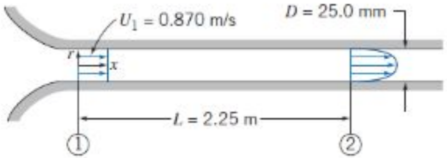
Recapitulation: Momentum Conservation (Integral Approach)

Air enters a duct, of diameter D = 25.0 mm, through a well-rounded inlet with uniform speed, $U_1 = 0.870$ m/s. At a downstream section where L = 2.25 m, the fully developed velocity profile is

$$\frac{u(r)}{U_c} = 1 - \left(\frac{r}{R}\right)^2 \tag{6}$$

The pressure drop between these sections is p_1-p_2 = 1.92 N/m^2. Find the total force of friction exerted by the tube on the air.



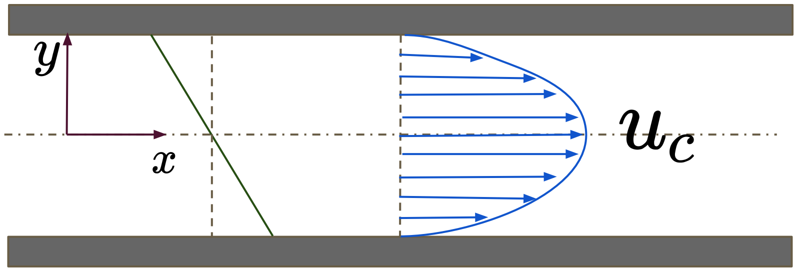
Question: How to determine the velocity profile of the fully developed flow: Differential Approach

ref:

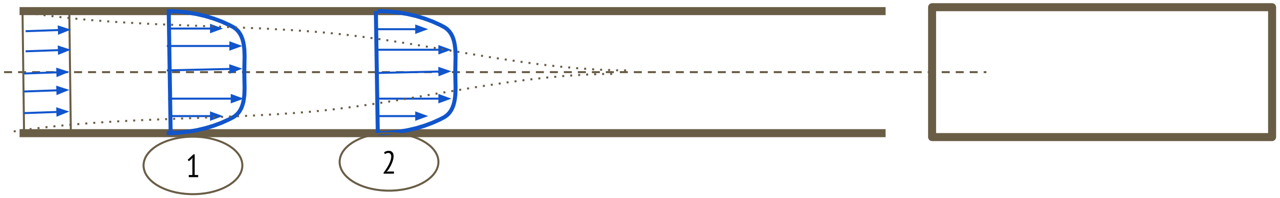
Momentum Conservation: Differential Approach

Ref: Chapter 8

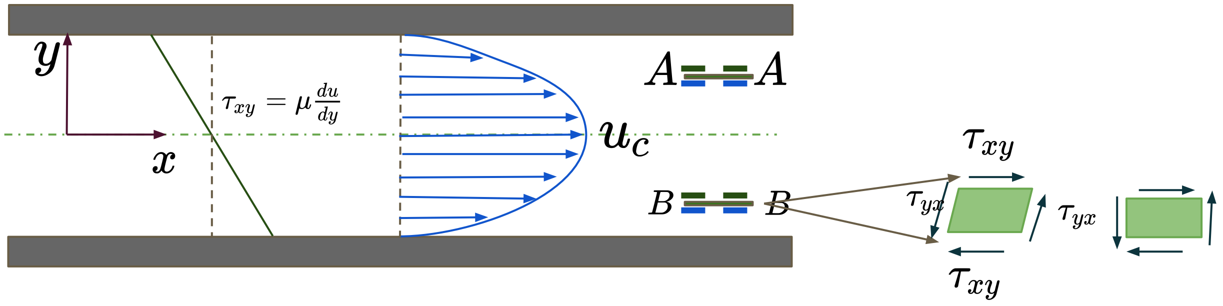
Flow between parallel plates (channel flow)



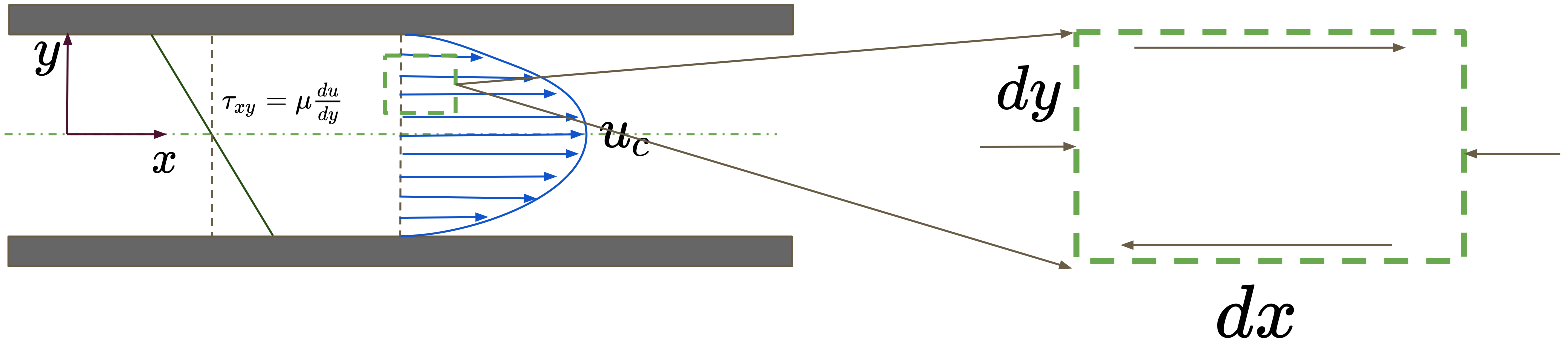
$$\tau_{xy} = \mu \frac{du}{dy}$$



Flow between parallel plates (channel flow)



Flow between parallel plates (channel flow)



$$\Sigma F_x = 0$$

$$\boxed{\left(p|_x - p|_{x+\Delta x}\right) dy(1) + \left(-\tau_{xy}|_y + \tau_{xy}|_{y+\Delta y}\right) dx(1) = 0}$$

$$\boxed{-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0}$$

$$\boxed{-\frac{\partial p}{\partial x} = -\frac{\partial \tau_{xy}}{\partial y}} \xrightarrow{\tau_{xy} = \mu \frac{du}{dy}} \boxed{-\frac{\partial p}{\partial x} = -\mu \frac{\partial^2 u}{\partial y^2}}$$

Assumptions:

1. Incompressible
2. Newtonian fluid
3. Steady state - › d/dt=0
4. Fully developed ->$du_x/dx = 0$
5. Two dimensional geometry ->d/dz=0
6. Laminar->u_y=0, u_z =0
7. g_x = 0

Flow between parallel plates (channel flow)

$$\mathrm{BCs}: @y = \pm\frac{h}{2}, u = 0$$

$$\boxed{-\frac{\partial p}{\partial x} = -\mu \frac{\partial^2 u}{\partial y^2}}$$

$$\boxed{-\frac{du}{dy} = \frac{1}{\mu}\left(-\frac{dp}{dx}\right) y + C_1}$$

$$\boxed{-u = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right) y^2 + C_1\, y + C_2}$$

$$\boxed{C_2 = -\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{h^2}{4}} \quad \boxed{C_1 = 0}$$

$$\boxed{-u = -\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{h^2}{4}\left(1 - \left(\frac{y}{\frac{h}{2}}\right)^2\right)}$$

<u>Assumptions:</u>

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2. Newtonian fluid
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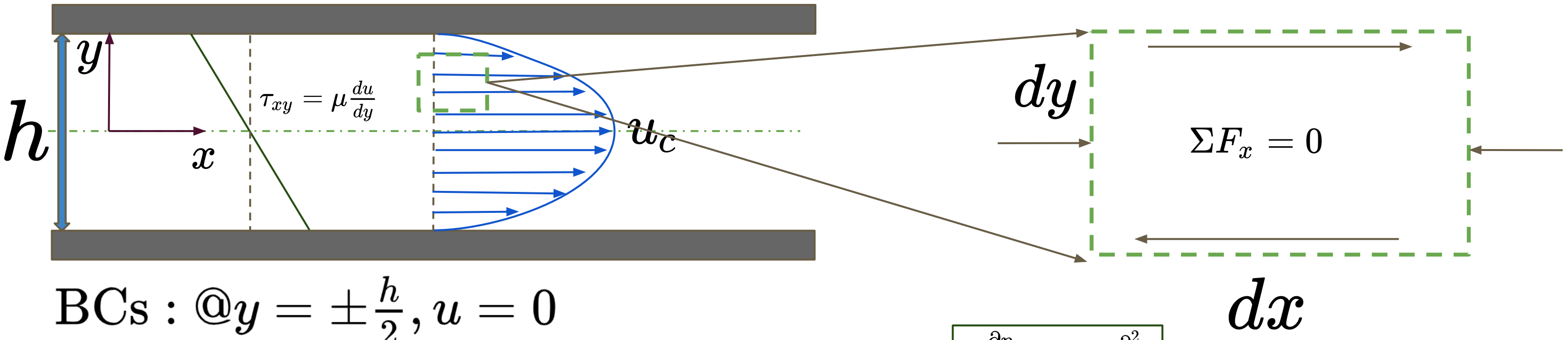
Flow between parallel plates (channel flow)



$$\mathrm{BCs}: @y = \pm\frac{h}{2}, u = 0$$

$$\boxed{-\frac{\partial p}{\partial x} = -\mu \frac{\partial^2 u}{\partial y^2}}$$

$$\boxed{-u = -\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{h^2}{4}\left(1-\left(\frac{y}{\frac{h}{2}}\right)^2\right)}$$

$$\boxed{u = \frac{h^2}{8\mu}\left(-\frac{dp}{dx}\right)\left(1-\left(\frac{y}{\frac{h}{2}}\right)^2\right)}$$

<u>Assumptions:</u>

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2. Newtonian fluid
3. Steady state - › d/dt=0
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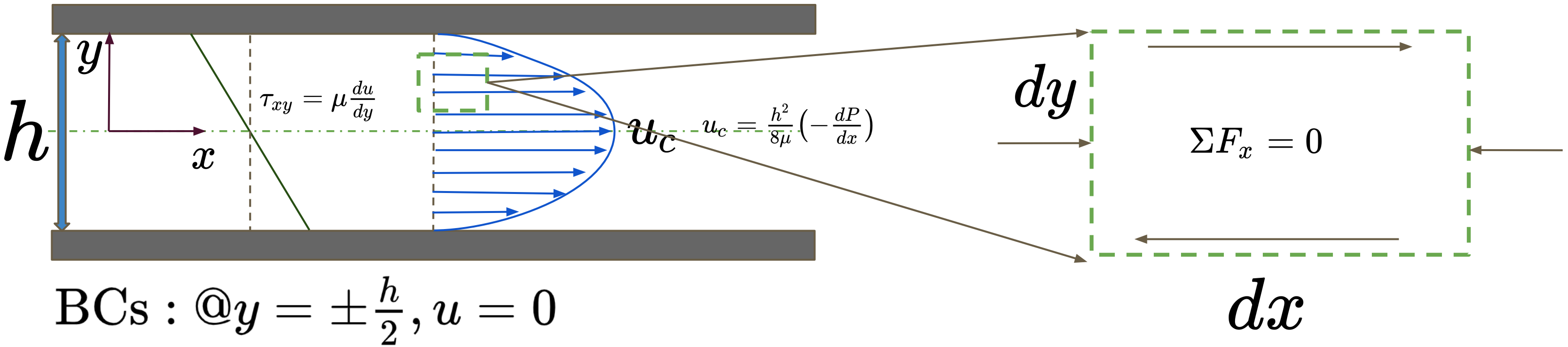
6. Laminar-›u_y=0, u_z =0
7. g_x = 0

Flow between parallel plates (channel flow): Centreline Velocity



$$\mathrm{BCs} : @y = \pm\frac{h}{2}, u = 0$$

Assumptions:

1. Incompressible
2. Newtonian fluid
3. Steady state - › d/dt=0
4. Fully developed ->$du_x/dx = 0$
5. Two dimensional geometry ->d/dz=0
6. Laminar->u_y=0, u_z =0
7. g_x = 0

$$\boxed{u = \frac{h^2}{8\mu}\left(-\frac{dp}{dx}\right)\left(1-\left(\frac{y}{\frac{h}{2}}\right)^2\right)}$$

$$\boxed{u_{\max} = u_c = \frac{h^2}{8\mu}\left(-\frac{dP}{dx}\right)}$$

$$\boxed{u(y) = u_c\left(1-\left(\frac{y}{\frac{h}{2}}\right)^2\right)}$$

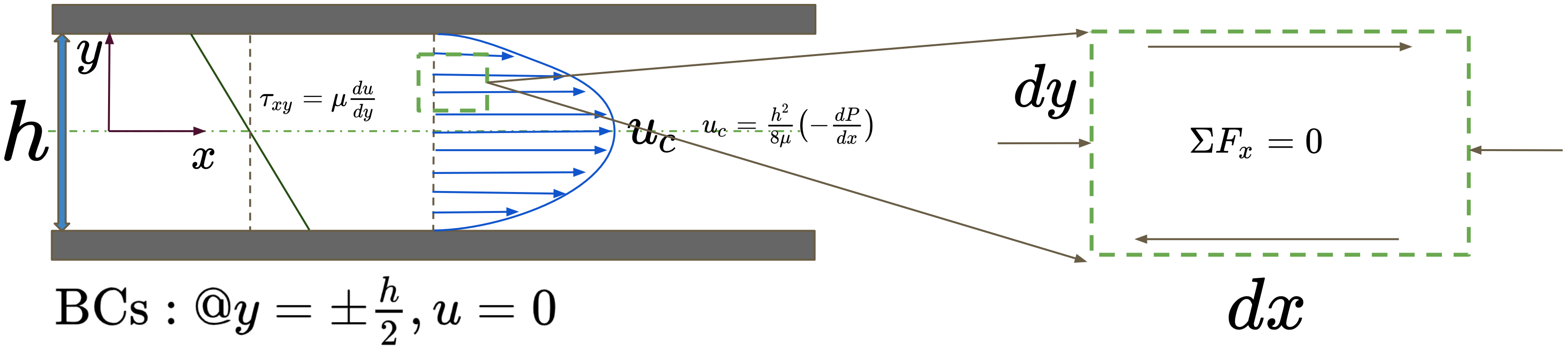
$$\boxed{u_c = \frac{3}{2}u_{\text{avg}}}$$

$$u_{\text{avg}} = \frac{1}{\frac{h}{2}}\int_0^{\frac{h}{2}} u(y)dy(1)$$

$$= \frac{1}{\frac{h}{2}}\int_0^{\frac{h}{2}} u_c\left(1-\left(\frac{y}{\frac{h}{2}}\right)^2\right)dy$$

$$= \frac{2}{3}u_c$$

Flow between parallel plates (channel flow): Pressure Drop



$$\mathrm{BCs}: @y = \pm\frac{h}{2}, u = 0$$

Assumptions:

1. Incompressible
2. Newtonian fluid
3. Steady state - › d/dt=0
4. Fully developed ->du_x/dx = 0
5. Two dimensional geometry ->d/dz=0
6. Laminar->u_y=0, u_z =0
7. g_x = 0

$$\boxed{u_{\max} = u_c = \frac{h^2}{8\mu}\left(-\frac{dP}{dx}\right)}$$

$$\boxed{u_c = \frac{3}{2}u_{\mathrm{avg}}}$$

$$\boxed{\left(-\frac{dP}{dx}\right) = \frac{\Delta P}{L}}$$

$$\boxed{\frac{\Delta P}{L} = \frac{8\mu u_c}{h^2}}$$

$$\boxed{\frac{\Delta P}{L} = \frac{12\mu u_{\mathrm{avg}}}{h^2}}$$

Flow through a pipe with circular cross-section

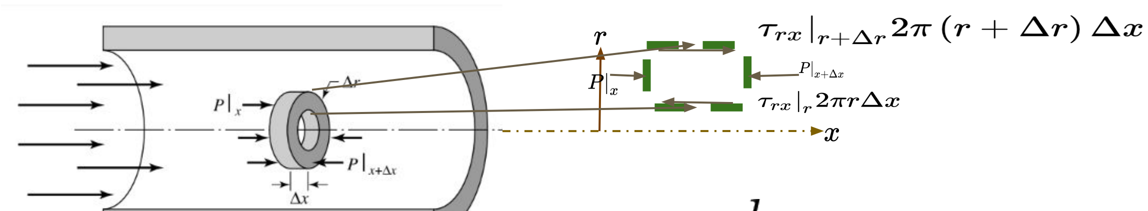


Figure ref: Welty, Wilkies

$$\tau_{xr} = \mu \frac{du_x}{dr}$$

Assumptions:

1. Incompressible
2. Newtonian fluid
3. Steady state - > d/dt=0
4. Fully developed ->$du_x/dx = 0$
5. Axisymmetric ->d/dΘ=0
6. Laminar->u_r=0, uΘ =0
7. g_x = 0

$$0 = +P\, 2\pi\, r\Delta r|_x - P\, 2\pi\, r\Delta r|_{x+\Delta x} - \tau_{xr} 2\pi r \Delta x|_r + \tau_{xr} 2\pi (r + \Delta r)\Delta x|_{r+\Delta r}$$

Outward normal positive convention

$$\boxed{-\frac{\partial P}{\partial x} + \frac{1}{r}\frac{\partial (r\tau_{rx})}{\partial r} = 0}$$

Solution

$$\boxed{\frac{d}{dr}\left(r\tau_{xr}\right) = -r\left(-\frac{dp}{dx}\right)} \leftarrow \tau_{xr} = \mu\frac{du_x}{dr}$$

$$\boxed{\frac{d}{dr}\left(r\,\mu\frac{du_x}{dr}\right) = -r\left(-\frac{dp}{dx}\right)}$$

$$\boxed{\left(r\,\mu\frac{du_x}{dr}\right) = -\frac{r^2}{2}\left(-\frac{dp}{dx}\right) + c_1}$$

$$\frac{du_x}{dr} = -\frac{r}{2\,\mu}\left(-\frac{dp}{dx}\right) + \cancel{\frac{c_1}{r}}$$

Solution has to be finite at r = 0

$$u_x(r) = -\frac{r^2}{4\,\mu}\left(-\frac{dp}{dx}\right) + c_2 \qquad c_2 = \frac{R^2}{4\,\mu}\left(-\frac{dp}{dx}\right)$$

$$\textbf{no slip bc} : u_x(r = R) = 0$$

$$\boxed{u_x(r) = \frac{R^2}{4\,\mu}\left(-\frac{dp}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)}$$

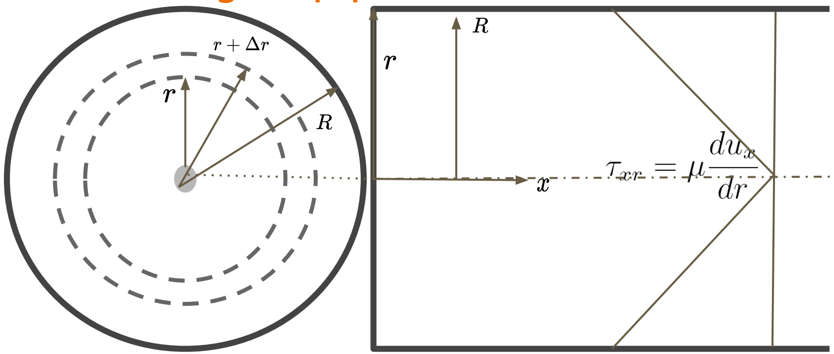
Assumptions:

1. Incompressible
2. Newtonian fluid
3. Steady state - > d/dt=0
4. Fully developed ->$du_x/dx = 0$
5. Axisymmetric ->d/dΘ=0
6. Laminar->u_r=0, uΘ =0
7. g_x = 0

$$\frac{dP}{dx} = \frac{1}{r}\frac{d}{dr}\left(r\tau_{rx}\right)$$

1. LHS is function of "r" alone
2. P is linear in "x"
3. dP/dx is independent of "r" from "r" momentum conservation

Flow through a pipe with circular cross-section & a channel



$$-\frac{\partial P}{\partial x} + \frac{1}{r}\frac{\partial (r\tau_{rx})}{\partial r} = 0$$

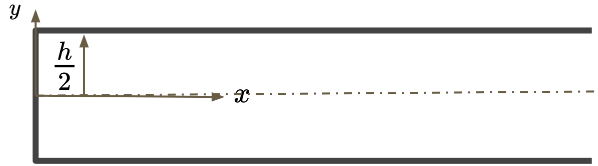
$$u_x(r) = \frac{R^2}{4\,\mu}\left(-\frac{dp}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)$$

$$u_{\max} = u_c = \frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)$$

$$u_x(r) = u_c\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$u_c = 2u_{\text{avg}}$$

$$u_{\text{avg}} = \frac{1}{\pi R^2}\int_0^R u(r)2\pi r dr$$
$$= \frac{2u_c}{R^2}\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr$$
$$= \frac{1}{2}u_c$$



$$\tau_{xy} = \mu\frac{du}{dy}$$

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$u = \frac{h^2}{8\mu}\left(-\frac{dp}{dx}\right)\left(1 - \left(\frac{y}{\frac{h}{2}}\right)^2\right)$$

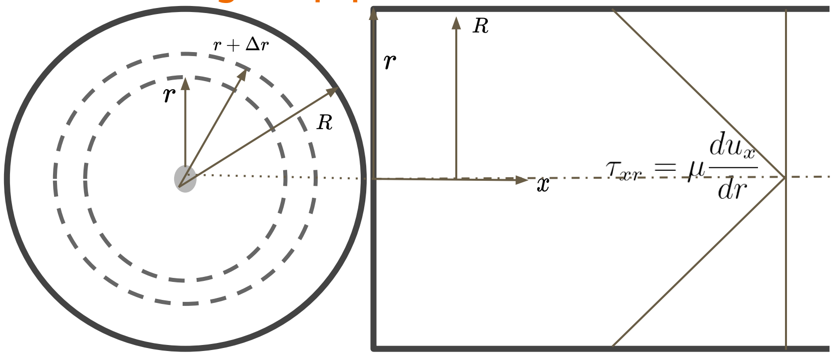
$$u_{\max} = u_c = \frac{h^2}{8\mu}\left(-\frac{dP}{dx}\right)$$

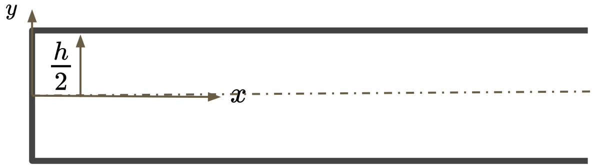
$$u(y) = u_c\left(1 - \left(\frac{y}{\frac{h}{2}}\right)^2\right)$$

$$u_c = \frac{3}{2}u_{\text{avg}}$$

$$u_{\text{avg}} = \frac{1}{\frac{h}{2}}\int_0^{\frac{h}{2}} u(y)dy(1)$$
$$= \frac{1}{\frac{h}{2}}\int_0^{\frac{h}{2}} u_c\left(1 - \left(\frac{y}{\frac{h}{2}}\right)^2\right) dy$$
$$= \frac{2}{3}u_c$$

Flow through a pipe with circular cross-section & a channel





$$\boxed{u_{\max} = u_c = \frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)}$$

$$\boxed{u_c = 2u_{\text{avg}}}$$

$$\boxed{\left(-\frac{dP}{dx}\right) = \frac{\Delta P}{L}}$$

$$\boxed{\frac{\Delta P}{L} = \frac{4\mu u_c}{R^2}}$$

$$\boxed{\frac{\Delta P}{L} = \frac{8\mu u_{\text{avg}}}{R^2}}$$

$$\boxed{\frac{\Delta P}{L} = \frac{32\mu u_{\text{avg}}}{D^2}}$$

$$\boxed{u_{\max} = u_c = \frac{h^2}{8\mu}\left(-\frac{dP}{dx}\right)}$$

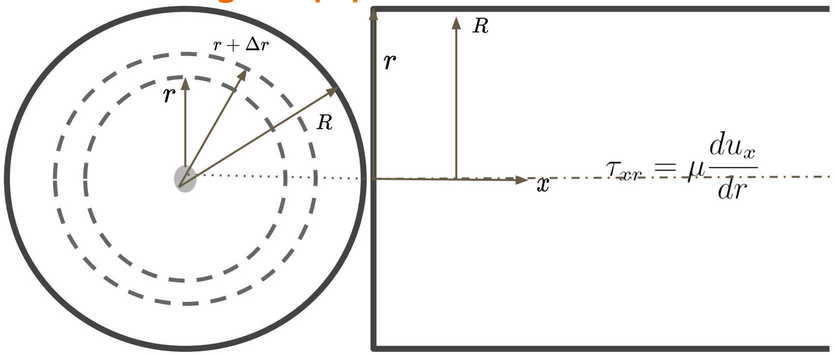
$$\boxed{u_c = \frac{3}{2}u_{\text{avg}}}$$

$$\boxed{\left(-\frac{dP}{dx}\right) = \frac{\Delta P}{L}}$$

$$\boxed{\frac{\Delta P}{L} = \frac{8\mu u_c}{h^2}}$$

$$\boxed{\frac{\Delta P}{L} = \frac{12\mu u_{\text{avg}}}{h^2}}$$

Flow through a pipe with circular cross-section



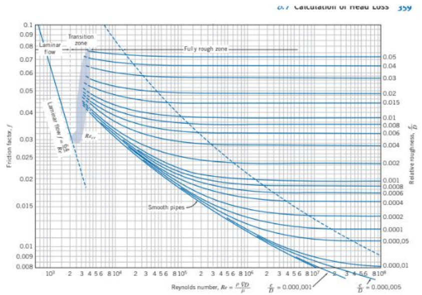
$$\boxed{\frac{\Delta P}{L} = \frac{32\mu u_{\text{avg}}}{D^2}}$$

$$\boxed{\frac{\Delta P}{\frac{1}{2}\rho u_{\text{avg}}^2} = \frac{64\mu}{\rho u_{\text{avg}} D}\frac{L}{D}}$$

$$\boxed{\frac{\Delta P}{\frac{1}{2}\rho u_{\text{avg}}^2} = \left(\frac{64\mu}{\rho u_{\text{avg}} D}\right)\left(\frac{L}{D}\right)}$$

Test yourself: Pressure drop in a pipe flow

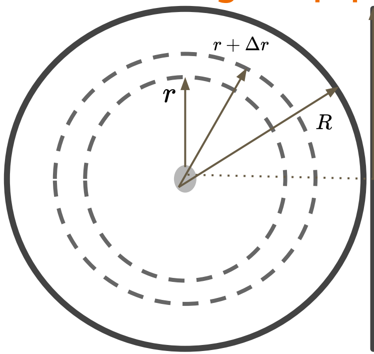
$$\Delta P = f_P\left(D, L, V, \mu, \rho_f, \epsilon\right)$$

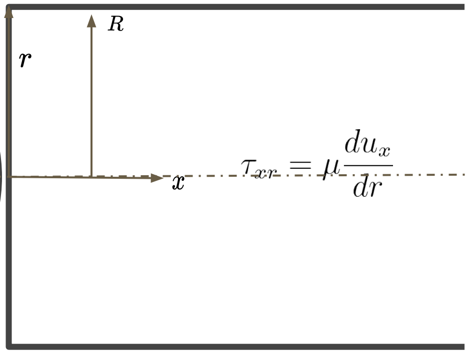


$$\Pi_1 = \frac{\Delta P}{\rho u_{\text{avg}}^2}$$

$$\Pi_2 = \frac{\mu}{D\rho u_{\text{avg}}} \qquad \Pi_3 = \frac{L}{D} \qquad \Pi_4 = \frac{\epsilon}{D}$$

Flow through a pipe with circular cross-section

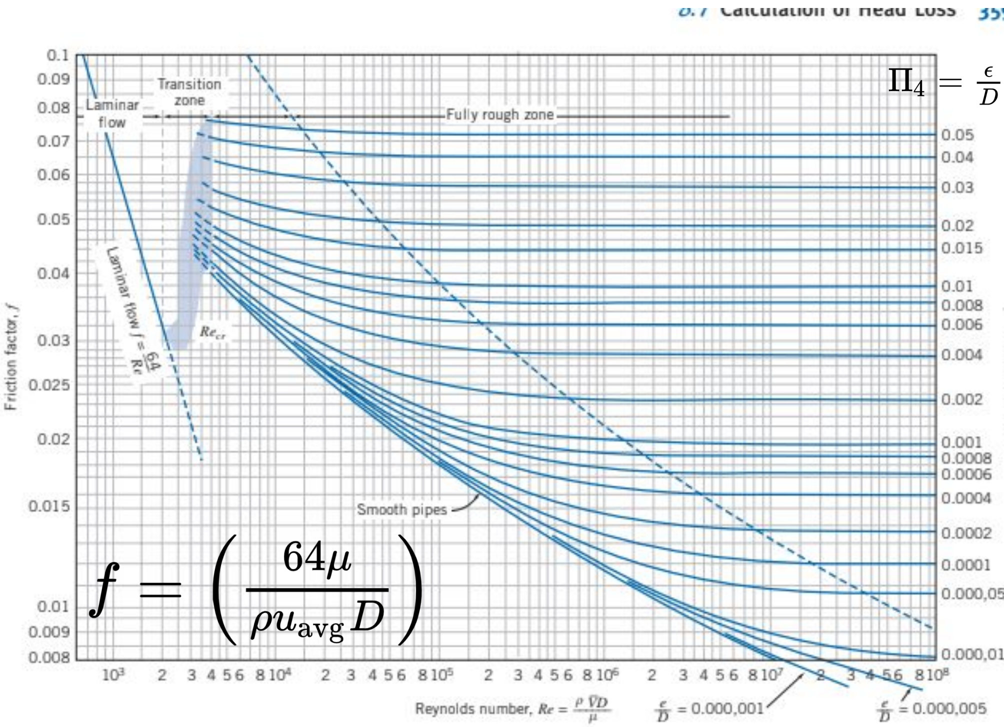




$$\boxed{\frac{\Delta P}{L} = \frac{32 \mu u_{\text{avg}}}{D^2}}$$

$$\frac{\Delta P}{\frac{1}{2}\rho u_{\text{avg}}^2} = \frac{64\mu}{\rho u_{\text{avg}} D} \frac{L}{D}$$

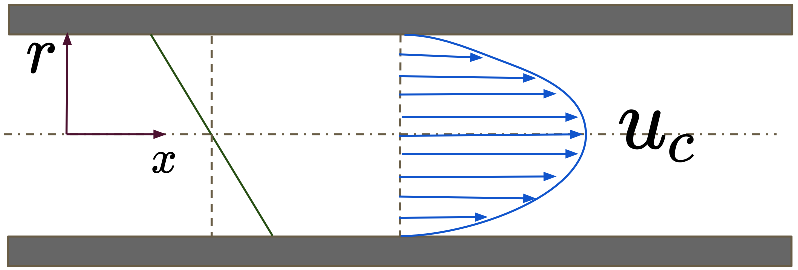
$$\frac{\Delta P}{\frac{1}{2}\rho u_{\text{avg}}^2} = \left(\frac{64\mu}{\rho u_{\text{avg}} D}\right)\left(\frac{L}{D}\right)$$

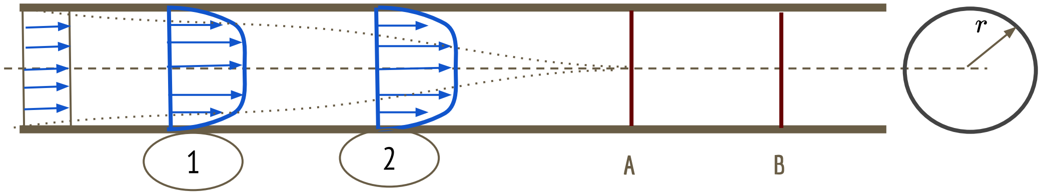


$$\Pi_4 = \frac{\epsilon}{D}$$

$$f = \left(\frac{64\mu}{\rho u_{\text{avg}} D}\right)$$

Flow through a pipe: Pressure Drop and Frictional Losses





Pointer to Application of Bernoulli Equation