# Sheet-1

1.) a.) Case 1 : 1st three are heads $\longrightarrow E_1$

$$P(E_1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Case 2 : 1st three are tails $\rightarrow E_2$

$$P(E_2) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$\therefore P(E) = \frac{1}{8} + \frac{1}{8} = \boxed{\frac{1}{4}}$$

b.) Case 1 : 1st 3 flips are same $\longrightarrow E_1$

$$P(E_1) = \frac{1}{4}$$

Case 2 : Last 3 flips are same $\longrightarrow E_2$
(equivalent to $E_1$)

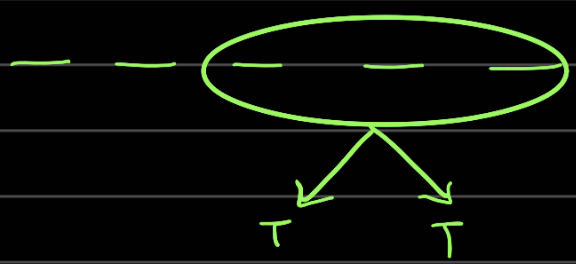
$$P(E_2) = \frac{1}{4}$$

Both cases include the case where all 5 flips are same $\left(\text{probability} = 2 \times \frac{1}{2^5} = \frac{1}{16}\right)$

$$\therefore P(E) = \frac{1}{4} + \frac{1}{4} - \frac{1}{16} = \boxed{\frac{7}{16}}$$

c.) Clearly, we must have 3H + 2T or 2H + 3T.

<u>Case 1 :</u> 3H + 2T




No. of ways $= {}^3C_2$
$= 3$

<u>Case 2 :</u> 3T + 2H $\longrightarrow$ same : ${}^3C_2 = 3$

$\therefore P(E) = 3+3 \ldots = \boxed{3 \ldots}$

$\therefore p(E) = \frac{[illegible]}{2^5} = \boxed{\frac{[illegible]}{16}}$

⊛ [PROBLEM 2 WILL NOT BE SOLVED HERE!]

3.) a.) $p(E) = \frac{6}{36} = \frac{1}{6}$

$E' \rightarrow$ 1st die lands on 4

$p(E') = \frac{1}{6} \times \frac{6}{6} = \frac{1}{6}$

$E' \cap E \equiv (4, 3)$

so $p(E' \cap E) = \frac{1}{36} = p(E)\,p(E')$

Hence, proved.

b.) $p(E) = \frac{6}{36} = \frac{1}{6}$

$E' \rightarrow$ 2nd die lands on 3

$p(E') = \frac{6}{6} \times \frac{1}{6} = \frac{1}{6}$

$E' \cap E \equiv (4, 3)$

so $p(E' \cap E) = \frac{1}{36} = p(E)\,p(E')$

Hence, proved.

4.) $p(\text{Choosing cabinet}) = \frac{1}{2}$

If we choose the cabinet whose both drawers have silver coin, $p(\text{finding silver coin}) = 1$, else for the other case, $p = 1/2$

$\therefore p\left({}^{\text{Both silver}}/\text{silver found}\right) = \frac{\frac{1}{2}\times 1}{\frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{2}}$

$= \boxed{2 [illegible]}$

$\boxed{\frac{}{3}}$

5.) a.) $p(\text{both gold} / \text{at least one gold})$

$\text{GG} \quad\quad \text{BG, GB, GG}$

$= \boxed{\frac{1}{3}}$

b.) Colour of one ball painted is independent of colour of other ball painted.

$\therefore\ p(\text{both gold} / \text{one is gold}) = \boxed{\frac{1}{2}}$

6.) a.) $p(\text{alive}) = 0.9 \times 0.85 + 0.1 \times 0.2$
$= \boxed{0.785}$

b.) $p(\text{No water} / \text{dead}) = \dfrac{0.1 \times 0.8}{0.1 \times 0.8 + 0.9 \times 0.15}$

$= \dfrac{8}{8 + 13.5}$

$= \dfrac{8}{21.5}$

$= \dfrac{16}{43}$

$\simeq \boxed{0.372}$

7.) a.) $m(t) = E(e^{tx})$

$= \int_0^\infty e^{tx} e^{-x}\, dx$

$= \int_0^\infty e^{(t-1)x}\, dx$

$= \dfrac{e^{(t-1)x}}{\phantom{x}} \Big|^{\infty}$

$$t-1 \Big|_0$$

$$= \boxed{\frac{1}{1-t}} \quad (\text{considering } t<1)$$

b.) $$E(X) = \int_0^\infty x e^{-x}\, dx$$

$$= -(x+1)e^{-x}\Big|_0^\infty$$
$$= \boxed{1}$$

$$\frac{dm}{dt}\Big|_{t=0} = \frac{1}{(1-t)^2}\Big|_{t=0} \quad \boxed{=1 = E(X)}$$

8.) a.) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 kx^2 dx + \int_1^{4/3} 1\, dx = 1$$

$$\Rightarrow \frac{k}{3} + \frac{1}{3} = 1$$

$$\Rightarrow \boxed{k=2}$$

b.) $$P = \int_0^1 2x^2 dx = \boxed{\frac{2}{3}}$$

c.) $$E(X) = \int_0^1 x \times 2x^2 dx + \int_1^{4/3} x \times 1\, dx$$

$$= \frac{1}{2} + \frac{1}{2} \times \left(\left(\frac{4}{3}\right)^2 - 1^2\right)$$

$$= \frac{8}{9} \simeq \boxed{0.889}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \int_0^1 x^2 \times 2x^2\,dx + \int_1^{4/3} x^2 \times 1\,dx - \frac{64}{81}$$

$$= \frac{2}{5} + \frac{64}{27} - 1 - \frac{64}{81}$$

$$= \frac{128}{81} - \frac{3}{5}$$

$$= \frac{397}{81}$$

$$\simeq \boxed{4.901}$$

9.) $E(X^2) - (E(X))^2$

$$= E(X^2) - 2(E(X))^2 + (E(X))^2$$

$$= E(X^2) - 2E(X)\,E(X) + (E(X))^2$$

$$= E(X^2 - 2X\,E(X) + (E(X))^2)$$

$$= E((X - E(X))^2) \geq 0 \quad (\because (X - E(X))^2 \geq 0)$$

Equality: $\boxed{X = E(X)}$ $\longrightarrow$ Sure shot constant data

10.) a.) $\boxed{P(X = k) = (1-p)^{k-1} p}$

b.) $E(X) = \sum_{k=1}^{n} k\,P(X = k)$

$$= \sum_{k=1}^{n} k(1-p)^{k-1} p$$

$= pS$

$= pS$

where, $S = 1 + 2(1-p) + 3(1-p)^2 + \cdots + n(1-p)^{n-1}$

$$\Rightarrow (1-p)S = (1-p) + 2(1-p)^2 + 3(1-p)^3 + \cdots + (n-1)(1-p)^{n-1} + n(1-p)^n$$

So $S-(1-p)S = pS$

$$= 1 + (1-p) + (1-p)^2 + \cdots + (1-p)^{n-1} - n(1-p)^n$$

$$\Rightarrow E(X) = \frac{1-(1-p)^n}{1-(1-p)} - n(1-p)^n$$

$$= \frac{1-(1-p)^n}{p} - n(1-p)^n$$

$$= \boxed{\frac{1}{p} - (1-p)^n\left(\frac{1}{p} + n\right)}$$

$$\boxed{\left(\lim_{n\to\infty} E(X) = 1/p\right)}$$