Digital Signal Processing Sound Assignment

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EP20BTECH11024

Contents

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.
    com/gadepall/
    EE1310/master/filter/codes/
    Sound_Noise.wav
```

http://tlc.iith.ac.in/img/sound/Sound_Noise.wav in the link given below.

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise. By observing spectrogram, it clearly shows that tonal frequency is under 4kHz. And above 4kHz only noise is present.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal
#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )
#sampling frequency of Input signal
sampl_freq=fs
#order of the filter
order=4
#cutoff frquency 4kHz
cutoff_freq=4000.0
#digital frequency
Wn=2*cutoff_freq/sampl_freq
# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')
#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output signal = signal.lfilter(b, a,
    input_signal)
#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play

the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following codes yields Fig. 3.2.

```
wget https://github.com/tj-devil/
    EE3900-2022/blob/main/codes/
    xnyn.py
wget https://github.com/tj-devil/
    EE3900-2022/blob/main/codes/
    xnyn.c
```

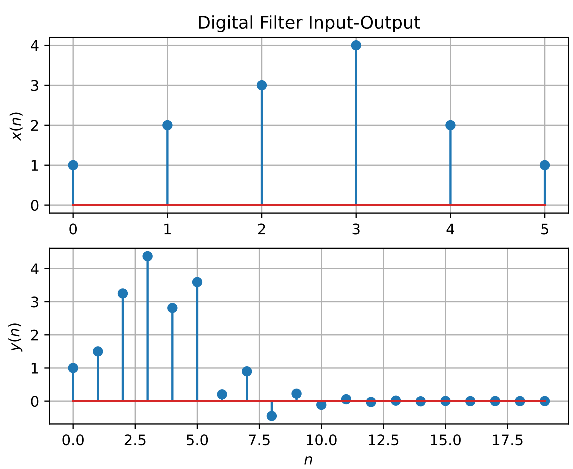

Fig. 3.2

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1} X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k} X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem (3.1).
Solution: From (3.1)

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Since, our $x(n)$ is of valid size with valid indices varying from 1 to 6. Therefore,

$$\mathcal{Z}\{x(n)\} = \sum_{n=1}^{6} x(n) z^{-n} \tag{4.7}$$

$$\mathcal{Z}\{x(n)\} = x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5} + x(6)z^{-6} \tag{4.8}$$

Which from (3.1) becomes,

$$\mathcal{Z}\{x(n)\} = 1 \cdot z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + 4 \cdot z^{-4} + 2 \cdot z^{-5} + 1 \cdot z^{-6} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2} z^{-1} Y(z) = X(z) + z^{-2} X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution: let $x(n) = a^n u(n)$,

$$\mathcal{Z}\{x(n)\} = \mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.20}$$

Using the formula for the sum of an infinite geometric progression, we get,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.21}$$

4.6 Let

$$H\left(e^{J\omega}\right) = H\left(z = e^{J\omega}\right). \tag{4.22}$$

Plot $|H\left(e^{J\omega}\right)|$. Is it periodic ? If so, find the period. $H(e^{J\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $x(n)$.

Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.23}$$

For $z = e^{\jmath\omega}$,

$$|H(e^{\jmath\omega})| = \left|\frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}}\right| \tag{4.24}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} \tag{4.25}$$

$$= \sqrt{\frac{2(1 + \cos 2\omega)}{\frac{5}{4} + \cos\omega}} \tag{4.26}$$

$$= \sqrt{\frac{2(2\cos^2\omega)}{\frac{5}{4} + \cos\omega}} \tag{4.27}$$

$$= \frac{4|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.28}$$

Thus,

$$\left|H\left(e^{\jmath(\omega+2\pi)}\right)\right| = \frac{4|\cos(\omega + 2\pi)|}{\sqrt{5 + 4\cos(\omega + 2\pi)}} \tag{4.29}$$

$$= \frac{4|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.30}$$

$$= |H(e^{\jmath\omega})| \tag{4.31}$$

and so its fundamental period is 2π.
The following code plots Fig. 4.6. And, further the it is periodic with a period of ~ 6.378.

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/dtft.py
```

Note - for a function to be periodic $f(t) = f(t+T)$ and here T be our period.

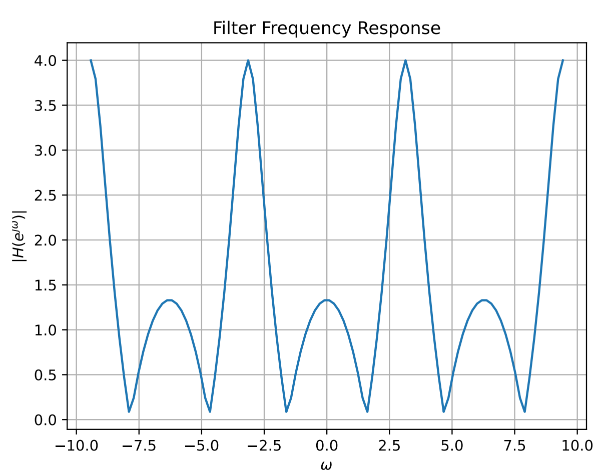

Fig. 4.6: $|H\left(e^{J\omega}\right)|$

4.7 Express $h(n)$ in terms of $H(e^{J\omega})$.
Solution: $h(n)$ can be expressed as follows:

$$H(e^{\jmath\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k} \tag{4.32}$$

However,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.33}$$

and so,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{j\omega n}d\omega \tag{4.34}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} h(k)e^{\jmath\omega(n-k)}d\omega \tag{4.35}$$

$$= \frac{1}{2\pi}2\pi h(n) = h(n) \tag{4.36}$$

which is known as the Inverse Discrete Fourier Transform. Thus,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.37}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}}e^{\jmath\omega n}d\omega \tag{4.38}$$

5 Impulse Response

5.1 Using long division, compute $h(n)$ for $n < 5$ from $H(z)$. **Solution:** We substitute $x := z^{-1}$, and write

$$\begin{array}{r} 2x - 4 \\ \frac{1}{2}x + 1 \overline{) \; x^2 \qquad + 1} \\ \underline{- x^2 - 2x} \\ - 2x + 1 \\ \underline{2x + 4} \\ 5 \end{array}$$

So,

$$H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.1}$$

$$= -4 + 2z^{-1} + 5\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.2}$$

$$= 1 - \frac{1}{2}z^{-1} + 5\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.3}$$

Now,

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} + 4\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.4}$$

$$= \sum_{n=-\infty}^{\infty} u(n)\left(-\frac{1}{2}\right)^n z^{-n} +$$

$$\sum_{n=-\infty}^{\infty} u(n-2)\left(-\frac{1}{2}\right)^{n-2} z^{-n} \tag{5.5}$$

Therefore, from (4.1),

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.6}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.7}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2). **Solution:** From (4.12),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.8}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.9}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify it theoritically. **Solution:** $h(n)$ can be written as-

$$h(n) = \begin{cases} 0 & n \le 0 \\ \left(-\frac{1}{2}\right)^n & 0 \le n < 2 \\ 5 * \left(-\frac{1}{2}\right)^n & n \ge 2 \end{cases} \tag{5.10}$$

for n $<$ 0,
$h(n)$ is constant and zero. Hence bounded.
for 0 $\ge$ n $<$ 2,
$h(n)$ is either 1 for n=0 or -0.5 for n=1. Hence, bounded.
for n $\ge$ 2,
$h(n)$ has a maxima at n=2 ,i.e., 1.25 nad minima at n=3, i.e., -0.625. Hence bounded.
Implying $h(n)$ is bounded over its domain.

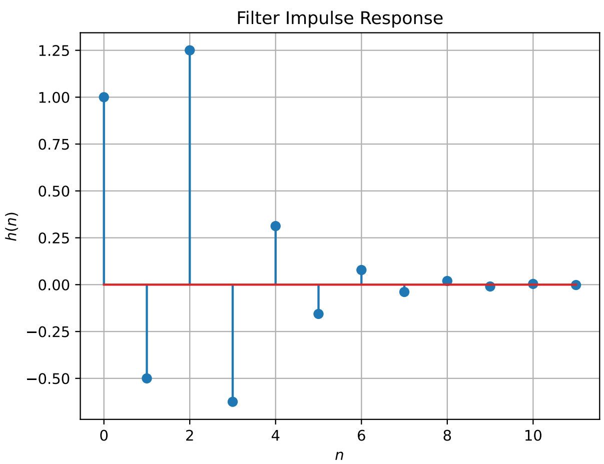

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

5.4 Convergent? Justify using the ratio test. For large n, we see that

$$h(n) = \left(-\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^{n-2} \quad (5.11)$$

$$= \left(-\frac{1}{2}\right)^n (4+1) = 5\left(-\frac{1}{2}\right)^n \quad (5.12)$$

$$\implies \left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} \quad (5.13)$$

and therefore, $\lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} <. 1$ Hence, we see that $h(n)$ converges.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.14)$$

Is the system defined by (3.2) stable for the impulse response in (5.7)? **Solution:** Note that

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.15)$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) = \frac{4}{3} \quad (5.16)$$

Hence, the given system is stable. The limit is verified at,

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/limithn.py
```

5.6 Verify the above result using a python code. **Solution:** The following code plots the Fig. (5.3).

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/hn.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.17)$$

This is the definition of $h(n)$. **Solution:** The following code plots the Fig. (5.7). Note, it is the same as Fig. (5.3).

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/hndef.py
```

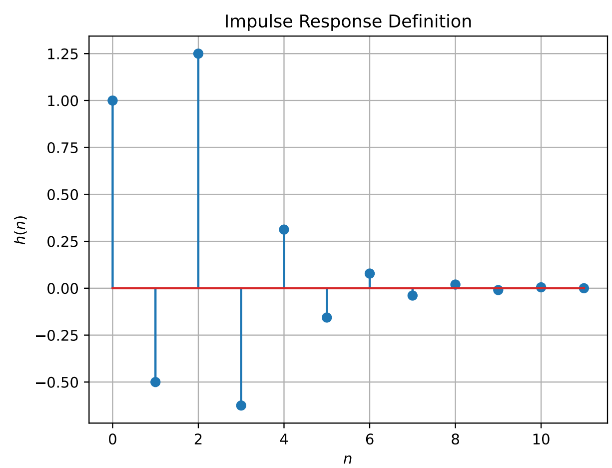

Fig. 5.7: $h(n)$ as the inverse of $H(z)$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.18)$$

Comment. The operation in (5.18) is known as *convolution*. **Solution:** The following code plots Fig. (5.8). Note that this is the same as $y(n)$ in Fig. (3.2).

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/ynconv.py
```

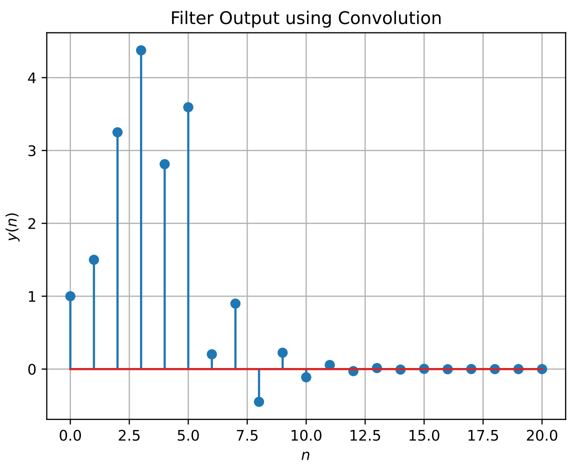

Fig. 5.8: $y(n)$ from the definition

5.9 Express the above convolution using a Toeplitz matrix **Solution:**

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{5.19}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & h_3 & h_2 & h_1 \\ 0 & . & . & . & h_3 & h_2 \\ 0 & . & . & . & 0 & h_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{5.20}$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.21}$$

Solution: From (5.18), we substitute $k := n - k$ to get

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)\, h(k) \tag{5.22}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \tag{5.23}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: The following code plots Fig.6.1

```
https://github.com/tj-devil/EE3900-2022/blob
    /main/codes/ynconv.py
```

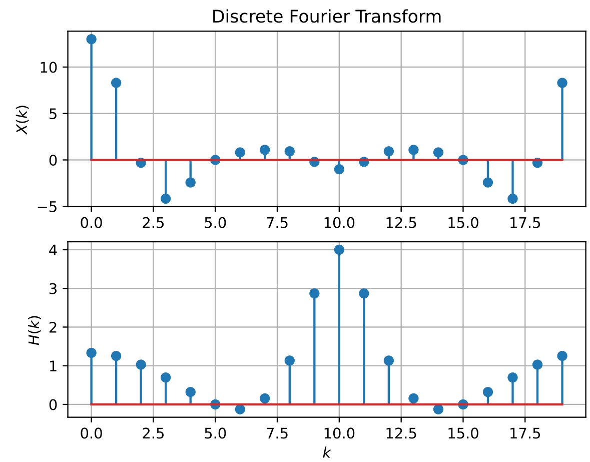

Fig. 6.1: Discrete Fourier Transform

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig.6.2

```
wget https://github.com/tj-devil/EE3900
    -2022/blob/main/codes/prodZ.py
```

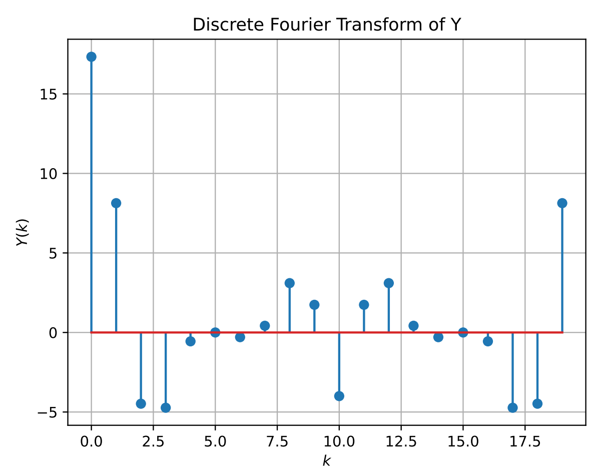

Fig. 6.2: Discret Fourier Transform of $Y(k)$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. (**??**) and computes $X(k)$ and $Y(k)$. Note that this is the same as $y(n)$ in Fig. (3.2). Download the code using

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/yndft.py
```

and execute it using

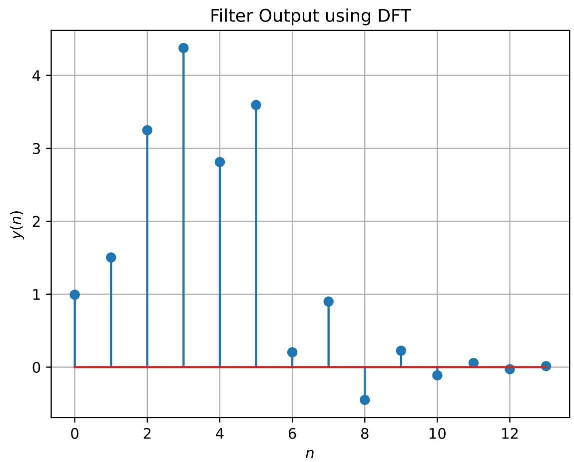

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the code from

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/ynifft.py
```

The values of $y(n)$ using all the three methods have been plotted on one stem plot for convenience. Note that there is very little difference in the values of $y(n)$.

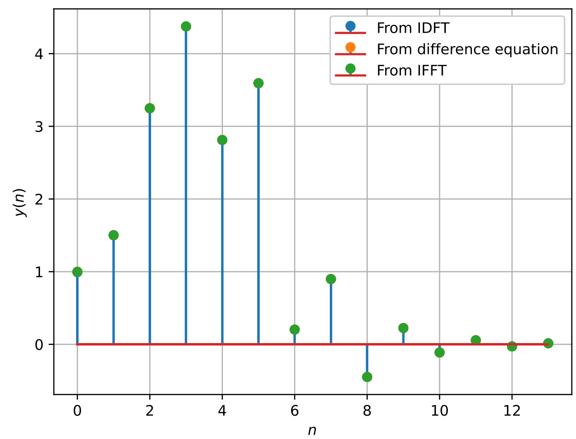

Fig. 6.4: $y(n)$ using FFT and IFFT

7 FFT

7.1 The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

7.2 Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\vec{F}_N = [W_N^{mn}], \quad 0 \leq m, n \leq N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\vec{F}_N$.

7.3 Let

$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$

7.4 The 4 point *DFT diagonal matrix* is defined as

$$\vec{D}_4 = diag \begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

7.5 Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N = e^{-j2\pi/N} \tag{7.8}$$
$$W_{N/2} = e^{-j2\pi*2/N} \tag{7.9}$$
$$W_{N/2} = \left(e^{-j2\pi/N}\right)^2 \tag{7.10}$$
$$W_{N/2} = W_{N/2}^2 \tag{7.11}$$
$$W_N^2 = W_{N/2} \tag{7.12}$$

7.6 Show that

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.13}$$

Solution: Observe that for $n \in \mathbb{N}$, $W_4^{4n} = 1$ and $W_4^{4n+2} = -1$. Using (7.7),

$$\vec{D}_2\vec{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.14}$$
$$= \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.15}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.16}$$
$$\implies -\vec{D}_2\vec{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.17}$$

and

$$\vec{F}_2 = \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \tag{7.18}$$
$$= \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{pmatrix} \tag{7.19}$$

Hence,

$$\vec{W}_4 = \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{pmatrix} \tag{7.20}$$
$$= \begin{bmatrix} \vec{I}_2\vec{F}_2 & \vec{D}_2F_2 \\ \vec{I}_2\vec{F}_2 & -\vec{D}_2F_2 \end{bmatrix} \tag{7.21}$$
$$= \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & \vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \tag{7.22}$$

Multiplying (7.22) by $\vec{P}_4$ on both sides, and noting that $\vec{W}_4\vec{P}_4 = \vec{F}_4$ gives us (7.13).

7.7 Show that

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.23}$$

Solution: Observe that for even N and letting $\vec{f}_N^i$ denote the i^{th} column of $\vec{F}_N$, from (7.16) and (7.17),

$$\begin{pmatrix} \vec{D}_{N/2}\vec{F}_{N/2} \\ -\vec{D}_{N/2}\vec{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \vec{f}_N^2 & \vec{f}_N^4 & \dots & \vec{f}_N^N \end{pmatrix} \tag{7.24}$$

and

$$\begin{pmatrix} \vec{I}_{N/2}\vec{F}_{N/2} \\ \vec{I}_{N/2}\vec{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \vec{f}_N^1 & \vec{f}_N^3 & \dots & \vec{f}_N^{N-1} \end{pmatrix} \tag{7.25}$$

Thus,

$$\begin{bmatrix} \vec{I}_2\vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{I}_2\vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix}$$
$$= \begin{pmatrix} \vec{f}_N^1 & \dots & \vec{f}_N^{N-1} & \vec{f}_N^2 & \dots & \vec{f}_N^N \end{pmatrix} \tag{7.26}$$

and so,

$$\begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N$$
$$= \begin{pmatrix} \vec{f}_N^1 & \vec{f}_N^2 & \dots & \vec{f}_N^N \end{pmatrix} = \vec{F}_N \tag{7.27}$$

7.8 Find

$$\vec{P}_4\vec{x} \tag{7.28}$$

Solution: We have,

$$\vec{P}_4\vec{x} = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix} = \begin{pmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{pmatrix} \tag{7.29}$$

7.9 Show that

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.30}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Writing the terms of X,

$$X(0) = x(0) + x(1) + \ldots + x(N-1) \tag{7.31}$$
$$X(1) = x(0) + x(1)e^{-\frac{j2\pi}{N}} + \ldots + \\ + x(N-1)e^{-\frac{j2(N-1)\pi}{N}} \tag{7.32}$$
$$\vdots$$
$$X(N-1) = x(0) + x(1)e^{-\frac{j2(N-1)\pi}{N}} + \ldots + \\ + x(N-1)e^{-\frac{j2(N-1)(N-1)\pi}{N}} \tag{7.33}$$

Clearly, the term in the m^{th} row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{\jmath 2mn\pi}{N}} \tag{7.34}$$

and so, we can represent each of these terms as a matrix product

$$\vec{X} = \vec{F}_N \vec{x} \tag{7.35}$$

where $\vec{F}_N = \left[e^{-\frac{-\jmath 2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

7.10 Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.37}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.38}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.39}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.40}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.41}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.42}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.43}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.44}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.46}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.47}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.48}$$

Solution: We write out the values of performing an 8-point FFT on $\vec{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n)e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.49}$$

$$= \sum_{n=0}^{3} \left(x(2n)e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x(2n+1)e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.50}$$

$$= X_1(k) + e^{-\frac{\jmath 2k\pi}{4}} X_2(k) \tag{7.51}$$

where $\vec{X}_1$ is the 4-point FFT of the even-numbered terms and $\vec{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \ge 4$,

$$X_1(k) = X_1(k-4) \tag{7.52}$$

$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.53}$$

we can now write out $X(k)$ in matrix form as in (7.36) and (7.37). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n)e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.54}$$

$$= \sum_{n=0}^{1} \left(x_1(2n)e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1)e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.55}$$

$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \tag{7.56}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.57}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.58}$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.60}$$

But observe that from (7.29),

$$\vec{P}_8\vec{x} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix} \tag{7.61}$$

$$\vec{P}_4\vec{x}_1 = \begin{pmatrix} \vec{x}_3 \\ \vec{x}_4 \end{pmatrix} \tag{7.62}$$

$$\vec{P}_4\vec{x}_2 = \begin{pmatrix} \vec{x}_5 \\ \vec{x}_6 \end{pmatrix} \tag{7.63}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

7.11 For

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.64}$$

compute the DFT using (7.30)
Solution: Download the Python code from

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/7.11.py
```

7.12 Repeat the above exercise using the FFT after zero padding $\vec{x}$.

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/7.1.py
```

7.13 Write a C program to compute the 8-point FFT.
Solution: The C code for the above two problems can be downloaded from

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/7.13.c
```

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: Download the source code by typing the next command

```
https://github.com/tj−devil/EE3900−2022/blob
    /main/codes/8.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b.
Solution: For the given values, the difference equation is

$$\begin{aligned} &y(n) - (2.52)\, y(n-1) + (2.56)\, y(n-2) \\ &- (1.21)\, y(n-3) + (0.22)\, y(n-4) \\ &= \left(3.45 \times 10^{-3}\right) x(n) + \left(1.38 \times 10^{-2}\right) x(n-1) \\ &+ \left(2.07 \times 10^{-2}\right) x(n-2) + \left(1.38 \times 10^{-2}\right) x(n-3) \\ &+ \left(3.45 \times 10^{-3}\right) x(n-4) \end{aligned} \tag{8.2}$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \tag{8.3}$$

$$= \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.4}$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.19),

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n-j) \tag{8.5}$$

Substituting the values,

$$\begin{aligned} h(n) &= [(-0.24 - 0.71\jmath)(0.56 + 0.14\jmath)^n \\ &+ (-0.24 + 0.71\jmath)(0.56 - 0.14\jmath)^n \\ &+ (-0.25 + 0.12\jmath)(0.70 + 0.41\jmath)^n \\ &+ (-0.25 - 0.12\jmath)(0.70 - 0.41\jmath)^n]u(n) \\ &+ \left(1.6 \times 10^{-2}\right)\delta(n) \end{aligned} \tag{8.6}$$

$$\begin{aligned} \implies h(n) &= (1.5)(0.58)^n \cos(n\alpha_1 + \beta_1) \\ &+ (0.55)(0.81)^n \cos(n\alpha_2 + \beta_2) \\ &+ \left(1.6 \times 10^{-2}\right)\delta(n) \end{aligned} \tag{8.7}$$

where

$$\tan\alpha_1 = 0.25 \tag{8.8}$$
$$\tan\beta_1 = 2.96 \tag{8.9}$$
$$\tan\alpha_2 = 0.59 \tag{8.10}$$
$$\tan\beta_2 = -0.48 \tag{8.11}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/8.2.1.py
```

The filter frequency response is plotted at

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/8.2.2.py
```

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We observe that for all i, $|p(i)| < 1$ and so, as $h(n)$ is the sum of many convergent series, we see that $h(n)$ converges and is bounded. From (4.1),

$$\sum_{n=0}^{\infty} h(n) = H(1) = \frac{\sum_{k=0}^{N} b(k)}{\sum_{k=0}^{M} a(k)} = 1 < \infty \tag{8.12}$$

Therefore, the system is stable. From Fig. (8.2), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (8.2).

```
wget https://github.com/tj−devil/EE3900
    −2022/blob/main/codes/8.2.3.py
```

The codes can be run all at once by typing a small shell script

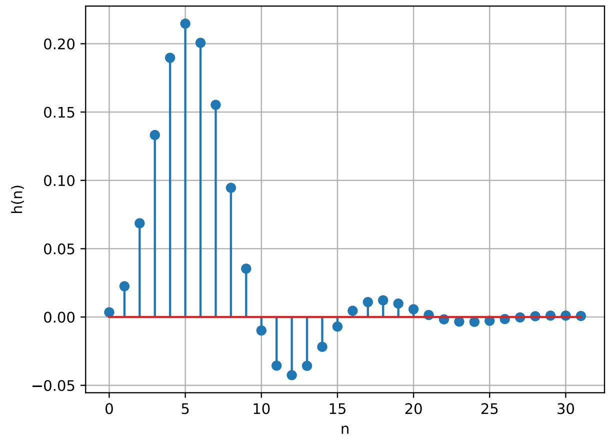

Fig. 8.2: Plot of $h(n)$

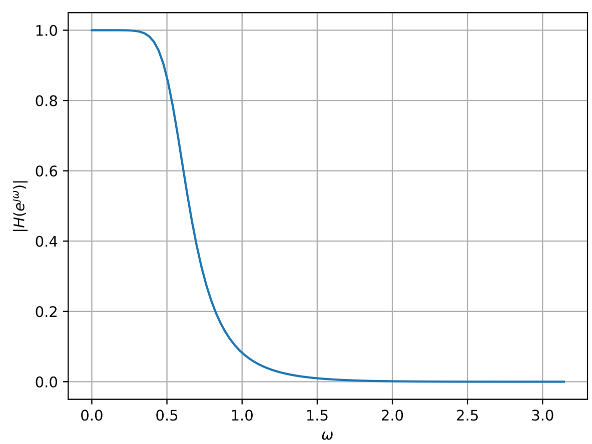

Fig. 8.2: Filter frequency response

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: A better filtering was found on setting the order of the filter to be 7.

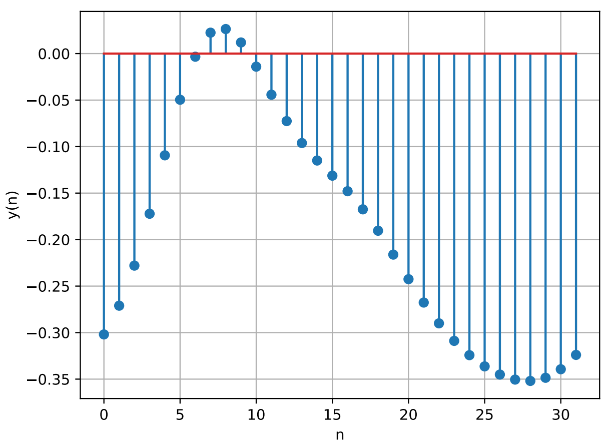

Fig. 8.2: Plot of $y(n)$