Practice Sheet-3

1) $$f(x_1, \dots, x_n \mid \mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp.\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} \exp.\left[\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2\right]$$

$$\Rightarrow \ln\left[f(x_1, \dots, x_n \mid \mu, \sigma)\right]$$

$$= -n\ln(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2$$

To find MLE for μ,

$$\frac{\partial}{\partial \mu}\left[-n\ln(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2\right] = 0$$

$$\Rightarrow -\frac{1}{2\sigma^2} \times -2\sum_{i=1}^{n} (x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n} (x_i-\mu) = 0$$

$$\Rightarrow \boxed{\mu = \frac{\sum_{i=1}^{n} x_i}{n}}$$

To find MLE for σ,

$$\frac{\partial}{\partial \sigma}\left[-n\ln(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2\right] = 0$$

$$\Rightarrow -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i-\mu)^2 = 0$$

$$\Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\Rightarrow \boxed{\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}}$$

2.) $f(x_1,\dots,x_n|\theta) = \frac{1}{\theta^n}$

To maximise f, θ should be as small as possible.
But to ensure $0 \le \text{probability} \le 1$, $\theta \ge \max.(x_1,\dots,x_n)$

$$\therefore \boxed{\theta = \max.(x_1, x_2, \dots, x_{n-1}, x_n)}$$

3.) $f(x_1,\dots,x_n|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

$$= \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}$$

$$\Rightarrow \ln f = -n\lambda + (\ln\lambda)\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\ln(x_i!)$$

Now, $\frac{d}{d\lambda}\ln f = 0$

$$\Rightarrow -n + \frac{\sum_{i=1}^{n}x_i}{\lambda} = 0$$

$$\Rightarrow \boxed{\lambda = \frac{\sum_{i=1}^{n}x_i}{n}}$$

4.) $f(x_1,\dots,x_n|\lambda) = \prod_{i=1}^{n}\lambda e^{-\lambda x_i}$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\Rightarrow \ln f = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i$$

Now, $\frac{d}{d\lambda} \ln f = 0$

$$\Rightarrow \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \boxed{\lambda = \frac{n}{\sum_{i=1}^{n} x_i}}$$

5.) $f(x_1, \ldots, x_n | p) = \prod_{i=1}^{n} (1-p)^{x_i - 1} p$

$$= (1-p)^{-n + \sum_{i=1}^{n} x_i} p^n$$

$$\Rightarrow \ln f = \left(-n + \sum_{i=1}^{n} x_i\right) \ln(1-p) + n \ln p$$

Now, $\frac{d}{dp} \ln f = 0$

$$\Rightarrow \frac{n - \sum_{i=1}^{n} x_i}{1-p} + \frac{n}{p} = 0$$

$$\Rightarrow \frac{n}{p} = \frac{\sum_{i=1}^{n} x_i - n}{1-p}$$

$$\Rightarrow \frac{1}{p} - 1 = \frac{\sum_{i=1}^{n} x_i}{n} - 1$$

$$\Rightarrow \boxed{p = \frac{n}{\sum_{i=1}^{n} x_i}}$$

6.) $f(x) = \begin{cases} e^{-(x-\theta)}, & x > \theta \\ 0, & x \le \theta \end{cases}$

$$f(x_1, \dots, x_n \mid \theta) = \prod_{i=1}^{n} e^{\theta - x_i}$$

$$= e^{n\theta - \sum_{i=1}^{n} x_i} \rightarrow \text{strictly} \uparrow \text{with } \theta$$

To maximise f, θ can be as large, but has to be smaller than any x_i $(i = 1, 2, \dots, n)$

So $\boxed{\theta = \min.\{x_1, x_2, \dots, x_{n-1}, x_n\}}$