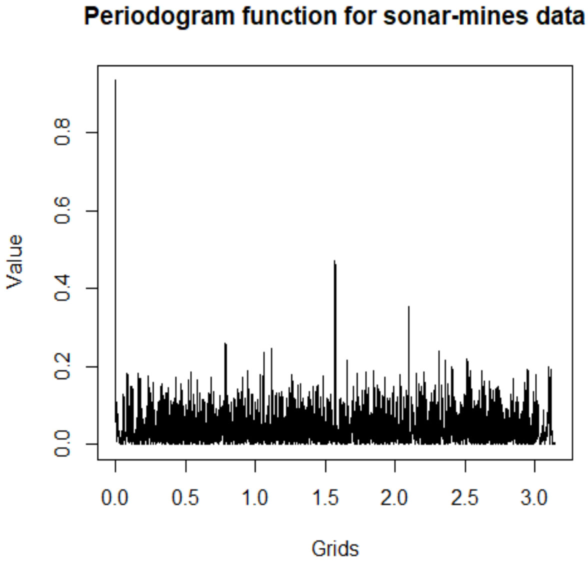


Figure 14: Periodogram function for Sonar-mines data

As discussed earlier we try to find the value of p by checking at which stage the residuals becomes stationary and we can fit a stationary process to it. It is observed that for the Elementary Chirp after fitting 8 components the residuals obtained from this model becomes stationary, and for the Chirp model after fitting 6 components the residuals obtained from this model becomes stationary. The residual plots for the 8 component Elementary Chirp model and 6 component Chirp model are given below;

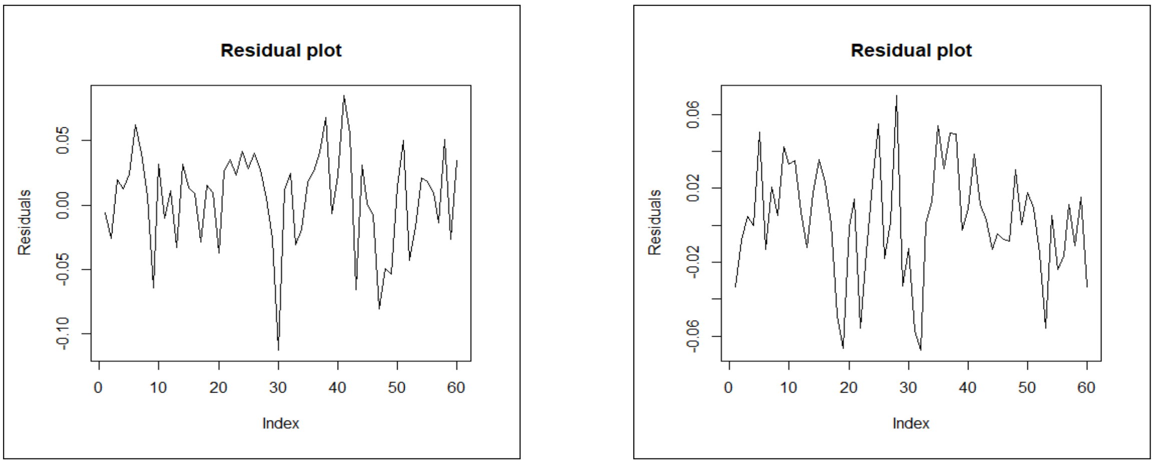


Figure 15: Left panel: Residual plot for Sonar-mines data obtained by fitting a 8 component Elementary Chirp model, Right panel: obtained by fitting a 6 component Chirp model.

From Figure (15) the plot is seeming to be stationary but still we perform the well known **Augmented Dickey-Fuller** test using the **adf.test** function in the **tseries** package in R. The p-value for the test performed on the residuals of the 8 component Elementary Chirp model is 0.02 and that of the 6 component Chirp model is 0.02485, thus at 5% level of significance we reject the null hypothesis and accept the alternative hypothesis which states that the residuals are stationary. Moreover we try to see which stationary process we can fit to the residuals and it is done by using **auto.arima** function in **forecast** package in R. Using this we see that the residuals obtained by fitting a 8 component Elementary Chirp model follows a AR(1) process

$$X_t = 0.2035\ X_{t-1} + \epsilon_t \tag{41}$$

and that of the 6 component Chirp model follows a MA(1) process

$$X_t = 0.3222\ \epsilon_{t-1} + \epsilon_t. \tag{42}$$

Now we observe how well the chosen model fits the data;

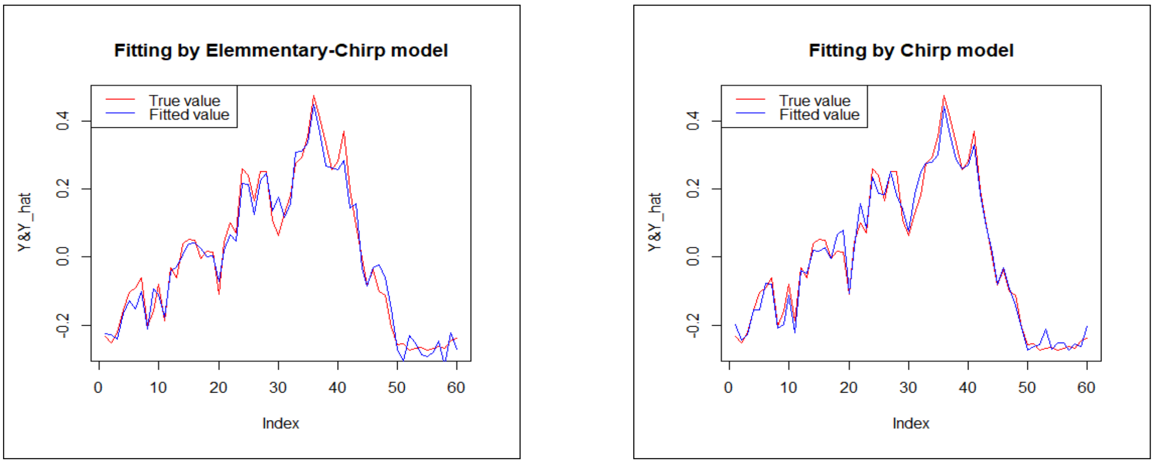


Figure 16: Left panel: Fitted signals from a 8 component Elementary Chirp model. Right panel: Fitted signals from a 6 component Chirp model for the Sonar-mines data.

From the above plot we can say that the observed signal (red) and the fitted signals (blue) matches quite well for both the models.

5.4 Visualisation for Sonar-rocks data

The Periodogram function for the Sonar-rocks data is given as below;

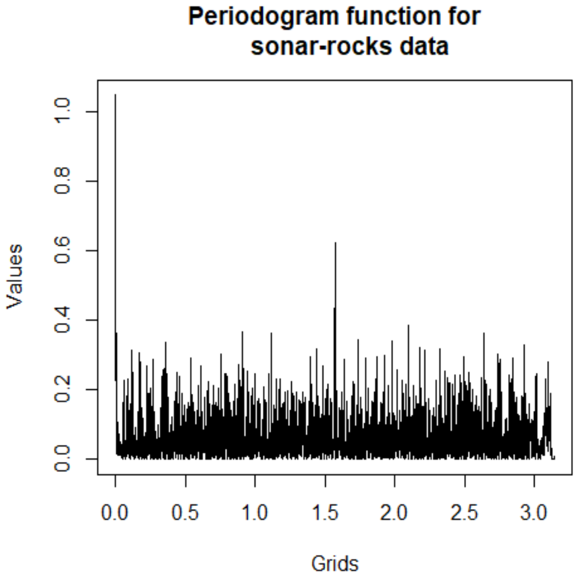


Figure 17: Periodogram function for sonar-rocks data.

It is observed that for the Elementary Chirp after fitting 8 components the residuals obtained from this model becomes stationary, and for the Chirp model after fitting 5 components the residuals obtained from this model becomes stationary for the sonar-rocks data. The residual plots for the 8 component Elementary Chirp model and 5 component Chirp model are given below;

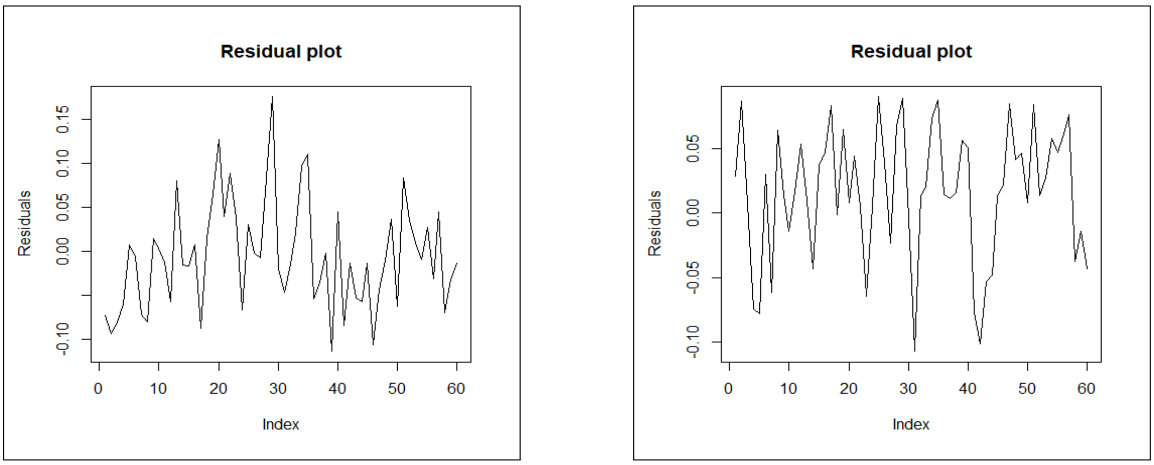


Figure 18: Left panel: Residual plot for Sonar-rocks data for 8 component Elementary Chirp model. Right panel: Residual plot for Sonar-rocks data for 5 component Chirp model.

The p-value for the test performed on the residuals of the 8 component Elementary Chirp model is 0.04352 and that of the 6 component Chirp model is 0.01904, thus at 5% level of significance we reject the null hypothesis and accept the alternative hypothesis which states that the residuals are stationary. Moreover we try to see which stationary process we can fit to the residuals and it is done by using **auto.arima** function in **forecast** package in R. Using this we see that the residuals obtained by fitting a 8 component Elementary Chirp model follows a AR(1) process

$$X_t = 0.2242\ X_{t-1} + \epsilon_t \tag{43}$$

and that of the 6 component Chirp model follows a MA(1) process

$$X_t = 0.3481\ \epsilon_{t-1} + \epsilon_t. \tag{44}$$

Now we observe how well the chosen model fits the data;

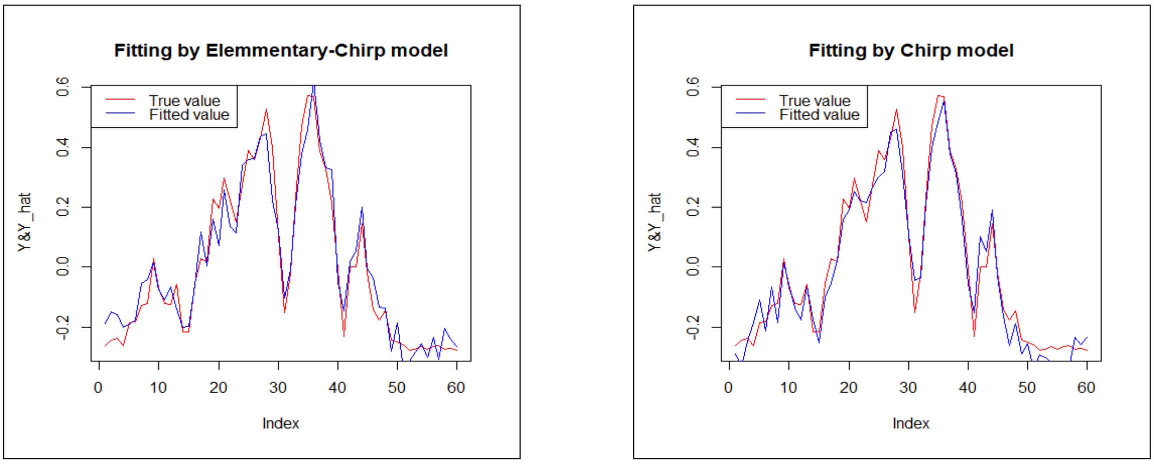


Figure 19: Left panel: Fitted signals from a 8 component Elementary Chirp model. Right panel: Fitted signals from a 5 component Chirp model .

From the above plot we can say that the observed signal(red) and the fitted signals(blue) matches quite well for both the models for the sonar-rocks data.

From the above analysis of both the datasets we note that for the Sonar-mines data we required a 8 component Elementary Chirp model and a 6 component Chirp model, so basically we need to estimate 24 parameters effectively for each of the model, so there is no reduction in the number of parameters hence both the models can be used but we prefer to fit the Elementary Chirp model due to its lesser model complexity than the Chirp model. Moreover for the Sonar-rocks dataset we observe a 8 component Elementary Chirp model fits the data well whereas for fitting a Chirp model we require 5 components, in this case for Elementary Chirp model we need to estimate 24 parameters, but for Chirp model we need to estimate 20 parameters, hence there is a reduction in the number of parameters, thus for Sonar-rocks we prefer to use the Chirp model.

6 Conclusion

In this project we have discussed two widely studied models in the field of Signal Processing namely, the Elementary Chirp model and the Chirp model. The Chirp model is a generalised model of the well known Sinusoidal model and the Elementary Chirp model is a special case of the Chirp model. Two estimation methods LS and approximate LS have been discussed briefly. Both the methods estimate the unknown parameters consistently. For the case of more than one component the sequential procedure is implemented for estimating the unknown parameters. The number of components p is assumed to be known in simulation and in data analysis it is estimated by exploratory analysis. A reasonable approach in estimating p may include any information theoretic criterion or cross-validation technique which is not addressed in this work.

7 Appendix

The following Lemmas will be required to find the asymptotic distribution of the estimators.

Lemma 1: If $\alpha, \beta \in (0, \pi)$, and $\alpha \neq \beta$, then except for countable number of points, the following results hold.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \cos(\beta t^2) = 0, \quad \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \sin(\beta t^2) = 0, \tag{45}$$

$$\lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \cos^2(\beta t^2) = \lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \sin^2(\beta t^2) = \frac{1}{2(k+1)}, \tag{46}$$

$$\lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \cos(\beta t^2)\sin(\alpha t^2) = 0. \tag{47}$$

In addition if $\alpha \neq \beta$, then for $k = 0, 1, 2, \ldots$

$$\lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \sin(\alpha t^2)\sin(\beta t^2) = \lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \cos(\alpha t^2)\cos(\beta t^2) = 0.$$

Lemma 2: If $\{X_t\}$ satisfies Assumption 1, then for $\beta \in (0, \pi)$

$$\sup_{\beta}\left|\frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k X_t \cos(\beta t^2)\right| \longrightarrow 0 \text{ a.s as } n \longrightarrow \infty,$$
$$\sup_{\beta}\left|\frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k X_t \sin(\beta t^2)\right| \longrightarrow 0 \text{ a.s as } n \longrightarrow \infty.$$

Theorem 1:- *Under Assumption 1 and 2 the limiting distribution of the least squares estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}^0$ is given by*

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^{\mathbf{0}})\boldsymbol{D}^{-1} \longrightarrow \boldsymbol{N}_3(\mathbf{0}, 2\sigma^2 c(\beta^0)\boldsymbol{\Sigma}^{-1}), \tag{48}$$

where $\boldsymbol{D}$ is a diagonal matrix

$$\boldsymbol{D} = diag(n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{5}{2}}),$$

$c(\beta^0)$ is given by

$$c(\beta^0) = \left| \sum_{j=-\infty}^{\infty} a_j e^{-i\beta^0 j^2} \right|^2$$

and $\mathbf{\Sigma}$ is given by

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 & \frac{B^0}{3} \\ 0 & 1 & \frac{-A^0}{3} \\ \frac{B^0}{3} & \frac{-A^0}{3} & \frac{A^{0^2}+B^{0^2}}{5} \end{bmatrix}, \tag{49}$$

and $\mathbf{\Sigma}^{-1}$ is given by

$$\mathbf{\Sigma}^{-1} = \frac{1}{4(A^{0^2}+B^{0^2})} \begin{bmatrix} 4A^{0^2}+9B^{0^2} & -5A^0B^0 & -15B^0 \\ -5A^0B^0 & 9A^{0^2}+4B^{0^2} & 15A^0 \\ -15B^0 & 15A^0 & 45 \end{bmatrix}, \tag{50}$$

Under the assumption of the i.i.d errors which is a special case of stationary error the value of $c(\beta^0) = 1$ as a result the distribution is given by

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^{\mathbf{0}})\boldsymbol{D}^{-1} \longrightarrow \boldsymbol{N}_3(\mathbf{0}, 2\sigma^2\mathbf{\Sigma}^{-1}). \tag{51}$$

where $\mathbf{\Sigma}^{-1}$ is given above.

Theorem 2:- *For one component Elementary Chirp model, under the assumption 1, the limiting distribution of $(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}^{\mathbf{0}})\boldsymbol{D}^{-1}$ is same as that of $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^{\mathbf{0}})\boldsymbol{D}^{-1}$ as $n \longrightarrow \infty$, where $\tilde{\boldsymbol{\theta}}$ is the ALSE of $\boldsymbol{\theta}^{\mathbf{0}}$ and $\hat{\boldsymbol{\theta}}$ is the LSE of $\boldsymbol{\theta}^{\mathbf{0}}$ and $\boldsymbol{D} = diag(n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{5}{2}})$.*

Theorem 3:- *Under Assumption 1 and 2 the limiting distribution of the least squares estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}^0$ for one component Chirp model is given by*

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^{\mathbf{0}})\boldsymbol{D}^{-1} \longrightarrow \boldsymbol{N}_4(\mathbf{0}, 2\sigma^2 c(\alpha^0, \beta^0)\mathbf{\Sigma}^{-1}), \tag{52}$$

where $\boldsymbol{D}$ is a diagonal matrix

$$\boldsymbol{D} = diag(n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{3}{2}}, n^{-\frac{5}{2}})$$

$c(\alpha^0, \beta^0)$ is given by

$$c(\alpha^0, \beta^0) = \left| \sum_{j=-\infty}^{\infty} a_j e^{-i(\alpha^0 j + \beta^0 j^2)} \right|^2$$

and $\mathbf{\Sigma}$ is given by

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 & \frac{B^0}{2} & \frac{B^0}{3} \\ 0 & 1 & \frac{-A^0}{2} & \frac{-A^0}{3} \\ \frac{B^0}{2} & \frac{-A^0}{2} & \frac{A^{0^2}+B^{0^2}}{3} & \frac{A^{0^2}+B^{0^2}}{4} \\ \frac{B^0}{3} & \frac{-A^0}{3} & \frac{A^{0^2}+B^{0^2}}{4} & \frac{A^{0^2}+B^{0^2}}{5} \end{bmatrix}, \tag{53}$$

and $\boldsymbol{\Sigma}^{-1}$ *is given by*

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{(A^{0^2} + B^{0^2})} \begin{bmatrix} A^{0^2} + 9B^{0^2} & -8A^0B^0 & -36B^0 & 30B^0 \\ -8A^0B^0 & 9A^{0^2} + B^{0^2} & 36A^0 & -30A^0 \\ -36B^0 & 36A^0 & 192 & -180 \\ 30B^0 & -30A^0 & -180 & 180 \end{bmatrix}, \tag{54}$$

Under the assumption of the i.i.d errors which is a special case of stationary error the value of $c(\alpha^0, \beta^0) = 1$ *as a result the distribution is given by*

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta^0})\boldsymbol{D}^{-1} \longrightarrow \boldsymbol{N}_4(\mathbf{0}, 2\sigma^2\boldsymbol{\Sigma}^{-1}). \tag{55}$$

where $\boldsymbol{\Sigma}^{-1}$ *is given above.*

Theorem 4:- *For one component Chirp model, under the assumption* 1*, the limiting distribution of* $(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta^0})\boldsymbol{D}^{-1}$ *is same as that of* $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta^0})\boldsymbol{D}^{-1}$ *as* $n \longrightarrow \infty$*, where* $\tilde{\boldsymbol{\theta}}$ *is the ALSE of* $\boldsymbol{\theta^0}$ *and* $\hat{\boldsymbol{\theta}}$ *is the LSE of* $\boldsymbol{\theta^0}$ *and* $\boldsymbol{D} = diag(n^{-\frac{1}{2}}, n^{-\frac{1}{2}}, n^{-\frac{3}{2}}, n^{-\frac{5}{2}})$

References

[1] Grover, R. Kundu, D. and Mitra, A., "On approximate least squares estimators of parameters on one dimensional chirp signal", *Statistics*, vol. 52, no. 5, 1060 - 1085, 2018.

[2] Mboup, M. and Adali, T., "A generalization of the Fourier transform an its application to spectral analysis of chirp-like signals", *Applied and Computational Harmonic Analysis*, Vol. 32, no, 2, 305-312, 2012.

[3] Kundu, D. and Nandi, S. "Parameter estimation of chirp signals in presence of stationary noise", *Statistica Sinica*, vol. 18, no. 1, 187 - 201, 2008.

[4] Kundu, D. and Nandi, S. "Estimating parameters of elementary chirp model using Modified Newton-Raphson algorithm", 2023, Preprint.

[5] Anjali Mittal, D.Kundu, Amit Mitra and Rhythm Grover "Estimation of the Elementary Chirp Model Parameters", IEEE Transactions on Aerospace and Electronic Systems, vol.59, 5218-5234, 2023.