MM 225 AI and Data Science

2024-25-1

Problems for practice 1

1. Five independent flips of a fair coin are made. Find the probability that
    a. the first three flips are the same;
    b. either the first three flips are the same, or the last three flips are the same;
    c. there are at least two heads among the first three flips, and at least two tails among the last three flips.
2. Write a python program to solve the problem 1 above.
3. A pair of fair dice is rolled. Let E denote the event that the sum of the dice is equal to 7.
    a. Show that E is independent of the event that the first die lands on 4.
    b. Show that E is independent of the event that the second die lands on 3.
4. Each of 2 cabinets identical in appearance has 2 drawers. Cabinet A contains a silver coin in each drawer, and cabinet B contains a silver coin in one of its drawers and a gold coin in the other. A cabinet is randomly selected, one of its drawers is opened, and a silver coin is found. What is the probability that there is a silver coin in the other drawer?
5. Each of 2 balls is painted black or gold and then placed in an urn. Suppose that each ball is colored black with probability 12, and that these events are independent.
    a. Suppose that you obtain information that the gold paint has been used (and thus at least one of the balls is painted gold). Compute the conditional probability that both balls are painted gold.
    b. Suppose, now, that the urn tips over and 1 ball falls out. It is painted gold. What is the probability that both balls are gold in this case? Explain.
6. You ask your neighbour to water a sickly plant while you are on vacation. Without water it will die with probability .8; with water it will die with probability .15. You are 90 percent certain that your neighbour will remember to water the plant.
    a. What is the probability that the plant will be alive when you return?
    b. If it is dead, what is the probability your neighbor forgot to water it?
7. Suppose that X has density function

$$f(x) = e^{-x}, \qquad x > 0$$

    a. Compute the $m(t) = E(e^{tX})$.
    b. Compute mean of X and $\frac{dm(t)}{dt}$ when t=0. Compare your results.
8. A machine makes a product that is screened (inspected 100 percent) before being shipped. The measuring instrument is such that it is difficult to read between 1 and $1\frac{1}{3}$ (coded data). After the screening process takes place, the measured dimension has density

$$f(x) = \begin{cases} kz^2 \; for \; 0 \le z \le 1 \\ 1 \; for \; 1 < z \le 1\frac{1}{3} \\ 0 \qquad Otherwise \end{cases}$$

   a. Find the value of k.
   b. What fraction of the items will fall outside the twilight zone (fall between 0 and 1)?
   c. Find the mean and variance of this random variable.

9. Argue that for any random variable X :

$$E[X^2] \ge (E[X\,])^2$$

   When does one have equality?

10. Independent trials, each of which is a success with probability p, are successively performed. Let X denote the first trial resulting in a success. That is, X will equal k if the first k −1 trials are all failures and the kth a success. X is called a geometric random variable. Compute
    a. P{X = k}, k = 1, 2, . . .;
    b. E[X].