MM 225 – AI and Data Science

Day 19: Hypothesis Testing 1

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Introduction

Consider a population with distribution function F(θ), where θ is unknown.

Let θ_0 be a specific known number.

Want to know whether θ = θ_0

In statistical parlance: want to test the <u>Null Hypothesis</u> that θ = θ_0 denoted by

$$H_0 : \theta = \theta_0$$

The Critical Region

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from F(θ)

Based on these n values we should decide if H_0 is true or not

Want to find region C in n-dimensional space so that H_0 is rejected if random sample $(X_1, X_2, \dots, X_n)$ lies in region C

Such a region C is called critical region.

Thus want to find a statistical test determined by the critical region C such that the test

Rejects H_0 if $(X_1, X_2, \dots, X_n) \in C$

And

Accepts H_0 if $(X_1, X_2, \dots, X_n) \notin C$

Resulting Errors

Want to test hypothesis $H_0 : \theta = \theta_0$

Critical region has been found

Two types of error can result

Decision \ Reality	H_0 **is true**	H_0 **is not true**
Accept H_0	good	Error Type II
Reject H_0	Error Type I	good

Also known as

- Type I error = false negative
- Type II error = False positive

Decision \ Reality	H_0 **is true**	H_0 **is not true**
Accept H_0	good	Error Type II
Reject H_0	Error Type I $\leq \alpha$	good

Would like to minimise both types of errors

Mathematically not possible

Classical approach is to

- fix α at the minimum possible level and,
- Set up test so that probability of Type I error of the test is ≤ α

Significance Level and Power of Test

$$P[type\ I\ error] \leq \alpha \quad \Rightarrow P[Reject\ H_0\ |H_0\ is\ true] \leq\ \alpha$$

$$\Rightarrow P[(X_1, X_2, \dots, X_n\ \in C)|H_0] \leq\ \alpha$$

α is called significance level of the test

1- $\boldsymbol{\alpha}$ is called confidence level of the test

$$P[type\ II\ error] =\ \beta \quad \Rightarrow \quad P[Accept H_0|H_0\ is\ not\ true] = \beta$$

$$\Rightarrow \quad P[Rejet\ H_0\ |H_0 is\ not\ true] = 1 - \beta$$

$$\Rightarrow P[(X_1, X_2, \dots, X_n\ \in C)|H_0\ is\ not\ true] = 1 - \beta$$

1- $\boldsymbol{\beta}$ is called Power of the test

Alternate Hypothesis

For testing the null hypothesis H_0 it is important to know what is meant by H_0 is not true?

Example: Let $H_0 : \theta = \theta_0$. When H_0 is not true, there are following three possibilities:

1. $\theta \neq \theta_0$
2. $\theta < \theta_0$
3. $\theta > \theta_0$

Thus, it is important to explicitly state the alternative hypothesis, generally denoted by H_A or H_1, as definition of critical region C depends on the alternate hypothesis

Classical approach

1. Fix level of significance α
2. Clearly state the null hypothesis and alternate hypothesis in terms of population parameter θ
3. Choose an appropriate estimator for θ using data $(X_1, X_2, \dots, X_n)$ say $\mathrm{d}(X_1, X_2, \dots, X_n) = \mathrm{d}(\boldsymbol{X})$
4. Define critical region C, where H_0 is rejected
5. Calculate $P[C \mid H_0] = \alpha$ to determine exact nature of C
6. Give decision

Case of N(μ, σ^2), when σ^2 is known

$X_1, X_2, \ldots, X_n \sim iid\ \mathrm{N}(\mu, \sigma^2)$, and σ^2 is known and μ is unknown

Want to test $H_0 : \mu = \mu_0$ vs. $H_A: \mu \neq \mu_0$ *(Two sided Alternative)*

1. Let α be fixed
2. $H_0 : \mu = \mu_0$ vs. $H_A: \mu \neq \mu_0$
3. It is shown that $E(\bar{X}) = \mu$, $\bar{X}$ is estimator
4. H_0 can be rejected if $\bar{X}$ is not in the close vicinity of μ_0, hence
$$C = \{X_1, X_2, \ldots, X_n \mid |\bar{X} - \mu_0| > c\}$$

Note that $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1), when\ H_0$ is true

Case of $N(\mu, \sigma^2)$, when σ^2 is known

4. H_0 can be rejected if $\bar{X}$ is not in the close vicinity of μ_0, hence

$$C = \{X_1, X_2, \dots, X_n \mid |\bar{X} - \mu_0| > c\}$$

Note that $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1), when\ H_0$ is true

5. $P\left[X_1, X_2, \dots, X_n \mid \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > c'\right] = \alpha$

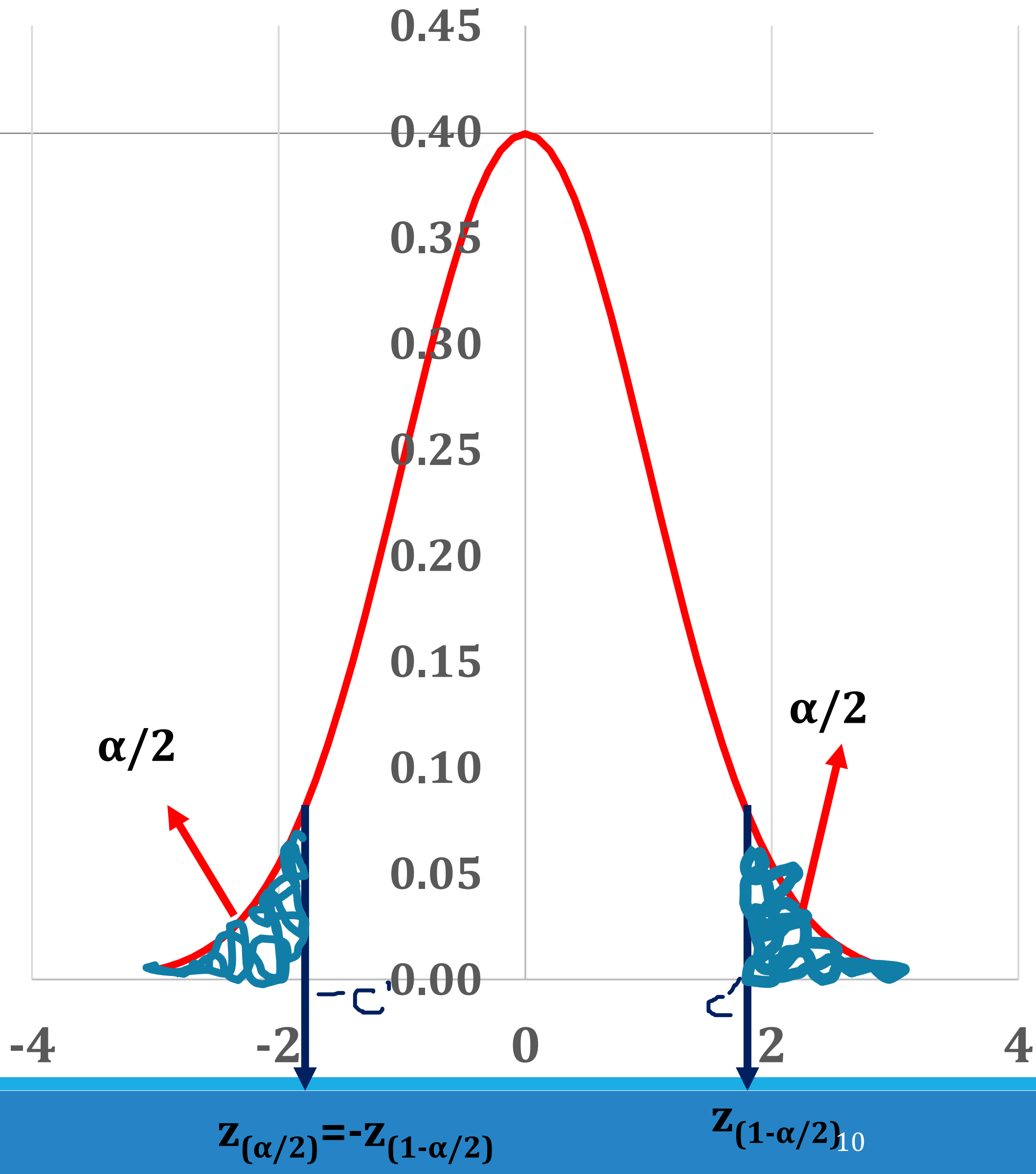
$$= P[Z > c'] = \frac{\alpha}{2}$$

$$\therefore c' = z_{1-\alpha/2}$$

6. If $\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > z_{1-\alpha/2}$ then reject H_0

Z = $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$ is called "TEST STATISTIC" and $z_{1-\alpha/2}$ is called critical value of the test

Shaded region is the critical region C



Example

An aerospace industry is interested in buying certain super alloy rods from a foundry. The industry has been told that the super alloy would have yield strength of 1110 MPa with standard deviation 110 MPa. The industry takes a random sample of size 100 from the supplied lot and finds that the average yield strength to be 1129 MPa. Should the company accept the supply?

μ_0 = 1110 MPa, σ = 110 MPa, n= 100 and $\bar{x}$ = 1129 MPa

Let us follow the steps of classical Hypothesis testing process

1. Let us fix α = 0.05

2. $H_0 : \mu = 1110$ MPa vs. $H_A : \mu \neq 1110$ MPa

3. Statistic of interest is $\bar{X}$ and $\bar{x} = 1129$

4. C = $\{X_1, X_2, \ldots, X_n \mid \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > z\}$ when H_0 is true

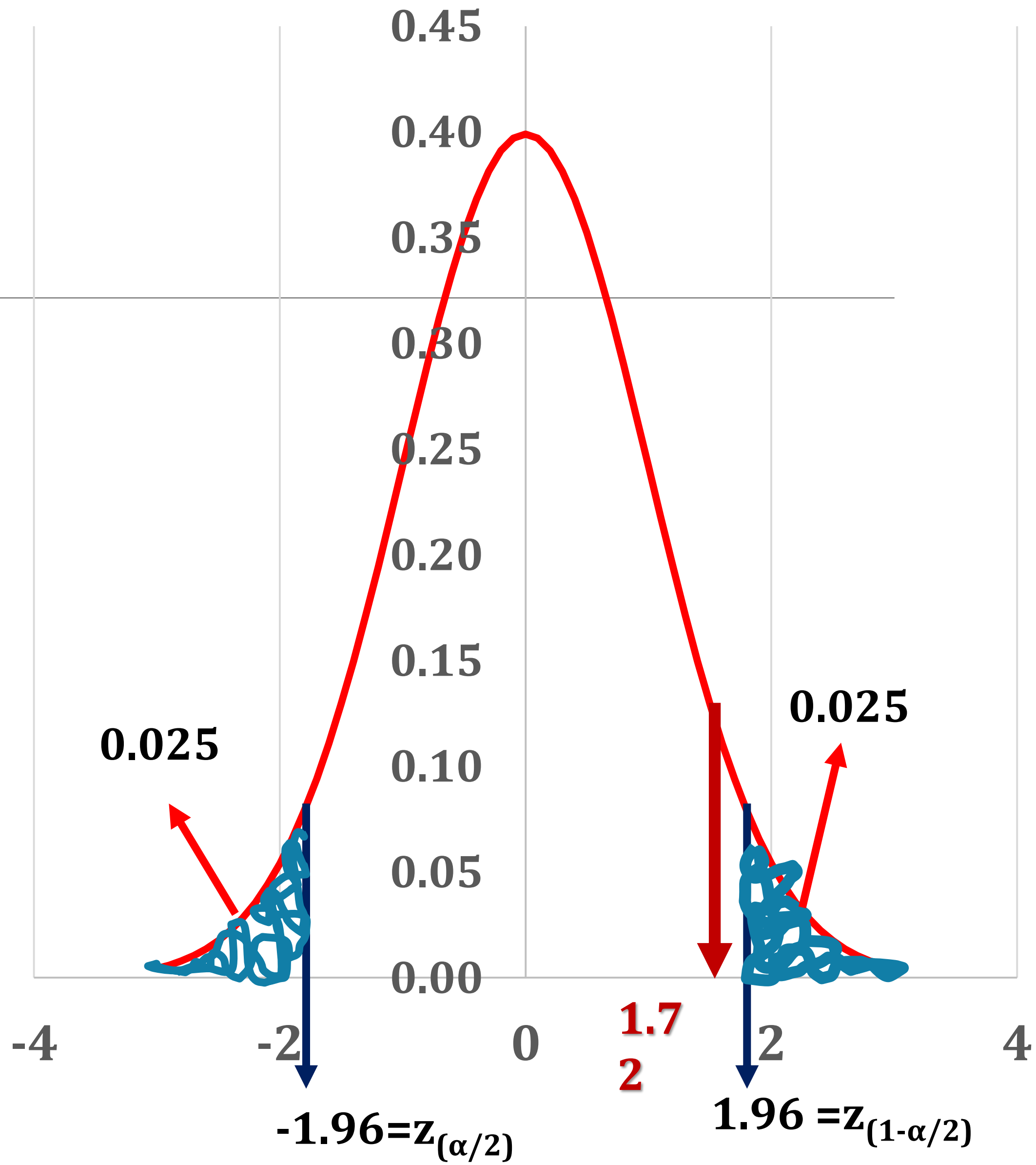
5. $P\left(\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > z\right) = \alpha = 0.05, \text{ as } Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

$$P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha/2}\right) = \frac{\alpha}{2} = 0.025$$

From Standard Normal table $z_{1-\alpha/2} = z_{0.975} = 1.96$ is the critical value

$\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} = \frac{1129-1110}{110}\sqrt{100} = 1.72$ is not > 1.96

6. Decision : H_0 cannot be rejected. So the industry should accept the supplied lot



Case of $\mathrm{N}(\mu, \sigma^2)$, when σ^2 is unknown

σ^2 *is known*

$X_1, X_2, \dots, X_n \sim iid\ \mathrm{N}(\mu, \sigma^2)$, and σ^2 is known and μ is unknown

Want to test $H_0 : \mu = \mu_0$ vs. $H_A: \mu \neq \mu_0$

1. Let α be fixed
2. $H_0 : \mu = \mu_0$ vs. $H_A: \mu \neq \mu_0$
3. It is shown that $E(\bar{X}) = \mu$, $\bar{X}$ is estimator
4. H_0 can be rejected if $\bar{X}$ is not in the close vicinity of μ_0, hence
$$C = \{X_1, X_2, \dots, X_n \mid |\bar{X} - \mu_0| > c\}$$

Note that $\boldsymbol{Z} = \frac{\bar{\boldsymbol{X}} - \boldsymbol{\mu_0}}{\boldsymbol{\sigma}/\sqrt{\boldsymbol{n}}} \sim N(0,1), when\ H_0$ is true

σ^2 *is unknown*

$X_1, X_2, \dots, X_n \sim iid\ \mathrm{N}(\mu, \sigma^2)$, and σ^2 is unknown and μ is unknown

Want to test $H_0 : \mu = \mu_0$ vs. $H_A: \mu \neq \mu_0$

1. Let α be fixed
2. $H_0 : \mu = \mu_0$ vs. $H_A: \mu \neq \mu_0$
3. It is shown that $E(\bar{X}) = \mu$, $\bar{X}$ is estimator
4. H_0 can be rejected if $\bar{X}$ is not in the close vicinity of μ_0, hence
$$C = \{X_1, X_2, \dots, X_n \mid |\bar{X} - \mu_0| > c\}$$

Note that $\mathbf{T} = \frac{\bar{\boldsymbol{X}} - \boldsymbol{\mu_0}}{\boldsymbol{s}/\sqrt{\boldsymbol{n}}} \sim t(n-1), when\ H_0$ is true

Case of $N(\mu, \sigma^2)$, when σ^2 is unknown

σ^2 is known

5. $P\left[X_1, X_2, \dots, X_n \mid \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > c'\right] = \alpha$

$$= P[Z > c'] = \frac{\alpha}{2}$$

$$\therefore c' = z_{1-\alpha/2}$$

6. If $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha/2}$ then reject H_0

σ^2 is unknown

5. $P\left[X_1, X_2, \dots, X_n \mid \left|\frac{\bar{X}-\mu_0}{s/\sqrt{n}}\right| > c'\right] = \alpha$

$$= P[t > c'] = \frac{\alpha}{2}$$

$$\therefore c' = t_{(n-1),\alpha/2}$$

6. If $\frac{\bar{X}-\mu_0}{s/\sqrt{n}} > t_{(n-1),\alpha/2}$ then reject H_0

Example: σ^2 is unknown

An aerospace industry is interested in buying certain super alloy rods from a Maxfoundry. The industry has been told that the super alloy would have yield strength of 1110 MPa. The industry takes a random sample of size 100 from the supplied lot and finds that the average yield strength to be 1129 MPa with standard deviation of 112 MPa. Should the company accept the supply?

μ_0 = 1110 MPa, n= 100 and $\bar{x}$ = 1129 MPa, s = 112 MPa

Let us once again follow the steps of classical Hypothesis testing process

1. Let us fix α = 0.05

2. H_0 : μ = 1110 MPa vs. H_A : $\mu \neq$ 1110 MPa

3. Statistic of interest is $\bar{X}$ = 1129

4. As σ^2 is unknown C = $\{X_1, X_2, \ldots, X_n \mid \left|\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\right| > c\}$ when H_0 is true

5. $P\left(\left|\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\right| > c\right) = \alpha = 0.05, as\ \ t = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \sim t_{(n-1)}$

$$P\left(\frac{\bar{X}-\mu_0}{S/\sqrt{n}} > t_{1-\alpha/2,(n-1)}\right) = \frac{\alpha}{2} = 0.025$$

From t probability table $t_{(n-1),1-\alpha/2} = t_{99,\ 0.975}$ $\approx$ 1.98 is the critical value

$$\frac{\bar{X}-\mu_0}{S/\sqrt{n}} = \frac{1129 - 1110}{112}\sqrt{100} = 1.69 \text{ is not} > 1.98$$

6. Decision : H_0 cannot be rejected. So the industry should accept the supplied lot

Thank you…