MM 225 – AI and Data Science

Day 10: Random Variable : Continuous

Instructors: Hina Gokhale, MP Gururajan, N. Vishwanathan

20 AUGUST 2024

Normal Distribution / Gaussian Distribution

Random variable X is said to follow Normal distribution, if its pdf takes following form

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \; for \; -\infty < x < \infty$$

Notation: $X \sim N(\mu, \sigma^2)$

$$E(X) = \mu \text{ and } Var(X) = \sigma^2$$

$Random\ Variable\ Z = \frac{X-\mu}{\sigma}\ has\ pdf$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}\ exp\left\{-\frac{z^2}{2}\right\}$$

is called Standard Normal distribution and Z is called standard normal variate with mean 0 and standard deviation 1

Hence $Z \sim N(0,1)$

Error Function and Normal Distribution

Error Function (Gauss error function) is defined as special function.

It occurs in partial differential equations describing diffusion, defined as

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{\{-t^2\}}\, dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{\{-t^2\}}\, dt$$

It can be seen that erf(x) describes probability of a normal random variable Y in the range [-x, x], where $Y \sim N(0, {}^{1}/_{2})$

Exponential Distribution

A random variable X is said to follow exponential distribution if its probability density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & for\ x > 0 \\ 0 & Otherwise \end{cases}$$

Notation: X~Exp(λ)

$$E(X) = {}^{1}/_{\lambda}$$

$$Var(X) = {}^{1}/_{\lambda^2}$$

Lack of Memory

$P[X > t_1 + t_2 | X > t_1] = P[X > t_2]$ or equivalently

$P[X < t_1 + t_2 | X > t_1] = P[X < t_2]$

Distributions derived from Normal distribution

Chi squared distribution with n degrees of freedom - $\chi^2(n)$

Student's t distribution

F distribution

Chi squared distribution - definition

Let $Z_1, Z_2, \dots, Z_n$ be random sample from N(0,1) then $W = \sum_{i=1}^{n} Z_i^2$ is said to have Chi-square distribution with n degrees of freedom

Notation: $W \sim \chi_n^2$

E(W) =n and Var(W) = 2n

Note that: $X_i^2 \sim \chi_1^2 \; for \; i = 1, 2, \dots, n$ and are independent

Hence if W $\sim \chi_n^2$ and V $\sim \chi_m^2$ and W and V are independent then

W+V $\sim \chi_{n+m}^2$

Chi squared distribution – pdf

If W ~ χ^2(n) then the pdf is given by

$$f(x) = \frac{x^{\left({}^{n}/_{2}-1\right)} e^{-{}^{x}/_{2}}}{2^{n}\Gamma\left({}^{n}/_{2}\right)}$$

Note that χ^2(n)≡Gamma($\frac{n}{2}, \frac{1}{2}$)

Student's t distribution

Let Z~ N(0, 1) and let W $\sim \chi_n^2$ and X and W are independent then

$$t = \frac{Z}{\sqrt{W/n}}$$

follows t distribution with n degrees of freedom

Note: $X_i \sim N(\mu, \sigma^2)$ $for\ i = 1, 2, \ldots, n$ is a random sample then

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \text{ and } (\mathrm{n}-1)\frac{S^2}{\sigma^2} \sim \chi_{n-1}^2 \ are\ independent\ then$$

$$\boldsymbol{t = \frac{\sqrt{n}(\bar{X}-\mu)/\sigma}{\sqrt{S^2/\sigma^2}} = \frac{\sqrt{n}\,(\bar{X}-\mu)}{S} \sim t(n-1)}$$

F distribution

Let W $\sim \chi_n^2$ and V $\sim \chi_m^2$ and are independent then

$$F = \frac{W/_{n}}{V/_{m}}$$

follows F distribution with (n, m) degrees of freedom

Example 1:

The lifetime of a color television picture tube is a normal random variable with mean 8.2 years and standard deviation 1.4 years. What percentage of such tubes lasts

(a) more than 10 years;

(b) less than 5 years;

(c) between 5 and 10 years?

Example 2

The number of years a radio functions is exponentially distributed with parameter $\lambda = 18$. If Sunita buys a used radio, what is the probability that it will be working after an additional 10 years?

Example 3

If a random variable $T \sim t(8)$ then find:

a) $P(T \geq 1)$

b) $P(T \leq 2)$

c) $P(-1 < T < 1)$

Thank you…..