Practice Sheet 2

1.) $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

a.) $P\{X < t_1 + t_2 / X > t_2\}$

$$= \frac{P\{X < t_1 + t_2 \cap X > t_2\}}{P\{X > t_2\}}$$

$$= \frac{\int_{t_2}^{t_1+t_2} \lambda e^{-\lambda x}\, dx}{\int_{t_2}^{\infty} \lambda e^{-\lambda x}\, dx}$$

$$= \frac{e^{-t_2\lambda} - e^{(-t_1-t_2)\lambda}}{e^{-t_2\lambda}}$$

$$= 1 - e^{-t_1\lambda}$$

$$P\{X < t_1\} = \int_0^{t_1} \lambda e^{-\lambda x}\, dx$$

$$= \left[-e^{-\lambda x}\right]_0^{t_1}$$

$$= 1 - e^{-\lambda t_1}$$

$$\therefore \boxed{P\{X < t_1 + t_2 / X > t_2\} = P\{X < t_1\}}$$

Hence, Proved.

b.) $E(X) = \int^{\infty} \lambda x e^{-\lambda x}\, dx$

$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$

$$= 0 + \left[\frac{-e^{-\lambda x}}{\lambda}\right]_0^\infty$$

$$= \boxed{\frac{1}{\lambda}}$$ Hence, proved

c) $$Var(X) = E(X^2) - (E(X))^2$$

$$= \underbrace{\int_0^\infty \lambda x^2 e^{-\lambda x}\,dx}_{} - \frac{1}{\lambda^2}$$

D.I. Table:

x^2	$+$	$e^{-\lambda x}$
$2x$	$-$	$-\frac{e^{-\lambda x}}{\lambda}$
2	$+$	$\frac{e^{-\lambda x}}{\lambda^2}$
		$-\frac{e^{-\lambda x}}{\lambda^3}$

$$= \left[-x^2e^{-\lambda x} - 2x\frac{e^{-\lambda x}}{\lambda} - \frac{2e^{-\lambda x}}{\lambda^2}\right]_0^\infty - \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$ Hence, proved.

2) Part 1: $$f(x) = \begin{cases} 0.05e^{-0.05x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$P\{X > 30000 / X > 10000\}$

$= P\{X > 20000\}$

$$= \int_{20000}^{\infty} 0.05\, e^{-0.05x}\, dx$$

$$= \left[-e^{-0.05x}\right]_{20000}^{\infty}$$

$$= \boxed{e^{-1000}}$$

Part 2 : $f(x) = \begin{cases} \frac{1}{40000}, & 0 \le x \le 40000 \\ 0, & \text{otherwise} \end{cases}$

$P\{X > 30000 / X > 10000\}$

$$= \frac{P\{X > 30000 \cap X > 10000\}}{P\{X > 10000\}}$$

$$= \frac{P\{P > 30000\}}{P\{X > 10000\}}$$

$$= \frac{\int_{30K}^{40K} \frac{1}{40K}\, dx}{\int_{10K}^{40K} \frac{dx}{40K}}$$

$$= \frac{10000}{30000} = \boxed{\frac{1}{3}}$$

3.) a.) $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$

$P\{X > 2\} = \int^{\infty} e^{-x}\, dx$

$$= \boxed{e^{-2}}$$

b.) $P\{x>3 / x>2\} = P\{x>1\}$

$$= \int_1^{\infty} e^{-x}\,dx$$

$$= \boxed{e^{-1}}$$

4.) Top 1% means 99%ile

z-score for 99%ile $\simeq 2.326$

So $X = \mu + z\sigma$

$$= 100 + 14.2 \times 2.326$$

$$\simeq 133.0292$$

So $\boxed{IQ \geq 133.0292}$

5.) $$p = \int_{1.19}^{1.21} \frac{1}{0.005\sqrt{2\pi}} \exp\left[\frac{1}{2}\left(\frac{x-1.2}{0.005}\right)^2\right] dx$$

Put $z = \dfrac{x-1.2}{0.005}$

$$p = \int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

$$= 2\int_{0}^{2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

$$= 2[0.9772 - 0.5]$$

$$= 2 \times 0.4772$$

$$= 0.9544$$

$\therefore$ Required %age $= (1-0.9544)\,100\,\%$

$$= \boxed{4.56\,\%}$$

6.) $f(x) = \Big\{ \ \frac{1}{\ }, \quad 0 \leq x \leq 30$

$$6.) \quad f(x) = \begin{cases} \frac{1}{30}, \\ 0, & \text{otherwise} \end{cases}$$

$$p(\text{wait} \geq 10\,\text{min}) = \int_{10}^{30} \frac{dx}{30} = \boxed{\frac{2}{3}}$$

$$p(t_{wait} > 25 / t_{wait} > 15)$$

$$= \frac{p(t_{wait} > 25 \cap t_{wait} > 15)}{p(t_{wait} > 15)}$$

$$= \frac{p(t_{wait} > 25)}{p(t_{wait} > 15)}$$

$$= \frac{\int_{25}^{30} \frac{dx}{30}}{\int_{15}^{30} \frac{dx}{30}} = \boxed{\frac{1}{3}}$$

7.) a.) $X = \boxed{\mathbb{N} \text{ (set of all natural numbers)}}$

b.) $\boxed{p\{X = k\} = (1-p)^{k-1} p}$

c.) $E(X) = \sum_{k=1}^{n} k(1-p)^{k-1} p$

$$\boxed{\begin{aligned} &= \frac{1}{p} - (1-p)^n \left(\frac{1}{p} + n\right) \\ &\lim_{n \to \infty} E(X) = \frac{1}{p} \end{aligned}}$$

d.) $\text{Var}(X) = E(X^2) - (E(X))^2$ ($n \to \infty$ is considered)

$$= \underbrace{\sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p}_{} - \frac{1}{p^2}$$

$$S = 1^2 + 2^2(1-p) + 3^2(1-p)^2 + \dots$$

$$(1-p)S = 1^2(1-p) + 2^2(1-p)^2 + \dots$$

so $pS = 1^2 + (1-p)(2+1) + (1-p)^2(3+2) + \dots$

$$= 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + \dots$$

$$pS(1-p) = 1-p + 3(1-p)^2 + 5(1-p)^3 + \dots$$

so $p^2 S = 1 + 2(1-p)[1 + (1-p) + (1-p)^2 + \dots]$

$$= 1 + \frac{2(1-p)}{p}$$

$$= \frac{2-p}{p}$$

$\therefore$ $\mathrm{Var}(X) = \frac{2-p}{p^2} - \frac{1}{p^2}$

$$= \boxed{\frac{1-p}{p^2}}$$

e.) $p\{X > k+m / X > m\}$ $(k, m \in \mathbb{N})$

$$= \frac{p\{X > k+m \cap X > m\}}{p\{X > m\}}$$

$$= \frac{p\{X > k+m\}}{p\{X > m\}}$$

$$= \frac{p \times [(1-p)^{k+m} + (1-p)^{k+m+1} + \dots]}{p \times [(1-p)^m + (1-p)^{m+1} + \dots]}$$

$$= \frac{(1-p)^{k+m}[1 + (1-p) + \dots]}{(1-p)^m[1 + (1-p) + \dots]}$$

$$= (1-p)^k = p\{x > k\}$$

Hence, memoryless probability is proved.