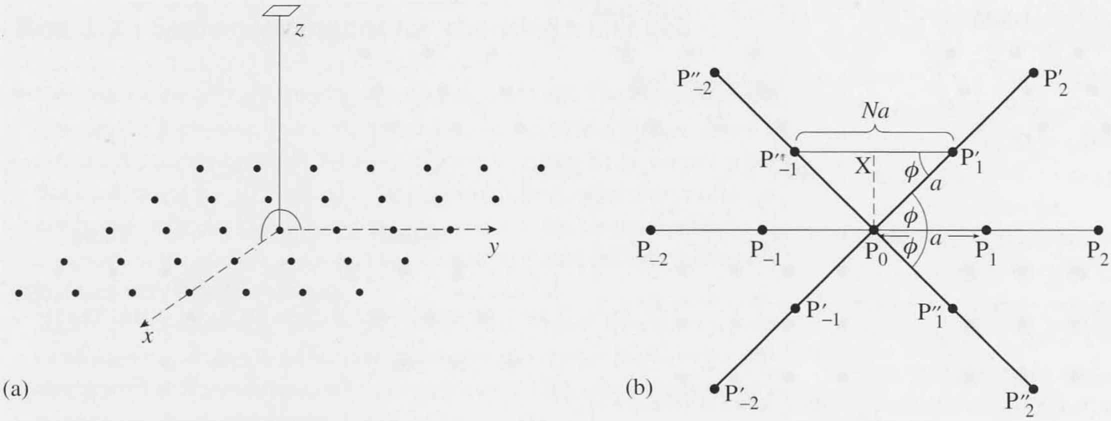

Fig. 3.11 (a) A rotation axis perpendicular to a lattice plane x, y. (b) Derivation of possible rotation angles for lattices. If lattice points along line P_i P_{-i} are rotated, new points P'_i P'_{-i} and P''_i P''_{-i} are produced.

Without too much difficulty, the types of possible rotation axes in crystals can be derived geometrically. Consider a lattice plane and, perpendicular to it, an n-fold rotation axis (Figure 3.11a). Symmetry requires that after a rotation of angle $\phi = 360^\circ/n$, all points of the rotated lattice plane coincide with points on the original lattice plane, and that after n rotations the lattice plane is again in the starting position. Now consider a line of points $P_{-2}P_{-1}$ P_0 P_1 $P_2 \ldots$ in the lattice plane with points spaced by a distance a (Figure 3.11b) and apply the symmetry rotation by an angle $\phi = 360^\circ/n$ in the counterclockwise direction, which repeats the line as P'_{-2} P'_{-1} P'_0 P'_1 $P'_2 \ldots$. The line continues to repeat after each rotational increment ϕ. (These lines are not plotted in Figure 3.11b.) Just before rotating back to the initial line again (rotation step $n-1$), we have a line P''_{-2} P''_{-1} P''_0 P''_1 $P''_2 \ldots$ that is at an angle of $-\phi$ to the initial line. P'_1 P''_{-1} are two lattice points defining a lattice line that is parallel to the original line. In order to satisfy the lattice condition, the distance P'_1 P''_{-1} has to be an integer multiple of the unit cell distance a. In the right triangle P_0 P'_1 X we calculate

$$\cos\phi = \frac{N \times \frac{a}{2}}{a} = N/2 \tag{3.1}$$

where N is an integer, and, since $|\cos\phi| \leq 1$, we find the following solutions for ϕ:

N	-2	-1	0	1	2
$\cos\phi$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
ϕ	180°	120°	90°	60°	$0^\circ = 360^\circ$
n-fold	2	3	4	6	1

This means that only 1-, 2-, 3-, 4-, and 6-fold rotation axes can occur in crystals. A lattice does not allow for axes with $n = 5$, 7, 8, or higher. A 1-fold rotation axis means no symmetry, since any object is brought to coincidence after a full 360° rotation.

This derivation for a two-dimensional lattice plane holds for three-dimensional lattices as well. Three-dimensional lattices are simply stacks of identical lattice planes, parallel to each other, with none or some displacement of corresponding points when viewed from above the planes.